Interlacements Essentials and Cover Time of Random Walks

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0 INTRODUCTION

Random walk is a stochastic process formed by successive independent, identically distributed random variables. We studied the vacant set left by some random walks on a discrete torus $(\mathbb{Z}/N\mathbb{Z})^d$ until it runs up to uN^d times, where u is a non-negative parameter that measures number of trajectories enter into the picture. The model of random interlacements is introduced by A.-S. Sznitman in the seminal paper [1]. The aim of this paper is to summarize the basic definitions and properties of random walk and random interlacements and their vacant set, and also investigate another variation of simple random walk, namely *element random walk*.

In chapter 1, we prepare a series of concepts and theorems for discrete-time stochastic processes that are essential for the discussion of random walks and random interlacements. In chapter 2, we define simple random walks on $\mathbb{Z}^d$ and look at its key properties. In chapter 3, we introduce the model of random interlacements and learn its important trait with respect to discrete lattice shifts. In chapter 4 and 5, we limit our discussion to a discrete torus in $\mathbb{Z}^d$. In chapter 4, we construct a connection between random walks and random interlacements. And finally in chapter 5, we investigate into elephant random walk and give some conjectures for its cover time.

1 SOME BASIC DEFINITIONS

Definition 1.1 (Basic concepts for discrete-time stochastic processes).

- A discrete-time real-valued *stochastic process* is a sequence of random variables $X_n : (\Omega, \mathscr{F}) \to (\mathbb{R}, \mathscr{B})$ indexed by $n \in \mathbb{Z}_+$, where $\mathscr{B}$ is the Borel σ-algebra generated on Ω. We write such sequences as $(X_n, n \geq 0)$, with the understanding that the time index n is always an integer.
- A *filtration* is a sequence of sub σ-algebras $(\mathscr{F}_n, n \geq 0)$ such that $\mathscr{F}_n \subseteq \mathscr{F}_{n+1} \subseteq \mathscr{F}$ for all $n \geq 0$.
- For a set $K \subset\subset \mathbb{Z}^d$, if the cardinality of K is less than ∞

$$\partial_{ext} K := \{x \in \mathbb{Z}^d \setminus K \ : \exists y \in K \ s.t. \mid x - y \mid_1 = 1\} \tag{1.1}$$

  is the *exterior boundary* of K. Here, $| \cdot |_1$ denotes the the 1-norm of $\cdot$ in $\mathbb{R}^d$. Any two vertices x, $y \in \mathbb{Z}^d$ are called *nearest neighbors* (in $\mathbb{Z}^d$) if $\mid x - y \mid_1 = 1$. Also,

$$\partial_{int} K := \{x \in K \ : \exists y \in \mathbb{Z}^d \setminus K \ s.t. \mid x - y \mid_1 = 1\} \tag{1.2}$$

  is the *interior boundary* of K.
- A random variable $\tau \in \{\mathbb{Z}_+ \cup \infty\}$ is a *stopping time* with respect to a filtration $(\mathscr{F}_n, n \geq 0)$ if $\{\tau = n\} \in \mathscr{F}_n$ for all $n \geq 0$.

Definition 1.2 (Markov chains).

- A process X_n is a *Markov chain* if, for any $y \in A$, A denotes a countable set, take any $n \geq 0$ and $m \geq 1$,

$$\mathbf{P}[X_{n+m} = y \mid X_0, \ldots X_n] = \mathbf{P}[X_{n+m} = y \mid X_n] \ a.s. \tag{1.3}$$

In words, a Markov chain process is a stochastic process that the probability for it to transit into any future state does not require any information about its previous trajectory, but depends solely on its current state.

- *Strong Markov property:* Let X_n be a Markov chain and τ be a stopping time with respect the the natural filtration of X_n, then for all x, $y_1, \ldots, y_k$ in a countable set A, it holds that

$$\mathbf{P}[X_{\tau+n_j} = y_j,\ j = 1, \ldots, k \mid \mathscr{F}_\tau,\ X_\tau = x] = \mathbf{P}_x[X_{\tau+n_j} = y_j,\ j = 1, \ldots, k] \quad (1.4)$$

2 RANDOM WALKS

In terminology of [2], we first define simple random walk and discuss its properties.

2.1 Simple Random Walk

Consider the measurable space $(\overline{RW}, \overline{\mathscr{RW}})$, where $\overline{RW}$ is the set of infinite $\mathbb{Z}^d$-valued sequences $w = (w_n)_{n \geq 0}$ and $\overline{\mathscr{RW}}$ is the σ-algebra generated on $\overline{RW}$ by coordinate maps $X_n : \overline{RW} \to \mathbb{Z}^d$, $X_n(w) = w_n$.
For $x \in \mathbb{Z}^d$, we consider the probability measure $\mathbf{P}_x$ on $(\overline{RW}, \overline{\mathscr{RW}})$ such that under $\mathbf{P}_x$, the random sequence $(X_n)_{n \geq 0}$ is a Markov chain on $\mathbb{Z}^d$ that starts at x and has transition probabilities

$$\mathbf{P}_x[X_{n+1} = y' \mid X_n = y] := \begin{cases} \dfrac{1}{2d}, & if \mid y - y' \mid_1 = 1 \\ 0, & otherwise \end{cases} \quad (2.1)$$

for all y, $y' \in \mathbb{Z}^d$. In words, X_{n+1} has the uniform distribution over the

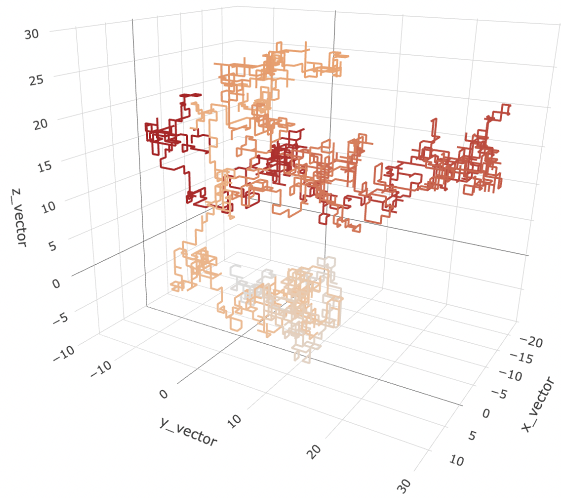


Figure 1: Simple Random Walk on Three Dimensions

nearest neighbors of X_n and reaches other vertices at zero probability. The random sequence $(X_n)_{n \geq 0}$ on $(\overline{RW}, \overline{\mathscr{RW}}, \mathbf{P})$ is called *simple random walk*

(SRW) on $\mathbb{Z}^d$ starting in x. Example of a simple random walk on three-dimensions is shown in Figure 1, which the color change represents time change respectively.

Now we define several key random times that is essential to our following discussion.

Definition 2.1.1 (entrance time, hitting time, exit time, last visit time).

- First Entrance Time:

$$H_A(w) := \inf\{n \geq 0 : X_n(w) \in A\} \tag{2.2}$$

- First hitting time:

$$\tilde{H}_A(w) := \inf\{n \geq 1 : X_n(w) \in A\} \tag{2.3}$$

- First exit time:

$$T_A(w) := \inf\{n \geq 0 : X_n(w) \notin A\} \tag{2.4}$$

- Last visit time:

$$L_A(w) := \sup\{n \geq 0 : X_n(w) \in A\} \tag{2.5}$$

Remark 2.1 If a random walk starts in $\partial_{int}K$ and leaves the set K, i.e. $X_0(w) \in \partial_{int}K$ and $X_1(w) \in \partial_{ext}K$, and it entries the set again at time $\tilde{t}$, its first entrance time $H_A(w)$ will be 0, while its first hitting time $\tilde{H}_A(w)$ will be $\tilde{t}$.

Remark 2.2 H_A is a stopping times with respect to a canonical filtration $\mathscr{F}_n$ since

$$\begin{aligned}\{H_A = n\} &= \{X_0 \notin K\} \cap \{X_1 \notin K\} \cap \cdots \cap \{X_{n-1} \notin K\} \cap \{X_n \in K\} \\ &\in \sigma(\{X_0,\ X_1, \ldots,\ X_n\})\end{aligned}$$

By similar proof, one can easily check that $\tilde{H}_A$, and T_A are also stopping times with respect to $\mathscr{F}_n$.

2.2 Green function

Definition 2.2.1 (Green function) A *Green function* of a simple random walk on $\mathbb{Z}^d$ is defined as

$$g(x,\ y) = \sum_{n\geq 0} \mathbf{P}_x[X_n = y] = \mathbf{E}_x\left[\sum_{n\geq 0} \mathbb{1}_{\{X_n = y\}}\right],\ x,y \in \mathbb{Z}^d. \tag{2.6}$$

Remark 2.3 By our definition, $g(x,\ y) > 1$ in all dimensions. Informally speaking, in the case $x = y$, we still count it as one visit. A Green function measures the expected number of SRW that starts in x and ends in y. By symmetry and translation invariance, it holds that $g(x,\ y) = g(y,\ x) = g(0,\ y - x)$. Thus we use the notation $g(x) := g(0,x)$ for simplicity.

The notion of Green function is closely related to an essential notion of SRW on transience and recurrence.

Definition 2.2.2 A simple random walk is *transient* if $\mathbf{P}_0[\tilde{H}_0 = \infty] > 0$, and is called *recurrent* otherwise.

In words, a simple random walk is transient if it has positive probability that never returns to its starting location, otherwise it is recurrent. Here we introduce a fundamental theorem of SRWs on intger lattices from George Pólya's result in [3].

Theorem 2.1 *SRW on $\mathbb{Z}^d$ is recurrent if $d \leq 2$ and transient if $d > 2$.*

In order to understand the notions of transience and recurrence better, we check one of the properties of the Green function.

Lemma 2.1 $g(0) = \mathbf{P}_0[\tilde{H}_0 = \infty]^{-1}$, in particular, SRW is transient if and only if $g(0) < \infty$.

Proof. The main idea is to employ *the last-visit decomposition*. For a set $K \subset\subset \mathbb{Z}^d$, for $i \in \mathbb{N}$, we denote $\theta_i(\omega)$ to be a segment of the random walk that start at time $t = i$. Also, let $H_K^{(i)}$ be the i-th entrance of the random walk into a set K. By using this notation, we can decompose entrance time and exit time. Thus,

$$H_K^{(k)} = H_K^{(k-1)} + \tilde{H}_0 \circ \theta_{H_K^{(k-1)}} \tag{2.7}$$

We first consider the case when $d \leq 2$. Using Theorem 3.1, we know SRW is transient, thus

$$\mathbf{P}[\tilde{H}_0 = \infty] = 0 \tag{2.8}$$

It suffices to prove

$$g(0) = \sum_{n=0}^{\infty} \mathbf{P}_0[X_n = 0] = \mathbf{E}_0\left[\sum_{n=0}^{\infty} \mathbb{1}_{\{X_n=0\}}\right] = \infty \tag{2.9}$$

Applying strong Markov property as addressed in 1.4, we can deduce

$$\begin{aligned}
\mathbf{P}_0[H_0^{(2)} < \infty] &= \mathbf{P}_0[\tilde{H}_0^{(1)} < \infty,\ \tilde{H}_0^{(1)} + \tilde{H}_0 \circ \theta_{\tilde{H}_0^{(1)}} < \infty] \\
&= \mathbf{P}_0[\tilde{H}_0 < \infty,\ \tilde{H}_0 \circ \theta_{\tilde{H}_0} < \infty] \\
&= \mathbf{E}_0\left[\mathbb{1}_{\{\tilde{H}_0 < \infty,\ \tilde{H}_0 \circ \theta_{\tilde{H}_0} < \infty\}}\right] \\
&= \mathbf{E}_0\left[\mathbf{E}_0[\mathbb{1}_{\{\tilde{H}_0 < \infty,\ \tilde{H}_0 \circ \theta_{\tilde{H}_0} < \infty\}} \mid \mathscr{F}_{\tilde{H}_0}]\right] \\
&= \mathbf{E}_0\left[\tilde{H}_0 < \infty,\ \mathbf{E}_{X_{\tilde{H}_0}}[\mathbb{1}_{\tilde{H}_0 < \infty}]\right]
\end{aligned}$$

Note that $\mathbf{E}_{X_{\tilde{H}_0}}[\mathbb{1}_{\tilde{H}_0 < \infty}] = \mathbf{P}_0[\tilde{H}_0 < \infty] = 1$, we know

$$\mathbf{P}_0[H_0^{(2)} < \infty] = \mathbf{P}_0[\tilde{H}_0 < \infty] = 1 \tag{2.10}$$

By mathematics induction, we conclude $\tilde{H}_0 < \infty$, $\tilde{H}_0^{(2)} < \infty$, $\tilde{H}_0^{(3)} < \infty, \ldots,$ $\tilde{H}_0^{(n)} < \infty$ $\mathbf{P}_0$-almost surely. So,

$$\sum_{n=0}^{\infty} \mathbb{1}_{\{X_n=0\}} = \sum_{k=1}^{\infty} \mathbb{1}_{\{H_0^{(k)} < \infty\}} + 1 \quad \mathbf{P}_0 \text{ - a.s.} \tag{2.11}$$

$$\sum_{k=1}^{\infty} \mathbb{1}_{\{H_0^{(k)} < \infty\}} \xrightarrow{\mathbf{P}_0 - a.s.} \infty \tag{2.12}$$

Thus we conclude $g(0) = \infty$.
Now we prove the other direction, consider the transient case. Again using (1.4), we get

$$\begin{aligned}
\mathbf{P}_0\left[\sum_{k=1}^{\infty} \mathbb{1}_{\{H_0^{(k)}<\infty\}} = \eta\right] &= \mathbf{P}_0\left[H_0^{(1)} < \infty,\, H_0^{(2)} < \infty, \ldots, H_0^{(n)} < \infty,\, H_0^{(n+1)} = \infty\right] \\
&= \mathbf{E}_0\left[\mathbf{E}_0\left[\mathbb{1}_{\{H_0^{(1)}<\infty,\ldots,H_0^{(n)}<\infty,\, H_0^{(n+1)}=\infty\}}\right]\right] \\
&= \mathbf{E}_0\left[\mathbf{E}_0\left[\mathbb{1}_{\{H_0^{(1)}<\infty,\ldots,H_0^{(n)}<\infty\}} \cdot \mathbb{1}_{\{\tilde{H}_0^{(n+1)}=\infty\}} \circ \theta_{H_0^{(n)}} \mid \mathscr{F}_{H_0^{(n)}}\right]\right] \\
&= \mathbf{P}_0[\tilde{H}_0 = \infty] \cdot \mathbf{P}_0[H_0^{(1)} < \infty, \ldots, H_0^{(n)} < \infty] \\
&= \mathbf{P}_0[\tilde{H}_0 = \infty] \cdot (1 - \mathbf{P}_0[\tilde{H}_0 = \infty])
\end{aligned}$$

Note that

$$\sum_{k=1}^{\infty} \mathbb{1}_{\{H_0^{(k)}<\infty\}} \sim Geom(\mathbf{P}_0[\tilde{H}_0 = \infty]) \tag{2.13}$$

$$g(0) = \mathbf{E}_0\left[\sum_{n=0}^{\infty} \mathbb{1}_{\{X_n=0\}}\right] = \mathbf{E}_0\left[\sum_{k=1}^{\infty} \mathbb{1}_{\{H_0^{(k)}<\infty\}} + 1\right] = \frac{1}{\mathbf{P}_0[\tilde{H}_0 = \infty]} < \infty \tag{2.14}$$

□

2.3 Equilibrium Measure and Capacity

We now focus on the case when dimension d is greater or equal to three, i.e. the random walk is transient. Fix a set $K \subset\subset \mathbb{Z}^d$ and $x \in \mathbb{Z}^d$, we can define the equilibrium measure and the capacity of K.

Definition 2.3.1 (equlibrim measure) For $K \subset\subset \mathbb{Z}^d$ and $x \in \mathbb{Z}^d$,

$$e_K(x) := \mathbf{P}_x[\tilde{H}_K = \infty] \cdot \mathbb{1}_{x\in K} = \mathbf{P}_x[L_K = 0] \cdot \mathbb{1}_{x\in K} \tag{2.9}$$

Definition 2.3.2 (capacity) The total mass of $e_K(x)$ for all $x \in K$ is called the *capacity* of K.

$$cap(K) := \sum_{x\in K} e_K(x) = e_K(K) = e_K(\mathbb{Z}^d) \tag{2.10}$$

Intuitively, $e_K(x)$ is not trivial when x lives in the interior boundary of K, $\partial_{int}K$. The following definition of normalized equilibrium measure gives us a probability measure on K.

Definition 2.3.3 (normalized equilibrium measure, also harmonic measure with respect to K) For any $K \subset\subset \mathbb{Z}^d$,

$$\tilde{e}_K(x) := \frac{e_K(x)}{cap(K)} \tag{2.11}$$

For more properties about simple random walk on $\mathbb{Z}^d$, one could read Lawler's book [4].

2.4 Lazy Random Walks

One more definition that will be essential to proof in Chapter 4 is *lazy random walks*, slightly modified from simple random walks. Simple random walks are periodic due to its construction. In fact, (X_{2i+1}) and (X_{2i}) will be disjoint subsets of $\mathbb{Z}^d$, $i \in \mathbb{N}_0$. The lazy random walks $(Y_n)_{n\geq 0}$ is defined as the Markov chain which has probability 0.5 to stay in its current state. Thus,

$$Y_{n+1} = \begin{cases} y_n + e_i, & \text{at probability } \dfrac{1}{4} \\ y_n - e_i, & \text{at probability } \dfrac{1}{4} \\ y_n, & \text{at probability } \dfrac{1}{2} \end{cases} \tag{2.12}$$

where e_i is selected uniformly at random from the standard basis of $\mathbb{Z}^d$, namely $(e_i)_{i=i}^d$.

It's also beneficial to construct lazy random walks directly from simple random walks. Define the sequence of $(\xi_n)_{n\geq 1}$ be i.i.d independent random variables such that $\xi_n \sim Ber(\frac{1}{2})$ with values in $\{0,1\}$. Then, let $S_0 = 0$, $S_n = \sum_{i=1}^n \xi_i$ for $n \geq 1$. $Y_n := X_{S_t}$ is the induced lazy random walk where S_n traces the number of steps when the walk changes it position.

The aim of introducing the concept of lazy random walks is to avoid periodicity of simple random walk. An important property of lazy random walk is its aperiodicity, starting from any position converges to the uniform measure on vertices of $(\mathbb{Z}/N\mathbb{Z})^d$. By the law of large numbers, a lazy random walk changes its position about $\frac{n}{2}$ times as $n \to \infty$. We will further discuss that in Chapter 4.

3 RANDOM INTERLACEMENT: BASIC DEFINITIONS AND PROPERTIES

We will start by introducing the random interlacement model in high dimensions $(d \geq 3)$ at level $u > 0$ as a random subset of $\mathbb{Z}^d$. Random interlacement in two dimensions are defined differently due to recurrence of SRW on two dimensions by Theorem 3.1.

3.1 Space of Subsets of $\mathbb{Z}^d$

In order to define random interlacements, we first need to put the one-to-one correspondence between the space $\{0,1\}^{\mathbb{Z}^d} (d \geq 3)$ and the subsets of $\mathbb{Z}^d$ in words. For every $\xi \in \{0,1\}^{\mathbb{Z}^d}$, the corresponding subset of $\mathbb{Z}^d$ is defined by

$$\mathscr{I}(\xi) = \{x \in \mathbb{Z}^d : \ \xi_x = 1\}$$

The process is to construct a configuration by randomly assigning zeros and ones to each element of $\mathbb{Z}^d$. A concrete example for a ξ in $\{0,1\}^{\mathbb{Z}^d}$ could be like Figure 2. Thus we can then define coordinate maps, local event and cylinder event.

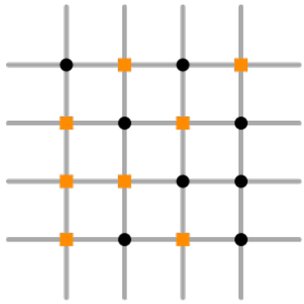


Figure 2: An example of an element in $\{0,1\}^{\mathbb{Z}^d}$

Definition 3.1.1 (coordinate map) Consider the space $\{0,1\}^{\mathbb{Z}^d}$ as the space of subsets of $\mathbb{Z}^d$. For $x \in \mathbb{Z}^d$ and $\xi \in \{0,1\}^{\mathbb{Z}^d}$, the function

$$\psi_x : \{0,1\}^{\mathbb{Z}^d} \to \{0,1\}$$
$$\psi_x(\xi) = \xi_x$$

are called *coordinate maps*.

Definition 3.1.2 (local event, cylinder event) For $K \subset \mathbb{Z}^d$, we denote by $\sigma(\psi_x,\ x \in K)$ the sigma-algebra on the space $\{0,1\}^{\mathbb{Z}^d}$ generated by the coordinate maps $\psi_x,\ x \in K$, and we define $\mathscr{F} = \sigma(\psi_x,\ x \in K)$.
If $K \subset\subset \mathbb{Z}^d$ and $A \in \sigma(\psi_x,\ x \in K)$, then we say that A is a *local event* with support K.
For any $K_0 \subseteq K \subset\subset \mathbb{Z}^d$, $K_1 = K \setminus K_0$, we say that

$$\{\forall x \in K_0 :\ \psi_x = 0;\ \forall x \in K_1 :\ \psi_x = 1\} = \{\mathscr{I} \cap K = K_1\} \tag{3.1}$$

is a *cylinder event* with base K.

Remark 3.1 The events $\{\psi_x = 1\}$, $x \in \mathbb{Z}^d$ could be considered as "building blocks" of any local event. A Local event with support K does not require every $x \in K$ to be mapped, but a cylinder event with base K precisely assigned every $x \in K$ by ψ_x to either 0 or 1. Thus, every local event is a finite disjoint union of cylinder events. For any $K \subset\subset \mathbb{Z}^d$, the sigma-algebra generated on ψ_x is atomic and has exactly $2^{|K|}$ atoms of form (3.1).

Example 3.1 Consider $K = \{(0,0),(0,1),(1,0),(1,1)\}$ and the local event $A = \{\psi_{(0,0)} = 1, \psi_{(1,0)} = 0\} \in \sigma(\psi_x, x \in K)$, we want to represent this event by cylinder events. Notice that $(1,1)$ and $(0,1)$ are not mapped to a specific value,

$$\begin{aligned}
A = &\left\{\psi_{(0,0)} = 1, \psi_{(1,0)} = 0\right\} \cap \left(\left\{\psi_{(0,1)} = 1, \psi_{(1,1)} = 1\right\}\right.\\
&\cup \left\{\psi_{(0,1)} = 1, \psi_{(1,1)} = 0\right\} \cup \left\{\psi_{(0,1)} = 0, \psi_{(1,1)} = 1\right\}\\
&\left.\cup \left\{\psi_{(0,1)} = 0, \psi_{(1,1)} = 0\right\}\right)\\
= &\left\{\psi_{(0,0)} = 1, \psi_{(1,0)} = 0, \psi_{(0,1)} = 1, \psi_{(1,1)} = 1\right\}\\
&\cup \left\{\psi_{(0,0)} = 1, \psi_{(1,0)} = 0, \psi_{(0,1)} = 0, \psi_{(1,1)} = 1\right\}\\
&\cup \left\{\psi_{(0,0)} = 1, \psi_{(1,0)} = 0, \psi_{(0,1)} = 1, \psi_{(1,1)} = 0\right\}\\
&\cup \left\{\psi_{(0,0)} = 1, \psi_{(1,0)} = 0, \psi_{(0,1)} = 0, \psi_{(1,1)} = 0\right\}.
\end{aligned}$$

Clearly, A could be expressed explicitly as a disjoint union of cylinder events. Based on such a map, we are now able to discuss the law that brings up the definition of random interlacements.

3.2 Random Interlacements

Definition 3.2.1 (random interlacements) For $u > 0$, we consider the one-parameter family of probability measures $\mathscr{P}^u$ on $(\{0,1\}^{\mathbb{Z}^d}, \mathscr{F})$ such that

$$\mathscr{P}^u[\mathscr{I} \cap K = \varnothing] = e^{-u\text{cap}(K)} \tag{3.2}$$

The random subset $\mathscr{I}$ of $\mathbb{Z}^d$ in $(\{0,1\}^{\mathbb{Z}^d}, \mathscr{F}, \mathscr{P}^u)$ is called *random interlacements* at level u.

Remark 3.2 The existence of the probability measure $\mathscr{P}^u$ is not obvious. The important idea that we also employed in later simulation is, the measure $\mathscr{P}^u$ also arises as the local limit of the trace of the first $\lfloor uN^d \rfloor$ steps of simple random walk with a uniform starting point on the d-dimensional torus $(\mathbb{Z}/N\mathbb{Z})^d$. One can formulate that construction by employing a random interlacement point process for segments of trajectories. Namely, it is a process of randomly dropping segments of trajectories into the space of a discrete torus.

The explicit expression for the probabilities of cylinder evensts is given by: for any $K_0 \subseteq K \subset\subset \mathbb{Z}^d$, $K_1 = K \setminus K_0$,

$$\mathscr{P}^u[\psi\,|_{K_0} \equiv 0, \psi\,|_{K_1} \equiv 1] = \mathscr{P}^u[\mathscr{I} \cap K = K_1] = \sum_{K' \subseteq K_1} (-1)^{|K'|} e^{-u\text{cap}(K_0 \cup K')} \tag{3.3}$$

Proof. The idea is to use the inclusion-exclusion formula.

$$\begin{aligned}
\{\mathscr{I} \cap K = K_1\} &= \{\mathscr{I} \cap K_0 = \varnothing\} \cap \bigcap_{x \in K_1} \{\mathscr{I} \cap \{x\} = \varnothing\}^c \\
&= \{\mathscr{I} \cap K_0 = \varnothing\} \setminus \left(\bigcup_{x \in K_1} \{\mathscr{I} \cap \{x\} = \varnothing\} \right)
\end{aligned}$$

Thus,

$$\begin{aligned}
\mathscr{P}^u[\{\mathscr{I} \cap K = K_1] &= \mathscr{P}^u \left[\{\mathscr{I} \cap K_0 = \varnothing\} \setminus \left(\bigcup_{x \in K_1} \{\mathscr{I} \cap \{x\} = \varnothing\} \right) \right] \\
&= \mathscr{P}^u[\mathscr{I} \cap K_0 = \varnothing] - \\
&\quad \mathscr{P}^u \left[\{\mathscr{I} \cap K_0 = \varnothing\} \cap \left(\bigcup_{x \in K_1} \{\mathscr{I} \cap \{x\} = \varnothing\} \right) \right] \\
&= \mathscr{P}^u[\mathscr{I} \cap K_0 = \varnothing] - \\
&\quad \mathscr{P}^u \left[\bigcup_{x \in K_1} \{\mathscr{I} \cap (K_0 \cup \{x\}) = \varnothing\} \right]
\end{aligned} \tag{3.4}$$

Here we can employ the inclusion-exclusion formula,

$$\begin{aligned}
&\mathscr{P}^u\left[\bigcup_{x\in K_1}\{\mathscr{I}\cap(K_0\cup\{x\})=\varnothing\}\right]\\
&=\sum_{J\neq\varnothing\subseteq K_1}(-1)^{|J|+1}\cdot\mathscr{P}^u\left[\bigcap_{x\in J}\{\mathscr{I}\cap(K_0\cup\{x\})=\varnothing\}\right]\\
&=\sum_{J\neq\varnothing\subseteq K_1}(-1)^{|J|+1}\cdot\mathscr{P}^u[\mathscr{I}\cap(K_0\cup J)=\varnothing] \qquad (3.5)
\end{aligned}$$

Thus from (3.4), we get

$$\begin{aligned}
\mathscr{P}^u[\{\mathscr{I}\cap K=K_1]&=e^{-u\mathrm{cap}(K_0)}+\sum_{K'\neq\varnothing\subseteq K_1}(-1)^{|K'|}e^{-u\mathrm{cap}(K_0\cup K')}\\
&=\sum_{K'\in K_1}(-1)^{|K|}e^{-u\mathrm{cap}(K_0\cup K')} \qquad (3.6)
\end{aligned}$$

□

3.3 Ergodicity

In this section, we'll explore the behavior of random interlacements under transformation. Finally, it's important to see that RI is ergodic with respect to lattice shifts.

Definition 3.3.1 (measurable) A transformation $T:\ \Omega\to\Omega$ is *measurable* if for any measurable set $A\in\mathscr{F}$, the preimage is again measurable, that is $T^{-1}(A)\in\mathscr{F}$.

Definition 3.3.2 (measure preserving) Let $\mathbf{P}$ be a probability measure on $(\Omega,\mathscr{F})$, a measure-preserving transformation T on $(\Omega,\mathscr{F},\mathbf{P})$ is an $\mathscr{F}$-measurable map $T:\ \Omega\to\Omega$ such that

$$\mathbf{P}[T^{-1}(A)]=\mathbf{P}[A] \text{ for all } A\in\mathscr{F} \qquad (3.7)$$

We also say that the transformation T preserves $\mathbf{P}$. If $\mathbf{P}$ satisfies (3.7), we say that the measure $\mathbf{P}$ is *invariant* under the transformation T. Such a measure-preserving transformation is called *ergodic* if all T-invariant events have $\mathbf{P}$-probability 0 or 1.
Now we want to check the measure-preserving transformation t_x on $(\{0,1\}^{\mathbb{Z}^d},\mathscr{F},\mathscr{P}^u)$ called canonical shift. RI under such transformation is ergodic. Ergodicity is important for us to further restrict our discussion on a finite subset in $\mathbb{Z}^d$ because it could be informally considered as a weaker statement of independence.

Definition 3.3.3 (canonical shift) A *canonical shift* is defined as

$$\begin{aligned}
&t_x:\ \{0,1\}^{\mathbb{Z}^d}\to\{0,1\}^{\mathbb{Z}^d}\\
&\psi_y(t_x(\xi))=\psi_{y+x}(\xi),\quad y\in\mathbb{Z}^d,\ \xi\in\{0,1\}^{\mathbb{Z}^d}
\end{aligned}$$

Also, for $K\subseteq\mathbb{Z}^d$, $K+x=\{y+x:\ y\in K\}$. It holds for any $x\in\mathbb{Z}^d$ and for any $u>0$, the transformation t_x preserves the measure $\mathscr{P}^u$. Finally, we reach the most essential theorem of this section, staking that random interlacements is ergodic with respect to the canonical shifts.

Theorem 3.1 *For any $u \geq 0$ and $0 \neq x \in \mathbb{Z}^d$, the measure-preserving transformation t_x is ergodic on* $(\{0,1\}^{\mathbb{Z}^d}, \mathscr{F}, \mathscr{P}^u)$. The proof of Theorem 3.1 is not obvious and could be found in Chapter two in [2].

Remark 3.3 The random interlacement model naturally reminds us of the traditional Bernoulli percolation model on $\mathbb{Z}^d$. Informally speaking, the two models could be viewed as a duality, where the traditional Bernoulli bond percolation defines the "open " and "closed" map on edges, and RI defines such maps on vertices.

4 RW AND RI ON A TORUS

In this section, we limit simple random walks on a discrete torus, $\mathbb{Z}_N^d = (\mathbb{Z}/N\mathbb{Z})^d$, focusing on higher dimensions when $d \geq 3$. Following the terminology of [2], we denote by $\varphi : \mathbb{Z} \to \mathbb{Z}_N^d$ for simplicity and avoiding dependence on N. Note that we use bold font instead of previous notation to distinguish vertices and subsets on the torus, i.e. $\mathbf{x} \in \mathbb{Z}_N^d$ and $\mathbf{K} \subset \mathbb{Z}_N^d$. By our notation, $(\varphi(X_n))_{n\geq 0} = (\mathbf{X}_t)_{t\geq 0}$ is the simple random walk on $\mathbb{Z}_N^d$ started from $\varphi(x)$.

4.1 Hitting of Subsets

An important result from [2] is that the local limit of the set of vertices in $\mathbb{Z}_N^d$ visited by the random walk up to time uN^d is given by random interlacement at level u, which brings us a connection between the two concepts.

Theorem 4.1 *For a given set $K \subset\subset \mathbb{Z}^d$,*

$$\lim_{N\to\infty} \mathbf{P}\left[\left\{\mathbf{X}_0, \ldots, \mathbf{X}_{\lfloor uN^d \rfloor}\right\} \cap \varphi(K) = \varnothing\right] = e^{-u\mathrm{cap}(K)}. \tag{4.1}$$

Recall from (3.2), the right hand side of (4.1) is exactly the probability that random interlacements at level u does not intersect K. This theorem gives us some information about if the simple random walk visits a subset of $\mathbb{Z}_N^d$ at time proportional to N^d. In fact, for $\delta \in (0, d)$, $N \geq 1$, and $n = \lfloor N^\delta \rfloor$,

$$\lim_{N\to\infty} \frac{N^d}{n} \cdot \mathbf{P}\left[\{\mathbf{X}_0, \ldots, \mathbf{X}_n\} \cap \varphi(K) \neq \varnothing\right] = \mathrm{cap}(K) \tag{4.2}$$

for any $K \subset\subset \mathbb{Z}^d$.

In words, we also have asymptotic formula for the probability that simple random walk visits a subset of $\mathbb{Z}_N^d$ after time much shorter than N^d. Based on these results, one can also deduce an asymptotic expression for the probability that lazy random walk visits a subset of $\mathbb{Z}_N^d$ in short time.

Lemma 4.1 For $\delta \in (0, d)$, $N \geq 1$, and $n = \lfloor N^\delta \rfloor$, for any $K \subset\subset \mathbb{Z}^d$,

$$\lim_{N\to\infty} \frac{N^d}{n} \cdot \mathbf{P}\left[\{\mathbf{Y}_0, \ldots, \mathbf{Y}_n\} \cap \varphi(K) \neq \varnothing\right] = \frac{1}{2} \cdot \mathrm{cap}(k) \tag{4.3}$$

Although we won't give the explicit proof here, we want to understand two key facts that are essential to the proof of (4.1).

i. Simple random walks is transient on $\mathbb{Z}^d$ on higher dimensions when $d \geq 3$.

ii. Lazy random walks on $\mathbb{Z}_N^d$ converges with high probability to its stationary measure in about $N^{2+\varepsilon}$ steps.

The first fact is introduced in Chapter 2. So we just want to understand the second fact regarding convergence of lazy random walks.

4.2 Mixing Property of Lazy Random Walk

Recall from Chapter 2 of the definitions of lazy random walks. Lazy random walks on a torus converge to a unique stationary distribution. In this section, we want to prove that after running for some time from t_1 to t_2, the trajectory of lazy randoms walks after t_2 becomes "nearly independent" with its trajectory before t_1, i.e. the mixing property of lazy random walks. First we define the error to analyze the convergence of lazy random walks to the uniform measure as the stationary distribution.

$$\varepsilon_n(N) := \sum_{\mathbf{y} \in \mathbb{Z}_N^d} \left| \mathbf{P}_0[\mathbf{Y}_n = \mathbf{y}] - \frac{1}{N^d} \right| \tag{4.4}$$

Remark 4.1 To check that lazy random walks converges to the uniform measure as stationary distribution, let π denote uniform distribution, μ a probability measure on $\mathbb{Z}_N^d$. $\pi(\mathbf{x})$ denotes the probability of a lazy random walk to start at $\mathbf{x}$. Note that for any $\mathbf{x} \in \mathbb{Z}_N^d$,

$$\begin{aligned}
\mathbf{P}_\mu[\mathbf{Y}_n = \mathbf{y}] &= \sum_{\mathbf{x} \in \mathbb{Z}_N^d} \mu(x) \cdot \mathbf{P}_\mathbf{x}[\mathbf{Y}_n = \mathbf{y}] = \frac{1}{N^d} \sum_{\mathbf{x} \in \mathbb{Z}_N^d} \mathbf{P}_\mathbf{x}[\mathbf{Y}_n = \mathbf{y}] \\
&= \sum_{\mathbf{x} \in \mathbb{Z}_N^d} \mathbf{P}_\pi[\mathbf{Y}_0 = \mathbf{x}] \mathbf{P}_\pi[\mathbf{Y}_n = \mathbf{y} \mid \mathbf{Y}_0 = \mathbf{x}] \\
&= \frac{1}{N^d} \sum_{\mathbf{x} \in \mathbb{Z}_N^d} \mathbf{P}_\pi[\mathbf{Y}_1 = \mathbf{y} \mid \mathbf{Y}_0 = \mathbf{x}] = \frac{1}{N^d}
\end{aligned}$$

Now we can state the mixing properties of lazy random walks.

Lemma 4.2 For $N \geq 1$, $1 \leq t_1 \leq t_2 \leq T$, $\mathscr{E}_1 \in \sigma(\mathbf{Y}_0, \ldots, \mathbf{Y}_{t_1})$ and $\mathscr{E}_2 \in \sigma(\mathbf{Y}_{t_2}, \ldots, \mathbf{Y}_T)$, we have

$$|\mathbf{P}[\mathscr{E}_1 \cap \mathscr{E}_2] - \mathbf{P}[\mathscr{E}_1] \cdot \mathbf{P}[\mathscr{E}_2]| \leq \varepsilon_{t_2 - t_1}(N) \tag{4.5}$$

Proof. The construction $\mathscr{E}_1 \in \sigma(\mathbf{Y}_0, \ldots, \mathbf{Y}_{t_1})$ here means $\mathscr{E}_1$ is an event depending only on the path up to time t_1. Similarly, $\mathscr{E}_2$ is an event depending only on the path after time t_2. For $x, y \in \mathbf{Z}_N^d$, define

$$f(\mathbf{x}) = \mathbf{P}[\mathscr{E}_1 \mid \mathbf{Y}_{t_1} = \mathbf{x}]$$

$$g(\mathbf{y}) = \mathbf{P}[\mathscr{E}_2 \mid \mathbf{Y}_{t_2} = \mathbf{y}]$$

It follows intuitively that

$$\mathbf{P}[\mathscr{E}_1] = \mathbf{E}\left[f(\mathbf{Y}_{t_1})\right] = \frac{1}{N^d} \cdot \sum_{x \in \mathbb{Z}_N^d} f(x)$$

$$\mathbf{P}[\mathscr{E}_2] = \mathbf{E}\left[g(\mathbf{Y}_{t_2})\right] = \frac{1}{N^d} \cdot \sum_{x \in \mathbb{Z}_N^d} g(x)$$

Thus,

$$\begin{aligned}\mathbf{P}[\mathscr{E}_1 \cap \mathscr{E}_2] &= \sum_{x \in \mathbb{Z}_N^d} \mathbf{P}[\mathbf{Y}_{t_1} = \mathbf{x}] \cdot \mathbf{P}[\mathscr{E}_1 \cap \mathscr{E}_2 \mid \mathbf{Y}_{t_1} = \mathbf{x}] \\ &= \frac{1}{N^d} \sum_{x \in \mathbb{Z}_N^d} \mathbf{P}[\mathscr{E}_1 \cap \mathscr{E}_2 \mid \mathbf{Y}_{t_1} = \mathbf{x}] \end{aligned} \tag{4.6}$$

Note that

$$\begin{aligned}\mathbf{P}[\mathscr{E}_1 \cap \mathscr{E}_2 \mid \mathbf{Y}_{t_1} = \mathbf{x}] &= \mathbf{E}\left[\mathbf{E}\left[\mathbb{1}_{\mathscr{E}_1} \cdot \mathbb{1}_{\mathscr{E}_2} \mid \mathbf{Y}_0, \dots, \mathbf{Y}_{t_1}\right] \mid \mathbf{Y}_{t_1} = \mathbf{x}\right] \\ &= \mathbf{E}\left[\mathbb{1}_{\mathscr{E}_1} \cdot \mathbf{E}\left[\mathbb{1}_{\mathscr{E}_2} \mid \mathbf{Y}_0, \dots, \mathbf{Y}_{t_1}\right] \mid \mathbf{Y}_{t_1} = \mathbf{x}\right] \\ &= \mathbf{E}[\mathbb{1}_{\mathscr{E}_1} \mid \mathbf{Y}_{t_1}] \cdot \mathbf{E}[\mathbb{1}_{\mathscr{E}_2} \mid \mathbf{Y}_{t_1} = \mathbf{x}] \\ &= f(\mathbf{x})\mathbf{P}[\mathscr{E}_2, \mathbf{Y}_{t_2} = \mathbf{y} \mid \mathbf{Y}_{t_1} = \mathbf{x}] \\ &= f(\mathbf{x}) \cdot \frac{\mathbf{P}[\mathscr{E}_2, \mathbf{Y}_{t_1} = \mathbf{x}, \mathbf{Y}_{t_2} = \mathbf{y}]}{\mathbf{P}[\mathbf{Y}_{t_1} = \mathbf{x}]} \\ &= f(\mathbf{x}) \cdot \frac{\mathbf{P}[\mathscr{E}_2 \mid \mathbf{Y}_{t_1} = \mathbf{x}, \mathbf{Y}_{t_2} = \mathbf{y}]\mathbf{P}[\mathbf{Y}_{t_1} = \mathbf{x}, \mathbf{Y}_{t_2} = \mathbf{y}]}{\mathbf{P}[\mathbf{Y}_{t_1} = \mathbf{x}]} \\ &= f(\mathbf{x})g(\mathbf{y})\mathbf{P}[\mathbf{Y}_{t_2} = \mathbf{y} \mid \mathbf{Y}_{t_1} = \mathbf{x}] \\ &= f(\mathbf{x})g(\mathbf{y})\mathbf{P}_{\mathbf{x}}[\mathbf{Y}_{t_2 - t_1} = \mathbf{y}] \end{aligned} \tag{4.7}$$

Now if we plugin back to (4.6), we get

$$\mathbf{P}[\mathscr{E}_1 \cap \mathscr{E}_2] = \frac{1}{N^d} \cdot \sum_{\mathbf{x}, \mathbf{y} \in \mathbb{Z}_N^d} f(\mathbf{x})g(\mathbf{y})\mathbf{P}_{\mathbf{x}}[\mathbf{Y}_{t_2 - t_1} = \mathbf{y}] \tag{4.8}$$

Therefore,

$$\begin{aligned} |\mathbf{P}[\mathscr{E}_1 \cap \mathscr{E}_2] - \mathbf{P}[\mathscr{E}_1] \cdot \mathbf{P}[\mathscr{E}_2] &= \mid \mathbf{E}[f(\mathbf{Y}_{t_1}) \cdot g(\mathbf{Y}_{t_2})] - \mathbf{E}[f(\mathbf{Y}_{t_1})] \cdot \mathbf{E}[g(\mathbf{Y}_{t_2})] \mid \\ &= \left| \frac{1}{N^d} \cdot \sum_{\mathbf{x}, \mathbf{y} \in \mathbb{Z}_N^d} f(\mathbf{Y}_{t_1})g(\mathbf{Y}_{t_2}) \left(\mathbf{P}_{\mathbf{x}}[\mathbf{Y}_{t_2 - t_1} = \mathbf{y}] - \frac{1}{N^d} \right) \right| \\ &\le \sup_{\mathbf{x} \in \mathbb{Z}_N^d} \sum_{\mathbf{y} \in \mathbb{Z}_N^d} \left| \mathbf{P}_{\mathbf{x}}[\mathbf{Y}_{t_2 - t_1} = \mathbf{y}] - \frac{1}{N^d} \right| = \varepsilon_{t_2 - t_1}(N) \end{aligned}$$

□

From this, one can actually generalize the result to further trajectories of the lazy random walk through induction.

Lemma 4.3 Fix $\mathbf{K} \subset\subset \mathbb{Z}_N^d$. For $0 \le s \le t$, let $\mathscr{E}_{s,t} = \{\{\mathbf{Y}_s, \dots, \mathbf{Y}_t\} \cap \mathbf{K} = \varnothing\}$. Then for any $k \ge 1$ and $0 \le s_1 \le t_1 \le \cdots \le s_k \le t_k$,

$$\left| \mathbf{P}\left[\bigcap_{i=1}^{k} \mathscr{E}_{s_i,\, t_i} \right] - \prod_{i=1}^{k} \mathbf{P}[\mathscr{E}_{s_i,\, t_i}] \right| \le (k-1) \cdot \max_{1 \le i \le k-1} \varepsilon_{s_{i+1} - t_i}(N). \tag{4.9}$$

And the whole proof for mixing properties of lazy randoms walks is finished.

4.3 Note: Proof Essentials

4.3.1 *Martingales*

The proof of (4.1) in [2] utilized Doob's submartingale inequality. Here we discuss the definition of martingales and its useful properties for consistency.

Definition 4.1 (martingales) A real-valued stochastic process X_n is called a $\mathscr{F}_n$*-martingale* if, for every $n \geq 0$,

i. $\mathbf{E}[| X_n |] < \infty$

ii. $X_n \in \mathscr{F}_n$

iii. $\mathbf{E}[X_{n+1} - X_n \mid \mathscr{F}_n] = 0.$

We also say such a process X_n is a martingale with respect to $\mathscr{F}_n$. Respectively, we can define *submartingales* and *supermartingales*.

Definition 4.2 (submartingales, supermartingales) For real-valued stochastic process X_n, if for every $n \geq 0$, i and ii in Definition 4.1 is fulfilled, is called a

$$\begin{cases} (\mathscr{F}_n)\text{-submartingale if, iii. } \mathbf{E}[X_{n+1} - X_n \mid \mathscr{F}_n] \geq 0 \\ (\mathscr{F}_n)\text{-supermartingale if, iii. } \mathbf{E}[X_{n+1} - X_n \mid \mathscr{F}_n] \leq 0 \end{cases}$$

A fundamental result of martingales is used frequently.

Theorem 4.2 (optional stopping theorem) Let X_n be a martingale, τ a finite stopping time and $0 < c < \infty$ is a constant, if at least one of the following conditions holds:

i. $\tau \leq c$ a.s.

ii. $\tau < \infty$ a.s. and $| X_{\min\{\tau,n\}} | \leq c$ for every $n \geq 0$

iii. $\mathbf{E}[\tau] < \infty$ and $\mathbf{E}[| X_n - X_{n-1} || \mathscr{F}_n] \leq c$ for every $n \geq 0$

then,

$$\mathbf{E}[X_{n+1}] = \mathbf{E}[\mathbf{E}[X_{n+1}\mathscr{F}_n]] = \mathbf{E}[X_n],\ (\mathbf{E}[X_\tau] = \mathbf{E}[X_0]) \tag{4.10}$$

For submartingales, if one of the conditions through i to iii in Theorem 4.2 is fulfilled, then respectively, $\mathbf{E}[X_\tau] \geq \mathbf{E}[X_0]$.

Theorem 4.3 (martingale convergence theorem)

- Assume that X_n is a martingale with $\sup_n \mathbf{E}[|X_n|] < \infty$, then there is an integrable random variable X such that $X_n \to X$ *a.s.* as $n \to \infty$.
- Every non-negative supermartingale converges.

Remark 4.2 The limit of the sequence $\mathbf{E}[|X_n|]$ exists due to the submartingale property, but it is not necessarily equal to $\mathbf{E}[|X|]$.

Lastly we state *Doob's inequality*, which could be viewed as a strengthening of Markov's inequality. If X_n is a non-negative submartingale, then

$$\mathbf{P}\left[\max_{k\leq n}(X_k) \geq c\right] \leq \frac{1}{c}\mathbf{E}[X_n] \tag{4.11}$$

4.3.2 Visualization of $\varphi(K)$ and $\varphi^{-1}(K)$

Another important idea used through the proof is canonical maps. The transformation is not obvious, but in order to show (4.1), it suffices to show that $\max_{0\leq t\leq n} \mid e_K(x,t) - \mathbf{e}_{\mathbf{K}}(\mathbf{x},t) \mid = 0$ as $N \to \infty$. (Intuitively, since visiting a set means that SRW must enter the set, and here we are discussing the projection of a random set on $\mathbb{Z}_N^d$, so one would consider the "projected" equilibrium measure. In order to connect it with random interlacement, we think of proving that the difference between the original equilibrium measure and the "projected" one differ only slightly when $N \to \infty$.)

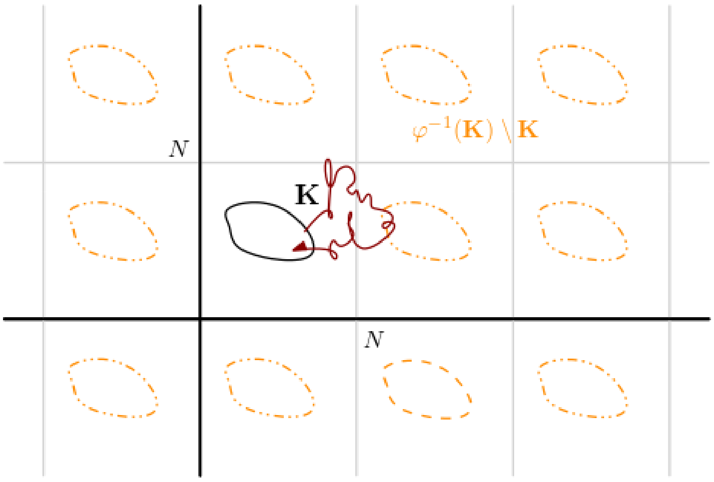


Figure 3: lattice shift of K on $\mathbb{Z}_N^2$ and its inverse image

From here, $\varphi(K)$ is denoted by $\mathbf{K}$ and φ^{-1} is the inverse map from $\mathbb{Z}_N^d$ to $\mathbb{Z}^d$. To understand these two notations better, a simple example is illustrated here. As shown in Figure 3, the black set represents $\varphi(K)$, and the orange sets represents $\varphi^{-1}(\mathbf{K}) \setminus \mathbf{K}$. In addition, let X_n be a SRW on $\mathbb{Z}^2$ that starts in $\partial_{int}\mathbf{K}$ and runs until time n (the red trajectory in Figure 3). Recall our definition of first hitting time $\tilde{H}_A$, here X_n is included in the set of events

$$\left\{X_n : \tilde{H}_{\varphi^{-1}(\mathbf{K})\setminus\mathbf{K}} \leq n\right\},$$

but does not belong to the set of events

$$\{X_n : \tilde{H}_{\mathbf{K}>n}\} \setminus \left\{X_n : \tilde{H}_{\varphi^{-1}(\mathbf{K})} > n\right\}.$$

Remark 4.3 Following this chapter, we also looked closely into the construction of random interlacements by a so called *random interlacement point process*, based on the understanding of Poisson Point Processes (PPP). We think of random interlacements as a collection of double infinite trajectories being randomly put into a torus, just as randomly dropping points into a space for PPP. Details of that process could be found in chapter 5 and 6 in [2].

5 COVER TIME OF RW ON TORUS

Finally, we look into a variation of random walk models, namely *Elephant Random Walk*, involving some reinforcement due to a certain memory of the process, quantified by a memory parameter p between 0 and 1. Based on [5] and [6], we focused on two-dimensions case and investigated cover time of random walks and elephant random walks of a discrete torus $\mathbb{Z}_n^2$ and simulated RWs and ERWs on $\mathbb{Z}_n^2$ with different size of N and p. Note that behaviors of random walks on two-dimensions is generally different comparing to their higher-dimensional counterparts due to its recurrence and strong correlation.

5.1 Definitions and Asymptotic Results

Introduced by Schütz and Trimper in [7], Elephant Random Walks on one dimension is a non-Markovian stochastic process referring to the saying that elephants can always remember where they have been. Here we introduce ERWs on two dimensions.

Let the random sequence $(X_n)_{n\geq 0}$ be a random walk on $\mathbb{Z}^2$. The random walks start uniformly at x_0 at t_0, and σ_{t+1} denotes a random variable that takes values from $\{(1,0), (-1,0), (0,1), (0,-1)\}$. At time t_1, the elephant chooses uniformly from the four directions to move forward.

$$X_1 = X_0 + \sigma_0,\ \sigma_0 \sim \mathscr{U}(\{(1,0),\ (-1,0),\ (0,1),\ (0,-1)\}) \tag{5.1}$$

For for the remaining steps $(X_{t+1})_1^{n-1}$,

$$X_{t+1} := \begin{cases} X_t + \sigma_i & \text{with probability } p \\ X_t + \sigma_{t+1} & \text{with probability } 1-p \end{cases} \tag{5.2}$$

where $\sigma_{t+1} \sim \mathscr{U}(\{(1,0), (-1,0), (0,1), (0,-1)\})$, and σ_i is chosen uniformly at random from $(\sigma_i)_0^t$. Following similar notation from section four, we use bold font to denote elephant random walks on torus.

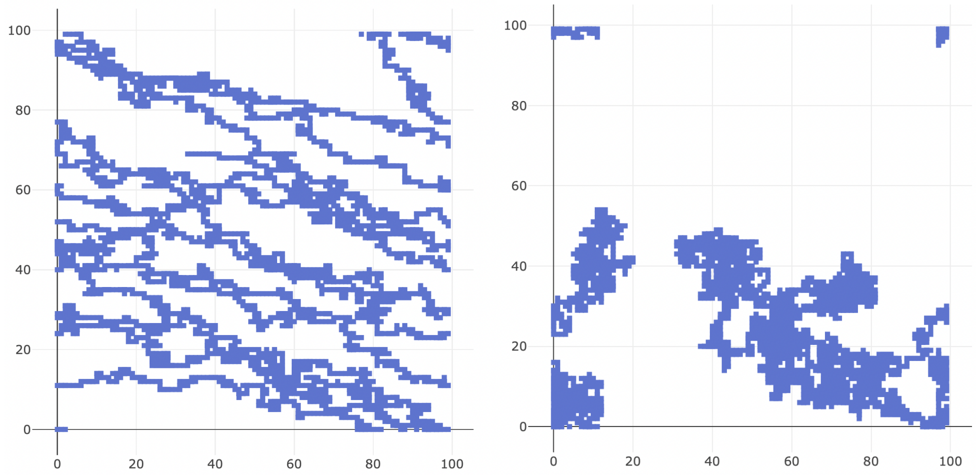
An example of EWR $(\mathbf{X}_n)$ on two dimensions discrete torus could look like Figure 4(a). Here, the size of torus is set at $n = 100$ and the walk runs up to 5000 times. As a comparison, simple random walk (i.e. EWR with $p = 0$) on $\mathbb{Z}_{100}^2$ is shown in Figure 4(b). Results in [6] shows that not only for EWRs on dimension two, but for all multi-dimensional elephant random walks, the influence of the memory parameter p on the random walks changes at the critical value

$$p_d = \frac{2d+1}{4d} \tag{5.3}$$

As addressed in (5.3), the critical p in $\mathbb{Z}_n^2$ should be 0.625. Depending on the value of p, ERWs could be can be categorized into diffusive ERWs, critical ERWs and superdiffusive ERWs.

Definition 5.1 (diffusive, critical, superdiffusive) Let $(\mathbf{X}_n)_{n\geq 0}$ be a multi-dimensional elephant random walk on discrete torus $\mathbb{Z}_n^d$, it is *diffusive* if $p < p_d$, *critical* if $p = p_d$, and *superdiffusive* if $p > p_d$.

Naturally, we are motivated to check the number of steps needed for an EWR to cover the whole torus. Such number of steps is defined by cover time.



(a) ERW with $p = 0.85$ (b) SRW

Figure 4: Elephant random walk on $\mathbb{Z}^2_{100}$ when $p = 0.85$ and $p = 0$ in 5000 steps

Definition 5.2 (cover time) $(\mathbf{X}_n)_{n\geq 0}$ be a random walk on $\mathbb{Z}^d_n$, *cover time* of the torus is

$$\mathcal{T}_n = \max_{\mathbf{x}\in\mathbb{Z}^d_n} H_{\mathbf{x}}(\mathbf{x}) \tag{5.4}$$

where $H_x(\mathbf{x})$ denotes entrance time to the site $\mathbf{x}$ as we defined in 2.2. By previous work in [6], it is proved that

$$\frac{\mathcal{T}_n}{n^2 \ln^2 n} \xrightarrow[n\to\infty]{\mathbf{P}} \frac{4}{\pi} \tag{5.5}$$

Generally there will be several last vacent points that takes the EWR a long time to reach. Following in the literature of [8], *late points* of random walks are concerned and they proved only stretched exponential decay of times which differ from the cover times only by a constant factor. Lower bound and upper bound of cover time is first given by the following theorems.

Theorem 5.1 *Assume that* $\gamma \in (0,1)$, *then for all* $\varepsilon > 0$, we have

$$\exp\left(-n^{2(1-\sqrt{\gamma})+\varepsilon}\right) \leq \mathbf{P}\left[\mathcal{T}_n \leq \frac{4}{\pi}\gamma n^2 \ln^2 n\right] \leq \exp\left(-n^{2(1-\sqrt{\gamma})-\varepsilon}\right) \tag{5.6}$$

for n large enough.

Theorem 5.1 *Assume that* $\gamma > 1$, *then for all* $\varepsilon > 0$, we have

$$n^{-2(\gamma-1)-\varepsilon} \leq \mathbf{P}\left[\mathcal{T}_n \geq \frac{4}{\pi}\gamma n^2 \ln^2 n\right] \leq n^{-2(\gamma-1)+\varepsilon} \tag{5.7}$$

for n large enough.

Additionally, for $\gamma \in (0,1)$, fix an arbitrary $\alpha \in (\sqrt{\gamma}, 1)$, consider simple random walk on the torus $\mathbb{Z}^2_n$ and divide the torus into boxes of size n^α, there exists $c = c(\alpha,\gamma) > 0$, $c' = c'(\alpha,\gamma) > 0$ such that by time $t = \frac{4}{\pi}\gamma n^2 \ln^2 n$, there are at least $cn^{2(1-\alpha)}$ boxes not completely covered with probability at least

$$1 - \exp\left(-c' n^{2(1-\alpha)}\right). \tag{5.8}$$

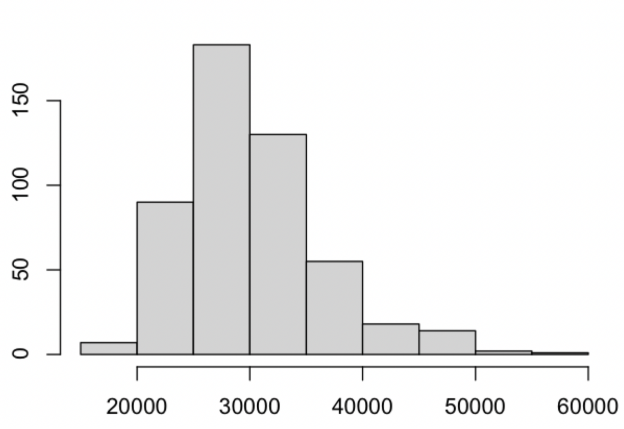


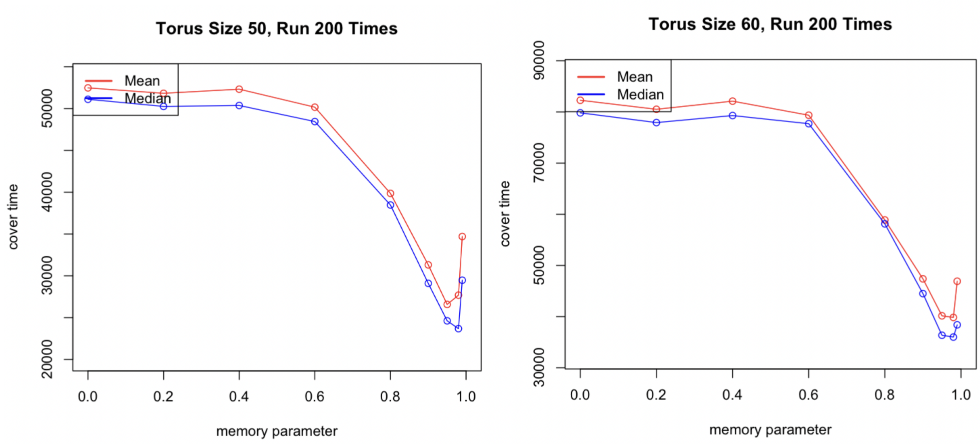
Figure 5: Histogram of cover time of simple random walk on $\mathbb{Z}_{50}^2$ for 500 trials

5.2 Implementation of SWR and EWR on $\mathbb{Z}_n^2$

Surprisingly there aren't much literature about cover times of elephant random walks on $\mathbb{Z}_n^2$. Based on asymptotic results in section 5.1, we are motivated to simulate simple random walks and elephant random walks on $\mathbb{Z}_n^2$ to confirm the results in [6] and construct a rough stretch for influence of memorization on cover time for future investigation.

We start by simulating simple random walks on $\mathbb{Z}_n^2$, i.e. elephant random walks when $p = 0$. We let the walk to run until it completely covers the torus, here of size $n = 50$. Since $(\mathbf{X}_t)_{t\geq 0}$ are i.i.d. random variables, by Law of Large Numbers, mean and median should give us a nice picture of cover time of SRWs when we repeat for large number of times. According to our simulation, the mean cover time of a simple random walk on $\mathbb{Z}_{50}^2$ is around 30151.69 and the median cover time is around 29045, as the distribution is shown in Figure 5. This result confirms the formula in (5.5).

Now we vary the memory parameter p and compare the cover time $\mathcal{T}_n$ at different p's, as shown in Figure 6 and Table 1. For fixed torus size n, cover



(a) Torus Size 50 (b) Torus Size 60

Figure 6: Mean and median cover time with respect to memory parameter

Table 1: Mean and Median of Cover Time For Different Size of Torus

p	40		50		60	
	mean	median	mean	median	mean	median
0.00	29768.52	29525.0	52465.99	51106.2	82285.32	79824.2
0.20	29929.55	28622.1	51818.55	50251.1	80517.99	77921.7
0.40	29494.57	28791.5	52312.52	50366.5	82109.38	79294.2
0.60	28593.09	27929.2	50162.05	48443.2	79361.77	77705.6
0.80	22590.21	21435.1	39860.43	38470.5	58882.81	58125.5
0.90	17935.65	17198.5	31312.68	29100.8	47377.38	44495.5
0.95	15601.31	14544.6	26583.77	24624.7	40153.21	36350.2
0.98	16063.24	13887.2	27693.42	23689.5	39840.40	35996.0
0.99	20680.24	15751.9	34702.23	29478.5	46909.86	38386.5

times remains large when $p \leq p_d$ and it decreases as p increases when $p \geq p_d$. In fact, it decreases to about half of the steps needed when $p = 0$ as $p \to 1$. The curve finally had a steep increase around $p = 1$, since the EWR could never cover the entire torus, $\mathcal{T}_n = \infty$.

From (5.5), we know that $\mathcal{T}_n$ grows faster than quadratically when n is large. Due to computational complexity, we only simulated ERWs up to $n = 60$. No conclusions could yet been given, but a new conjecture could be that for ERWs with p larger than critical value, there could be a lower bound for its cover time on $\mathbb{Z}_n^2$.

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