(1) Make sure A part matrix has only negative eigenvalues. The largest eigenvalue is -0.1.

$$A = \begin{bmatrix} -2.1 & 1 & 0 & 0 & \dots & 0 \\ 1 & -2.1 & 1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 1 \end{bmatrix} \tag{1}$$

We attempt to have a stable as the following energy, we cut the first [0,20] data off, only keep the stationary part.

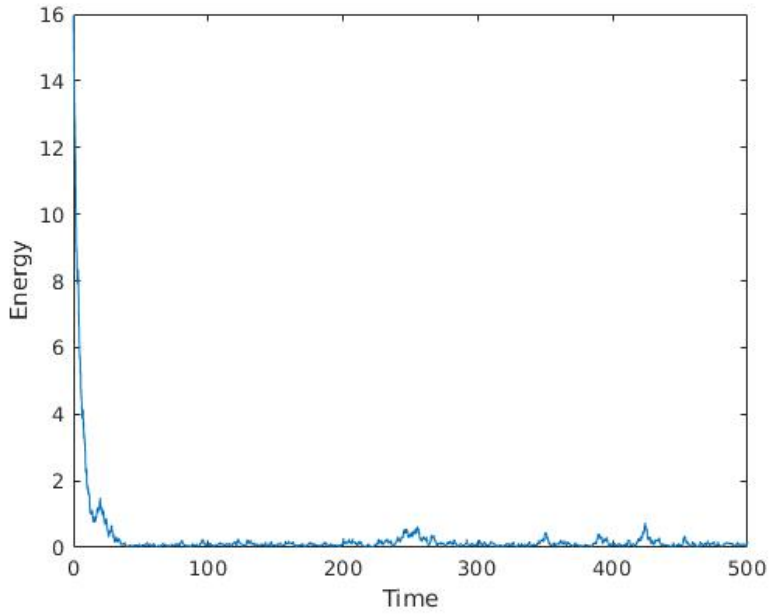


Figure 1: Energy

(2)Same Trend, different Δt

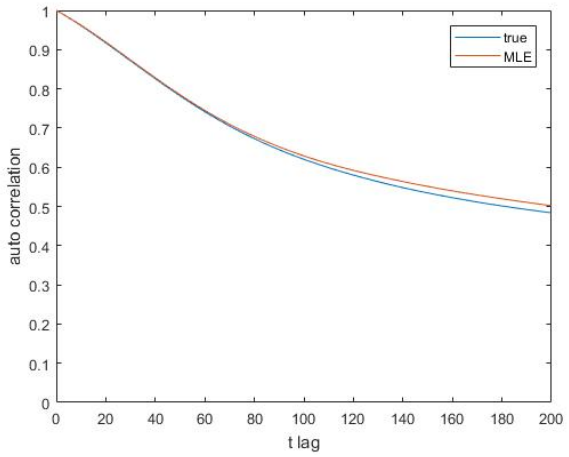
Fixed Trend=500. Fix the calculation time step = 0.001 in the generation, store data with different frequency (5,10,50), so that we have observation with different $\Delta t = 0.005, 0.01, 0.05$. In the auto-correlation plot, keep the maximum time lag is 1, such that $1 = 200 \times 0.005 = 100 \times 0.01 = 20 \times 0.05$.

- $\Delta t = 0.005$
- $\Delta t = 0.01$
- $\Delta t = 0.05$

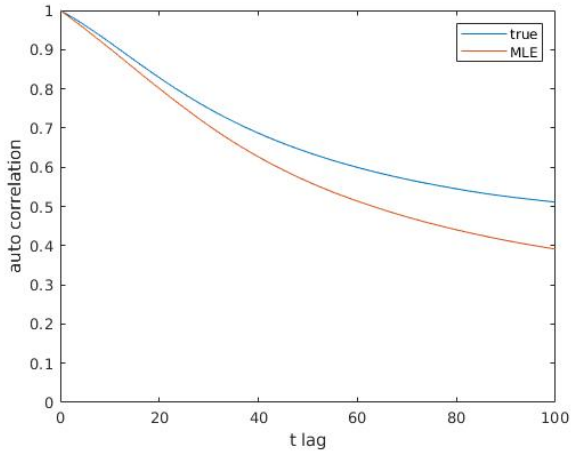
Figure 2: Energy

	mean	variance	the 3rd moment	the 4th moment
Trend=500; dt=0.005; true	-0.0116	0.0070	0.0001	0.0001
Trend=500; dt=0.005, MLE	-0.0213	0.0071	0.0002	0.0002
Trend=500;dt=0.01, true	-0.0246	0.0075	0.0002	0.0002
Trend=500;dt=0.01, MLE	0.0080	0.0055	0.0000	0.0001
Trend=500; dt=0.05,true	-0.0007	0.0072	0.0001	0.0002
Trend=500; dt=0.05,MLE	-0.0172	0.0050	0.0000	0.0001

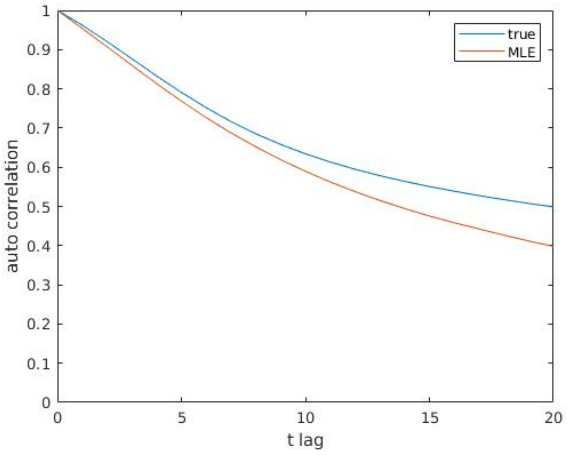
Table 1: statistics



(a) Trend= 500,dt=0.005



(b) Trend= 500,dt=0.01



(c) Trend= 500,dt=0.05

Figure 3: Same Trend, different dt

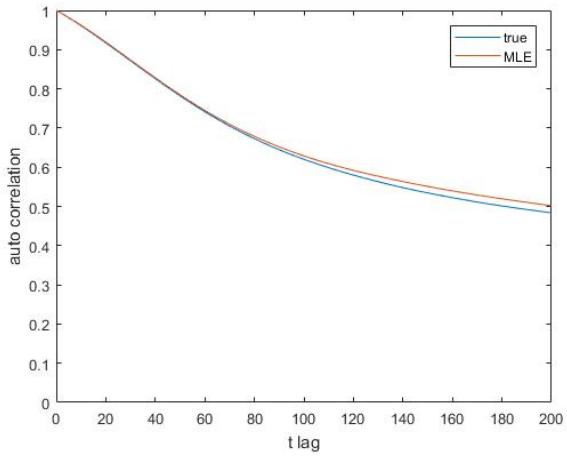
(3)Same Data Size, different Δt

It may also cause by the reduced data size. If we increase the Trend with smaller Δt, to have same data size

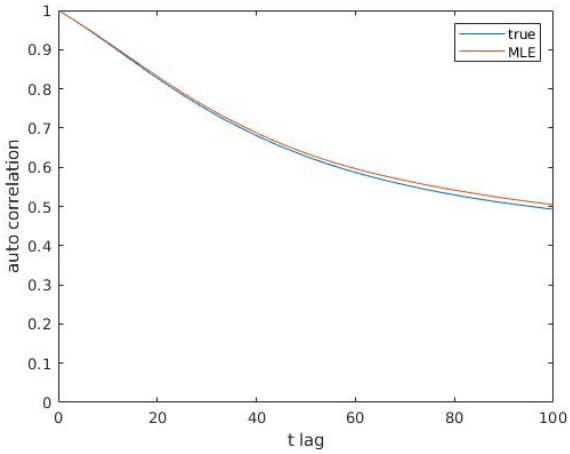
$N = 10^5 = \frac{500}{0.005} = \frac{1000}{0.01} = \frac{5000}{0.05}$.

	mean	variance	the 3rd moment	the 4th moment
Trend=500; dt=0.005; true	-0.0116	0.0070	0.0001	0.0001
Trend=500; dt=0.005, MLE	-0.0213	0.0071	0.0002	0.0002
Trend=1000;dt=0.01, true	0.0073	0.0067	0.0000	0.0001
Trend=1000;dt=0.01, MLE	-0.0058	0.0066	0.0000	0.0001
Trend=5000; dt=0.05,true	-0.0047	0.0066	-0.0000	0.0001
Trend=5000; dt=0.05,MLE	-0.0036	0.0054	-0.0000	0.0001

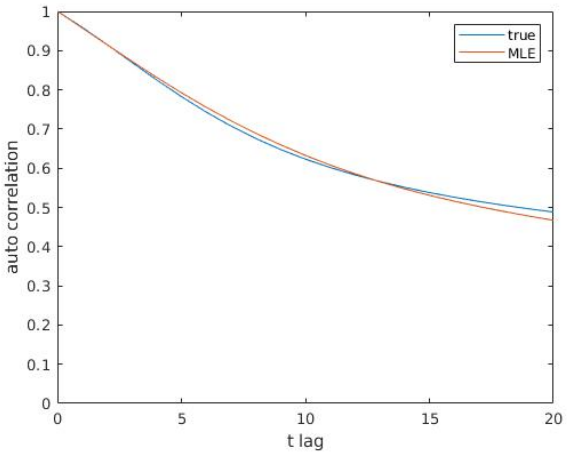
Table 2: statistics



(a) Trend= 500,dt=0.005

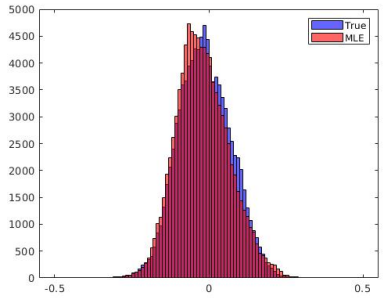


(b) Trend= 1000,dt=0.01

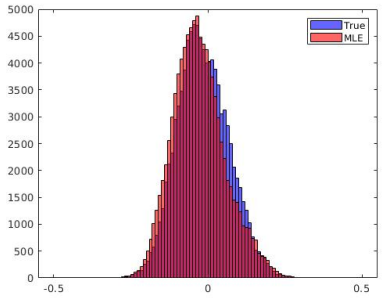


(c) Trend= 5000,dt=0.05

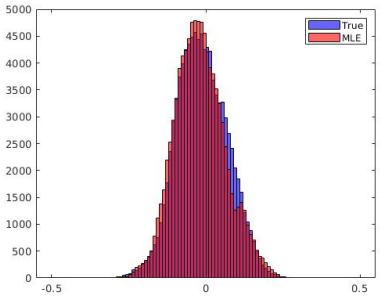
Figure 4: Same data size, different dt



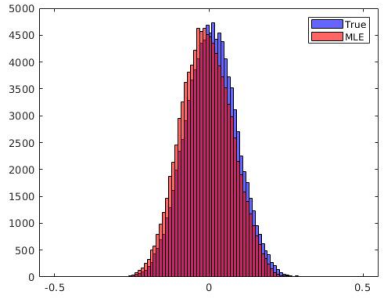
(a) x(1),Trend= 500,dt=0.005



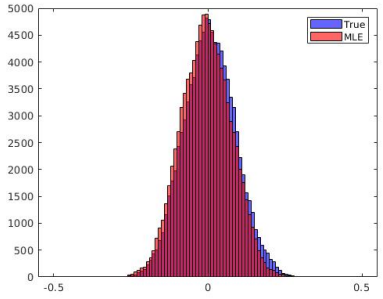
(b) x(8),Trend= 500,dt=0.005



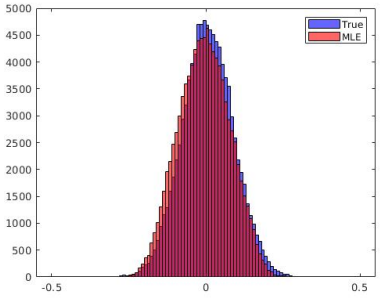
(c) x(16),Trend= 500,dt=0.005



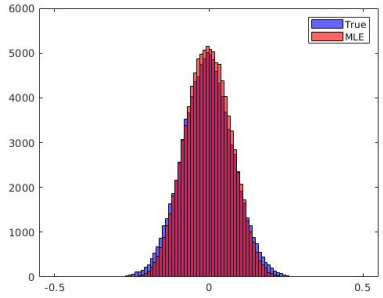
(d) x(1),Trend= 1000,dt=0.01



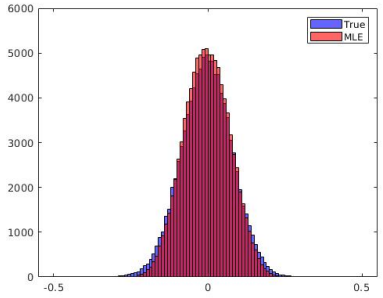
(e) x(8),Trend= 1000,dt=0.01



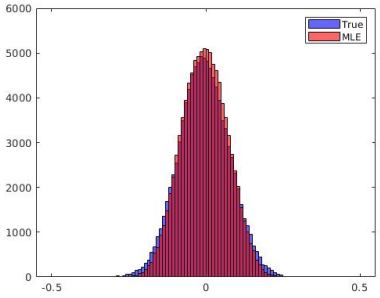
(f) x(16),Trend= 1000,dt=0.01



(g) x(1),Trend= 5000,dt=0.05

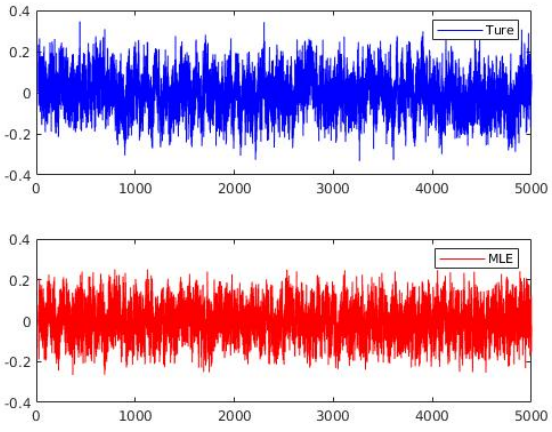


(h) x(8),Trend= 5000,dt=0.05

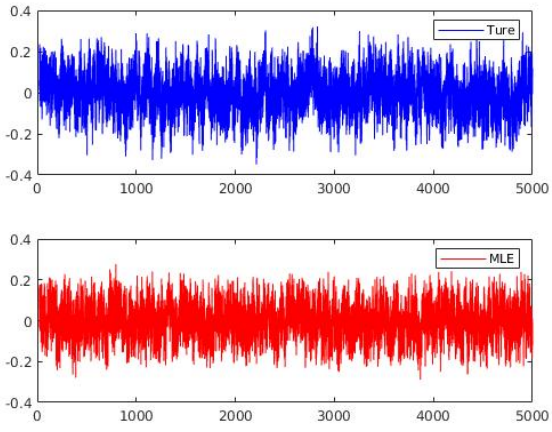


(i) x(16),Trend= 5000,dt=0.05

Figure 5: histogram



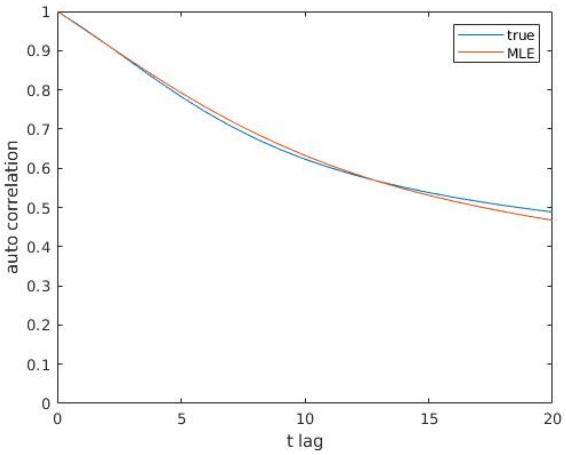
(a) x(1),Trend= 5000,dt=0.05



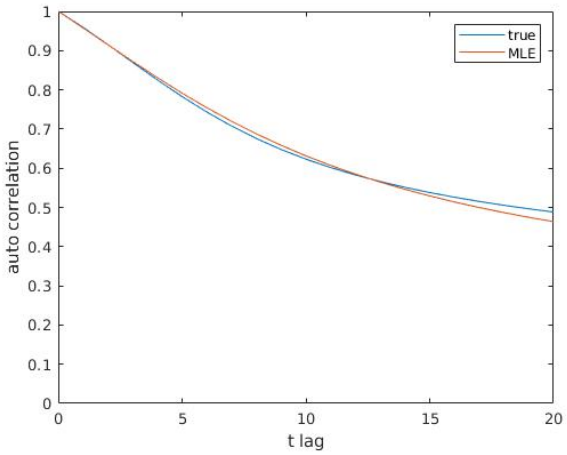
(b) x(8),Trend= 5000,dt=0.05

Figure 6: some example of Trend=5000,dt=0.05

(4) for the case Trend=5000, dt=0.05, we print out the true parameter for data generation, the user-defined initial guess and the MLE result. Also to check the optimization is unique, try different initial guess, which is $0.9 \times true$, do the Maximum likelihood estimation again. we get very similar.



(a) Trend= 5000,dt=0.05,initial=user-defined guess



(b) Trend= 5000,dt=0.05, initial=0.9*true

Figure 7: different initiation

The detail of parameters is attached.