Math 245 course note

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May 2021

Contents

1 Sum and direct sum

1.1 Intro

1. The field we are going to use in this course including:
    - R
    - C
    - Z_p = modules with prime p
2. **Vector space** (V) have the following property:
    - $\vec{x} + \vec{y} \in V$
    - $\vec{x} + \vec{y} = \vec{y} + \vec{x}$
    - $(\vec{x} + \vec{y}) + \vec{z} = \vec{x} + (\vec{y} + \vec{x})$
    - $\vec{0} \in V$
    - $-\vec{x} \in V$
    - $c\vec{x} \in V$
    - $c(\vec{x} + \vec{y}) = c\vec{x} + c\vec{y}$
    - $(c + d)\vec{x} = c\vec{x} + d\vec{x}$
    - $c(d\vec{x} = (cd)\vec{x}$
    - $1\vec{x} = \vec{x}$
3. Some example including
    - $F^n, n \geq 1$ which is a n-tuples
    - $M_{m,n}(F)$ which is m * n matrices
    - $M_n(F)$ which is n * n matrices
    - $F[t]$ polynomial in t with degree = ∞
    - $F_n[t]$ polynomial in t with degree = n
    - $C(R, R)$ continuoues from R to R

1.2 Sum and direct Sums

1. **Sum**

   let V be a vector space

   $U_1, U_2 \dots U_n \subseteq V$ are subspaces

   $\sum_{k=1}^{n} U_k = Sum = \{\sum_{p=1}^{n} \vec{u_p} : \vec{u_p} \in U_k\}$

   Note: $\sum_{k=1}^{n} U_k$ is a subspace of V

2. **Internal Direct Sum**

   If the following 2 condition is saticified

   - V $= \sum_{k=1}^{n} U_k$
   - $\forall \vec{v} \in V$ can be written **uniquely** as $\vec{v} = \sum_{p=1}^{n} \vec{u_p}$, for some $\vec{u_p} \in U_k$

   then V is **internal direct sum** of $U_1, U_2 \dots U_n$
   V $= \oplus_{k=1}^{n} U_k$

3. Proposition Let V be vector space $U_1, U_2 \dots U_n \subseteq V$ subspace

   V $= \oplus_{k=1}^{n} U_k \iff$

   - V $= \sum_{k=1}^{n} U_k$
   - if $\sum_{p=1}^{n} \vec{u_p} = \vec{0}$, for some $\vec{u_p} \in U_k$, then $\forall p, \vec{u_p} = \vec{0}$ linear independence

4. Corollary

   Let $U_1, U_2 \subseteq V$, are subspace then $V = U_1 \oplus U_2 \iff$

   - $V = U_1 + U_2$
   - $U_1 \cap U_2 = \{\vec{0}\}$

5. **External Direct Sum**

   Let $U_1, U_2 \subseteq F$ be vector spaces

   then the External Direct sum of them $U_1 \oplus U_2 = \{(\vec{u_1}, \vec{u_2}) : \vec{u_1} \in U_1, \vec{u_2} \in U_2\}$

   with the following property

   - $(\vec{u_1}, \vec{u_2}) + (\vec{u_1'}, \vec{u_2'}) = (\vec{u_1} + \vec{u_1'}, \vec{u_2} + \vec{u_2'}) \in U_1 \oplus U_2$
   - let $\lambda \in F$ $\lambda(\vec{u_1}, \vec{u_2}) = (\lambda\vec{u_1}, \lambda\vec{u_2}) \in U_1 \oplus U_2$

   Note:

   (a) $F^n = F \oplus F \oplus \ldots \oplus F$ n times

   (b) if $V = U_1 \oplus \ldots \oplus U_n$ is external, it is also internal

   (c) $U_i' \subseteq V$ where $U_i' = \{(\vec{0}, \vec{0}, \ldots \vec{u_i}, \vec{0} \ldots \vec{0}) \in V\}$

6. Proposition

   if $U, V \subseteq W$ are subspaces, B, C are basis
   where $B = \{\vec{b_1} \ldots \vec{b_n}\} \subseteq U$, $C = \{\vec{c_1} \ldots \vec{c_n}\} \subseteq V$ and $W = U \oplus V$
   then $B \cup C$ is a basis for W work for any number of B, C

2 Matrix Representations of Linear Maps

2.1 Basic notation of Matrix Representation

1. Space of linear map of V and W, V and U are vector space,
   $\zeta(V, W) = \{T : V \to W\}$
   or if $V = W$, $\zeta(V) = \{T : V \to V\}$

2. Now let $T \in \zeta(V)$ and a baiss $B = \{\vec{v_1} \dots \vec{v_n}\}$ for V For $1 \leq k \leq n$, we write
   $T(\vec{v_k}) = \sum_{i=1}^{n} a_{ik}\vec{v_i} = a_{1k}\vec{v_1} + \dots a_{nk}\vec{v_n}$
   for $a_{ik} \in F$

3. The matrix representation of T, relevent to basis B
   with $[T]_B = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$ For $\vec{v} \in V$ with $v = b_1\vec{v_1} + \dots + b_n\vec{v_n}$
   by linearity $T(\vec{v})$
   $= T(b_1\vec{v_1} + \dots + b_n\vec{v_n})$
   $= b_1T(\vec{v_1}) + \dots + b_nT(\vec{v_n})$

4. Suggestion: Look for a "nice basis for V say B,
   that make $[T]_B$ diagonal or a upper triangluar one

2.2 Invariant

1. **Invariant**
   For $T \in \zeta(V)$, a subspace $U \subseteq V$ is invariant for T if:
   - if $\forall \vec{u} \in U, T(\vec{u}) \in U$
   - or $\{T(\vec{u}) : \vec{u} \in U\} \subseteq U$
2. Example: Trivial invariant
   For any $T \in \zeta(V)$ the subspaces $0 \subseteq V$ and V are both invariant for T
3. Finding Non-trivial invariant Suppose :
   - $T \in \zeta(W)$, W is a vector space and $W = U \oplus V$ for $U, V \subseteq W$
   - U is an invariant of $T \rightarrow \forall \vec{u} \in U, T(\vec{u}) \in U$
   - $B = \{\vec{u_1} \ldots \vec{u_m}\}$, and $C = \{\vec{v_1} \ldots \vec{v_n}\}$ be bases for U and V

   then $D = B \cup C$ is a basis for W by proposition of external direct sum

   then $[T]_D =$ a $(m+n \times m+n)$ matrix $= \begin{pmatrix} a_{11} & \ldots & a_{1m} & \ldots & \ldots \\ \vdots & & & & \\ a_{m1} & \ldots & a_{mm} & & \\ 0 & \ldots & 0 & & \\ \vdots & & & & \\ 0 & \ldots & 0 & & \end{pmatrix}$

   $T(\vec{u_1}) \in U$ since $\vec{u_1} \in U$ and U is invarient of T
   Note: $\vec{u_1} = \sum_{i=1}^{n} a_{i1} \vec{u_i}$

   similarly to $\forall 1 \leq k \leq m, T(u_k)$
4. Conclusion:
   Since U is T invariant
   We can get the nice block upper triangular form matrix representation of T

3 Isomorphism

3.1 Definition of Isomorphism

1. **Isomorphism** Let V, W be vector spaces
   A linear map $T : V \to W$ is an isomophism if it is a bijection
   we write $V \cong W$

2. **Bijection** from Math 239
   Let A and B be two set
   $T : A \to B$ is bijection if

   - $\forall a, a^{'} \in A$, if $T(a) = T(a^{'})$ then $a = a^{'}$ one-to-one
   - $\forall b \in B, \exists a \in A$ such $f(a) = b$ onto

   Or $\exists W : B \to A$ such

   - $\forall a \in A, W(T(a)) = a$
   - $\forall b \in B, T(W(b) = b$

3. Fact
   If $T : V \to W$ is an isomorphism then
   $T^{-1} : W \to V$ always exist (by Bijection) is also:

   - Linear
   - Bijection
   - Isomorphism

4. Idea:
   Isomorphic vector spaces are "the same" up to a relabeling.
   Specificlly, suppose:

   - $T : V \to W$ is an isomorphism
   - $B = \{v_i\}$ be a basis for V

   Then $\{w_i = T(v_i)\}$ is a baiss for W.
   In particular, note that for $v \in V$, written as
   $v = a_1 v_{i1} + \cdots + a_n v_{in}$
   Then by linearty, $T(v) = a_1 w_{i1} + \cdots + a_n w_{in}$

4 Quotients

1. **Equivalance relation** ($\sim$)
   Let $U \subseteq V$ be a subspace, V is a vector space.
   $\vec{v_1} \sim \vec{v_2}$ if $\vec{v_1} - \vec{v_2} \in U$
   it has the following property:
   $\forall v_1, v_2, v_3 \in V$

   - **Reflexive**: $\vec{v_1} \sim \vec{v_1}$
   - **Symmetric**: $(\vec{v_1} \sim \vec{v_2}) \Leftrightarrow (\vec{v_2} \sim \vec{v_1})$
   - **Transitive**: $\vec{v_1} \sim \vec{v_2}$, $\vec{v_2} \sim \vec{v_3} \rightarrow \vec{v_1} \sim \vec{v_3}$

2. **Equivalance class** of $\vec{v} \in V$
   is $\vec{v} + U$

   - $= \{\vec{v} + \vec{u} : \vec{u} \in U\}$
   - $= \{\vec{v'} \in V : \vec{v'} \sim \vec{v}\}$

   Note: Every element have a unique Equivalance class

3. **Quocient space**
   Let $U \subseteq V$ be a subspace, the quotient space V/U is a new vector space is the set $\{\vec{v} + U : \vec{v} \in V\}$ with

   - $(\vec{v_1} + U) + (\vec{v_2} + U) = \vec{v_1} + \vec{v_2} + U$
   - $\lambda(\vec{v} + U) = \lambda\vec{v} + U$

4. Motivation:
   when we consider V/U, we collapsing U by set every element in U to $\vec{0}$
   Hence, $V/\{\vec{0}\} = V$
   Taking Quotient "undoes" direct sum

5. Proposition
   If $W = U \oplus V$ for $U, V \subseteq W$
   then, $W/U \cong V$ (W/U and V are bijection)

   - Proof:
     Let $T : W/U \to V$ be a Isomorphism
     where $T(\vec{w} + U) = \vec{v}$ where $\vec{w} = \vec{u} + \vec{v}$ for $\vec{v} \in V, \vec{u} \in U$
     - Proof of onto:
       For $\vec{v} \in V$, $T(\vec{v} + U) = \vec{v}$ since $\vec{v} = \vec{0} + \vec{v}, \vec{0} \in U$
     - Proof of one-to-one:
       Suppose $T(\vec{w_1} + U) = T(\vec{w_2} + U)$
       write $\vec{w_1} = \vec{u_1} + \vec{v_1}, \vec{w_2} = \vec{u_2} + \vec{v_2}$ for $\vec{u_1}, \vec{u_2} \in U, \vec{v_1}, \vec{v_2} \in V$
       Then $T(\vec{w_1}) = \vec{v_1}, T(\vec{w_2}) = \vec{v_2}$
       $\to \vec{v_1} = \vec{v_2}$
       $\to \vec{w_1} - \vec{w_2}$
       $= (\vec{u_1} + \vec{v_1}) - (\vec{u_2} + \vec{v_2})$
       $= (\vec{u_1} - \vec{u_2}) \in U + (\vec{v_1} - \vec{v_2}) = 0$
       $\to \vec{w_1} - \vec{w_2} \in U$
       $\to \vec{w_1} \sim \vec{w_2}$ or $\vec{w_1} + U = \vec{w_2} + U$

     Hence, T is a Isomorphism

6. Fact:
   If $W = U \oplus V$, B be a basis for V,
   then $v_i + U \subseteq W/U$ is a basis for W/U
   To see this, note that $\vec{v_i} + U = T^{-1}(\vec{v_i})$
   Suppose $W = U \oplus V$ for $U, V \subseteq W$
   Let $T \in \zeta(W)$ and U is T-invariant
   then, T **induces** a quotient linear map $\overline{T} \in \zeta(W/U)$
   $\overline{T}(\vec{v} + U) = T(\vec{v}) + U, \vec{v} + U \in W/U$
   $\overline{T}$ is linear

7. **Induced quotient map $\overline{T}$**
   For vector space $W = U \oplus V$, $U, V \subseteq W$
   $T \in \zeta(W)$ such that U is T-invariant, the induced quotient map $\overline{T} \in \zeta(W/U)$
   $\overline{T}(\vec{w} + U) = T(\vec{w}) + U, \vec{w} + U \in W/U$
   It has the following property:

   - It is linear

8. Matrix representation of $\overline{T}$
   Let $W = U \oplus V$, $T \in \zeta(W)$ as above, so U is T-invariant and $\overline{T} \in \zeta(W/U)$
   Let $\{\vec{u_1} \dots \vec{u_m}\}$ and $\{\vec{v_1} \dots \vec{v_n}\}$ be basis for U and V, then:

   - $\{\vec{u_1}, \dots, \vec{u_m}, \vec{v_1}, \dots, \vec{v_n}\}$ is a basis for W
   - matrix representation of T is $[T] = \begin{pmatrix} A & ? \\ 0 & B \end{pmatrix}$
   - where $A = [T|_u]$ is the restricient to U
   - $B = [\overline{T}]$ where $\overline{T} \in \zeta(W/U)$
     where $B = (B_{ij})_{i,j=1}^n$
   - $det(T) = det(A)det(B)$
   - $\{\vec{v_1} + U, \dots, \vec{v_n} + U\}$ is a basis for W/U
   - $T(\vec{v_i}) = \vec{u} + \sum_{k=1}^n b_{ki}\vec{v_k}$
   - $\overline{T}(v_i + U) = \sum_{k=1}^n b_{ki}(\vec{v_k} + U)$

5 Invarient Subspaces

1. Important fact related to eigenvector of $T \in \zeta(V)$
   $Ker(T)$ and $Ran(T)$ are both invariant
   Let $T \in \zeta(V)$ and $U \subseteq V$ is invariant, $dim(U) = 1$, $B(U) = \{\vec{u}\}$
   Then, $\vec{u} \neq 0$ so $\vec{u}$ is a eigenvector for T with eigenvalue λ
   In addition, if $\vec{u} \in V$ is a eigenvector for T, span($\vec{u}$) is a one dimensional invariant subspace for T

2. Polynomial of matrix rep of Linear transformation
   Let $T \in \zeta(V)$ let $p \in F[t]$
   then $p(T) = a_0 I_v + a_1 T + a_2 T^2 + \ldots + a_n T^n$

3. Little theorem
   If $T_1 \ldots T_n \in \zeta(V)$, then let $T = T_1 T_2 \ldots T_n$ is not invertible iff one of T_i is not invertible

4. Theorem
   For $V \subseteq C$ dim(V) is finite, $\forall T \in \zeta(V)$ have an eigenvector.and 1-dim invariant subspace

   - Proof of the theorem
     Let $\dim(V) =$ n, and $0 \neq \vec{v_0} \in V$
     Then the set,$\{\vec{v}, T\vec{v}, T^2\vec{v}, \ldots, T^n\vec{v}\}$ is always a linear dependent set since

     – If all element are distinct, there are $n + 1$ element, and the dim(V) = n, then it is dependent
     – If there are some element that are same, the it is clrearly dependent

     So, there exist $a_0, a_1, \ldots, a_n \in C$, such that

     $$a_0(T^0 = I_n)\vec{v} + a_1T\vec{v} + \ldots + a_mT^m\vec{v} = 0(1)$$

     where $m \leq n$ and $\forall i, a_i \neq 0$ Hence Let $p \in C[t]$ where $p(t) = a_0 + a_1t + \ldots + a_mt^m$
     Then (1) is quivalent to $p(T)\vec{v} = 0$, since $p \in C[t]$, then by fundemantal theorem of algebra, we can factor p as

     $$p(t) = (t - \lambda_1) \ldots (t - \lambda_m)$$

     Then
     $$p(T) = (T - \lambda_1 I_v) \ldots (T - lambda_m I_v)$$
     Since $p(T)V = 0$ by little theorem
     Some $T - lambda_i I_v$ is not invertible, so $Ker(T - \lambda_i I_v) \neq 0$ by rank-nullity theorem
     Sp $\exists \vec{u} \in V$, such that $(T - \lambda_i I_v)\vec{u} = 0 \rightarrow Tu = \lambda_i \vec{u}$
     Hencem $\vec{u}$ is an eigenvector for T

6 Triangularization

6.1 The theorem

1. Recall: Diagonizeable
   A matrix $A \in M_n(F)$ is upper triangular, where $\forall i > j, A_{ij} = 0$
   Lower triangular is defined similarly

2. **Triangularization Theorem**
   For $T \in \zeta(V)$ and $\dim(V) =$ finite, $V \subseteq C$
   then there is a basis for V which the maxtirx representation of T is upper triangular.

   - Proof of the Triangularization Theorem by induction
     - Base Case
       If dim(V) = 1, then it is already upper triangular since 1×1 is always upper triangular
     - Inductinve Step
       Assume result holds $\forall n \geq 2$, up to $n-1$, we want to prove the result holds for $dim(V) = n$
       T has an eigenvector $\vec{u}$ by the result from last lecture, and has a 1-dim invariant subspace U, and this is non-trivial since $dim(V) \geq 2$
       Let $U = span(\vec{u})$, By extention theorem
       we can extend $\{\vec{u}\}$ in to a basis $\{\vec{u}, \vec{v_1}, \ldots, \vec{v_n}\}$ for V
       Let $U^{'} = span(\{\vec{v_1} \ldots \vec{v_n}\})$, then $V = U \oplus U^{'}$
       Consider $\overline{T} \in \zeta(V/U)$ where $U^{'} \sim V/U$
       Then $sim(V/U) = n-1$ we can get a basis $\{w_1+U, \ldots, w_{n-1}+U\}$ for V/U, which $\overline{T}$ has a upper triangular matrix
       Thus $\{w_1, \ldots, w_{n-1}\}$ is a basis for $U^{'}$, and $\{u, w_1, \ldots, w_{n-1}\}$ is a bsis for V.
       THe matrix representation of T has the following form:

$$\begin{pmatrix} * & * \\ 0 & [\overline{T}]_{\{\vec{w_1}+U, \ldots, \vec{w_{n-1}}+U\}} \end{pmatrix}$$

       Hence, this matrix is upper triangular, which means the theorem is true.

6.2 Application of the theorem

3. Theorem
   Let $A \in M_n(F)$ be upper triangular, then $det(A) = A_{11} \times A_{22} \times A_{33} \ldots A_{nn}$

4. Corollary
   Let $A \in M_n(F)$ be upper triangles, then eigenvalus of A, listed with multiplicity
   are $A_{11}, A_{22}, \ldots, A_{nn}$

   - Mulplicity = number of one eigenvalue shows up

7 Cayley - Hamilton Theorem

1. Review
   For $V \subseteq F$, $T \in \zeta V$, then characteristic polynomial of T is

   $$P_T(t) = det(T - tI_v) \in F[t]$$

   where t is eigenvalue of T

2. **Cayley-Hamilton Theorem** Let V be a vector space, $T \in \zeta(V)$, then

   $$P_T(T) = 0$$

   In particular, this hold for any field F

   - Proof of the theorem
     - Assume we are working on the filed C
     - Then Let $B = \{\vec{v_1}, \vec{v_2}, \dots, \vec{v_n}\}$ where $[T]_B$ is upper triangular. by triangularization Theorem
     - And $[P_T(T)]_B = P_T([T]_B)$
     - It is suffices to check $P_T([T]_B) = 0$, and since $P_{[T]_B}(t) = P_T(t)$
     - We can assume T is upper triangular matrix where $T_{ii} = \lambda_i$
     - And λ_i are eigenvalus of T, so the roots of $P_T(t)$
     - Hence, $P_T(t) = (t - \lambda_1) \dots (t - \lambda_n)$
     - Note L $= T - \lambda_i I_v$ is a upper triangular matrix that have the property of $L_{jj} = \lambda_j - \lambda_i$ and there is a 0 is the middle
     - Hence $P_T(T) = (T - \lambda_1 I_v) \dots (T - \lambda_n I_v)$
       * Lemma:
         Let $A_1, A_2, \dots A_n \in M_n(F)$ be upper triangular, with $A_{1_{11}} = 0$
         Then $A_1 A_2 \dots A_n = 0$
       * Prove by induction:
         · Base case:
           $A_1 = 0$,yea it is ture
         · Hypothesis:
           Assume it is true for n-1 which $A_1 A_2 \dots A_{n-1} = 0$

- Inductive steps:

  Consider $A_i = \begin{pmatrix} B_i & * \\ 0 & * \end{pmatrix}$ Note B_i is upper triangular, and $B_{i_{ii}} = 0$

  Hence $A_1 \dots A_{n-1} = \begin{pmatrix} B_1 \dots B_{n-1} & * \\ 0 & * \end{pmatrix} = \begin{pmatrix} 0 & * \\ 0 & * \end{pmatrix}$

  Hence $A_1 \dots A_n = \begin{pmatrix} 0 & * \\ 0 & * \end{pmatrix} \begin{pmatrix} * & * \\ 0 & 0 \end{pmatrix}$

  $$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Hence, the lemma is true.

And the theorem is true

8 Jordan Form

8.1 Intro

1. Math 146: Diagonalization
   Not every linear transformation is diagonalizable

2. (Upper) Triangularization
   Can answer all question since the top part might be very complex

3. Similar for linear transformation
   Let $A \in \zeta(V)$ and $B \in \zeta(W)$ if A is similar to B if$\exists P \in \zeta(V, W)$ such $B = PAP^{-1}$
   Note: S is an Isomomorphism, V and W have bijection relationship between them

8.2 Jordan from

1. **Nilpotent**
   Let $T \in \zeta(V)$ is nilpotent if $\exists n \in N, T^n = 0$ The order of T is n

   - if T is nilpotnet, then it is not invertible
   - Ex.
     $$N = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$
     is a nilpolent of order 3
   - In addition, $N(\vec{e_1}) = 0, N(\vec{e_2}) = \vec{e_1}, N(\vec{e_3}) = \vec{e_2}$
   - Note, if $N \in M_n(F)$ is left or right shift, it is a nilpotnet with order n.

2. Left shift
   If $N \in zeta(V)$ and $N(\vec{v_1}) = 0, N(\vec{v_i}) = \vec{v_{i-1}}$

3. Right shift
   If $N \in zeta(V)$ and $N(\vec{v_n}) = 0, N(\vec{v_i}) = \vec{v_{i+1}}$

4. Direct sum decompostion

   Let V be a vector space, $V = V_1 \oplus \ldots \oplus V_n$ be direct sum devompostion of V

   For $T_1, \ldots T_n$ with $T_i \in \zeta(V_i)$, the direct sum of $T \in \zeta(V) = T_1 \oplus \ldots T_n$

   Hence, $T(\vec{v}) = T_1(\vec{v_1}) + \ldots + T_n(\vec{v_n})$ and $\vec{v} = \vec{v_1} + \ldots + \vec{v_n}$

   Note: Let $B_1, \ldots B_n$ are bases for $V_1, \ldots, V_n$, then $B = B_1 \cap \ldots \cap B_n$ is a basis for $V = V_1 \oplus \ldots \oplus V_n$

   Then The matrix representation of $[T]_B = \begin{pmatrix} [T_1]_{B_1} & 0 & 0 \\ 0 & \ldots & 0 \\ 0 & 0 & [T_n]_{B_n} \end{pmatrix}$

   In addition, For $T \in \zeta(V)$ if $V = V_1 \oplus \ldots \oplus V_n$ for $V_1 \ldots V_n \subseteq V$ and each V_i is invariant for T

   Then Let $T_i = T/V_i$, $T = T_1 \oplus \ldots \oplus T_n$

   This is because if $\vec{v} \in V$ decomposes as $\vec{v} = \vec{v_1} + \ldots + \vec{v_n}$ for $\vec{v_i} \in V_i$ then

   $T(\vec{v}) = T(\vec{v_1} + \ldots + \vec{v_n})$

   $= T(\vec{v_1}) + \ldots + T(\vec{v_n})$

   $= T|_{V_1}(\vec{v_1}) + \ldots + T|_{V_n}(\vec{v_n})$

   $= T_1(\vec{v_1}) + \ldots + T_n(\vec{v_n})$

   and $T_i \in \zeta(V_i)$ by the invariance of V_i of T

5. Sum of Right shift
   Let V be a vector space, $N \in \zeta(V)$ and N is Nilpotent, then N is a direct sum of right shifts
   Specifically, there are subspaces $V_1, \ldots, V_k \subseteq V$ invariant for N
   Such $V = V_1 \oplus \ldots \oplus V_k$ and $N = N_1 \oplus \ldots \oplus N_k$ where $N_i = N/V_i$

   - Proof of the Theorem
     The theorem is equivalent to show that there is a basis for V of the form $\{\vec{v_1}, N\vec{v_1}, \ldots, N^{a_1-1}\vec{v_1}, \vec{v_2} \ldots \ldots N^{a_k-1}\vec{v_k}\}$
     Note: V_i will be span $\{\vec{v_i} \ldots N^{a_i-1}\vec{v_i}\}$, clearly each V_i is invariant for N, and restriction $N|_{V_i}$ will be a right shift
   - We will use induction on $n = dim(V)$
     - Base case: n = 0, then it is trivial since nothing is in the vector space
       n = 1, let $V_i = \{0\}$, then everything times 0 is zero, so it is invariant
     - Inductive hypothesis: Suppose $n \geq 2$ assume the result holds for nilpotent maps acting on vector spaces of dim at most n
       Then if N = 0, it is trival. So assume $N \neq 0$
       Since N is nilpotent, it is not invertible, so by the rank - nullity theorem. $dim(ran(N)) < n$. Also, $ran(N)$ is invariant for N.
       Apply the induction hypothesis, to $N|_{ran(N)}$, we get a bsis for $ran(N)$ in the form of $\{\vec{v_1}, \ldots, N^{b_1-1}\vec{v_1} \ldots N^{b_e-1}\vec{v_e}\}$
       Then $N^{b_i}\vec{v_i} = 0 \ \forall i$
       Since $\vec{v_i} \in ran(N), \exists \vec{u_i} \in V$ such that $N\vec{u_i} = v_i \forall i$
       Note $N^{b_i-1}\vec{v_i} \in Ker(N) \forall i$ and linear independent
       Hence , by extention theorem, can extend to a basis $\{N^{b_1-}\vec{v_1}, N^{b_l-1}\vec{v_l}, \vec{w_1}, \vec{w_m}\}$

6. Claim: $\{\vec{u_1}, N\vec{u_1} \ldots, N^{b_1}\vec{u_1}, \ldots, N^{b_l}\vec{u_l}, \vec{w_1}, \ldots, \vec{w_m}\}$ is a basis for V since $N\vec{w_i} = 0$
   proof is waiting

9 Inner Produce space

9.1 Intro

1. Norm
   For $z = (z_1, \dots, z_n) \in C^n$, the norm of z is $||z|| = \sqrt{\sum_{i=1}^{n} |z_i|^2}$
   where $|z_i|^2 = z_i * \overline{z_i}$ and $\overline{z_i}$ is conjugate
   Note: if $z \in R^n$ then $||z|| = \sqrt{z_1^2 + \dots + z_n^2}$

2. Intuation
   R^n and C^n are vector spacese with addtional structure:

   - Length/distances
   - Angles

   Inner produce space is a vector space with this additional structure.

3. Relationship between angle and length
   For $x, y \in R^2$ let $z = x - y = (x_1 - y_1, x_2 - y^2)$
   By cos law, $||z||^2 = ||x||^2 + ||y||^2 - 2||x|| \times ||y|| \times cos(\theta)$
   $(x_1 - y_1)^2 + (x_2 - y_2)^2 = x_1^2 + x_2^2 + y_1^2 + y_2^2 - 2||x|| \times ||y|| \times cos(\theta)$
   Hence, $||x|| \times ||y|| cos(\theta) = x_1 y_1 + x_2 y_2$
   If x and y are unit vector, $cos(\theta) = x_1 y_1 + x_2 y_2$

9.2 Inner Produce

1. Euclidean inner product on C^n
   $<\vec{x}, \vec{y}> = x_1\overline{y_1} + \ldots + x_n\overline{y_n}$
   Note: $<\vec{z}, \vec{z}> = ||z||^2$

2. Inner Produce
   For a complex vector space V, an inner produce on V is a function $<*, *>$: $V \times V -> C$ such that

   - Positivity: $<\vec{v}, \vec{v}> \geq 0 \; \forall \vec{v} \in V$
   - Non-degeneracy: $<\vec{v}, \vec{v}> \neq 0 \iff \vec{v} = 0$
   - Linearty in First component: $<\vec{u} + \lambda\vec{v}, \vec{w}> = <\vec{u}, \vec{w}> + \lambda <\vec{v}, \vec{w}>$, $\forall \vec{u}, \vec{v} m \vec{w} \in V$
   - Conjugate linear in second component: $<\vec{u}, \vec{v} + \lambda\vec{w}> = <\vec{u}, \vec{v}> + \overline{\lambda} <\vec{u}, \vec{w}>$, $\forall \lambda \in C$
   - Symmetry: $<\vec{v}, \vec{w}> = \overline{<\vec{w}, \vec{v}>}$

   We call the vector space V, equipped $<*, *>$ is an inner product space.

3. Norm
   For a inner product space, the norm ——*.*——:V-¿R is defined by $||v|| = \sqrt{<\vec{v}, \vec{v}>}$ where $v \in V$

10 Polar Decomposition

1. Basic Idea
   For inner product space V and $T \in \zeta(V)$, the adjoint $T^* \in \zeta(V)$ should be viewed as higher - dimensional generalization of complex conjugation

2. Dictionary

   - $z \in R \iff \overline{z} = z$ is related to T is self-adjoint ie. $T^* = T$
   - z is purely imaginary $\iff \overline{z} = -z$ is related to T is skew-adjoint ie. $T^* = -T$
   - z has length 1 ie lies on complex unit cricle $\iff \overline{z}z = 1$ is related to T is unitary ie. $T^*T = TT^* = I_v$

3. Note
   If $U \in \zeta(V)$ is unitary, then every eugenvalue $\lambda \in C$ satisifies $|\lambda| = 1$
   To see this, recall U is normal and if $\vec{v} \in V$ is an eigenvector for U, then $\overline{\lambda}$ is an eigenvalue for U^* with $\vec{v}$
   Hence $< U\vec{v}, U\vec{v} >=< \lambda\vec{v}, \lambda\vec{v} >= \lambda\overline{\lambda}||v||^2 = |\lambda|^2||v||^2$
   But also, $< U\vec{v}, U\vec{v} >=< U^*U\vec{v}, \vec{v} >=< \vec{v}, \vec{v} >= ||v||^2$, hence we get what we need

4. Def
   Let V be an inner porduct space, $T \in \zeta(V)$, and
   $Re(T) = \frac{T+T^*}{2}$, $Im(T) = \frac{T-T^*}{2i}$
   Note Re(T) and Im(T) are self-adjoint

5. Proposition
   Let V be a inner product space, $T \in \zeta(V)$, then
   $T = Re(T) + iIm(T)$, and this decomposition is unique

6. Unitaries and Orthonormal Bases (This is a iff statement)
   Let V, W be inner product spaces, $\{\vec{v_1}, \ldots, \vec{v_n}\}$, $\{\vec{w}_1, \ldots, \vec{w_n}\}$ be orthonormal basis
   Define $U \in \zeta(V,W)$ by $U(\vec{v_i}) = \vec{w_i}$, then U is unitary.

   - To show this, recall from A3, U is unitary if $||U\vec{v}|| = ||\vec{v}||$ all $\vec{v} \in V$ (ie, U is an isometry) and U is invertible which $U^{-1} = U^*$
     Now $< U(a_1\vec{v_1} + \ldots + a_n\vec{v_n}), U(a_1\vec{v_1} + \ldots + a_n\vec{v_n}) >$
     $=< a_1\vec{w_1} + \ldots + a_n\vec{w_n}, a_1\vec{w_1} + \ldots + a_n\vec{w_n} >$
     $= ||a_1\vec{w_1} + \ldots + a_n\vec{w_n}||^2$
     $= |a_1|^2 + \ldots + |a_n|^2$
     $= ||a_1\vec{v_1} + \ldots + a_n\vec{v_n}$
     Hence it is isometry
   - Suppose we start with unitary
     Let $\{\vec{v_1}, \ldots, \vec{v_n}\}$ be an orthonormal basis for V. Then $\{U\vec{v_1}, \ldots, U\vec{v_n}\} \in W$ is an orthonormal basis for W.
     Since U is invertible, it is a basis. To see that is is orthonormal, note
     $< U\vec{v_i}, U\vec{v_j} >=< U^*U\vec{v_i}, \vec{v_j} >=< \vec{v_i}, \vec{v_j} > = 1$ or 0

7. Polar decomposition
   For $z \in C$, the polar decomposition of $z = re^{i\theta}$ for $r \geq 0, \theta \in [0, 2\Pi)$
   Note $r = |z|, e^{i\theta} = \frac{z}{|z|}$
   So letting $u = e^{i\theta}$, then $z = u|z|$

8. Theorem (polar decomposition)
   Let V be a inner profuct space, $T \in \zeta(V)$. Then there exists a unitary $U \in \zeta(V)$ such that
   $T = U|T|$, where $|T| = (T^*T)^{\frac{1}{2}}$
   Note T^*T is self-adjoint and positive and there is a unique S such $S^2 = T^*T$

   - Proof:
     Observe $||(|T|\vec{v}||^2 =< |T|\vec{v}, |T|\vec{v} >=< (T * T)^{\frac{1}{2}}\vec{v}, (T * T)^{\frac{1}{2}}\vec{v} >$
     $=< T * T\vec{v}, \vec{v} >$ by adjoint
     $=< T\vec{v}, T\vec{v} >$by adjoint
     $= ||T\vec{v}||^2$
     Hence $|||T|\vec{v}|| = ||T\vec{v}||$ for all $\vec{v} \in V$. In particular, note this implies the range of $|T|$ and T have same dim
     Define a unitary $U_1 : Ran(|T|)- > Ran(T)$ by $U_1|T|\vec{v} = T\vec{v}$
     And U_1 is unitary follows from above since U_1 os an isometry and is invertible since they have the saem dim
     Now we extend U_1 to be unitary on V, where
     $V = Ran(|T|) \oplus Ran(|T|) = Ran(T) \oplus Ran(T)$ note one is the projection
     Let $\{e_1, \ldots, e_n\}$, $\{f_1, \ldots, f_n\}$ be orthonormal bases
     Define $U_2 : Ran(|T|)^*- > Ran(T)$ by $U_2e_I = f_i$
     Then U_2 is unitary since it maps on orthonormal basis to an orthonormal basis.
     Lastly, Define $U : V- > V$ by $U_{\vec{v}} = U_1\vec{v_1} + U_2\vec{v_2}$ where $\vec{v} = \vec{v_1} + \vec{v_2}$, $\vec{v_1} \in Ran(|T|)$ and $\vec{v_2} \in theprojection$
     Then U is unitary since
     $||U_v||^2 = ||U_1\vec{v_1} + U_2\vec{v_2}||^2 = ||U_1\vec{v_1}||^2 + ||U_2\vec{v_2}||^2$
     $= ||\vec{v_1}||^2 + ||\vec{v_2}||^2 = ||\vec{v_1} + \vec{v_2}||^2 = ||\vec{v}||$
     Hence it is unitary