CO 456 course note

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Sep 2022

Contents

1 NIM

1. Terms
   A game have multiple **positions**
   In particular, it has a **starting position**
   A player perform a **move** out of a set of allowed moves
   Executing a move generates an **option** of the current position

2. Impartial game
   - The game must determined
   - Normal play: The first who have no move loses
   - Misere : the first who can't move winss

3. Positions
   - winning position
     optimal play by the player guarantees a win
   - Losing position
     optimal play by their opponent guarantees a loss

4. Lemma
   for NIM, (n,m) is winning $\iff m \neq n$

5. Simpler
   A game H that is reachable from G by some moves is defined to be simpler than G

6. Lemma
   For any game G, either player 1 or 2 have a winning strategy

7. Sum
   Assume G have options $G_i, i \leq q$, H have options $H_i, i \leq l$
   $G + H = \{G_i + H : 1 \leq i \leq q\} \cup \{G + H_j : 1 \leq j \leq l\}$

8. Lemma, copycat principle
   FOr any game, G+G is losing

9. Equivalent $\equiv$
   if $\forall$ game J: $G + J$ is losing $\iff H + J$ is losing, then $G \equiv H$

10. Theorem of $=$
    - $G \equiv G$
    - $G \equiv H \iff H \equiv G$
    - $G \equiv H$ and $H \equiv K => G \equiv K$

11. Main Lemma
    G is a losing position $\iff$ G $\equiv *0$

12. Generalization of copycat principle
$G \equiv H \iff G + H \equiv *0$

13. Lemma

    - For each option G' of G, there is an option of H $\equiv$ to G'
    - For each option H' of H, there is an option of G $\equiv$ to H'

    Then $G \equiv H$

14. Lemma
$*n \equiv *m \iff n = m$

15. Theorem
Assume n_i are distinc power of 2
Then $*(\sum n_i) \equiv \sum(*n_i)$

16. Lemma bitwise xor
if $p, q \leq 2^a$, then $p \bigoplus q \leq 2^a$

17. Balanced and unbalanced

    - balanced if $a_1 \bigoplus a_2 \cdots = 0$
    - unbalanced if $a_1 \bigoplus a_2 \ldots \neq 0$

18. Theorem
A NIM is losing $\iff$ it is balanced

19. MEX
give a subset $S \subseteq N$, mex(S) = smallest element of $N - S$

20. MEX rule
if G is an impartial game which options equivalanet to $\{*s : s \in S\}$
then $G \equiv *(mex(S))$

21. Corollary
For every impartial game G, there exists a natural number $n \in N$ such that $G \equiv *n$