AMSC808N/CMSC828V

Processes on complex networks

Keywords:
network growth,
preferential attachment,
power-law degree distribution,
random failure vs attack,
percolation

Maria Cameron

References

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Growth and preferential attachment lead to power-law degree distribution

A.-L. Barabasi and R. Albert (1999)

- Observed that numerous real-world networks exhibit power-law degree distribution $p_k \sim k^{-\gamma}$
- Argued that this is the result of two factors: (1) growth and (2) preferential attachment
- Proposed a simple growth model leading to $p_k \sim k^{-3}$

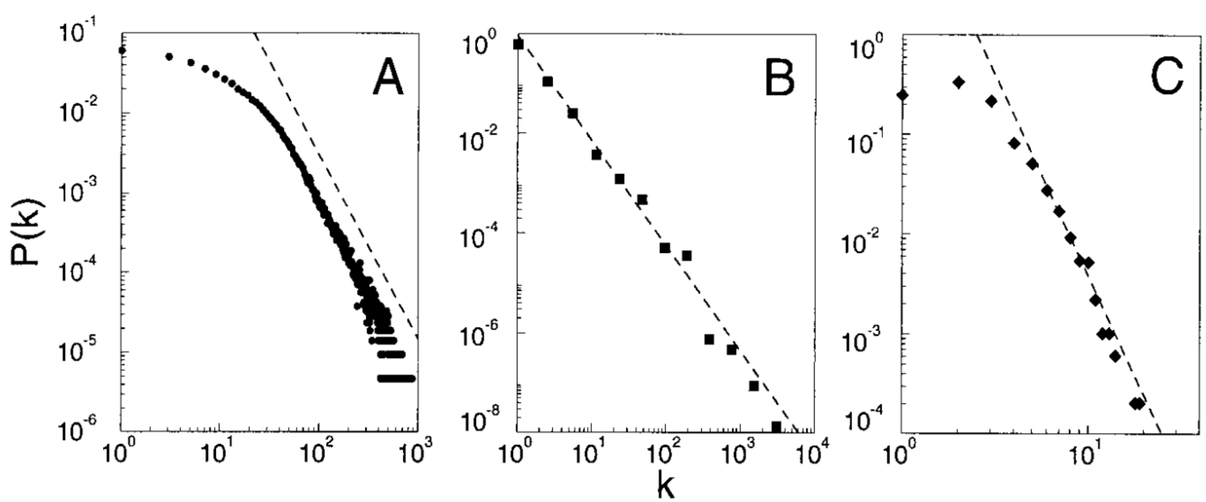
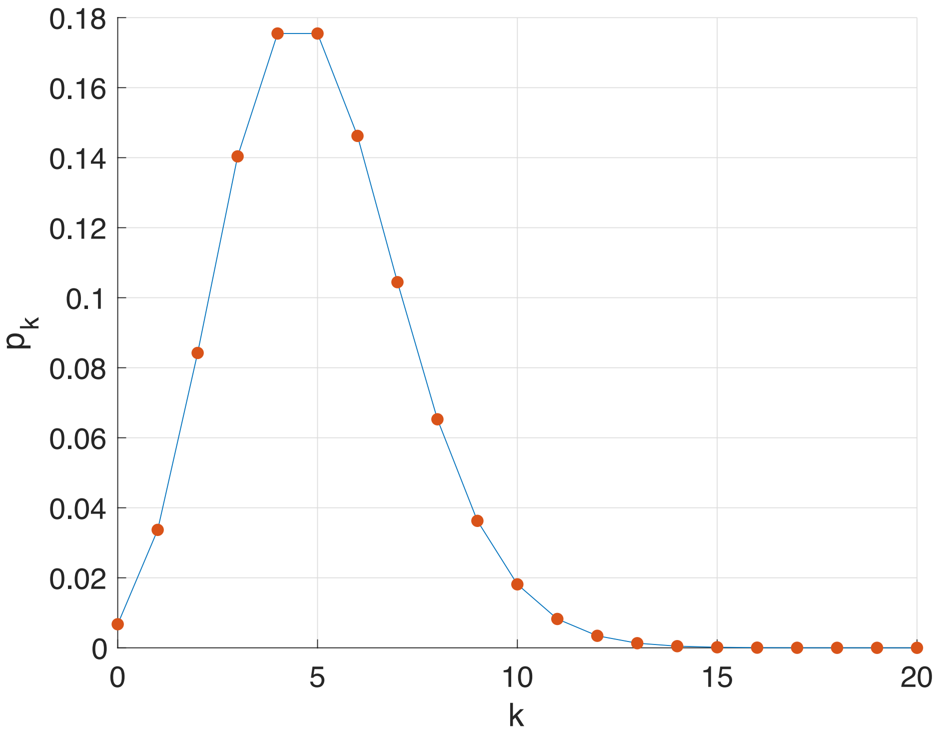
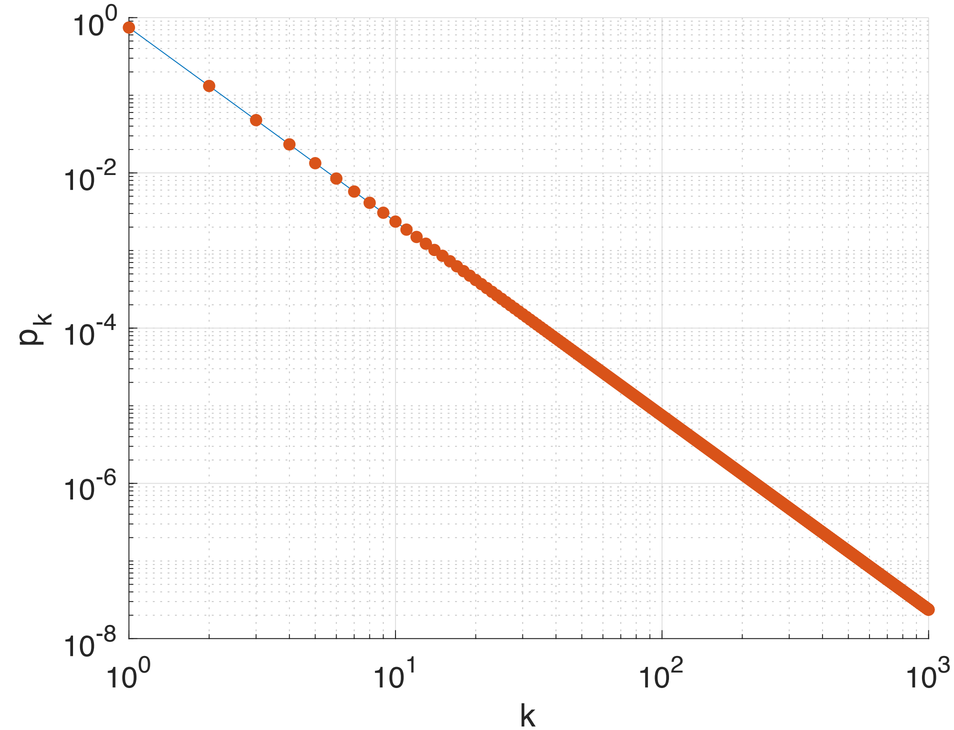


Fig. 1. The distribution function of connectivities for various large networks. **(A)** Actor collaboration graph with $N = 212{,}250$ vertices and average connectivity $\langle k \rangle = 28.78$. **(B)** WWW, $N = 325{,}729$, $\langle k \rangle = 5.46$ (6). **(C)** Power grid data, $N = 4941$, $\langle k \rangle = 2.67$. The dashed lines have slopes (A) $\gamma_{actor} = 2.3$, (B) $\gamma_{www} = 2.1$ and (C) $\gamma_{power} = 4$.

Poisson vs power-law



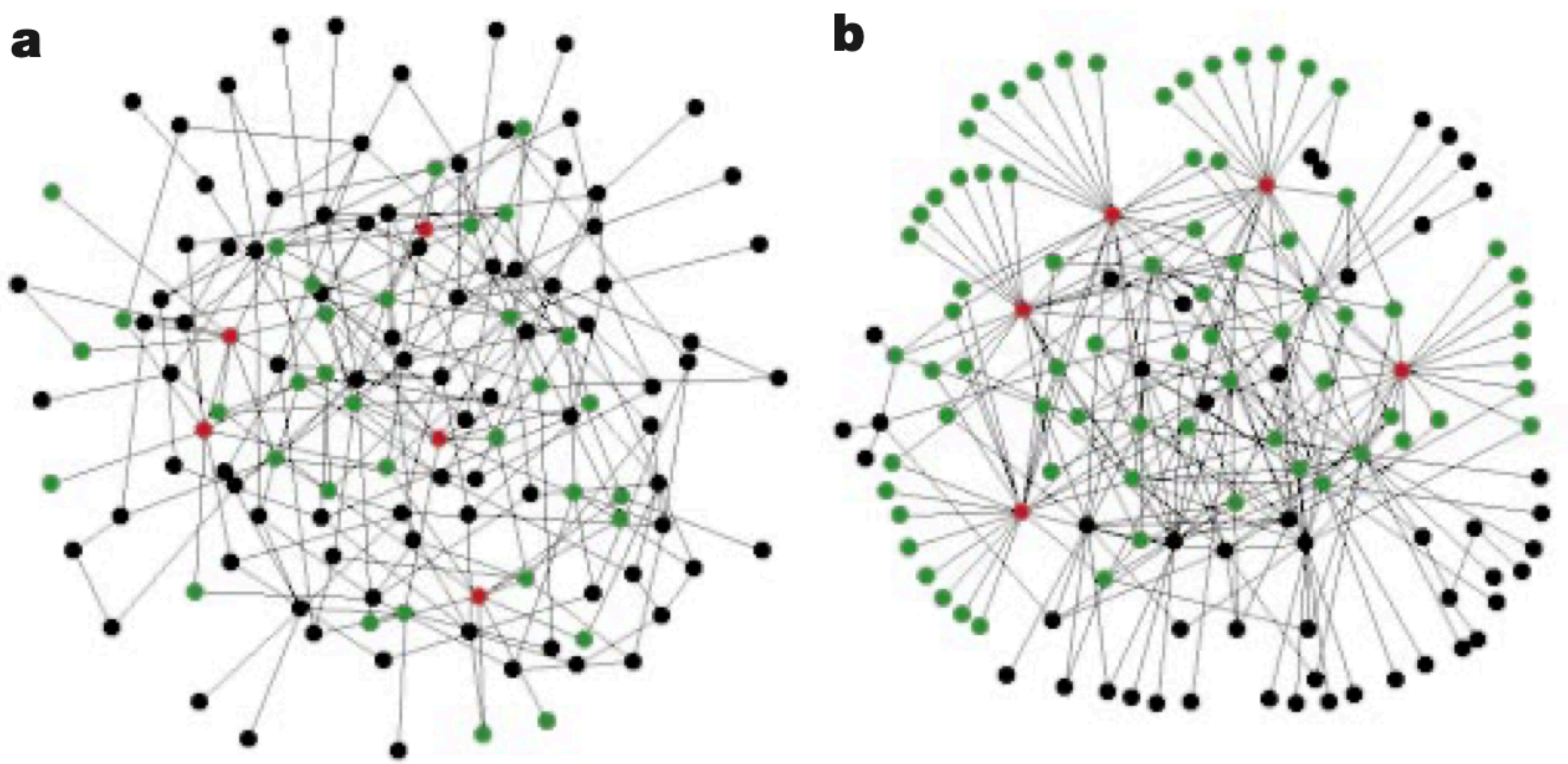
Poisson distribution is sharply peaked at $z = \langle k \rangle$, indicating that there is a characteristic scale for k.



Power-law distribution does not have a characteristic scale.

Exponential vs power-law networks

Figure is from R. Albert, H. Jeong, and A.-L. Barabasi



Exponential

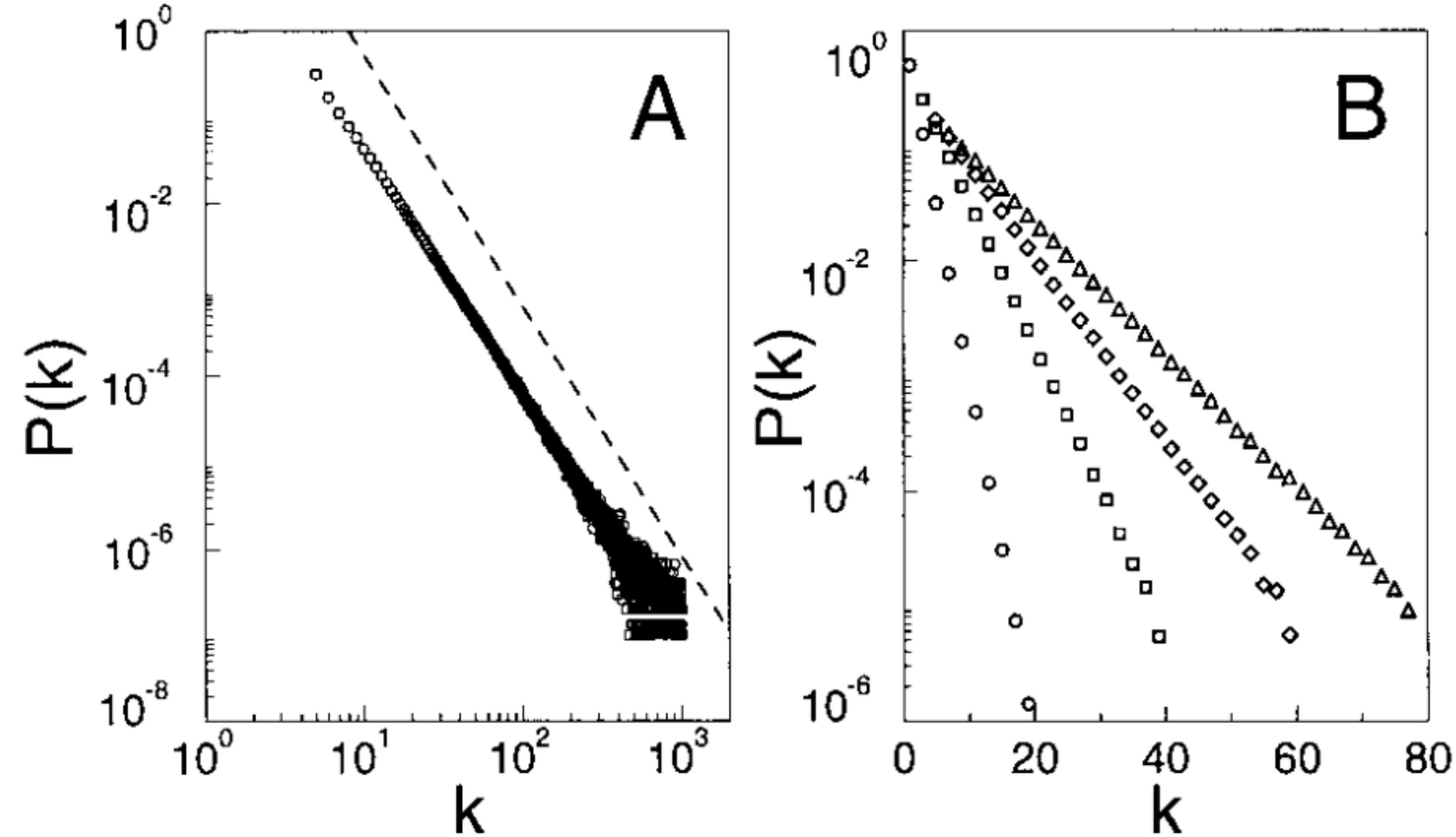
Scale-free

Red: 5 nodes with highest degree, green: their first neighbors.

Barabasi-Albert growth model

Preferential attachment

- **Start** with m vertices and no edges
- **Step 1**: add a vertex and link it to all vertices.
- **Step 2, 3, 4, ...**: add a vertex with m edges and link it to m different vertices. The probability that at step t the new vertex will be linked to vertex i is $P(k_i) = k_i/\Sigma_j k_j$, where k_i is the degree of vertex i.
- After t steps, there will be $m + t$ vertices and mt edges.



(A) The power-law connectivity distribution at t 150,000 (circles) and t 200,000 (squares) as obtained from the model, using $m_0 = m = 5$. The slope of the dashed line is   2.9.

(B) The exponential connectivity distribution for model A, in the case of $m_0 = m = 1$ (circles), $m_0 = m = 3$ (squares), $m_0 = m = 5$ (diamonds) , and $m_0 = m = 7$ (triangles).

Emergence of a power law

A.-L. Barabasi and R. Albert, (1999)

Make time continuous to facilitate calculations

The rate at which a vertex acquires edges is $\dfrac{dk_i}{dt} = \dfrac{k_i}{2t}$. Initially, $k_i(t_i) = m$.

Justification: the rate must be proportional to k_i and all rates must sum up to m.

$$\sum_i \frac{dk_i}{dt} = \frac{1}{2t} \sum_i k_i = \frac{2mt}{2t} = m$$

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$$P\left[t_i > \frac{m^2 t}{k^2}\right] = 1 - P\left[t_i \leq \frac{m^2 t}{k^2}\right] = 1 - \frac{m^2 t}{k^2(t+m)}$$

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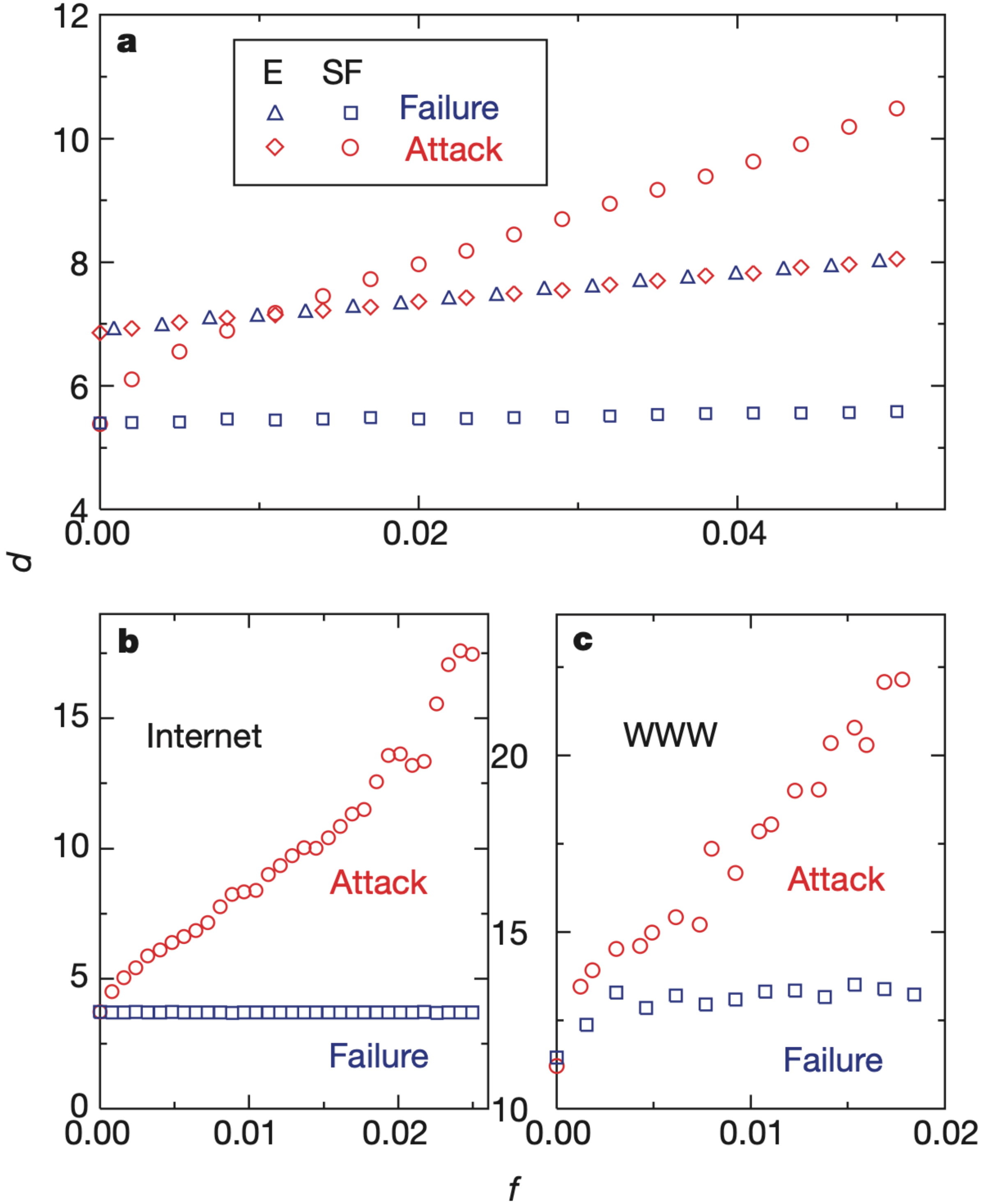
Now, find the pdf: $p(k) = \frac{\partial P[k_i(t) < k]}{\partial k} = \frac{2m^2 t}{k^3(t+m)} \to \frac{2m^2}{k^3}$.

Error and Attack tolerance

R. Albert, H. Jeong, and A.-L. Barabasi

- Two types of random networks: **Poisson** and **scale-free**
- Two types of disturbances: **random failures** and **targeted attacks**.
- **Poisson random graphs** are ***equally tolerant*** to **random failures** and **targeted attacks**.
- **Scale-free random graphs** are ***highly tolerant*** to **random failures** but ***extremely vulnerable*** to **targeted attacks**.



E = “Exponential” = “Poisson” = “Erdos-Renyi”

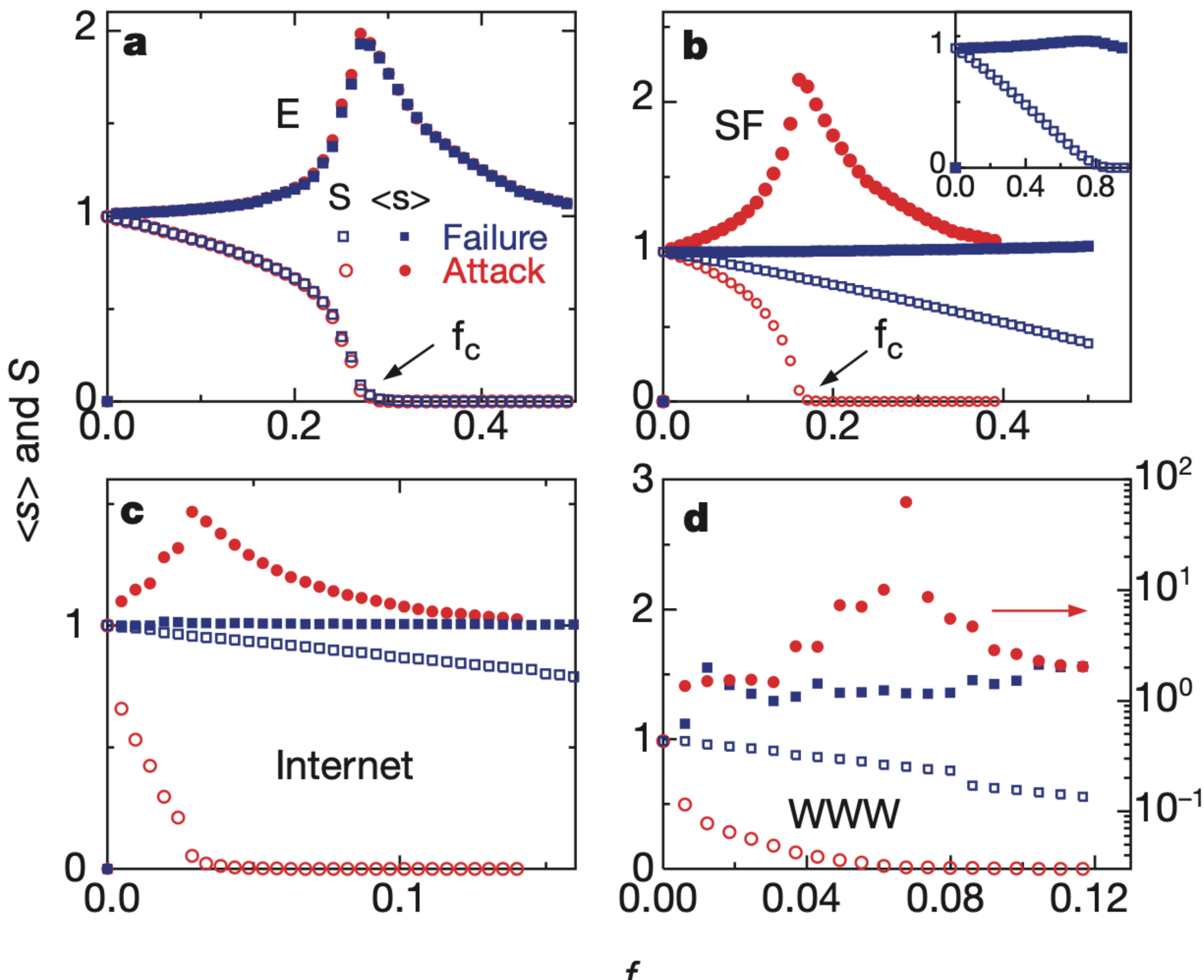
SF = “Scale-free” = “Power law”

d = the average length of the shortest path

f = fraction of removed nodes

Failure = removal of randomly picked nodes

Attack = removal of nodes of highest degree



a
E
S <s>
Failure
Attack
f_c
b
SF
f_c
c
Internet
d
WWW
<s> and S
f

Resilience to random breakdowns

Cohen, Erez, ben Avraham, Havlin, 2000

- Recall the criterion for the phase transition from no giant component to its existence

$$0 = z_2 - z_1 = \langle k^2 \rangle - 2\langle k \rangle, \quad \text{or} \quad \kappa := \frac{\langle k^2 \rangle}{\langle k \rangle} = 2$$

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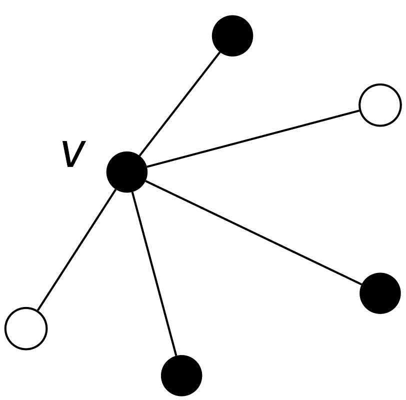
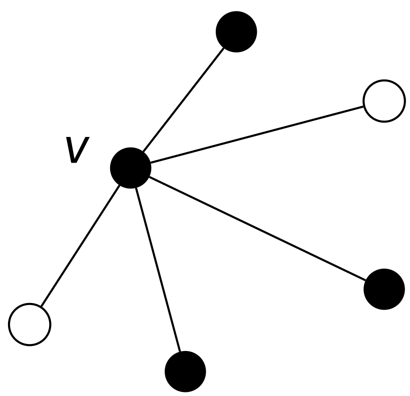
- Imagine that each node is destroyed with probability *p*.



k_0 = the original number of first neighbors of *v*

k = the number of first neighbors of *v* that are not destroyed

$$P'(k) = \sum_{k_0=k}^{\infty} P(k_0) \binom{k_0}{k} (1-p)^k p^{k_0-k}$$



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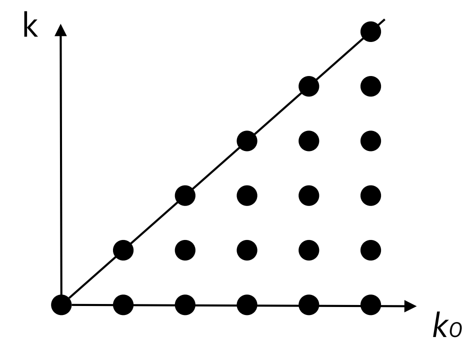

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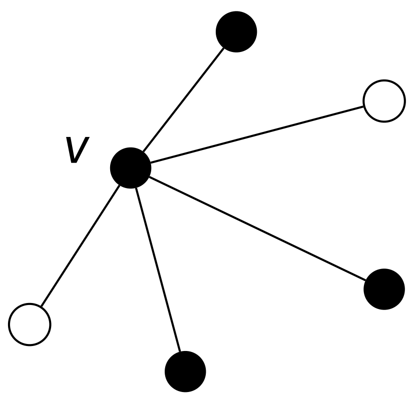

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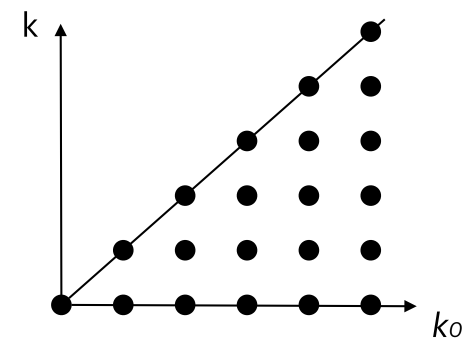

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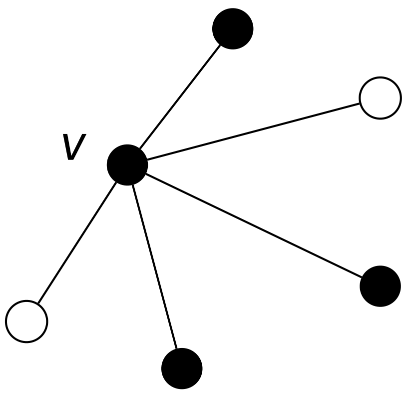

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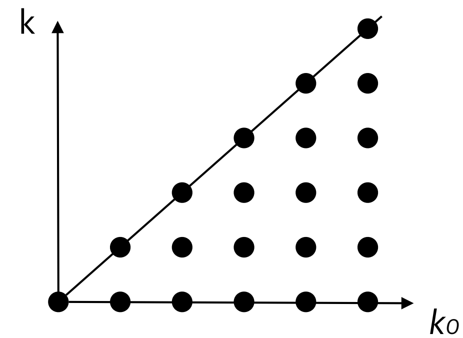

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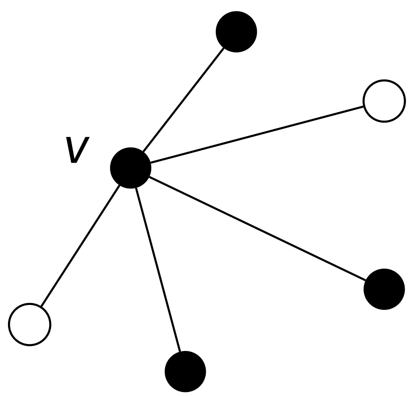

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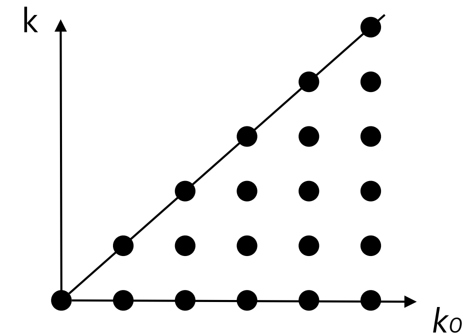

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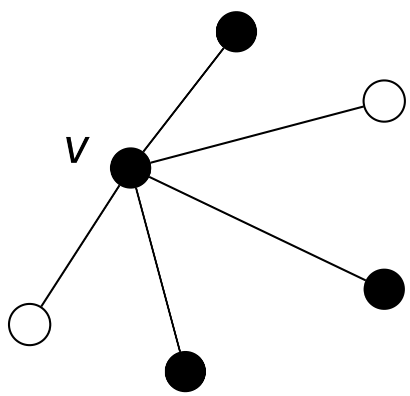

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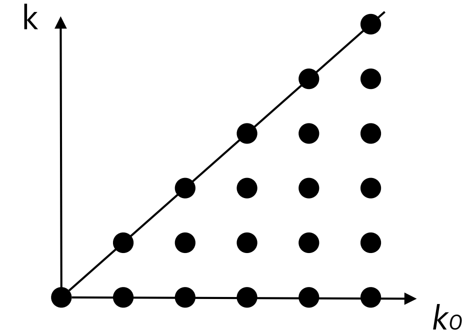

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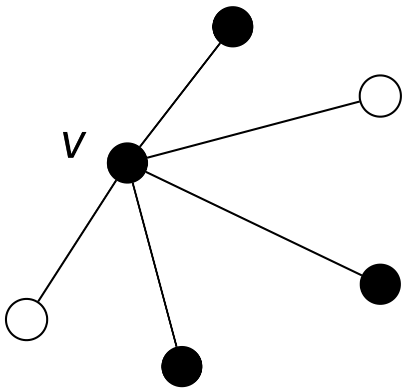

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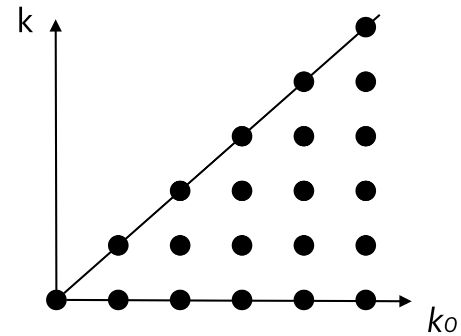

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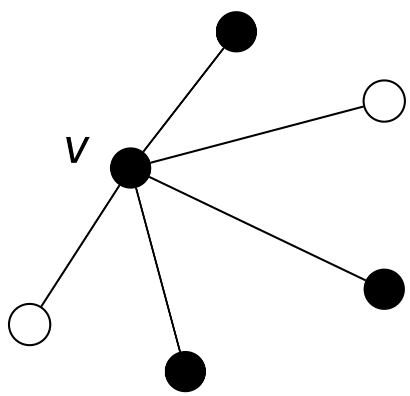

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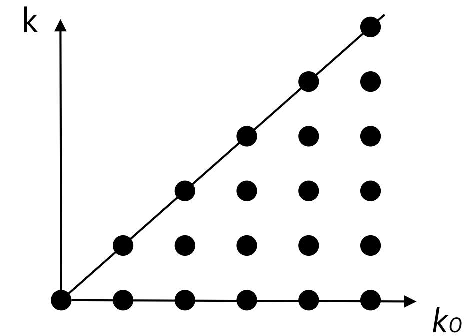

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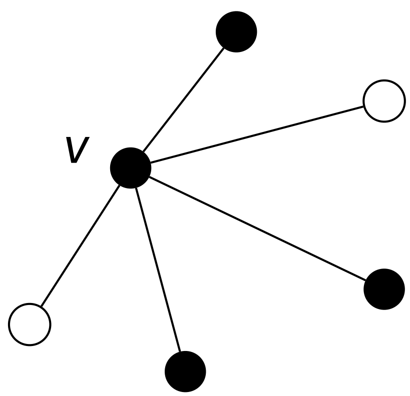

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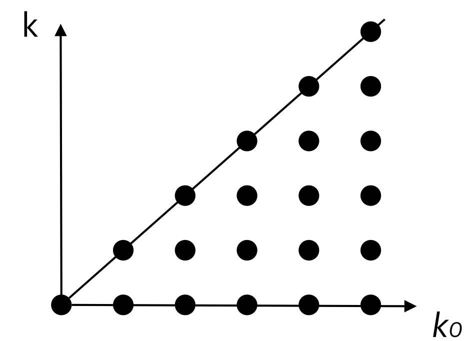

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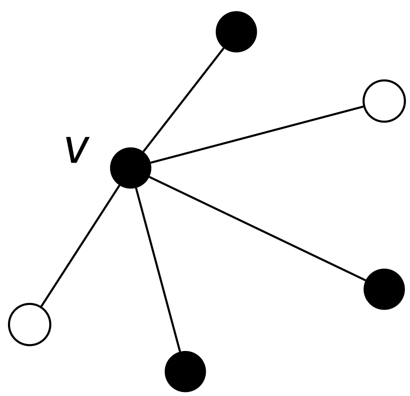

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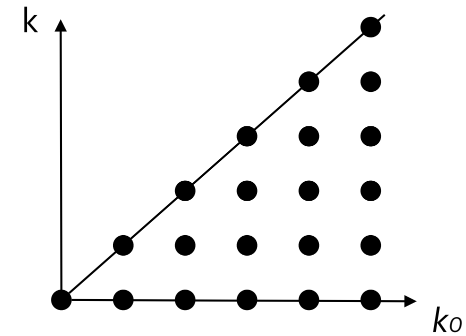

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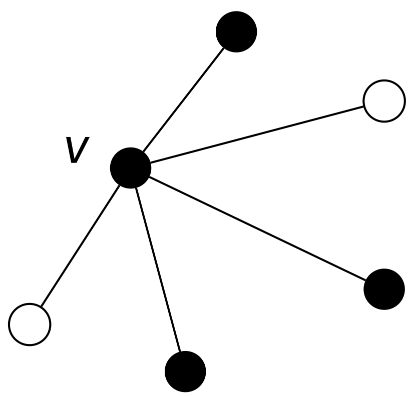

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k = the number of first neighbors of *v* that are not destroyed

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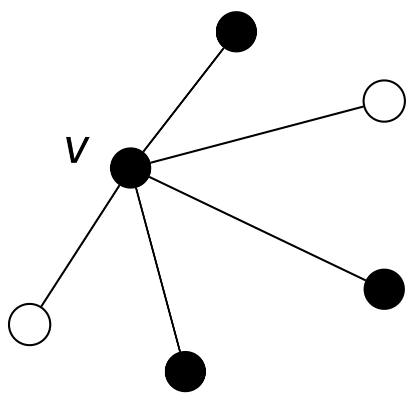

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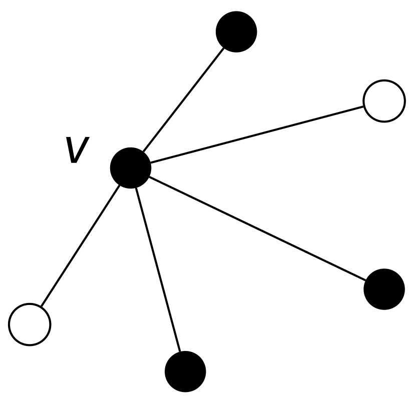
Critical probability of failure

$$\kappa = \kappa_0 (1-p_c) + p_c = 2$$

$$\boxed{p_c = 1 - \frac{1}{\kappa_0 - 1}}$$



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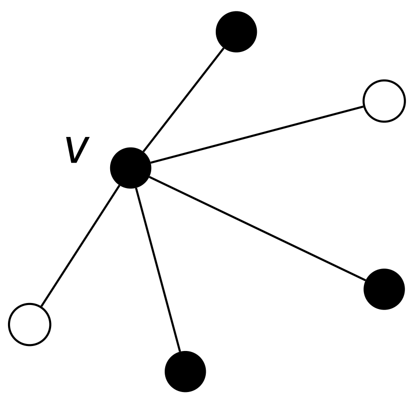

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Random failures in random graphs with power-law degree distribution

$$P(k_0) = ck_0^{-\alpha}, \quad k_0 = m, m+1, \ldots, K$$

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$$\int_m^K ck_0^{-\alpha} dk_0 = \left[c(1-\alpha)k_0^{1-\alpha}\right]_m^K = c(1-\alpha)[K^{1-\alpha} - m^{1-\alpha}] = 1$$

Hence $\qquad c \approx \dfrac{m^{\alpha-1}}{\alpha-1}$

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Estimate maximal vertex degree for a finite network with *N* nodes

$$\int_K^\infty P(k_0) dk_0 = \frac{1}{N}$$

I.e., the probability that a node has at least K first neighbors is 1/N, i.e., we expect to have at most one such a node.

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$$\int_K^\infty P(k_0)dk_0 = c(\alpha-1)K^{1-\alpha} = \left(\frac{m}{K}\right)^{\alpha-1} = \frac{1}{N}$$

$$\boxed{K = mN^{1/(\alpha-1)}}$$

Hence $K \to \infty$ as $N \to \infty$.

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$$p_c = 1 - \frac{1}{\kappa_0 - 1}$$

Most real-world networks:

$$2 < \alpha < 3$$

$$\text{Hence} \quad \kappa_0 \to \infty \quad \text{as} \quad K \to \infty$$

Random failures in random graphs with power-law degree distribution

$$P(k_0) = ck_0^{-\alpha}, \quad k_0 = m, m+1, \dots, K$$

$$p_c = 1 - \frac{1}{\kappa_0 - 1}$$

Most real-world networks: $2 < \alpha < 3$

Hence $\kappa_0 \to \infty$ as $K \to \infty$ which is caused by $N \to \infty$.

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Callaway, Newman, Strogatz, Watts (2000)

- The failure probability is allowed to depend on degree: the probability that a vertex of degree k survives (is occupied) is q_k.

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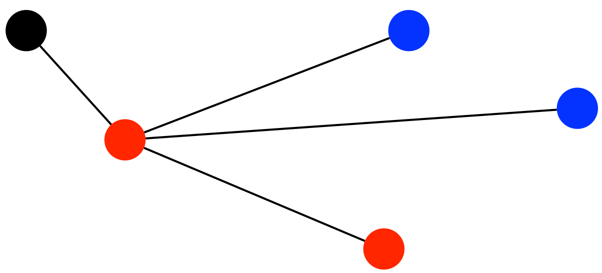
- Note that if $q_k = q$ for all k, then $q = 1 - p$ from Cohen et al.
- Method of generating functions is used. Result from Cohen et al. is rederived and refined.
- Disappearance of the giant component is shown for targeted attack removing highest degree nodes.

Spread of epidemic disease on network

M. Newman (2002)

SIR model: Susceptible → Infecting → Removed
(L. Reed, W. H. Frost, 1920s, unpublished)

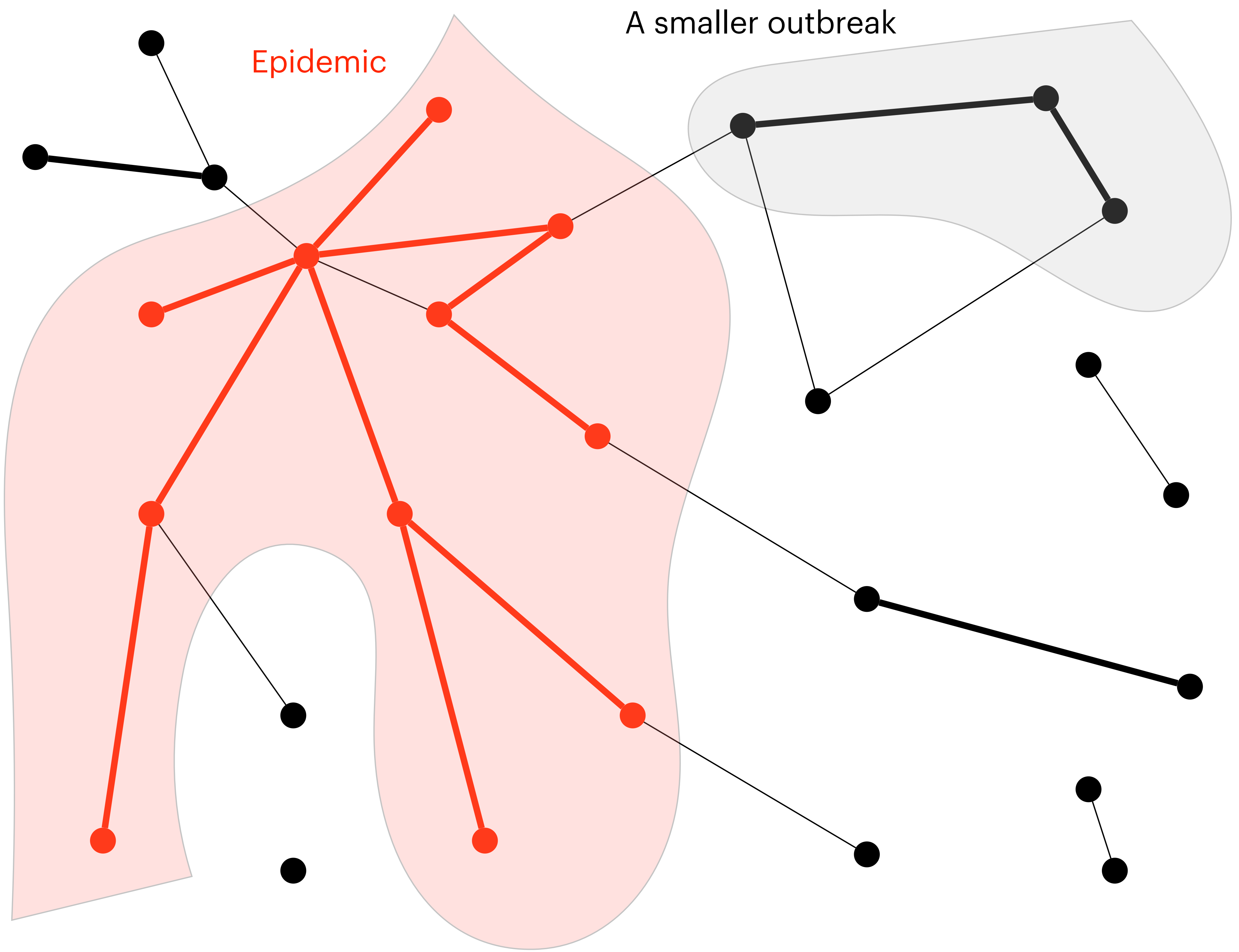
$$\frac{ds}{dt} = -\beta is, \quad \frac{di}{dt} = \beta is - \gamma i, \quad \frac{dr}{dt} = \gamma i \qquad s + i + r = 1$$



r = rate of disease-causing contacts
τ = duration of being infecting

$$T = 1 - e^{-r\tau} = \text{transmission rate}$$

Grassberger (1983): Mapping on the bond percolation problem:
each edge is transmitting with probability T.


A smaller outbreak
Epidemic

$$G_0(x) = \sum_{k=0}^{\infty} p_k x^k = \text{generating function for degree distribution}$$

$$G_1(x) = \sum_{k=0}^{\infty} q_k x^k = \sum_{k=0}^{\infty} \frac{(k+1)p_{k+1}}{\sum_{j=0}^{\infty} jp_j} x_k = \frac{G_0'(x)}{z}$$

$$= \text{generating function for the excess degree distribution}$$

$$\begin{aligned} G_0(x;T) &= \sum_{m=0}^{\infty} \sum_{k=m}^{\infty} p_k \begin{pmatrix} k \\ m \end{pmatrix} T^m (1-T)^{k-m} x^m \\ &= \sum_{k=0}^{\infty} p_k \sum_{k=0}^{m} \begin{pmatrix} k \\ m \end{pmatrix} (xT)^m (1-T)^{k-m} \\ &= \sum_{k=0}^{\infty} p_k (1-T+xT)^k = G_0(1+(x-1)T) \\ &= \text{generating function for distribution of transmitting edges adjacent to a node} \end{aligned}$$

$$\begin{aligned} G_1(x;T) &= G_1(1+(x-1)T) \\ &= \text{generating function for distribution of transmitting edges adjacent to a node} \end{aligned}$$

arrived at by a randomly chosen edge

$$H_1(x;T) = xG_1(H_1(x;T);T) = \text{generating function for the size of transmitting cluster reached from a randomly chosen edge}$$

$$H_0(x;T) = xG_0(H_1(x;T);T) = \text{generating function for the size of transmitting cluster reached from a randomly chosen vertex}$$

$$P_s(T) = \frac{1}{s!}\left.\frac{d^s H_0}{dx^s}\right|_{x=0} = \frac{1}{2\pi i}\oint \frac{H_0(\zeta;T)}{\zeta^{s+1}}d\zeta$$

Recipe for finding the distribution of cluster sizes numerically

$$= \text{probability that transmitting cluster has size } s$$

$$\langle s\rangle = H_0'(1;T) = 1 + G_0'(1;T)H_1'(1;T)$$

$$= \text{average outbreak size}$$

$$H_1'(1;T) = 1 + G_1'(1;T)H_1'(1;T) = \frac{1}{1 - G_1'(1;T)}$$

$$\langle s\rangle = H_0'(1;T) = 1 + \frac{G_0'(1;T)}{1 - G_1'(1;T)} = 1 + \frac{TG_0'(1)}{1 - TG_1'(1)}$$

$$= \text{average outbreak size}$$

If T is below the epidemic threshold

Critical transmission:

$$\boxed{T_c = \frac{1}{G_1'(1)} = \frac{G_0'(1)}{G_0''(1)} = \frac{\sum_{k=1}^{\infty} kp_k}{\sum_{k=1}^{\infty} k(k-1)p_k}}$$

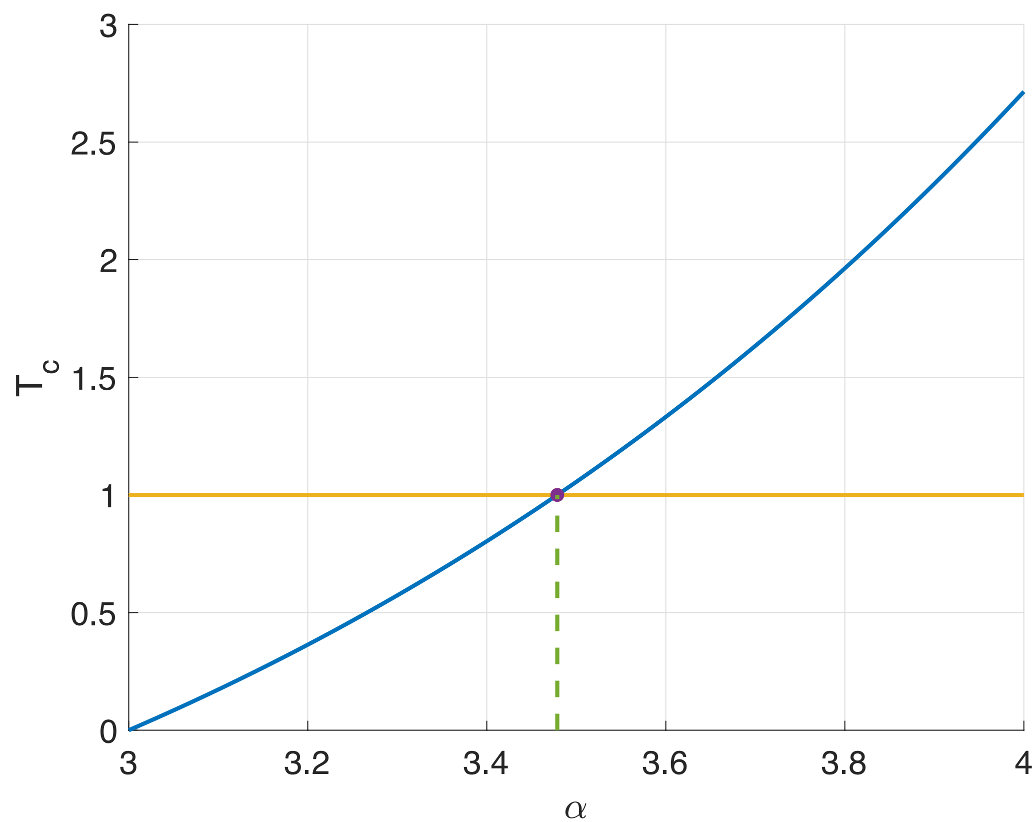
- for $T > T_c$ we have a giant component connected by transmitting edges (an epidemic);
- for $T < T_c$ all components are small (no epidemic).

Critical transmission probability for power law degree distribution

$$p_k = \frac{k^{-\alpha}}{\zeta(\alpha)}, \quad \zeta(\alpha) = \sum_{k=1}^{\infty} k^{-\alpha} = \text{Riemann zeta function}$$

$$T_c = \frac{\sum_{k=1}^{\infty} k p_k}{\sum_{k=1}^{\infty} k(k-1) p_k} = \frac{\zeta(\alpha-1)}{\zeta(\alpha-2) - \zeta(\alpha-1)}$$

- If $\alpha \leq 3$ then $T_c = 0$, hence, there is always an epidemic.
- If $3 < \alpha < \alpha_c \approx 3.4788$, then $0 < T_c < 1$, hence, there is epidemic threshold.
- If $\alpha \geq \alpha_c \approx 3.4788$, no epidemic can occur unless $T = 1$.



- For $T>T_c$, we redefine H_0 as the generating function for outbreaks other than the giant component.
- Note: we cannot use H_0 for the giant cluster as the "no loop" assumption no longer holds.

$$H_0(1;T) = \sum_{s=1}^{\infty} P_s(T) = 1 - S(T), \quad S(T) = \text{fraction in the giant component}$$

$$H_0(1;T) = G_0(u;T), \quad \text{where} \quad u = H_1(1;T)$$

$$H_1(1;T) = G_1(H_1(1;T);T), \quad \text{hence we get an equation for } u: \quad u = G_1(u;T)$$

The quantity u is the probability that the vertex at the end of a randomly chosen edge remains uninfected during an epidemic i.e., that it belongs to one of the finite components.

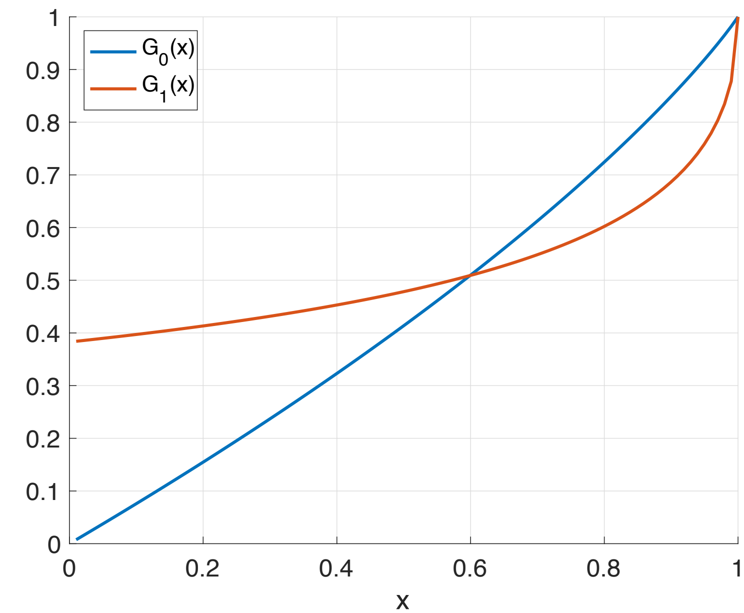
G_0, G_1, u, and S for power law degree distribution

$$Li_\alpha(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^\alpha} = \text{polylogarithm}$$

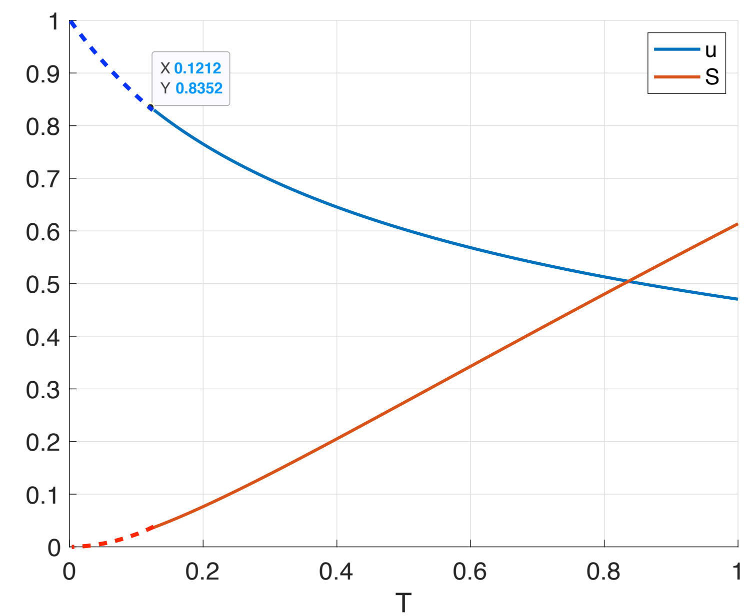
```
function SIR()
close all
fsz = 16;
% power law degree distribution p_k = k^(-a)/zeta(a)
a = 2.5;
G0 = @(x)polylog(a,x)/polylog(a,1);
G1 = @(x)polylog(a-1,x)./(x*polylog(a-1,1));
x=linspace(0,1,100);
%
figure(1);
hold on;
grid;
plot(x,G0(x),'Linewidth',2)
plot(x,G1(x),'Linewidth',2)
legend('G_0(x)','G_1(x)');
xlabel('x','Fontsize',fsz);
set(gca,'Fontsize',fsz)
%
% critical transissibility = 0, hence, there is always an epidemic
nt = 100;
t = linspace(0,1,nt); % transimissibility
u = zeros(nt,1);
S = zeros(nt,1);
for i = 1 : nt
    T = t(i);
    u(i) = fzero(@(x)G1(1-T+T*x)-x,0.3);
    S(i) = 1 - G0(1-T+T*u(i));
end
figure(2);
hold on;
grid;
plot(t,u,'Linewidth',2)
plot(t,S,'Linewidth',2)
legend('u','S');
xlabel('T','Fontsize',fsz);
set(gca,'Fontsize',fsz)
end
```

$$G_0 = \frac{Li_\alpha(x)}{Li_\alpha(1)}$$

$$G_1 = \frac{G_0'(x)}{G_0'(1)} = \frac{Li_{\alpha-1}(x)}{xLi_{\alpha-1}(1)}$$



u = probability that a vertex at the end of a random edge stays uninfected duding the epidemic;
S = fraction in the giant component.



The quantity u is the probability that the vertex at the end of a randomly chosen edge remains uninfected during an epidemic i.e., that it belongs to one of the finite components.

The probability that a vertex does not become infected via one of its edges is

$$v = 1 - T + Tu,$$

which is the sum of the probability (1-T) that the edge is non-transmitting, and the probability Tu that it is transmitting but connects to an uninfected vertex. The total probability of being uninfected if a vertex has degree k is v^k, and the probability of having degree k given that a vertex is uninfected is

$$\frac{p_p v^k}{\sum_{k=0}^{\infty} p_k v^k}. \quad \text{This distribution is generated by} \quad \frac{G_0(vx)}{G_0(v)}.$$

The average vertex degree outside the giant component:

$$z_{\notin Giant} = \frac{d}{dx}\frac{G_0(vx)}{G_0(v)}\bigg|_{x=1} = \frac{vG_0'(v)}{G_0(v)} = \frac{vzG_1(v)}{G_0(v)}$$

Recall that $G_1(x;T) = G_1(1-T+xT)$. Hence $G_1(v) = G_1(u;T) = u$.

Also recall that $G_0(v) = G_0(1-T+Tu) = G_0(u;T) = 1-S(T)$.

Hence $z_{\notin Giant} = \dfrac{vzG_1(v)}{G_0(v)} = \dfrac{(1-T+Tu)u}{1-S(T)}z$

The average vertex degree inside the giant component:

$$z_{\in Giant} = \frac{d}{dx}\frac{G_0(x)-G_0(vx)}{G_0(1)-G_0(v)}\bigg|_{x=1} = \frac{vG_0'(v)}{G_0(v)} = \frac{1-vG_1(v)}{1-G_0(v)}z = \frac{1-u(1-T+Tu)}{S}z$$

Mean degrees for the power law degree distribution

$$p_k = \frac{k^{-\alpha}}{\zeta(\alpha)}, \quad \zeta(\alpha) = \sum_{k=1}^{\infty} k^{-\alpha} = \text{Riemann zeta function}$$

$$z = \frac{Li_{\alpha-1}(1)}{Li_{\alpha}(1)} = \text{the mean degree}$$

$$\alpha = 2.5$$