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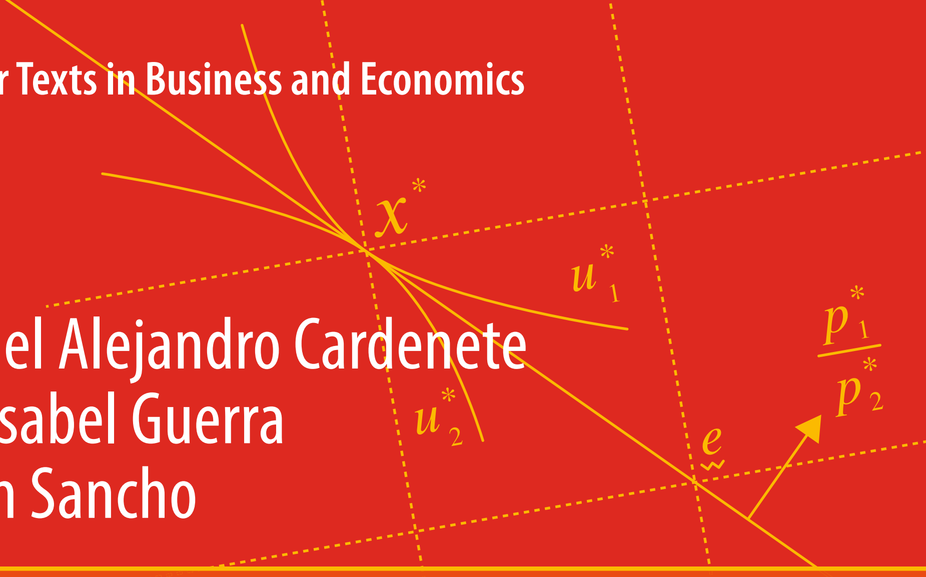

Applied General Equilibrium

An Introduction

Second Edition

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An Introduction

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ISSN 2192-4333 ISSN 2192-4341 (electronic)
Springer Texts in Business and Economics
ISBN 978-3-662-54892-9 ISBN 978-3-662-54893-6 (eBook)
DOI 10.1007/978-3-662-54893-6

Library of Congress Control Number: 2017942831

Printed on acid-free paper

This Springer imprint is published by Springer Nature
The registered company is Springer-Verlag GmbH Germany
The registered company address is: Heidelberger Platz 3, 14197 Berlin, Germany

Foreword

The economic theory of general equilibrium is rooted in Walras (1874) and reaches its formal development, as we know it today, with the fundamental contributions of Arrow and Debreu (1954). The development of this theory has been nurtured by the efforts of some very bright economists and mathematicians who have refined the many different facets arising from it. However, it is the pioneering work of Professor H. Scarf (1969) that has moved this abstract theory to the field of real empirical problems. Scarf posed and solved the problem of computing the equilibrium (relative prices and quantities) of the so-called Walrasian model. Professors J. Shoven and J. Whalley, two of Scarf's students, are the first to demonstrate the power of the empirical analysis developed from the theory, when they set out to analyze the economic impact of some fiscal reforms in the US economy.

Since then, the area and discipline that we know in the literature as Applied General Equilibrium analysis has not stopped growing. The numerous papers exploring empirical applications of the basic theory of General Equilibrium may appear to have very little in common. They raise basic questions and issues in areas as diverse as Public Finance (what is the impact of a tax reform on the welfare of a country's households and on government revenues?), Trade (what are the economic gains resulting from a trade agreement between two or more countries or regions, and how they are distributed?), Environmental Economics (how do green taxes and/or limiting emission permits impact CO_2 emissions?), Regional and Urban Economics, etc. However, appropriate quantitative answers to these questions can all be found thanks to the *instruments* and *methodology* provided by Applied General Equilibrium. We emphasize here the words "instruments" and "methodology" because this is what these papers devoted to quite different issues have in common.

Those who wish to learn about the questions posed and the answers obtained in the works mentioned above, and many others, need know neither the instruments nor the methodology. Much the same as we do not need to be a composer, a conductor, or a musician to listen to, and enjoy, the melodies that they create. However, for those who want to play an active role in studying these and other interesting economic issues and would like as well to be able to come up with answers and other interesting questions of their own, I would say they will be lucky to find this book in their hands, and I would advise them to read it carefully. In this book, they will find the tools and the methodological ingredients needed to create and play their own tune. And I say this because attentive readers of this book can hopefully end up being composer, conductor, and musician all at the same time.

To begin with, we can find the basic elements of general equilibrium theory (in Chap. 2), which prepares us to rigorously develop and build the appropriate models needed for answering our questions. Then we also note that readers are introduced to the ins and outs of Applied General Equilibrium by way of simple basic models (in Chap. 3), which progressively become more elaborate (in Chap. 4) until they reach a considerable level of complexity (in Chap. 5). In Chap. 6, readers are confronted with the actual data of the economy and learn how it should be used when a researcher wishes to tackle a relevant problem. Unfortunately, data for an economy is usually scattered and a first and unavoidable task is to compile and organize the data so that it has the economic consistency that the analysis requires. The rules of consistency and format for an economy's data, and which are necessary and quite useful for the task of developing Applied General Equilibrium models, have been formalized in what is known in the literature as a Social Accounting Matrix. What these matrices are, how they are constructed, and how they can be used for model building are also discussed in Chap. 6. Readers will find here the fundamental bond that connects the theoretical models with the data using a procedure called calibration. Finally, in Chap. 7 readers can "listen to the melodies" of some real examples that are the result of applying the methodology and the instruments they have been learning in previous chapters.

Upon finishing the book, readers should have sufficient preparation to start to compose and perform their own tunes. But even if they do not feel confident enough to do it right away, I am sure that the value added and compensation for their efforts will not be in vain. It is much more interesting to hear a melody when we can read the score! I am confident that readers will find that this book provides them with both accessible and rigorous preparation for mastering the instruments and the basic methodology of this wide literature, allowing them to now play their own music.

Universitat de Barcelona
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Antonio Manresa

Preface to the Second Edition

We are happy to make available to the general public this second edition of our book. The first edition has had a 5-year run since its publication in 2012, and additions, adaptations, and a few corrections were in order. After using the book to prepare classes and presentations, it became clear to us that a few changes would be helpful in making the contents of the book wider in reach while at the same time maintaining its self-contained focus. The fact that the first edition was self-contained was not casual but a carefully planned fact that, we believe, is what makes this book a bit different from other worthy contending books in the field. New developments and new results keep appearing and they certainly enrich the field. But more, per se, is not always better. It is therefore necessary to open the lens, but we had better do it in a selective way that should fit adequately within the established structure of the book. Hence, all the selected new material appears in one of the current chapters and we do not add any new chapters.

Experience indicates that it would be useful to offer more information regarding the properties of equilibrium and its implications for policy analysis. The idea of multiplier matrices linking resource allocation with external-to-the-model impulses initiated, for instance, in government decisions is therefore incorporated in Chap. 2, both in linear and nonlinear versions of the standard general equilibrium model. Linear versions of equilibrium models have been and still are immensely popular and of widespread use for policy assessment and for regional, national, and inter-country structural analysis. Erroneously, however, linear models are sometimes presented as competing—hence essentially different—models with those of standard general equilibrium models. We show they belong to the same breed of models and what distinguishes them is just a subset of assumptions regarding the description of the technology and the availability of resources. These specific assumptions are not a small matter, certainly, and analytical results will be bound to be different, but nonetheless this does not make them unrelated models.

We like general equilibrium models because, among other things, they are able to provide welfare conclusions. Whenever and however an economy is shocked, resource allocation will readjust demands and supplies and in the process the previous situation of agents will change. Two questions are relevant here. One is if the economy as a whole has moved to a better or worse equilibrium state. The second one has to do with winners and losers in the distribution game. Even if in terms of aggregate indicators we could conclude that the economy is performing better (or worse), the fact remains that some individuals may benefit from the change in equilibrium while others may see their state deteriorating. In the new Chap. 5, we introduce the theoretical highlights that will give us the tools to conclude sensible things regarding welfare, both economy wide and individual wise.

We economists thrive when we have access to good and abundant data and we languish when there is a lack of it. We need data because it is fundamental for the implementation of empirically based economic analysis. When data is scarce, sometimes we really need to be able to produce it. In the new Chap. 6, we explain a class of projection algorithms that yields consistent data when we face incomplete information regarding the statistical structure of the economy. Incomplete data is projected so as to closely resemble known but older data while incorporating whatever partial and new information that is available. This type of procedures, believe it or not, may save your day at some point in your career as an economist. You had better pay attention to it.

The economy is a complex structure and the government even more so. Policy decisions by the government are always subject to trade-offs. Choosing what looks like a favorable course of action may have adverse effects on other social objectives. If the government raises payroll taxes today to finance retirement pensions tomorrow, which looks like a good thing to do, we will immediately find that the raised tax will affect unemployment, which is bad. When labor becomes more expensive to hire, the tendency will be for firms to hire fewer workers—an inevitable consequence of the law of demand. For the trade-offs associated with many desirable but competing criteria, the clearer the pros and cons, the better. In Chap. 7, we will tell you how we can combine the nature of general equilibrium analysis with multi-criteria programming and thus making multiple and competing social goals as compatible as possible, under the constraints that the economy certainly imposes, of course. Also in Chap. 7, we introduce a few comments, even if briefly, on the characteristics of some dynamic versions of general equilibrium models. This is a book devoted to showing you how to build, from scratch, a general equilibrium model that represents an economy at a point in time. But time itself flows and economic magnitudes change over the course of time following specific rules that derive from the underlying structure of the economy. A dynamic model tries to capture this evolution over time. We will not get deep into them, and we only aim at making you aware of the role that time plays within a general equilibrium framework. In the learning process, the building of point-in-time models comes first; we usually refer to these versions as static general equilibrium models. Only when you achieve mastery with static models should you consider moving into building dynamic ones.

We have also expanded the list of questions and exercises and undertaken minor, usually hard to pinpoint, changes and additions in the text. We are confident that, with all these major and minor extra materials, the new version will definitely improve on the first edition and will hopefully provide a better reading and learning experience. Lastly, we thank the publisher, Springer, for being open to back this new edition and persuade us to carry it out. We also thank keen readers who have detected and pointed out small errors and typos. All have been amended, and we now hope that none (or very few) are left. Encouraging book reviews of the first edition by K. Turner (Economic Systems Research, vol. 24(4), 2012) and M. Alvarez (Review of Regional Studies, vol. 42, 2012) are also gratefully acknowledged.

Barcelona 2017

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Ferran Sancho is Professor in the Department of Economics at the Universitat Autònoma de Barcelona since 1992. His articles have appeared in the *International Economic Review*, the *European Economic Review*, the *Review of Economics and Statistics*, *Economic Theory*, *Economic Modelling*, and the *Journal of Forecasting*, among others. He has visited the University of California at Berkeley in 1992 and

2003, the Federal Reserve Bank of Minneapolis in 1992, the Universidad Pablo de Olavide in 2010, the Universidad Loyola Andalucía in 2016, and the Universitat de Barcelona in 2017. He has also been Chair of the Department of Economics, Vice-chancellor, and President of the Universitat Autònoma de Barcelona.

1 Introduction

Perhaps one of the most exciting aspects of economics is its ability to provide inner views of the functioning of the economic system that are not apparent at all to the layman and, sometimes not even so, to many of the economics trained professionals. The real world economy is a rather complex and badly understood system, but nowadays economists know much more about it than economists from 20 or 30 years ago. And of course this is as it should be for the sake of the advancement in science. Despite all the accumulated knowledge and all the new advances, economics remains a kind of mysterious affair for the common folk, notwithstanding that if very few people pretend to know anything from astrophysics or the medical sciences, aficionados are not scarce as far as economic chat goes. Since the equivalent of the penicillin, or the more modest aspirin, has yet to be discovered in economics, people find much room for intellectual and friendly speculation on how the economy—as our social body—can be put under our firm control.

A crucial aspect that helps to explain why economic developments are sometimes difficult to understand, let alone foresee, is the interdependent nature of the economic system. As such, it is composed by pieces that interact among each other in complicated and quite often non-obvious ways and the overall composite effect of those interactions is not transparently perceived. Economic theory is helpful in providing sound grounds for the understanding and interpretation of economic phenomena with the so called *caeteris paribus* clause (e.g., all else being equal, an increase in the price of ale will reduce the quantity of ale demanded). This way of thinking, also known as partial equilibrium analysis, is rigorous but limited. It is rigorous because it instills a much needed methodological discipline in economic reasoning (in fact, in all scientific reasoning). As the assumptions involving a restricted set of circumstances are made clear and the causal links among the relevant variables are identified, economists learn valuable information concerning the mechanism linking stimuli and response implicit in the studied phenomena. The approach is limited, however, because in an actual economy 'all else' is not equal

M.A. Cardenete et al., *Applied General Equilibrium*, Springer Texts in Business and Economics, DOI 10.1007/978-3-662-54893-6_1

and the sum of the parts do not always yield an accurate and coherent representation of the total.

The limitations of partial equilibrium analysis would not be all that important if we were to content ourselves with qualitative appraisals about the general direction of how one economic variable is influenced and in turn influences other economic variables. Economics, however, is a social science that pretends to be useful for understanding the real world and for acting upon it via policies. Therefore only in very specific instances the *caeteris paribus* approach will be a sufficient empirical tool for drawing conclusions or suggesting policy recommendations. Clearly, better instruments are necessary. Fortunately, economics does have at its disposal a small but powerful arsenal of tools that if judiciously used can provide sensible answers to complex problems.

We are concerned in this book with general equilibrium analysis, one of the analytical tools that can address the question of how to integrate the smaller parts of the economic system in a comprehensive economy-wide setting. By general equilibrium analysis we mean the neoclassical approach started by Walras (1874a, b) and culminated by Arrow and Debreu (1954), among others. The basic tenets of the neoclassical approach are that economic agents are rational, prices are flexible and, henceforth, markets tend to clear. This is by no means the only approach to modeling an economy-wide system but its appeal arises from the distinct advantage of laying its foundations at the individual level of consumers and firms.

A real world economy contains many consumers, also referred to as households, and firms. Consumers make decisions on what goods and services to buy and in what amounts; simultaneously they also decide how much labor services they wish to offer in the marketplace and, if they possess property rights on capital, they also want to obtain a return on the amount of capital they provide to firms for production. Firms plan their production schedules to satisfy demand and in the process obtain a profit. These decisions are independently taken on a daily basis by hundreds of thousands of individuals so the question arises of how they interact among each other to jointly give rise to a system that, most of the time, performs in an acceptable manner. On the aggregate, consumers do see their needs satisfied and firms indeed obtain profits, although the picture may not be rosy for each and all of the economic agents involved. Some people cannot find work and some firms have to shut down for lack of profitable business.

Economic theory has quite a bit to say on how markets are organized and how production meets the needs of consumption. Economic knowledge is built with the use and help of models, which are simple representations of the way agents behave and interact. Even at their simplest possible level, however, models are complicated structures and more so when models deal with the economic system as a whole. With the help of the well-established theory of general equilibrium, economists have designed quite complex, large-scale models based on actual economic data, models that have been proven to be dependable tools for analyzing a variety of policy issues. This type of modeling approach is known as Applied general equilibrium analysis. It is also commonly referred to as Computational general equilibrium (*CGE*), since these applied models rely on computational techniques to be solved.

While these models have been incorporated in mainstream economic analysis and their use has expanded both in depth and scope, we feel, however, that there is a lack of bibliographical sources that can be used in teaching at the senior undergraduate and graduate levels. Most if not all of the published general equilibrium literature deals with the analysis of specific issues in specialized research papers. These papers, valuable as they are, leave out however the dirty details of model design and construction: data gathering and data reconciliation, parameter specification, choice of functional forms, computer implementation, and others. Beginners in the area have to struggle deciphering material and reading between lines to grasp the modeling efforts beneath brief model descriptions and lengthy discussion of results. Our aim in writing this monograph is to be as explicit as possible with the details but at the same time trying to avoid impenetrable complexity. Any student of economics from her senior year up should be able to follow these lectures; all that is required is a good background in microeconomics and basic optimization and, of course, the time and willingness to devote some sustained effort in the learning process.

We have observed a tendency, which we definitely disavow, consisting in developing economic models of the 'black-box' variety using packaged software. These so-called modelers may be able to assemble a working model but they have little or no real inkling of its inner workings or of its internal economic logic. It is of course much more costly to build a model from the base, thinking of the relevant issues and policies that merit analysis, designing the modeling blocks that are needed for a rigorous analysis of those selected issues, assembling and organizing the data, preparing the computer code, running the simulations, and finally evaluating and interpreting the results. Frankly, all this is not trivial at all. The temptation to rely on prepackaged software is there for the taking. But we believe that what characterizes a good economist is that she ought to be able to understand, and explain, all that has been done step by step and leaving nothing aside. Needless to say we do not wish to transform an economics student into a full policy analyst, it is simply too complicated a task for a monograph like this. The focus is on the economics of general equilibrium and its empirical modeling, and not so much on the programming of computer code. There are good options for programming, but we particularly like *GAMS* (General Algebraic Modeling System, see Brooke et al. 1988). Some examples that can be executed using the demo version of the *GAMS* software will be provided but be aware that this is not the place for learning *GAMS*. We encourage students to use the free and generously available online resources to throw themselves wholeheartedly into acquiring the basics of a programming language. The rewards, we hope, will be plentiful and will provide successful economics students with a distinctive skill that will promote their employability and will enhance the quality of their professional profile.

2 An Overview of General Equilibrium Theory

2.1 Agents, Behavior and Markets

As economists we are usually interested in how production is organized and in how whatever is produced is eventually distributed among consumers. All these activities take place within specific institutions we know as markets. What condition markets' outcomes, i.e. prices of goods and services and quantities traded, are agents' behavioral characteristics and the market mechanisms that emanate from them, namely, the so-called law and supply and demand. It is common to distinguish two large and distinct groups of agents—households and firms. Each of these groups plays a different role in the marketplace and in the whole economic system as well, depending on the particular type of commodity being traded.

The role of firms is to organize and accomplish the production process of commodities or services, which are then supplied and sold in markets. For production to take place, firms become demanders and buyers of primary (or non-produced, like labor or capital) and non-primary (or produced, like iron or energy) goods and services that will be subjected to the transformation process we call production. The goods that enter into the production process are referred to as inputs while the goods that result are called outputs. In playing this common role, firms are assumed to share a common grand objective: of all possible production plans, they will choose the one that yields the greater return to the firm. For return we mean the difference between the income associated to selling the production plan and the expenditure in inputs needed to make that particular plan possible. This behavioral assumption is basic in economics and is called profit maximization. Nonetheless, the specific way these production activities are undertaken may differ among firms, the reason being that not all firms share or can effectively use the same technology. By technology we mean all the available ways, or recipes, for some good or service to be produced. Electricity producing firms, for instance, aim at generating their output in the most profitable way for them but it is clear that different technological options exist. While some firms will produce electricity using plants of nuclear power, other firms may use coal, water, solar or wind power.

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The role of households is to demand commodities so that they can satisfy their consumption needs. At the same time, households own labor and capital assets that they offer to the firms that need them to carry out the production process. In deciding how much to demand of each good and how much to offer of their assets, consumers value the return they would obtain from the possible consumptions of goods that can be made possible by the sale of their assets. This judgment takes place in terms of an internal value system we call preferences o utility. The assumption that households select the consumption schedule that best fits their value system, under whatever their available income makes possible, is called preference or utility maximization. Utility is a classical concept in economics and even though it is unnecessary in modern microeconomics, it is retained because it happens to be equivalent, and much more convenient to use, than preferences. As is the case with firms, we can find within the households' group different ways of achieving the same objectives, mainly defined by individual preferences, or more simply utility levels. As we well know, there are different ways of feeling content. Some of us prefer riding a bike while enjoying the scenery. Instead, others take pleasure in relaxing at home watching a science-fiction movie, reading a best-selling novel, or listening to classical music. Similarly, even when sharing the same basic preferences, some consumers might be wealthier than others and this will most likely lead to consuming a larger amount of each commodity.

What happens when these two groups of agents interact in the marketplace? Despite the abovementioned heterogeneity in preferences, income levels and technology, we can aggregate—for each possible set of prices—all agents' decisions in two market blocks or sides, i.e. the demand block and the supply block. Households and firms might be either behind the demand or the supply side. In the market for primary inputs, for instance, firms demand labor and capital services while households supply them. In the market for non-primary inputs, or intermediate commodities, some firms are demanders and other firms are suppliers. Lastly, in the markets for goods and services oriented to final uses households demand these commodities for consumption while firms supply them. Households and firms are both demanders and suppliers. Once the two sides of the market are well-defined, the next question is whether these two blocks can reach a "trade agreement" based on a mutually compatible price. In other words, a price at which the specific amount being demanded coincides with the specific amount being produced. This type of coincidental situations, with the matching of demand with supply, is referred to as market equilibrium. In fact, market equilibrium is all about finding prices that are capable of yielding this type of agreement between the two market blocks.

There are other types of economic situations that condition the way prices are set and thus how markets agreements are achieved. Besides agents' behavior, for instance, the number of market participants and the way information is shared among them constitute key determinants for markets' mechanisms and, consequently, for the equilibrium outcome in the markets. Depending on the rules shaping how equilibrium prices are set, markets can be perfectly competitive or imperfectly competitive. In the first case, when markets are perfectly competitive, prices are publicly known by all participants and they have no way to individually exert any influence over those prices. A condition for this is to have large markets. When the number of participants in a market is large, then each one of them is negligible given the size of the market and no one can have an individual influence on prices. We will say that all agents participating in these markets are price takers. It is also worth noting that when a market is considered to be perfectly competitive, there is an implicit assumption that has to do with the production technology and, more specifically, with the link between the unit costs and the scale of production. It is postulated that in perfectly competitive economies this link is constant, in other words, production exhibits Constant Returns to Scale. Decreasing returns to scale are ruled out since it is assumed that replication of any production plan is always possible, provided there are no indivisibilities. Finally, the presence of increasing returns to scale, as a possible alternative in technology, is incompatible with perfectly competitive markets since price-taking behavior would imply that firms incur in losses. In this case, the setting of non-competitive prices by firms opens some room for strategic behavior leading to a rich constellation of possible market organizations. In this introductory book, we will focus our attention exclusively on competitive markets.

Perfectly competitive markets are very appealing since they turn out to have nice properties in terms of welfare. The result that prices equal marginal rates of transformation is a necessary condition and, under convex preferences and convex choice sets, a sufficient one for optimality. Therefore, the way equilibrium prices are set in perfectly competitive markets relies on an efficient mechanism that could not be present with non-convexities in the production set (see Villar 1996). The equilibrium outcome of perfectly competitive markets is known as Walrasian equilibrium in honor of the French mathematical economist Léon Walras (1834–1910), a leading figure of the "marginalist" school at that moment.

The equilibrium outcome of a particular competitive market can be analyzed essentially from two perspectives: from a partial equilibrium perspective or from a general equilibrium perspective. Partial equilibrium implies analyzing one market in isolation from all other markets. This is in fact the very definition of partial equilibrium analysis where only direct effects are taken into account while omitting possible indirect and induced or feedback impacts that occur simultaneously in other interrelated markets. In doing so, as we indicated in the introduction of this book, we make use of the *caeteris paribus* assumption. The general equilibrium approach, however, considers the economy as a closed and interdependent system of markets where equilibrium prices and quantities are the result of all kind of economy-wide interactions, that is, in an equilibrium there is a reflection of the role played by all direct, indirect and induced effects.

Once we have commented on the basic flavor of how market institutions work and, more specifically, have described the essentials of the Walrasian approach to economics, we devote the following two sections of this chapter to present the normative and positive aspects of Walrasian equilibrium theory. We will use the simple case of a pure exchange economy because of its transparency and also because almost everything which is conceptually relevant to the analysis can be discussed using this setup. Technical details will be, for the most part, omitted since we want to focus in the issues rather than in the mathematics.

2.2 Positive Analysis of the Walrasian Equilibrium

2.2.1 Walras' Law and the Walrasian Equilibrium: Definitions

Economists are very much interested in ascertaining how the market allocation problem is solved, i.e. which are the laws and the conditions that regulate the way markets work. In this section we describe these mechanisms in the context of Walrasian markets. In doing so, we will confine our attention to the simplest case of a pure exchange economy, or Edgeworth box economy, where no production possibilities are considered. In this economy the role of agents is simply to trade among themselves the existing stock of commodities. In this pure exchange economy there is a finite number of N different commodities (listed as $i = 1, 2, \ldots, N$) and H different consumers or households or, more generally, agents (listed as $h = 1, 2, \ldots, H$). A vector such as $x = (x_1, \ldots, x_i, \ldots x_N) \in \mathbb{R}^N$ will represent a listing of quantities of those N goods. Each commodity is valued with a non negative unit of market value that we call price and represent by a vector $p = (p_1, \ldots, p_i, \ldots, p_N) \in \mathbb{R}^N_+$. Since production possibilities are not considered in this economy, there are initial commodity endowments. Each agent h initially owns a part of these endowments, i.e. $e_h \in \mathbb{R}^N_+$, that along with prices define for all agents their initial wealth and thus their budget constraint.

Agents want to consume goods and this is only possible from the existing level of endowments. These consumptions are a reflection of the preference system of agents once feasibility via prices is established. We will assume that for each agent h preferences in consumption for commodity i are reflected through a non-negative demand function $\chi_{ih}(p, e_h)$ that is continuous and homogenous of degree zero in prices. Each of these individual commodity demands are the result of a well-defined restricted optimization problem, but we bypass the details here. The most common way is to assume that preferences are utility representable and thus demand functions are derived from solving the utility maximization problem of every agent, subject to the restriction that the only possible consumptions are those within the agent's budget constraint.

Since the analysis of this pure exchange economy is done from a general equilibrium perspective, we need to impose some "laws" to make this economy become a closed and interdependent system. Therefore, we first notice that the following restriction is satisfied:

$$\sum_{i=1}^{N} p_i \cdot \chi_{ih}(p, e_h) = \sum_{i=1}^{N} p_i \cdot e_{ih} \;\; \forall h = 1, \ldots, H \tag{2.1}$$

Restriction (2.1) says that for each agent the value of the total consumption of available resources should be equal to the value of the owned wealth, or endowment. Consequently, this restriction will be also verified, by summation for all agents, at the level of the whole economy. It is known as Walras' law:

$$\sum_{h=1}^{H}\sum_{i=1}^{N} p_i \cdot \chi_{ih}(p, e_h) = \sum_{h=1}^{H}\sum_{i=1}^{N} p_i \cdot e_{ih} \tag{2.2a}$$

and with some algebraic rearrangement:

$$\begin{aligned}\sum_{i=1}^{N} p_i \sum_{h=1}^{H} (\chi_{ih}(p, e_h) - e_{ih}) &= \sum_{i=1}^{N} p_i \left(\sum_{h=1}^{H} \chi_{ih}(p, e_h) - \sum_{h=1}^{H} e_{ih}\right) \\ &= \sum_{i=1}^{N} p_i \cdot (\chi_i(p) - e_i) = 0 \end{aligned} \tag{2.2b}$$

We have added up all individual demands and endowments and introduced market demand $\chi_i(p)$ and market supply e_i, or total endowment, for each good i. We now introduce the market excess demand function as their difference, that is, $\zeta_i(p) = \chi_i(p) - e_i$. The final and most compact expression for Walras' law will read:

$$\sum_{i=1}^{N} p_i \cdot \zeta_i(p) = 0 \tag{2.2c}$$

This equality is telling us something quite interesting: the value of market excess demands equals zero at all prices, whether or not they are equilibrium prices. Thus, Walras' law is a necessary condition for markets to be in equilibrium but it is not sufficient.

For a set of prices to become an equilibrium in this closed and interdependent economic system the sufficient condition, using market demand and supply, reads as:

$$\zeta_i(p^*) = \chi_i(p^*) - e_i = 0 \tag{2.3}$$

for all markets $i = 1, 2,\ldots,N$ and $p^* > 0$.

A Walrasian equilibrium is defined as an allocation of the available level of goods and a set of prices such that condition (2.3) is fulfilled. The allocation is given by the demand functions at the equilibrium prices. Notice that (2.3) is a system of N equations (one for each of the N goods) with N unknowns (one for each of the N prices). This condition of exact equality between demand and supply is in fact a bit stronger than needed. We use it here because it helps in simplifying the presentation of the issues without delving unnecessarily into some technicalities. All that is required, in fact, is that demand is no greater than supply at the equilibrium prices, and hence it is possible in principle that some of the goods are free goods, but then their price should be zero. We restrict ourselves to exact equality and positive prices for all goods as expressed in (2.3), which implicitly requires some type of monotonicity property on consumers' preferences. This desirability assumption, by the way, justifies as well that the budget constraints (2.1) hold as equalities and so by implication does Walras' law.

A key corollary that stems from Walras' law is that *"if all markets but one are cleared, then the remaining market must also be cleared"*. So, to determine equilibrium prices in this economy, we just need to pick up $N-1$ of the equations from (2.3) and find a solution to that reduced size system. But notice that now we have more variables to determine (N) than available independent equations ($N-1$). An alternative way of presenting this corollary is by noticing that if there is excess demand in a certain market i ($\zeta_i(p) > 0$) then there must be an excess supply in some other market j ($\zeta_j(p) < 0$), or the other way around.

For a better understanding of the implications of Walras' law and the definition of Walrasian equilibrium, we move now to the details of a more specific example. We will consider an Edgeworth box economy with two agents, i.e. $h = (1,\ 2)$, and two commodities, i.e. $i = (1,\ 2)$. Agents' preferences in this economy will follow the pattern of the well-known Cobb–Douglas utility functions, with $u_1(c_{11}, c_{21}) = c_{11}^{\alpha^1} \cdot c_{21}^{1-\alpha^1}$ for agent 1 and $u_2(c_{12}, c_{22}) = c_{12}^{\alpha^2} \cdot c_{22}^{1-\alpha^2}$ for agent 2 (Cobb and Douglas 1928). In the example c_h is the consumption vector for each of the two agents, $h = 1, 2$. Their consumption possibilities will be limited by the aggregate stocks form their initial property of endowments, so for agent 1 we write $e_1 = (e_{11}, e_{21})$ while for 2 we have $e_2 = (e_{12}, e_{22})$.

The commodity demand functions consistent with solving the utility maximization problem for agent 1 are:

$$
\begin{aligned}
c_{11} &= \chi_{11}(p, e_1) = \frac{\alpha^1 \cdot (p_1 \cdot e_{11} + p_2 \cdot e_{21})}{p_1} \\
c_{21} &= \chi_{21}(p, e_1) = \frac{(1-\alpha^1) \cdot (p_1 \cdot e_{11} + p_2 \cdot e_{21})}{p_2}
\end{aligned}
\tag{2.4}
$$

Similarly for the case of agent *2*:

$$
\begin{aligned}
c_{12} &= \chi_{12}(p, e_2) = \frac{\alpha^2 \cdot (p_1 \cdot e_{12} + p_2 \cdot e_{22})}{p_1} \\
c_{22} &= \chi_{22}(p, e_2) = \frac{(1-\alpha^2) \cdot (p_1 \cdot e_{12} + p_2 \cdot e_{22})}{p_2}
\end{aligned}
\tag{2.5}
$$

We now write, for this simple Cobb–Douglas economy, the two equilibrium equations corresponding to expression (2.3) above:

$$\frac{\alpha^1 \cdot \left(p_1^* \cdot e_{11} + p_2^* \cdot e_{21}\right)}{p_1^*} + \frac{\alpha^2 \cdot \left(p_1^* \cdot e_{12} + p_2^* \cdot e_{22}\right)}{p_1^*} = e_{11} + e_{12}$$
$$\frac{(1-\alpha^1) \cdot \left(p_1^* \cdot e_{11} + p_2^* \cdot e_{21}\right)}{p_2^*} + \frac{(1-\alpha^2) \cdot \left(p_1^* \cdot e_{12} + p_2^* \cdot e_{22}\right)}{p_2^*} = e_{21} + e_{22} \tag{2.6}$$

Remember though that the corollary of Walras' law implies one of these two equations is redundant. We can focus in solving either one of them, and discard the other one. Let's check the solution from using the first equation. With a little bit of algebra we would find:

$$\frac{p_2^*}{p_1^*} = \frac{(1-\alpha^1) \cdot e_{11} + (1-\alpha^2) \cdot e_{12}}{\alpha^1 \cdot e_{21} + \alpha^2 \cdot e_{22}} \tag{2.7}$$

Notice that if p^* solves the equation, so does $\kappa \cdot p^*$ where κ refers to any positive real number. Any set of prices that fulfils condition (2.7) will clear both markets. The reader can try and solve the equilibrium set of prices using instead the second equation in expression (2.6). No surprise here, the same ratio of prices will be found. This statement can be checked graphically in the Edgeworth box economy of Fig. 2.1 above where the slope of the budget sets for both agents is given by the price ratio p_1^*/p_2^*. Each agent maximizes his or her utility at the relative price p_1^*/p_2^* and their individual demands are at the same time exactly compatible with the available supply of goods determined by the initial endowment point e.

Notice that only relative equilibrium prices can be determined. If we wish to fix all prices at some absolute level, arbitrary of course, we need some "reference" price fixed from outside. This reference price, or unit of value measurement, is usually called the *numéraire*. With an adequate selection of numéraire, all prices are expressed as relative distances to it, both when determining the initial or benchmark equilibrium scenario and when exploring possible changes in this economy in response to variations, for instance, of the initial level of endowments. We will come back to this issue later on along Chap. 3.

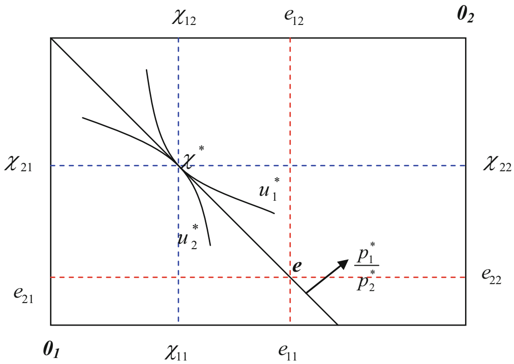

Fig. 2.1 Walrasian equilibrium in a pure exchange economy

2.2.2 Existence and Uniqueness of Walrasian Equilibrium

Walras (1874a, b) was the first to present the equilibrium in a set of markets as the solution of a system of equations, reflecting how commodities are allocated between agents or groups of agents for a specific set of prices. Nevertheless, Walras did not provide any formal proof of the existence of the solution for this system, and in addition he presupposed the solution to be unique.[1] Later on, during the thirties, the seminal work of Wald (1951) and in a more integrated way, the analysis carried out by Arrow and Debreu (1954) contributed to show under which specific but quite general conditions this market system of equations had a solution relevant for both *"descriptive and normative economics"* (op. cit., 1954, page 265).

When we think in terms of the pure exchange economy described in the previous section, proving the existence of a general equilibrium is equivalent to prove that there is at least one set of prices, in the non-negative price set, that makes all commodity excess demands equal to zero. By defining general equilibrium prices and allocations in this way, it becomes natural to use theorems from topology to provide a proof of the existence of equilibrium. And within the arsenal of topology's theorems, we find that fixed point theorems have been quite appropriate and powerful mathematical tools for this purpose. The two main theorems that were instrumental to Arrow and Debreu in answering the existence question in Walrasian general equilibrium theory are Brouwer's (1911–1912) and Kakutani's (1941) fixed point theorems, the latter being an extension of the former. Since Brouwer's theorem can be stated with far less mathematical apparel than Kakutani's, we will present it here. First, the unit simplex S is defined as the subset of points p in $\mathbb{R}_+^N$ such that $p_1 + p_2 + \ldots + p_N = 1$ and from the definition we can easily see that S is a closed and convex set. For general equilibrium theory, as in Debreu (1952) and Arrow and Debreu (1954), the proper statement of the theorem takes the following form.

Brouwer's Fixed Point Theorem Let $\boldsymbol{\varphi}(p)$ be a continuous function $\boldsymbol{\varphi}: S \to S$ with $p \in S$, then there is a $p^* \in S$ such that $\boldsymbol{\varphi}(p^*) = p^*$.

In Fig. 2.2 we illustrate the theorem for the simple situation of a continuous real valued function defined in the unit interval, a case where the theorem is almost graphically self-evident. As we can also see in this Figure, this theorem implies that if the function $\boldsymbol{\varphi}(\cdot)$ is continuous, its graph cannot go from the left edge to the right edge without intersecting the diagonal at least once. That intersection is a fixed

[1]Walras (1874a, b) was wrong when posing and facing the question about the uniqueness of equilibrium. In fact, aggregate excess demand function and not only individual excess demand functions must present additional properties for the equilibrium to become unique and stable, since not all the properties of the later are inherited into the former. This is the so-called Sonnenschein–Mantel–Debreu Theorem (1973, 1974, 1974). The concept of uniqueness is also related to the concept of stability (Arrow and Hurwicz 1958, 1959). An excellent exposition of uniqueness and stability of equilibrium for pure exchange economies and economies with production can be found in Elements of General Equilibrium Analysis (1998), Chap. III, written by Timothy J. Kehoe.

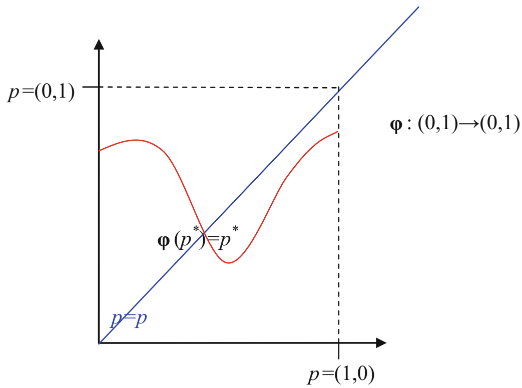

Fig. 2.2 Illustration of Brouwer's Fixed Point Theorem in the unit interval

point. In the picture there is just one fixed point but the reader can easily visualize that should the graph of the function intersect the diagonal a second time, then it must necessarily do it a third time. In words, there would be in this case a finite and odd number of fixed points. This possibility is in fact a very remarkable result that holds for so-called *regular* economies. In the spirit of Fig. 2.2, an economy is called regular when the mapping $\boldsymbol{\varphi}(\cdot)$ only and always intersects the diagonal. Expressed in economics terms it says that in regular economies there will always be when a finite but odd number of equilibrium price configurations. Another outstanding result is that almost all economies are regular. Non-regular economies can none-theless be devised but in probabilistic terms they would be observed with zero probability. The theory of regular economies is due to Debreu (1970).

The fixed point refers in fact to the equilibrium set of prices. In proving the existence of a fixed point, and thus making the theorem applicable to the economic problem of existence, we need first a continuous function that maps the set of prices into itself. Furthermore, this continuous function should verify the version of Walras' law in expression (2.3). It seems then that one good candidate function should refer to a continuous transformation of the excess demand functions, since a continuous transformation of continuous functions is also continuous. Remember also that excess demand functions are homogenous of degree 0 in prices, and thanks to this prices can be transformed into points belonging to the unit simplex S using the normalization $\sum_{i=1}^{N} p_i$.

How can the continuous function $\boldsymbol{\varphi}(\cdot)$ be defined?. The literature has provided alternative ways of defining this mapping (Gale 1955; Nikaido 1956; Debreu 1956), but a function $\boldsymbol{\varphi}(\cdot)$ with a possible economic interpretation should always be preferred. To this effect, the mapping $\boldsymbol{\varphi}(\cdot)$ could be viewed as a "price adjustment function" mimicking the way the so-called Walrasian auctioneer would adjust initial prices following the law of demand and supply until equilibrium prices are achieved (Gale 1955), that is, the fixed point is determined. The Walrasian auction-eer is just a symbolic way to represent the way markets are supposed to adjust.

Consider this function:

$$\boldsymbol{\varphi}_i(p) = \frac{p_i + \max[0, \zeta_i(p)]}{1 + \sum_{j=1}^{N} \max\left[0, \zeta_j(p)\right]} \tag{2.8}$$

The function in (2.8) is indeed a price adjustment function and it is known as the Gale–Nikaido mapping. To see how this price adjustment function works, take an initial set of prices p. If such p were an equilibrium, then for all i $\zeta_i(p) = 0$ and by the definition in (2.8) we would have $\boldsymbol{\varphi}_i(p) = p_i$, since all the max operators would be zero. In other words, if p is an equilibrium then p is also a fixed point of the mapping in (2.8). Now, we will not be usually so lucky and when picking up a price vector p it will most likely not be an equilibrium price. This means that in some market i its excess demand function in (2.3) will not be zero. If $\zeta_i(p) > 0$ then the numerator of (2.8) indicates that the price of good i should be increased precisely by the value of the positive excess demand. The denominator ensures that the new adjusted price remains in the simplex. Notice also that by summation for all i in (2.8) we would find that $\sum_{i=1}^{N} \boldsymbol{\varphi}_i(p) = 1$ and so price adjustments take always place in the simplex S. If instead $\zeta_i(p) < 0$ then by the corollary of Walras' law there has to be another market with positive excess demand, say market k where $\zeta_k(p) > 0$. The adjustment rule now acts modifying the price of good k, and so on. The Walrasian auctioneer will raise prices wherever there is a positive excess demand in order to make the consumption of those goods less attractive to households. Since prices are relative when raising the price p_k for the good with a positive excess demand, the auctioneer is implicitly reducing p_i in the market with a negative excess demand.

The idea behind the adjustment function, or Walrasian auctioneer, is that markets work to smooth out the differences between demand and supply. These price adjustments will cease when all excess demand functions are zero for some positive price vector p^*:

$$\zeta_i(p^*) = 0 \quad \forall i = 1, 2, \ldots, N \tag{2.9}$$

which is the definition of equilibrium prices in our pure exchange economy.

The next step is to show that the mapping $\boldsymbol{\varphi}(p)$ in expression (2.8) has a fixed point and that the fixed point is indeed an equilibrium. The first part is just an application of Brouwer's theorem since the mapping $\boldsymbol{\varphi}(p)$ satisfies all of its requirements. It is continuous, it is defined on the simplex S and its images are all in the simplex S. As a consequence there is a vector p^* such that $\boldsymbol{\varphi}(p^*) = p^*$. The second part involves showing that for this vector p^* we indeed have $\zeta_i(p^*) = 0$ for all i. Since $\boldsymbol{\varphi}_i(p^*) = p_i^*$ we can replace it in expression (2.8):

$$p_i^* = \frac{p_i^* + \max[0, \zeta_i(p^*)]}{1 + \sum_{j=1}^{N} \max\left[0, \zeta_j(p^*)\right]} \tag{2.10}$$

Rearranging terms and simplifying we would find:

$$p_i^* \cdot \sum_{j=1}^{N} \max\left[0, \zeta_j(p^*)\right] = \max[0, \zeta_i(p^*)] \tag{2.11}$$

Multiply now by the excess demand function for good i and add all of them up to obtain:

$$\left(\sum_{i=1}^{N} p_i^* \cdot \zeta_i(p^*)\right) \cdot \left(\sum_{j=1}^{N} \max\left[0, \zeta_j(p^*)\right]\right) = \sum_{i=1}^{N} \max\left[0, \zeta_i^2(p^*)\right] \tag{2.12}$$

By Walras' law the left-hand side of (2.12) is necessarily zero and so must therefore be the right-hand side. From here it must follow that all $\zeta_i(p^*) = 0$. Should one of them not be zero, say $\zeta_k(p^*) \neq 0$, then $\zeta_k^2(p^*) > 0$ and the right hand side could not be zero, a contradiction.

We have seen that an equilibrium price vector p^* is a fixed point for the mapping $\boldsymbol{\varphi}(p)$ and a fixed point p^* for this mapping is an equilibrium for the economy. This is a result often overlooked or forgotten but very relevant since it says that the existence theorem and the fixed point theorem are in fact equivalent statements.

We now review the question of the number of equilibria. We commented before, while reviewing Fig. 2.2, that such a number will always be finite and odd for the vast majority of economies. This issue is very relevant in fact for the appropriateness of the method of comparative statistics, which is routinely used in applied general equilibrium models for the evaluation of specific policies (Kehoe 1985, 1991). The exploration of the economy-wide impacts of policies is undertaken comparing the initial equilibrium (i.e. the benchmark) with the equilibrium that would ensue after the policy change is enacted and absorbed (i.e. the counterfactual). If the equilibrium is unique for each set of structural and policy parameters defining the economy and it behaves continuously, then comparative statics make sense. We can compare the two equilibria and from their comparison we can extract valuable information on how the policy affects the economy. If multiple equilibria are possible, however, we can get into methodological trouble. The reason is that a change in parameters may move the economy to one of the different equilibria with no a priori information on which one will actually be, nor if other policy changes will yield a transition to a different equilibrium. In short, in the presence of non-uniqueness it would be difficult to really know what we are comparing. Notice also that when an economy has a unique equilibrium point for each possible parameter configuration, then it is both globally and locally unique and it does not matter for comparative statics if we perform an experiment where a parameter change is large or small. Anything goes and it goes nicely. When an economy has multiple equilibria, the next question is whether those equilibrium points are locally unique. If they are, comparative statics would still make sense, provided changes are small enough to be contained in a close neighborhood of the initial equilibrium configuration. With large parameter changes, however, comparative statics is methodologically risky since we do not really know where the economy may be jumping to.

The theoretical conditions that guarantee the uniqueness of equilibrium have been widely explored by many economists (Wald 1936; Arrow and Hahn 1971; Kehoe 1980, 1985, 1991; Kehoe and Mas-Colell 1984; Kehoe and Whalley 1985; Mas-Colell 1991). For a pure exchange economy, uniqueness of equilibrium can be assured if the aggregate excess demand functions satisfy the so-called gross substitutability property. This property rules out all type of complementarities in demand. Translating it into our notation it means that if we have two price vectors, p and p', such that for some good j we have $p'_j > p_j$ but $p'_i = p_i$ for all $i \neq j$ then $\zeta_i(p') > \zeta_i(p)$ for $i \neq j$. This would imply that if the price of one commodity j increases then the excess demand of the other commodities $i \neq j$ must increase too; using derivatives we would write $\partial \zeta_i(p)/\partial p_j > 0$, and vice-versa. This condition of gross substitutability is sufficient but not necessary for the globally uniqueness of equilibrium. The flavor of the proof can be seen straightforwardly by considering two different equilibrium prices, call them p' and p, that are not proportional (otherwise they would just be different normalizations of each other and we would be done). If they are equilibrium prices then it will be the case that, for all goods $i, \zeta_i(p') = \zeta_i(p) = 0$. Without loss of generality (via price normalization if need be) we can look for the good k such that that $p'_k = p_k$ while $p'_j \geq p_j$ with strict inequality for some of the $j \neq k$. In this case, say $\bar{j}$, raise the price from $p'_{\bar{j}}$ to $p_{\bar{j}}$ and by the gross substitutability property the excess demand of good k will increase, that is, $\zeta_k(p') > \zeta_k(p)$, which is a violation of the equilibrium condition. Thus, prices cannot be different.

In our pure exchange economy, a sufficient condition for gross substitutability would be that the elasticity of substitution in consumption for each agent is larger or equal than one (Kehoe 1992). Which additional assumptions do we need if the gross substitutability assumption is not fulfilled, for example when the elasticity of substitution is lower than one?. One such property on the excess demand functions is the extension to them of the weak axiom of revealed preference, a concept originally applied to individual choice. We will omit the details here and refer the reader to Kehoe (1992, 1998).

Uniqueness, global or at least local when there are multiple equilibria, is fundamental from the perspective of the type of applied models we will be building and dealing with in the next chapters. Since these models are commonly used to evaluate actual policy changes, we must make sure that the comparison of equilibria makes methodological sense. The theoretical conditions in the uniqueness literature may be too strong for applied models and some testing of these models is called for. In this direction, Kehoe and Whalley (1985) have performed extensive numerical calculations searching for multiple equilibria in applied general equilibrium models of the *USA* and Mexico using 1 and 2-dimensional search grids, respectively. They report that no such multiplicity has been found. Even though their conclusion is not based on theoretical arguments, the fact that the examined applied models are quite standard provides researchers with reassurance that uniqueness is the common situation in empirical analysis. This reinforces the reliability of the routine comparative statics exercises, at least from an empirical perspective.

2.3 Normative Properties of Walrasian Equilibrium

After reviewing some of the basic results concerning the positive aspects of Walrasian general equilibrium theory, we now turn to the description of its normative aspects. The nice welfare properties of the competitive equilibrium are what have made this concept so appealing to generations of economists. These welfare properties can be summarized with the help of the two well-known fundamental theorems of welfare economics (Arrow 1951). The first of these theorems states that any Walrasian allocation is a Pareto efficient allocation as well. The second theorem says that any Pareto efficient allocation can be implemented as a Walrasian allocation with the use of appropriate lump-sum transfers, i.e. transfers based on a reshuffling of the initial endowments, thus not wasting any of the economy's initial resources.

An allocation is said to be Pareto efficient if there is no other feasible allocation where at least one agent is strictly better off while nobody else is worse off. In the context of our Edgeworth box economy, the first theorem implies that if the economy is in a Walrasian equilibrium, then there is no alternative feasible allocation at which every agent in this economy is at least as well off and some agent is strictly better off. In other words, there is no way for the agents of this economy to collectively agree to move to a different feasible allocation. If they moved from the market equilibrium, somebody would certainly be worse off.

In our pure exchange economy, agents are rational in the sense that they will participate in the exchange process and trade using their initial endowments as long as that makes them at least as well as not trading. Among all the Pareto efficient allocations, the one marked as a in Fig. 2.3 makes agent 1 indifferent between trading and not trading, with allocation b playing the same role for agent 2. The curve that connects point a with b is known as the contract curve and represents all the feasible allocations at which all agents do at least as well as in their initial endowments. All the allocations in the contract curve represent a subset of the Pareto efficient allocations, i.e. the line that connects point O_1 and O_2 in Fig. 2.3. The Walrasian auctioneer will adjust prices till reaching equilibrium prices, i.e. p_1^*/p_2^* where excess demands are zero. Note that since the equilibrium allocation χ^* lies on the contract curve, this allocation is Pareto efficient.

We now provide a quick proof of the first welfare theorem. Let us consider that there is another feasible allocation $\hat{\chi}$ that Pareto dominates the Walrasian allocation χ^*. This would imply that at the equilibrium prices p^*:

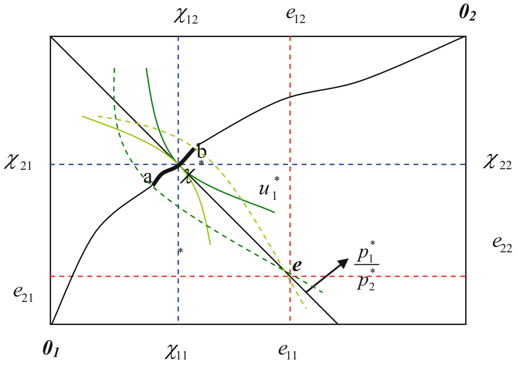

Fig. 2.3 First Welfare Theorem

$$\sum_{i=1}^{N} p_i^* \widehat{\chi}_{ih} \geq \sum_{i=1}^{N} p_i^* \chi_{ih}^* \text{ for all } h \in H \tag{2.13}$$

and

$$\sum_{i=1}^{N} p_i^* \widehat{\chi}_{ih'} > \sum_{i=1}^{N} p_i^* \chi_{ih'}^* \text{ for at least one agent } h' \in H \tag{2.14}$$

The reason is that if $\widehat{\chi}$ is at least as good as χ^* then it must be the case that $\widehat{\chi}$ is beyond the budget sets of all agents, otherwise being at least as good it would have been selected. Adding up (2.18) and (2.19) for the set of agents we obtain:

$$\sum_{h=1}^{H} \sum_{i=1}^{N} p_i^* \widehat{\chi}_{ih} > \sum_{h=1}^{H} \sum_{i=1}^{N} p_i^* \chi_{ih}^* = \sum_{h=1}^{H} \sum_{i=1}^{N} p_i^* e_{ih} \tag{2.15}$$

Expression (2.15) is in fact a contradiction since it implies that $\widehat{\chi}$ could not be a feasible allocation.

We should point out a major distinction between the concepts of Pareto efficiency and Walrasian equilibrium. While Pareto efficiency specifies a subset of allocations which depend on total endowments and individual preferences alone, a Walrasian equilibrium relates to two set of variables, allocations and prices, and depends on individual endowments and individual preferences. In this more stringent sense, it is clear that not all Pareto efficient allocations will be a Walrasian equilibrium. Nonetheless, with appropriately designed transfers of endowments, we can transform any Pareto efficient allocation into a Walrasian equilibrium, provided some basic technical assumptions (such as convexity in preferences, and in technology when production possibilities are present) are satisfied. This is what is stated in the second welfare theorem.

We reproduce in Fig. 2.4 above two Pareto efficient allocations, χ^* and χ'^*. They can also be seen as Walrasian allocations conditional to two different

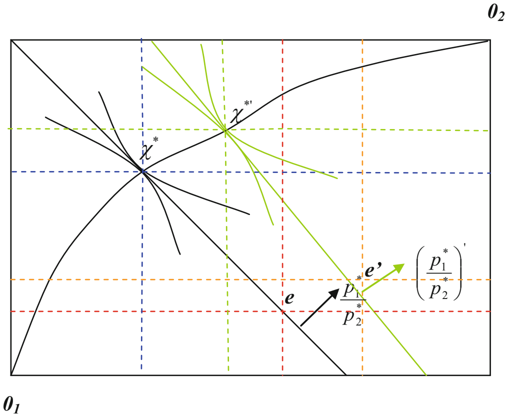

Fig. 2.4 Second Welfare Theorem

distributions of the total endowment in the economy, i.e. $e \neq e'$. Using Fig. 2.4 we can visualize the essence of what lies behind the second welfare theorem. Suppose an hypothetical central planner, or some social welfare function, deems allocation χ'^{*} to be socially more desirable than the current market allocation χ^{*}. Suppose also the planning authority wants to reach allocation χ'^{*} through the workings of the market. What should this authority do to reach this goal? The actual details do not matter but the second welfare theorem would suggest the redistribution of initial endowments from e to e', for instance, and then let the market work out the new equilibrium for the new endowment distribution. Any such redistribution of the physical assets of the economy is called a lump-sum transfer, i.e. $t = e' - e$ in our example of Fig. 2.4, since total endowments in this economy remained unchanged. Notice too that e' in Fig. 2.4 is just one of the very many possible redistribution schemes. The message here is that there is quite a bit of room in the design of redistributive policy actions and that efficiency and equity considerations can be separated.

2.4 Equilibrium Properties in Linear Economies

We now turn our attention to a representation of the economy that assumes a simple property relating inputs with outputs, namely linearity. This representation has a number of advantages, mainly the transparency that governs the economic relations among variables, the ability to undertake straightforward policy simulation exercises and, most importantly, the ease of interpretation of results. All in all, a linear economy is a beginner's dream. In a general setup, a simple private economy will be comprised by H households, K primary factors of production and N commodities and firms. Factors are owned by households and are offered by them to firms as inputs. Firms have no market power and use these factors along with material goods to produce output which is, in turn, used by households to satisfy their demands and by other firms as material inputs in their productive processes. Firms pay households for the use of their factors of production and households use this income to purchase their demand for consumption. Let us further simplify this setting by assuming that there is a single household in this economy (i.e. the set H contains a unique element) and that this household only owns one type of primary factor (again, this makes the set K to contain one element). We will refer to this unique resource as 'value-added' and will denote it by VA. With one household, aggregate consumption demand, CD, coincides with this household's individual demand c_h. Therefore there is no need to keep explicit the label h that identifies this unique household and we proceed to omit it for the time being. Consumption demand includes a given non-negative amount of each of the n goods that the economy produces. In vector terms we have $CD = c = (c_1, c_2, \ldots, c_N)$.

Each of the N firms, listed as $i = 1, 2, \ldots, N$, produces output y_i using value-added VA_i and material goods in amounts y_{ji}, with y_{ji} being the amount of good j used by firm i in the production of its output level y_i. Total output in the economy is given by a vector Y that lists the production of each and every of the N firms, namely $Y = (y_1, y_2, \ldots, y_N)$. This output vector is produced so that demand for consumption c can be satisfied. However this level of consumption demand is formulated, this demand is going to be satisfied. With the sale of its output y_i to the household who demands c_i and to other firms which demand y_{ij}, firm i obtains income that allows said firm i to purchase all the productive resources that it requires for the implementation of its production process. These inputs include material goods y_{ji} and value-added VA_i. In the productive balance, total income will be equal to total expenditure for each firm i. Therefore the following holds true for each $i = 1, 2, \ldots, N$:

$$p_i y_i = p_i c_i + \sum_{j=1}^{N} p_i y_{ij} = VA_i + \sum_{j=1}^{N} p_j y_{ji} \tag{2.16}$$

The first equality in expression (2.16) indicates the sources of income for firm i, whereas the expression to the right of the second equality indicates the cost structure of the firm. It must be noted that this balance expression will always be fulfilled since in the absence of market power no pure profits are possible. We now separate the expression in two parts, the first one responding to income generation with the second one referring to the cost structure. Thus:

$$p_i y_i = p_i c_i + \sum_{j=1}^{N} p_i y_{ij} \tag{2.17}$$

$$p_i y_i = VA_i + \sum_{j=1}^{N} p_j y_{ji} \tag{2.18}$$

Expression (2.17) can be further simplified by noticing that the price p_i is in fact redundant and can be eliminated to have:

$$y_i = c_i + \sum_{j=1}^{N} y_{ij} \tag{2.19}$$

Now things become interesting since we can algebraically transform expression (2.19) in such a way that it transcends its so far descriptive nature to become a powerful economic equation. As we will see shortly, this step requires of course of an assumption. Notice first that it is trivially true that $y_{ij} = y_{ij}(y_j/y_j)$, provided that $y_j > 0$. Hence it will also be true that $y_{ij} = (y_{ij}/y_j)y_j$. The ratio (y_{ij}/y_j) indicates the average quantity of good i used in the production of all the units in y_j. Being an average, however, this quantity may or may not coincide with actual quantity of i being used in the production of a given unit of j. But if it does happen to be the case that each and all units of j use the same amount, then we can take this average quantity as a coefficient representing the production technology in firm j. The average ratio (y_{ij}/y_j), being independent of the total output level y_j, becomes a technological coefficient. No matter how many units of good j are being produced, all of them require exactly the same amount of good i as input. Let us denote this constant and non-negative coefficient by $a_{ij} = (y_{ij}/y_j)$. If this constancy property is global and applies to all the goods being produced in the economy, then Expression (2.19) becomes:

$$y_i = c_i + \sum_{j=1}^{N} y_{ij} = c_i + \sum_{j=1}^{N} a_{ij} y_j \tag{2.20}$$

What we are saying here is that if the production of each and all units of j require the same amounts of good i, then of course the production of 1 unit of j will require the quantities in the vector $(a_{1j}, a_{2j}, \ldots, a_{Nj})$, 2 units will require twice as much $2 \times (a_{1j}, a_{2j}, \ldots, a_{Nj})$ and so on and so forth. In general, y_j units will require $y_j \times (a_{1j}, a_{2j}, \ldots, a_{Nj})$ quantities of the different goods as material inputs. Additionally, production of y_j also needs the use of the primary factor value-added in amount VA_j. As before, we can transform the identity $VA_j = VA_j(y_j/y_j) = (VA_j/y_j)y_j$ into an equation provided the average use of value-added, VA_j/y_j, is independent of the number units of good j being produced. The average thus becomes a value-added technical coefficient, $v_j = VA_j/y_j$, whose interpretation is straightforward. Each unit of output j will require the application of v_j units of value-added. The complete description of the technology for producing (one unit of) good j is given by the vector $(a_{1j}, a_{2j}, \ldots, a_{Nj})$ and the scalar v_j.

We are therefore in a world where proportionality is the rule, i.e. the well-known property of Constant Returns to Scale (*CRS*) of microeconomics (Varian 1992, Chap. 1). In addition to proportionality, notice that each firm has a unique technological recipe to generate output from inputs and no substitution among inputs is possible. This characterization of production, namely, *CRS* plus fixed coefficients, is what we economists refer to as a 'linear' economy. The methodological analysis of this type of economies is frequently known as input–output or interindustry analysis (Leontief 1966; Miller and Blair 2009).

The linear technology we have just introduced has a nice representation is terms of a standard production function of microeconomic theory. The so-called Leontief production function takes the form:

$$y_j = \min\left(\frac{VA_j}{v_j}, \frac{y_{1j}}{a_{1j}}, \frac{y_{2j}}{a_{2j}}, \ldots, \frac{y_{Nj}}{a_{Nj}}\right) \tag{2.21}$$

Since production will only take place in the efficient point of any of the isoquants representing the production function in (2.21) (why so? Otherwise firms would be paying a bill for inputs greater than the minimal amount that yields the desired level of output). In the efficient point, it is always the case that $y_j = VA_j/v_j = y_{ij}/a_{ij}$ for all $i = 1, 2, \ldots, N$ and we can see how we would recover the constant averages that define the technical coefficients of the linear economy. An alternative, and quite handy, representation of the technological relationships in the economy is facilitated via matrix algebra. Indeed, if we position each vector $(a_{1j}, a_{2j}, \ldots, a_{Nj})$ as the j column of a non-negative matrix A we will obtain:

$$A = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{N1} \\ a_{21} & a_{22} & \cdots & a_{N2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{N1} & a_{N2} & \cdots & a_{NN} \end{bmatrix} \tag{2.22}$$

We can now rewrite expression (2.20) in matrix terms as:

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_N \end{bmatrix} + \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{N1} \\ a_{21} & a_{22} & \cdots & a_{N2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{N1} & a_{N2} & \cdots & a_{NN} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix} \tag{2.23}$$

Or in more compact notation:

$$Y = c + AY \tag{2.24}$$

where Y and c are the (non-negative column) vectors $Y = (y_1, y_2, \ldots y_N)$ and $c = (c_1, c_2, \ldots, c_N)$, respectively. If we now look at expression (2.24) as an equation with unknown Y it is legitimate to wonder if it is solvable. Using linear algebra the solution can be easily seen to be:

$$Y = (I - A)^{-1} c \tag{2.25}$$

Notice that a sufficient condition for the solution Y to be non-negative is that the inverse matrix $(I - A)^{-1}$ exists and both this matrix and the consumption vector

c are non-negative. Clearly vector c will be non-negative since negative consumptions are ruled out. As for matrix $(I-A)^{-1}$, known in the literature as the Leontief inverse, it will exist and have the required non-negative sign provided, in turn, that the non-negative matrix A representing the economy's technology satisfies a certain productivity condition.[2] If this is the case, then for *any* possible consumption vector $c \geq 0$, expression (2.25) yields an equilibrium solution in total output that we now represent parametrically as $Y(c) \geq 0$. This is the level of output that allows the economy to satisfy the consumption demand c formulated by the household and the intermediate demand $AY(c)$ needed by firms to implement the production plan.

The attentive reader will have observed, however, that we have so far said nothing regarding how a given consumption vector c materializes. Let us give this question some thought now. In actual economic reality, consumption demand is the result of households allocating income for purchasing, at market prices, goods sold by firms. Income, in turn, consists of the receipts that households obtain from selling their primary resources, usually in the form of labor and capital services, to producers. Firms require these services to put into effect the production plans that will fulfill households' demand for consumption. And it is thanks to the goods that firms sell to households that firms obtain the revenue that allows them to hire labor and capital services. Here you can see the circular flow of income at work. Quantities demanded by households (for consumption) and by firms (for labor and capital services) depend on prices for goods and for primary services, respectively. Equally, quantities of services supplied by households to firms and quantities of goods supplied by firms to households will depend too on market prices. This is the economic interplay of demand and supply that characterizes a market economy and it is an interplay that it is synchronized through prices. In equilibrium, prices will conveniently adjust so as to equate all demands and supplies, for goods and primary services. This is the essence and the object of study of general equilibrium that we pursue in this text. Before getting there, let us focus in the linear economy and let us ignore away the fact that c does depend on income and prices.

Expression (2.24) can be seen as an equilibrium condition in quantities between the total output Y being supplied by firms and total demand $c+AY$, the latter consisting of consumption demand c and intermediate demand AY. Notice also that output Y must be considered as 'gross' output, inclusive of everything being produced, be it for final use c or intermediate use AY, whereas 'net' output is whatever output is left to be delivered for final use in terms of consumption demand once the required production process has taken place. In this case, net output is simply $c = Y - AY$. In the linear economy, no matter what level of consumption demand is formulated, the economy will be able to carry out the requisite

[2]Some technicalities regarding the maximal eigenvalue (or Frobenius root) of the non-negative matrix A must be satisfied for the solvability of the linear equation, specifically the maximal eigenvalue should be <1. The needed property for the eigenvalue has an economical interpretation in terms of the capacity of the economy to produce a positive surplus of some goods. See Nikaido (1972), Chap. 3.

production and expression (2.25) provides the unequivocal answer. Implicitly, the economy does not seem to be subject to any resource constraints. Whatever households demand for consumption happens to be, it can be produced and delivered. It is therefore relevant to make explicit this implicit situation: in the linear economy there is an unlimited pool of productive resources.[3] Alternatively, and more restrictively as well, expression (2.25) can be interpreted as yielding the equilibrium quantities solution as though no resource constraints were binding.

Using the same assumptions regarding the properties of the technology expression (2.18), which describes the cost structure of each sector, can be transformed by simple division by y_i into the following:

$$p_i = (VA_i/y_i) + \sum_{j=1}^{N} p_j\left(y_{ji}/y_i\right) = v_i + \sum_{j=1}^{N} p_j a_{ji} \tag{2.26}$$

In explicit matrix terms:

$$\begin{aligned}[p_1, p_2, \ldots, p_N] = {} & [v_1, v_2, \ldots, v_N] \\ & + [p_1, p_2, \ldots, p_N]\begin{bmatrix} a_{11} & a_{21} & \cdots & a_{N1} \\ a_{21} & a_{22} & \cdots & a_{N2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{N1} & a_{N2} & \cdots & a_{NN} \end{bmatrix}\end{aligned} \tag{2.27}$$

With the usual notational conventions for (the row) vectors of prices and value-added coefficients, the compact notation version of (2.27) becomes:

$$p = v + pA \tag{2.28}$$

We now proceed to go from the description of the unitary cost structure in expression (2.28) to an expression regarding the level of equilibrium prices using the same productivity property of matrix A that allowed us to derive equilibrium quantities in (2.25). In this case:

$$p = v(I - A)^{-1} \tag{2.29}$$

Prices will be non-negative provided the Leontief inverse matrix exists and is non-negative and the value-added coefficient vector is non-negative as well. In fact, under productivity of the technology matrix A, equilibrium prices will always exist for *any* possible vector $v \geq 0$.

For a linear economy, the complete description of the equilibrium relationships for quantities and prices is fully given by expressions (2.24) and (2.28) whereas

[3]In microeconomic terms, primary factors are supplied with infinite elasticity at the current market prices. Another common, but surely more vague, terminology in the literature for such a situation is to designate it as an 'excess supply' of factors.

expressions (2.25) and (2.29) give us the explicit equilibrium solutions. Notice also that, given the technology embodied in A, in the quantity side of the economy gross output Y is an 'explained' or 'endogenous' variable whereas consumption demand c is an 'unexplained' or 'exogenous' variable. Consumption demand is the driving force in this part of the equilibrium conditions. Similarly, in the value side of the economy, prices p are the explained or endogenous variables with unitary value-added v being the unexplained or exogenous variable that drives price formation under the input–output technology A.

Let us now check three interesting economic properties of equilibrium in the linear economy. Consider in the first place expression (2.25) once again and recall that it tells us equilibrium quantities Y for *any* non-negative vector c. Thus a very particular non-negative vector c could be the vector with a 1 in position j and 0 in all other entries. Applying this especial vector in (2.25) would yield gross output exactly equal to the j-th column of matrix $(I-A)^{-1}$. If we set $M=(I-A)^{-1}$, the j-th column of M provides summary information of how gross output in all sectors must respond to accommodate a consumption demand of 1 unit of good j. Since technical coefficients are fixed in the linear economy, the matrix M itself is fixed too. Each of its columns is therefore a unitary, scaled down representation of the equilibrium relationships in the economy and evaluates how a unitary injection in consumption demand for the j good affects the production status in all sectors of the economy and how this translates into adjusted output levels in all sectors. It is because of this translation property that matrix M is commonly known as the 'multiplier' matrix. The linearity built in Eq. (2.25) makes it easy to derive its equivalent counterpart in differential terms. Indeed, just take differentials and obtain:

$$dY = (I-A)^{-1}dc = Mdc \tag{2.30}$$

Here we now directly represent the quantity equilibrium relationships in terms of changes in consumption demand dc and that the effect that it brings about in gross output dY. In terms of partial derivatives the elements of matrix M would be: $m_{ij}=\partial Y_i/\partial c_j$. Similar 'multiplier' considerations apply to expression (2.29) which can be written in differential terms as:

$$dp = dv(I-A)^{-1} = dvM \tag{2.31}$$

We leave the interpretation of the multiplier mechanism acting in the price equation as an exercise for the reader.

Secondly, take expression (2.24) and pre-multiply it by equilibrium prices p from (2.29) and take (2.28) and post-multiply it by equilibrium quantities Y in (2.25). In the first case we would have:

$$v(I-A)^{-1}Y = v(I-A)^{-1}c + v(I-A)^{-1}AY \tag{2.32}$$

Recalling from (2.25) that $Y = (I - A)^{-1}c$ and rearranging a little bit yields:

$$vY = v(I - A)^{-1}(Y - AY) = pc \tag{2.33}$$

In the second case we take (2.28) and post-multiply by the solution in (2.25) to obtain:

$$p(I - A)^{-1}c = v(I - A)^{-1}c + pA(I - A)^{-1}c \tag{2.34}$$

As before, using the solution $p = v(I - A)^{-1}$ and rearranging gives:

$$pc = (p - pA)(I - A)^{-1}c = vY \tag{2.35}$$

In both instances we have found that $pc = vY$. This equality tells us that in equilibrium total expenditure pc will necessarily be equal to total income vY, one of the macroeconomic identities that we know must hold true in national accounting. In other words, with $pc = vY$ we are verifying that Gross Domestic Product (*GDP*) has a unique value whether it is calculated from the income side or from the expenditure side of the economy.

A third observation that applies to the equilibrium magnitudes in the linear economy is that quantities (expressions 2.24 or 2.25) do not depend on prices and prices (expressions 2.28 and 2.29) do not depend on quantities. The dichotomy between the independent ways that equilibrium quantities and equilibrium prices are determined is a property of linear economies that very much simplifies and facilitates the analysis but at the same time marks the limits of its capacities. The message that we read from the properties of the linear economy is that we can look at the quantity side of the economy without worrying too much on what is going on at the value side, and vice versa. As a first approximation to discern some economic phenomena this approach may perhaps be enough; also under certain situations where idle primary resources, labor and capital services, are abundant and thus their prices are not subject to upward pressures. A full and comprehensive appraisal, however, will always require taking into consideration the mutual interplay of quantities and prices that general equilibrium theory posits and that we know takes place in most economic situations.

2.5 Differential Properties of Equilibrium

The concept of the multiplier matrix is a very appealing one since it allows us to focus on a very specific set of information that captures the underlying general equilibrium effects in a very synthetic way. With the properties that characterize the linear economy, the multiplier matrix turns out to be a constant and its calculation follows from simple linear algebra. Linear economic models turn out to be theoretically simple and easily implementable in empirical terms when data is available. Nonetheless, linear models miss some of the action that takes place in the market.

Along the way we have cursorily mentioned some of these limitations. It is appropriate to take stock of the most relevant ones. For instance, linear models do not take into account that primary factors are usually restricted in supply. If so, any increase in the demand for factors by firms will exert an upward pressure on their prices making production more costly and pushing upwards as well final prices for goods. This will make consumers poorer in real income terms, hence negatively affecting consumption demand. On the other hand, a higher retribution for primary factors will be a countervailing effect on consumers' real income. The net outcome, affected by income as well as by price effects in demand, is therefore unclear and a more comprehensive analysis that captures the interplay of demand and supply is certainly needed.

Linearity itself is also a restrictive assumption. With fixed technological coefficients any effects on factor prices will not be translated into an adjusted demand for factors via technological substitution. The structure of the Leontief production function (2.21) completely prevents any such substitution. Regardless of prices, under this production function the same proportion of inputs is always chosen. This strict and unique selection does not seem reasonable under all circumstances. Firms need some room to fine-tune their selection of inputs when their prices change and we know that in reality they usually have some technological room to maneuver. If this is the case, once again substitution effects will be a contributing factor in determining resource allocation. In other words, the world has quite a few nonlinearities.

These considerations should cool down a little bit any exaggerated enthusiasm we may have regarding linear economies. The most prominent victim of nonlinearities is the multiplier matrix M since it cannot be used 'as is' in a non-linear world. If factors prices affect the choice of technique by firms, the technological recipes will not be fixed. The matrix A may not be fixed, or the value-added vector v may be an aggregation of labor and capital whose composition reacts to the price of labor and the price of capital. More on this will be studied in the subsequent chapters of this book devoted to general equilibrium. For now, all we need is to reach the conclusion that the multiplier matrix M from the linear economy is of no use under the standard nonlinearities of general equilibrium and the restrictions that downward sloping demand functions and upward sloping supply functions exert. Hence, we need a different 'multiplier' tool for when prices p and quantities Y are not dichotomous.[4]

2.5.1 General Equilibrium Multipliers

Let us consider a (non-linear) economy whose equilibrium consists of N prices $p = (p_1, p_2, \ldots, p_N)$ and N quantities $Y = (Y_1, Y_2, \ldots, Y_N)$. Prices and quantities are the endogenous to the model, or explained, variables. The equilibrium is therefore

[4]The idea of a general multiplier matrix in a nonlinear setting is due to Robinson and Roland-Holst (1988).

described by a total of $2N$ variables. For notational simplicity, let us consider N exogenous or unexplained variables $u = (u_1, u_2, \ldots, u_N)$; these quantities may represent, for instance, the level of public investment or exports demand in each of the N goods and their levels are taken from decisions that rest outside the model. Let $\varphi : R^{2N+N} \to R^{2N}$ be the function that describes in its fixed points the equilibrium state in the economy:

$$\varphi(Y, p, u) = (Y, p) \tag{2.36}$$

We now split the equilibrium function φ into its two projections φ^y and φ^p:

$$\begin{aligned} \varphi^y(Y, p, u) &= Y \\ \varphi^p(Y, p, u) &= p \end{aligned} \tag{2.37}$$

Each of the two projections contains N equations so that in total we have $2N$ unexplained variables and $2N$ equilibrium equations.[5] Assuming that function φ is differentiable, we proceed to perform comparative statics in expression (2.37) considering an exogenous change du in the unexplained variable u. The goal is to obtain a relationship that shows how these changes in the unexplained variables trigger changes in the equilibrium levels of the endogenous variables. Taking derivatives we would obtain:

$$\begin{aligned} dY &= \frac{\partial \varphi^y(Y, p, u)}{\partial Y} dY + \frac{\partial \varphi^y(Y, p, u)}{\partial p} dp + \frac{\partial \varphi^y(Y, p, u)}{\partial u} du \\ dp &= \frac{\partial \varphi^p(Y, p, u)}{\partial Y} dY + \frac{\partial \varphi^p(Y, p, u)}{\partial p} dp + \frac{\partial \varphi^p(Y, p, u)}{\partial u} du \end{aligned} \tag{2.38}$$

We simplify notation by setting $M_{yy} = \partial \varphi^y(Y, p, u)/\partial Y$, $M_{yp} = \partial \varphi^y(Y, p, u)/\partial p$, and so on. Solving for dp in the second expression of (2.38) (and assuming of course that all the involved mathematical operations are doable) we find:

$$dp = \left(I - M_{pp}\right)^{-1} \left(M_{py} dY + M_{pu} du\right) \tag{2.39}$$

We substitute this result in the first expression of (2.38):

$$\begin{aligned} dY &= M_{yy} dY + M_{yp} dp + M_{yu} du = \\ &= M_{yy} dY + M_{yp} \left(I - M_{pp}\right)^{-1} \left(M_{py} dY + M_{pu} du\right) + M_{yu} du = \\ &= \left(M_{yy} + M_{yp} \left(I - M_{pp}\right)^{-1} M_{py}\right) dY + \left(M_{yp} + \left(I - M_{pp}\right)^{-1} M_{pu}\right) du \end{aligned} \tag{2.40}$$

[5] Recall in fact that Walras' Law implies that one of this $2N$ equations is redundant and only $2N - 1$ variables can be determined within the model. Hence the need of fixing one of the variables to balance the account between independent equations and unknowns. Since only relative prices matter the selection is a price that works as *numéraire*.

Finally, we now solve for dY in (2.40) to obtain:

$$dY = \left(I - \left(M_{yy} + M_{yp}\left(I - M_{pp}\right)^{-1} M_{py}\right)\right)^{-1} \left(M_{yu} + M_{yp}\left(I - M_{pp}\right)^{-1} M_{pu}\right) du = \\ = \mathbf{M}(Y, p, u) du \tag{2.41}$$

Notice first that $\mathbf{M}(Y, p, u)$ is nothing but a $N \times N$ matrix and that this matrix captures the complex interplay chain that takes place between endogenous prices and quantities and among them and the unexplained variable u through the partial derivative matrices M_{yy}, M_{yp}, etc. It is worth remarking that some of these effects respond to cross effects between variables (take $M_{yp}, M_{py}, \ldots$), other effects correspond to own effects (M_{yy}, M_{pp}) and still other effects respond to amplifier loops (the simplest being $(I - M_{pp})^{-1}$). Observe too that the matrix $\mathbf{M}(Y, p, u)$ can be seen as a multiplier matrix. Indeed, for any of its entries we have:

$$\frac{\partial Y_i}{\partial u_j} = \mathbf{m}_{ij}(Y, p, u) \tag{2.42}$$

However, this matrix, unlike the multiplier matrix M in the linear economy, is not constant. Its values depend on the equilibrium magnitudes (Y, p) which in turn depend on the values of the exogenous variable u. A multiplier matrix in the general equilibrium setting does exist but it is not directly observable and, in addition, its values are equilibrium dependent. The most we can do is to estimate it for a given configuration of the unexplained variable. Instead of a fixed multiplier matrix M we have a continuous multiplier function in matrix format $\mathbf{M}(Y, p, u)$.

A noteworthy fact relating the results for the general equilibrium model with those of the linear economy stems when we recall that in the linear setup prices and quantities are independent of each other. In the terminology of this section this would mean that the partial derivatives between prices and quantities would be the null matrix, i.e. $M_{yp} = M_{py} = \mathbf{0}$. Insert this condition in expression (2.41) and verify that it reduces to:

$$dY = \left(I - M_{yy}\right)^{-1} M_{yu} du \tag{2.43}$$

This simplified expression yields back the quantity equilibrium Eq. (2.30), when we express it in differential terms $dY = (I - A)^{-1} dc$, with $M_{yy} = A$ and $M_{yu} du = dc$. The full chain of interactions in (2.41) are substantially more complex than those that expression (2.30) captures. It cannot be otherwise given the more general approach and the more comprehensive layers of interactions present in non-linear models versus those of linear models.

An image is always a great helping aid and "worth a thousand words", as the saying goes. Take a look at Fig. 2.5. It depicts in a simplified way how the general equilibrium model responds to a change in the exogenous variable u. The first reaction on prices p and quantities Y takes place through matrices M_{pu} and M_{yu},

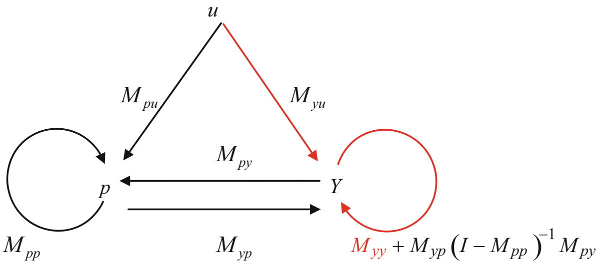

Fig. 2.5 Graphical visualization of circuits of influence

respectively, as the straight arrows originating in u signal. In turn, price effects repeatedly self-multiply over the loop M_{pp} represented by the circular arrow around p. They are also influenced by cross quantity-to-prices effects represented by M_{py}. Likewise, the initial kick off effect on quantities is itself directly multiplied by the chain reaction M_{yy} around the loop circling Y and indirectly from quantities-to quantities by way of looped prices via the combined effect that is measured by $M_{yp}(I - M_{pp})^{-1}M_{py}$. The role exerted by cross effects M_{py} from quantities-to-prices closes the description.

The red arrows show the circuits of influence that would remain in the linear model, when price and quantity determination are unrelated. The external change in u would work only through M_{yu} and the looped effects would interact exclusively via quantities on quantities from M_{yy}. Recall to this effect the structure of Eq. (2.43). In the present discussion of the different layers of influence that underlay the determination of equilibrium, we have not been completely forthcoming—and willingly so. In fact, the reader will have probably realized how we have resorted again and again to use the partial equilibrium framework as an attempt to approach and understand what we know is nothing but a general equilibrium setting. We should by now be fully aware that all general equilibrium magnitudes are determined simultaneously but we need to somehow sequence the explanations (*first* this happens, *then* this other effect takes place, etc.) to make some sense of how prices and quantities behave. There is nothing wrong with using this recourse provided we do it with the necessary discipline and rigor and we do the appropriate contextualization. In fact, we should be deeply thankful to partial equilibrium for it gives us a solid platform for discussing and interpreting economic facts.

Finally, departing from expression (2.38) and repeating all of the steps but now solving for dp instead of dY, we could derive a multiplier matrix that would capture general equilibrium price effects in response to changes in the exogenous variable u. It would be the generalized counterpart of the price Eq. (2.31). We leave this derivation to the reader as an exercise.

2.6 Summary

General equilibrium theory has provided economics with a very sound set of powerful tools and has instilled discipline of though and analytical rigor to countless generations of economists. The influence of general equilibrium has been pervasive even beyond its original and traditional microeconomics setting. Modern macroeconomics, for instance, is general equilibrium in an aggregate, dynamic time set-up. Industrial organization has also evolved from its partial equilibrium beginnings and has adopted some of the general equilibrium flavor. International trade theory is nothing but general equilibrium between countries or regions.

An actual economy, after all, is composed of numerous agents, markets and institutions, all of them interacting in the social medium and general equilibrium theory gives us insights as to what can we expect, in economic terms, from such a complex interaction. Theoretical models are of course ideal constructs and they are developed using assumptions that sometimes are hard to fathom in practical terms. The transition from these ideal models to more mundane applied models is justified, on the one hand, by the well-established body of theoretical results and, on the other hand, by the need to have operational tools which are able to explore the intricacies of complex issues; in other words, by the need to evaluate the actual economic policies being implemented in the real world by all levels of government. As we will see in the next chapters, applied general equilibrium is one very powerful tool to this effect.

In this chapter we have attempted to present, in an admittedly brief way, the main properties of the Walrasian equilibrium concept and provide some insights on their theoretical relevance, from both a positive and normative perspective. We have presented a linear version of the equilibrium model that is simpler, hence more transparent and operational but also with less scope, than the full-fledged model. The linear model allows for the elementary derivation of the so-called multiplier matrices. These matrices are pieces of information that encapsulate the essential interactions taking place in the economy regarding quantity and price formation. Using the tool of comparative statics to explore how equilibrium magnitudes depend on external to the model variables, we have also shown that the linear model is just a particular case of the general equilibrium model whenever prices and quantities are unrelated. In the general setting multiplier matrices still exist but, unlike the linear case, they are not constant and their values depend on the specific equilibrium under scrutiny. Even though the Walrasian theoretical model is just that, a theoretical model, it is nonetheless the point of departure and provides the necessary blueprint for carrying out sound and well-founded applied economic analysis and it is the basis for most if not all of the modeling extensions that have been developed in the literature.

2.7 Questions and Exercises

1. Consider an exchange economy with two goods ($i = 1, 2$) and two consumers ($h = 1, 2$). Both consumers have preferences represented by Leontief utility functions:

$$u_h(c_{1h}, c_{2h}) = \mathrm{Min}\left(\frac{c_{1h}}{\alpha_h^1}, \frac{c_{2h}}{\alpha_h^2}\right)$$

with endowments $e_h = (e_{1h}, e_{2h})$.

(a) Derive the Leontief demand functions and verify they take the form (hint: you can't use calculus here):

$$c_{ih} = \chi_{ih}(p, e_h) = \frac{\alpha_h^i \sum_{j=1}^2 p_j \cdot e_{jh}}{\sum_{j=1}^2 \alpha_h^j p_j}$$

(b) Assume values $\alpha_1^1 = \alpha_1^2 = 1; \alpha_2^1 = 1, \alpha_2^2 = 0.5; e_1 = (1, 0), e_2 = (0, 1)$. Show that an equilibrium in positive prices does not exist for this particular economy (hint: use an Edgeworth's box). Show also that if an equilibrium exists it will have to be one of the type $\left(p_1^*, p_2^*\right) = (1, 0) = (0, 1)$. Explain what is peculiar about this equilibrium.

2. Consider these two Leontief production functions for two material inputs and one primary factor:

$$Y_1 = \mathrm{Min}\left(\frac{VA_1}{0.5}, \frac{y_{11}}{0.2}, \frac{y_{21}}{0.7}\right)$$

$$Y_2 = \mathrm{Min}\left(\frac{VA_2}{0.2}, \frac{y_{12}}{0.5}, \frac{y_{22}}{0.3}\right)$$

(a) Write the input–output matrix A.
(b) Calculate the multiplier matrix $M = (I - A)^{-1}$ using the inversion routines built in Excel (or in any other computational software that you are familiar with and use) and check that matrix A is productive (think well how to do this).
(c) We know that to satisfy a vector of consumption demand c the economy has to produce a total output of $Y = Mc$. Because of Eq. (2.24) we know that c is a part of Y so we can try to compute the needed production in a sequential way. First is c and then what intermediate output you need to make c to appear. Let us call it $Y_0 = c$. The required intermediate output will be $Y_1 = AY_0$. But Y_1 has itself to be produced and this needs an intermediate output level of $Y_2 = AY_1 = A(AY_0) = A^2c$. In general, in the k-th stage an amount $A^k c$ of intermediate output is needed. Adding up all rounds we would obtain all the required production in all of the stages:

$$Y_0 + Y_1 + Y_2 + \ldots. = c + Ac + A^2c + \ldots. = \left(I + A + A^2 + \ldots\right)c$$

Verify (take $k = 5$) that this matrix sum approximates the multiplier matrix M and that the approximation improves as more terms are added.

3. Derive the general price multiplier matrix that would be the sister expression to (2.41) but now relating dp to du.

A Simple General Equilibrium Model 3

The model we describe in this chapter is a skeleton model upon which more in depth specifications will be developed later. Our purpose is to lay out in a very transparent way the main structure of the basic model and we do so by omitting almost all the details that would play a significant role in dealing with real life policy applications. The simple model corresponds to an economy with no government and no foreign trade. The absence of government means that no economic activity is taxed and no government spending is allowed. In a similar vein, all goods are produced and consumed within the country with no domestic goods being diverted abroad as exports and no foreign goods being hauled into the country as imports. The only economic agents are households (or consumers) and firms (or producers).

3.1 Households

There are H households in this economy, and each one of them is characterized by a pair that describes their characteristics in terms of preferences and initial property over goods, $\{u_h, e_h\}$. Here u_h is a utility function and e_h a vector that describes the initial endowment of goods in the hands of agent h. We distinguish between goods for consumption or intermediate use, of which there are N, and factors of production (say, various types of labor and capital) which consumers are endowed with and whose number is K. The complete commodity space is, therefore, $\mathbb{R}^N \times \mathbb{R}^K$ and a vector (c, x) in $\mathbb{R}^N \times \mathbb{R}^K$ will denote a typical bundle with c representing the N entries for produced goods and x being the K entries for primary factors or non-produced goods. With this notation, the endowment vector e_h of consumer h is a point in $\mathbb{R}^K$ since we assume no initial endowments of produced goods.

Each consumer h faces a set of prices $p = (p_1, p_2, \ldots, p_N)$ for consumption goods and $\omega = (\omega_1, \omega_2, \ldots, \omega_K)$ for factors. The problem of the consumer is to obtain the optimal consumption bundle and this is accomplished by solving the constrained utility maximization problem:

M.A. Cardenete et al., *Applied General Equilibrium*, Springer Texts in Business and Economics, DOI 10.1007/978-3-662-54893-6_3

$$Max\ u_h(c_h, x_h)\ \text{ subject to } \sum_{i=1}^{N} p_i \cdot c_{ih} + \sum_{j=1}^{K} \omega_k \cdot x_{kh} = \sum_{k=1}^{K} \omega_k \cdot e_{kh} \tag{3.1}$$

where c_{ih} is the amount of good i consumed by h and x_{kh} is what we will call, for lack of a better generic name, household's h use of factor k. The latter is easily interpreted as leisure when we refer to labor but no clear-cut interpretation is available for capital. In practice, however, this will not cause any interpretation difficulties since in all cases capital *per se* will not be utility producing.

We shall assume the usual properties of the utility function u_h for this problem to have a solution.[1] Given the vector of prices (p, ω), the optimal solution can be written as:

$$\text{Demand for consumption}: \ c_h(p,\omega), \text{with } c_h(p,\omega) \in \mathbb{R}^N$$

$$\text{Supply of factors}: \ s_h(p,\omega) = e_h - x_h(p,\omega), \text{with } s_h(p,\omega) \in \mathbb{R}^K \tag{3.2}$$

Now, if a factor k does not enter the utility function as an argument, then the consumer is better off by devoting all of the income to buy the utility producing goods. This enlarges the feasible set and allows the consumer to move to higher level indifference curves. In this case $x_{kh}(p,\omega) = 0$ and consumer h will be inelastically offering all the endowment of factor k, i.e. $s_{kh}(p,\omega) = e_{kh}$.

3.2 Firms

There are as many firms as consumption goods, N, and each firm produces one and only one good. Let x_{kj} denote the amount of factor k used by firm j and let y_j be its output. The available technology to the firm, embodied in a production function F_j that allows substitution among factors, can be written as:

$$y_j = F_j\left(x_{1j}, x_{2j}, \ldots, x_{Kj}\right) \tag{3.3}$$

We shall assume that F_j exhibits Constant Returns to Scale (*CRS*) (Varian 1992, Chap. 1). In addition, the production of each unit of output requires the use of raw materials as inputs. These inputs are goods that need to be produced and belong to the same classification as the consumption goods. They are in fact the same goods as the previously mentioned N consumption goods and what distinguishes them is their use by households (as consumption) or firms (as intermediate inputs). Let us further assume that the unitary requirements for these intermediate inputs are also independent of the scale of output, that is, for each y_j units of output firm j needs to use an amount $y_{ij} = a_{ij} \cdot y_j$ of good i. Here a_{ij} are non-negative coefficients that describe the relation between output and inputs in the production of good j. We will

[1] More specifically, these functions are continuously differentiable, quasi-concave, monotone increasing.

refer to them, from now on, as input–output coefficients. Notice that there is no substitution among material or non-primary inputs.

An alternative, but fully equivalent, specification of the production technology states that factors generate or produce a composite factor, called 'value-added' (VA), which in turn combines in fixed proportions with the material inputs to yield output. The production function is described henceforth in two stages. The first stage is a classical Leontief (1966) production function of the type:

$$y_j = \min\left(\frac{VA_j}{v_j}, \frac{y_{1j}}{a_{1j}}, \frac{y_{2j}}{a_{2j}}, \ldots, \frac{y_{Nj}}{a_{Nj}}\right) \tag{3.4}$$

where v_j is a coefficient that represents the minimal amount of the composite factor needed to produce one unit of output j, and y_{ij} are the requirements of good i to obtain y_j units of j. In the second stage we have a production function f_j that maps primary factors into value-added allowing for factor substitution:

$$VA_j = f_j\left(x_{1j}, x_{2j}, \ldots, x_{Kj}\right) \tag{3.5}$$

By construction the coefficient v_j satisfies $v_j = VA_j/y_j$. Therefore:

$$VA_j = v_j \cdot y_j = v_j \cdot F_j\left(x_{1j}, x_{2j}, \ldots, x_{Kj}\right) \tag{3.6}$$

and the relationship between the functions F_j in (3.3) and f_j in (3.5) is simply $f_j = v_j \cdot F_j$.

The economic problem of the firm is to select the profit maximizing level of output given commodity prices p, factor prices ω and the available technology. The firm's revenue for output level y_j is $I_j = p_j \cdot y_j$ whereas total cost includes payments to factors and payments to other firms supplying material inputs:

$$C_j\left(y_j\right) = \sum_{k=1}^{K} \omega_k \cdot x_{kj} + \sum_{i=1}^{N} p_i \cdot y_{ij} \tag{3.7}$$

The cost minimizing factor combination to produce y_j units of output is the solution of the problem:

$$Min \ \sum_{k=1}^{K} \omega_k \cdot x_{kj} + \sum_{i=1}^{N} p_i \cdot y_{ij} \text{ subject to : } y_j = F_j\left(x_{1j}, x_{2j}, \ldots, x_{Kj}\right) \tag{3.8a}$$

Given the level of output y_j, the amount of the required composite factor is obtained through the coefficient v_j (i.e. $VA_j = v_j \cdot y_j$) while the input–output coefficients determine the amount of material inputs (i.e. $y_{ij} = a_{ij} \cdot y_j$). This simplifies the above problem to:

$$Min \sum_{k=1}^{K} \omega_k \cdot x_{kj} \text{ subject to: } VA_j = f_j\left(x_{1j}, x_{2j}, \ldots, x_{Kj}\right) \tag{3.8b}$$

The solution of this problem yields the conditional factors demand of firm j for factor k, $x_{kj} = x_{kj}(\omega; VA_j)$. Plugging these functions into the objective function we obtain the cost function for the composite factor:

$$C_j\left(\omega; VA_j\right) = \sum_{k=1}^{K} \omega_k \cdot x_{kj}\left(\omega; VA_j\right) \tag{3.9}$$

Constant Returns to Scale imply that the total cost of generating VA_j units of the composite factor can be multiplicatively split into two elements, namely, the cost of a unit of value-added and the physical amount of value-added[2]:

$$C_j\left(\omega; VA_j\right) = C_j(\omega; 1) \cdot VA_j \tag{3.10}$$

The unit cost $C_j(\omega; 1)$ is, in fact, a price index for the composite good value-added. We will be also denoting this price index by $pva_j(\omega)$ (for price of value-added). Since absolute levels do not matter in determining the optimal mix of factors (can you think why?), from expressions (3.7) and (3.9) we have:

$$\begin{aligned} C_j\left(y_j\right) &= C_j\left(\omega; VA_j\right) + \sum_{i=1}^{N} p_i \cdot y_{ij} = C_j(\omega; 1) \cdot VA_j + \sum_{i=1}^{N} p_i \cdot y_{ij} \\ &= pva_j(\omega) \cdot VA_j + \sum_{i=1}^{N} p_i \cdot y_{ij} \end{aligned} \tag{3.11}$$

We can now define average or unit cost ac_j and using the technical coefficients in (3.4) write:

$$ac_j = \frac{C_j\left(y_j\right)}{y_j} = \frac{pva_j(\omega) \cdot VA_j + \sum_{i=1}^{N} p_i \cdot y_{ij}}{y_j} = pva_j(\omega) \cdot v_j + \sum_{i=1}^{N} p_i \cdot a_{ij} \tag{3.12}$$

Finally, the profit function is:

$$\prod_j \left(y_j\right) = p_j \cdot y_j - ac_j \cdot y_j = \left(p_j - ac_j\right) \cdot y_j \tag{3.13}$$

and profit maximization requires $p_j = ac_j$. At any output level the firm is making zero profits, and thus the output level cannot be determined independently of demand. Under *CRS*, ac_j is also the familiar marginal cost, and we see that the standard textbook condition "price = marginal cost" is satisfied.

[2]This is a well-known result from micro-theory. See Varian (1992), Chap. 5.

It is convenient at this point to introduce some new notation in matrix form. If A denotes the matrix of input–output coefficients a_{ij}, V is a diagonal matrix with diagonal elements defined as $v_{jj} = v_j$, and $pva(\omega)$ stands for the vector of unitary sectoral value-added, then combining expression (3.12) with the zero profit condition we can write:

$$p = pva(\omega) \cdot V + p \cdot A \tag{3.14}$$

which gives us a very compact representation of the basic price equation of the model. Sometimes an alternative expression for the price equation may be more appropriate. Remember that $pva(\omega)$ is the optimal (minimum) cost for producing one unit of value-added of each good whereas total minimum cost is given by expression (3.9). An application of a fundamental microeconomics result known as Shephard's lemma (1970) to the cost function $C_j(\omega; VA_j)$ allows us to obtain the conditional demand function for factor k:

$$\frac{\partial C_j\left(\omega; VA_j\right)}{\partial \omega_k} = x_{kj}\left(\omega; VA_j\right) \tag{3.15}$$

Dividing and multiplying by VA_j we obtain:

$$x_{kj}\left(\omega; VA_j\right) = \frac{x_{kj}\left(\omega; VA_j\right)}{VA_j} \cdot VA_j = b_{kj}(\omega) \cdot VA_j \tag{3.16}$$

Notice that $b_{kj}(\omega)$ can be seen as the optimal amount of factor k required to produce one unit of j's value-added at factor prices ω. Being a ratio of input (factor k) over output (value-added j), it is a technical coefficient. The set of coefficients $b_{kj}(\omega)$ are, in fact, variable technical coefficients since they depend on the values taken up by the vector of factor prices ω. We now combine expressions (3.9) and (3.10):

$$C_j\left(\omega; VA_j\right) = C_j(\omega; 1) \cdot VA_j = \sum_{k=1}^{K} \omega_k \cdot x_{kj}\left(\omega; VA_j\right) \tag{3.17}$$

Since $pva_j(\omega) = C_j(\omega; 1)$ we obtain:

$$\begin{aligned} pva_j(\omega) &= \frac{1}{VA_j} \cdot \sum_{k=1}^{K} \omega_k \cdot x_{kj}\left(\omega; VA_j\right) = \sum_{k=1}^{K} \omega_k \cdot \left(\frac{x_{kj}\left(\omega; VA_j\right)}{VA_j}\right) = \\ &= \sum_{k=1}^{K} \omega_k \cdot b_{kj}(\omega) \end{aligned} \tag{3.18}$$

A quick substitution into (3.14) yields the equivalent price equation:

$$p_j = v_j \cdot \sum_{k=1}^{K} \omega_k \cdot b_{kj}(\omega) + \sum_{i=1}^{n} p_i \cdot a_{ij} \tag{3.19}$$

When substitution is possible, the technical coefficients are price dependent as we can see in expression (3.19).

3.3 Equilibrium

The basic model outlined above describes how demands and supplies are determined and their dependence on prices. An equilibrium for this economy corresponds to the standard Walrasian concept: a situation in which at a given set of commodity and factor prices all producers maximize profits, all consumers maximize their utility, and all markets for goods and factors clear. All agents, therefore, do as well as possible within their individual constraints when facing equilibrium prices while at the same time aggregate feasibility constraints are satisfied too.

On the demand side, goods are demanded by consumers to satisfy their consumption needs and by firms to carry out their production plans. In addition, firms also demand those primary factors that are necessary to generate output. On the supply side, we have firms offering goods and households offering all or part of their endowment of factors. Market demand and supply are derived by summation of the individual demand and supply schedules.

The vector of consumption demand by households is obtained as:

$$CD(p, \omega) = \sum_{h=1}^{H} c_h(p, \omega) \tag{3.20}$$

Using the firms' technologies embodied in A we derive total intermediate demand for good i as:

$$ID_i = \sum_{j=1}^{N} a_{ij} \cdot y_j \tag{3.21}$$

Observe two things here that are relevant. First, intermediate demand does not depend on prices; this is due to the Leontief specification (i.e., fixed coefficients) of the inputs requirement technology. Second, ID_i is the product of the i—row of the input–output matrix A with the vector of outputs $Y = (y_1, y_2, \ldots, y_N)$, expressed in column format. Therefore, the (column) vector ID of intermediate demands can be written as:

$$ID = A \cdot Y \tag{3.22}$$

Total demand for goods $TD(p, \omega)$ is therefore:

$$TD(p, \omega) = CD(p, \omega) + A \cdot Y \tag{3.23}$$

When firm i is producing y_i units of output, the fixed coefficients technology requires the use of $v_i \cdot y_i$ units of value-added. Hence the total amount of each factor k required to fulfill the production plan y_i is given by:

$$z_{ki}(\omega; y_i) = x_{ki}(\omega; VA_i) = x_{ki}(\omega; v_i \cdot y_i) = b_{ki}(\omega) \cdot v_i \cdot y_i \tag{3.24}$$

From here it can be readily seen that the vector Z of total factors demand by firms at factor prices ω and output levels Y has the following matrix expression:

$$Z(\omega; Y) = B(\omega) \cdot V \cdot Y \tag{3.25}$$

where $B(\omega)$ is a $K \times N$ matrix of variable factor coefficients, V is the $N \times N$ diagonal matrix of unitary valued-added requirements, and Y is an N-dimensional column vector of outputs.

On the supply side, household h wishes to provide the vector $s_h(p, \omega)$ of factors as a result of his utility maximization problem (recall expression (3.2) above). Summation over all H consumers will yield the K-dimensional vector of market factor supplies:

$$S(p, \omega) = \sum_{h=1}^{H} (e_h - x_h(p, \omega)) = \sum_{h=1}^{H} e_h - \sum_{h=1}^{H} x_h(p, \omega) = e - X(p, \omega) \tag{3.26}$$

where e stands for the vector of total endowments and $X(p, \omega)$ represents market demand for the use of factors by households (leisure, for instance, in the case of labor).

The *CRS* assumption on firms' technologies implies that any amount of output is optimal from the firms' perspective, since any output level will give rise to zero economic profits. Hence, as long as price equals marginal cost, each and every of the firms will adapt its output level to match households' demand for consumption plus other firms' demand for intermediate inputs. Once all the price dependent demand and supply schedules for goods and factors have been specified, the neoclassical equilibrium concept involves a set of prices for goods and factors (p^*, ω^*) and a vector of output levels Y^* such that the following holds:

$$\begin{aligned}
&\text{(i) Supply} = \text{Demand in all markets for goods}:\\
&\qquad Y^* = TD(p^*, \omega^*, Y^*) = CD(p^*, \omega^*) + A \cdot Y^*\\
&\text{(ii) Supply} = \text{Demand in all factor markets:}\\
&\qquad S(p^*, \omega^*) = Z(\omega^*; Y^*)\\
&\text{(iii) All firms make zero profits:}\\
&\qquad p^* = pva(\omega^*) \cdot V + p^* \cdot A
\end{aligned} \tag{3.27}$$

It is interesting and informative to count the number of equations and of variables. Since $Y \in \mathbb{R}^N$, we have N equations embodied in the first set of equations, one for each of the produced goods; the second set of equilibrium conditions describe market clearing for K factors, while the zero profit condition applies to all N firms in the economy. In total, there are $2N+K$ equations in our equilibrium system. As for variables, they number $2N+K$ too, N of which correspond to prices for goods p, K to prices for factors ω, and N to output levels Y. The system of equations in (3.27) that represents an economic equilibrium has therefore the same number of equations as unknowns. It is a square system. This is no guarantee, however, for the equilibrium system to have a solution. Indeed, given our assumptions on preferences and technology, all demand functions are homogeneous of degree zero in prices. The implication that follows is well known in microeconomic theory: only relative prices matter and at most $2N+K-1$ prices can be determined. On the other hand, market demand functions satisfy the aggregate budget constraint known as Walras' law that we commented in Chap. 2 (see also Varian 1992, Chap. 17, for details) which takes here this specific form:

$$p \cdot [CD(p, \omega) - (I - A) \cdot Y] + \omega \cdot \left[Z(\omega, Y) - S\left(p, \omega\right)\right] = 0 \tag{3.28}$$

An implication of Walras' law is that one of the market clearing conditions is redundant and the equilibrium system contains only $2N+K-1$ independent equations. To determine all variables in the system, in particular all $N+K$ prices, some unit of measurement has to be chosen. In the economist's jargon such a unit of reference is commonly referred as the *numéraire*. The simplest way to proceed is to add a new condition to the equilibrium system that fixes one price or even establishes a normalization among a set of prices.

The equilibrium system (i) to (iii) involving $2N+K$ equations and $2N+K$ unknowns can, in fact, be reduced in dimensionality to K equations and unknowns. Given a factor price vector ω, condition (iii) yields:

$$p = pva(\omega) \cdot V \cdot (I - A)^{-1} \tag{3.29}$$

and as long as the matrix A is productive,[3] the inverse $(I - A)^{-1}$ will be non-negative and commodity prices will also be non-negative. Let us represent the dependence of p on ω by $p = p(\omega)$. The pair $(p(\omega), \omega)$ determines consumption demand $CD(p(\omega), \omega)$ and from condition (i) above and expression (3.23) we obtain:

$$Y = (I - A)^{-1} \cdot CD(p(\omega), \omega) \tag{3.30}$$

As before, the output levels Y will be non-negative for any ω if A is a productive matrix. Let now $Y = Y(\omega)$ denote the dependence of output levels on the given factor price vector ω. Observe that if for the chosen ω and the reduced form levels $p = p(\omega)$ and $Y = Y(\omega)$ condition (ii) holds, i.e. the factor markets are in equilibrium,

[3] Recall from the previous chapter that productivity of A is related to its maximal eigenvalue being <1. See Nikaido (1972), Chap. 3.

then the complete equilibrium system has been determined. In other words, if a given ω solves the K equations in K unknowns in (ii)

$$S(p(\omega),\omega) = B(\omega)\cdot V\cdot Y(\omega) \tag{3.31}$$

it also solves (i) and (iii) !!. This suggests a possible strategy in the search for equilibrium. We can form a problem of reduced dimensionality and from its solution ω^* obtain the remaining variables p^* and Y^*. This approach may have clear advantages in the computational stage of model solution since it would allow us to reduce the search space from a $2N+K$ dimension to a much smaller K dimension.

3.4 A Simple Example

There is no better way to fix the analytical ideas above than with an example. We shall try to keep it simple and transparent and for this purpose we will assume that there only are four economic agents, namely, two consumers and two firms. This small size economy will be enough to appraise the issues around general equilibrium modeling. Each firm produces a distinct good using two factors of production-labor and capital- and intermediate goods. Factors are owned by consumers who sell their services to firms to obtain the income that will allow them to finance their consumption purchases. Both consumers have standard Cobb–Douglas preferences defined only on the two produced goods. For instance:

$$\begin{aligned} u_1(c_{11},c_{21}) &= c_{11}^{\beta_{11}}\cdot c_{21}^{\beta_{21}} = c_{11}^{0.3}\cdot c_{21}^{0.7} \\ u_2(c_{21},c_{22}) &= c_{21}^{\beta_{21}}\cdot c_{22}^{\beta_{22}} = c_{21}^{0.6}\cdot c_{22}^{0.4} \end{aligned} \tag{3.32}$$

The endowment vectors e_h for households ($h = 1, 2$) are as follows:

$$\begin{aligned} e_1 &= (e_{11},e_{21}) = (30,20) \\ e_2 &= (e_{12},e_{22}) = (20,5) \end{aligned}$$

If we wish we can interpret the first entry in each vector e_h as labor ($k = 1$) and the second one as capital ($k = 2$).

The technology of the two firms in the economy is fully described by the input–output matrix A and the value-added production functions VA. We will assume the following values for A:

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 0.20 & 0.50 \\ 0.30 & 0.25 \end{pmatrix}$$

whereas Cobb–Douglas aggregators are also used to represent the substitution possibilities between labor and capital in both firms:

$$
\begin{aligned}
VA_1 &= \mu_1 \cdot x_{11}^{\alpha_{11}} \cdot x_{21}^{\alpha_{21}} = \mu_1 \cdot x_{11}^{0.8} \cdot x_{21}^{0.2} \\
VA_2 &= \mu_2 \cdot x_{12}^{\alpha_{12}} \cdot x_{22}^{\alpha_{22}} = \mu_2 \cdot x_{11}^{0.4} \cdot x_{21}^{0.6}
\end{aligned}
\tag{3.33}
$$

where μ_j represents a parameter related to the units in which factors and value-added are measured. The final step is to choose the coefficients v_j that state the amount of value-added that combines in fixed proportions with intermediate inputs. We will assume these coefficients to take values $v_1 = 0.5$ and $v_2 = 0.25$. The complete production functions can now be written as:

$$
\begin{aligned}
y_1 &= \min\left(\frac{VA_1}{0.5}, \frac{y_{11}}{0.2}, \frac{y_{21}}{0.3}\right) \\
y_2 &= \min\left(\frac{VA_2}{0.25}, \frac{y_{12}}{0.5}, \frac{y_{22}}{0.25}\right)
\end{aligned}
\tag{3.34}
$$

with VA_j depending on x_{ij} according to the Cobb–Douglas expressions (3.33). The description of preferences and technology completely characterizes the physical side of this simple economy. When we embed the economy with the behavioral assumptions of utility and profit maximization we further obtain demand and supply functions for goods and factors and the equilibrium relationships can be established.

Consider consumers in the first place. Since they only obtain utility from consumption the budget constraint reduces, for $h = 1, 2$ to:

$$
p_1 \cdot c_{1h} + p_2 \cdot c_{2h} = \omega_1 \cdot e_{1h} + \omega_2 \cdot e_{2h} \tag{3.35}
$$

Given prices p and ω the solution of the utility maximization problem with Cobb–Douglas preferences yields the demand for good i by consumer h as:

$$
c_{ih}(p, \omega) = \beta_{ih} \frac{\omega_1 \cdot e_{1h} + \omega_2 \cdot e_{2h}}{p_i} \tag{3.36}
$$

and total consumption demand by households, $CD(p, \omega)$ as defined in expression (3.20), can be specified. Let us now turn our attention to firms. Here we need to derive demand for factors and demand for intermediate goods. Solving problem (3.8a, 3.8b) would give us the cost functions for valued-added, which in our example can be seen to take the form:

$$
C_j(\omega, VA_j) = \frac{1}{\mu_j} \cdot \alpha_{1j}^{-\alpha_{1j}} \cdot \alpha_{2j}^{-\alpha_{2j}} \cdot \omega_1^{\alpha_{1j}} \cdot \omega_2^{\alpha_{2j}} \cdot VA_j \tag{3.37}
$$

The parameter μ_j relates to the units of measurement and this gives us some freedom to choose units and therefore μ_j. For convenience let us choose μ_j such that:

$$
\frac{1}{\mu_j} \cdot \alpha_{1j}^{-\alpha_{1j}} \cdot \alpha_{2j}^{-\alpha_{2j}} = 1
$$

This selection of units simplifies considerably the notation and gives us a more compact expression for the cost functions:

$$C_j(\omega, VA_j) = \omega_1^{\alpha_{1j}} \cdot \omega_2^{\alpha_{2j}} \cdot VA_j \tag{3.38}$$

We use now Shephard's lemma to derive the conditional demand for factors and the variable technical coefficients. A little algebra (do it as an exercise) shows that the demand for labor ($k = 1$) by firm j is given by:

$$\frac{\partial C_j(\omega, VA_j)}{\partial \omega_1} = x_{1j} = \alpha_{1j} \cdot \left(\frac{\omega_2}{\omega_1}\right)^{\alpha_{2j}} \cdot VA_j \tag{3.39}$$

with the labor technical coefficient in firm j being:

$$b_{1j}(\omega) = \alpha_{1j} \cdot \left(\frac{\omega_2}{\omega_1}\right)^{\alpha_{2j}} \tag{3.40}$$

Similar computations would allow us to obtain the conditional demand for capital ($k = 2$) and the technical coefficient for capital services (make sure to write them down also as an exercise). These price-dependent coefficients for labor and capital along with the unitary value-added requirements yield demand for factors conditional on output levels Y as stated in expression (3.25). Given any vector of output levels Y, intermediate requirements depend on the input–output matrix A and are computed as $A{\cdot}Y$. This completes the description of the demand side of the economy.

On the supply side, the aggregate endowment of each factor determines the availability of primary factors to firms. Firms will adjust their supply of output to satisfy households demand and intermediate demand by the firms themselves. With all demand and supply functions specified we could explicitly write the equilibrium conditions (i) and (ii) in (3.27). For condition (iii)—the zero profit condition—we use the price index for a unit of value-added, $pva_j(\omega)$, by setting $VA_j = 1$ in (3.38). The complete equilibrium system with 6 equations (2 equilibrium conditions for goods, 2 equilibrium conditions for primary factors and 2 price-cost equations) and 6 unknowns $(p_1, p_2, \omega_1, \omega_2, y_1, y_2)$ appears in detail in Appendix 1 at the end of this chapter.

3.5 The Numerical Solution

The willing reader can verify that the equilibrium equations for our simple economy are satisfied for the following set of output levels and prices:

$$\left(y_1^*, y_2^*\right) = (100, 100)$$
$$\left(p_1^*, p_2^*\right) = (1, 1)$$
$$\left(\omega_1^*, \omega_2^*\right) = (1, 1)$$

One may well wonder if it is purely coincidental that all prices have the nice property of being unitary. It is not. We have deliberately chosen the parameters and coefficients that describe initial endowments, preferences and technology for this to happen. This is of course not necessary but it is clearly convenient to have all the equilibrium values in a format which is easily remembered and, most important, handy to use when the need for comparisons arises. Appendix 2 provides an example of programming code in *GAMS* that would allow us to solve the model.[4]

With the equilibrium allocation Y^*, and equilibrium prices (p^*, ω^*) we can easily compute additional information about the equilibrium values for several relevant economic magnitudes. From the individual demand functions (3.36) we obtain total private aggregate consumption:

$$CD^* = \sum_{i=1}^{2} p_i^* \cdot \sum_{h=1}^{2} c_{ih}(p^*, \omega^*) \tag{3.41}$$

Since the model does not include a government or foreign sector and no investment activity takes place, Gross Domestic Product (*GDP*), from the spending point of view, is equal to private aggregate consumption. Similarly, we can compute payments to labor and capital services. Equilibrium in the labor and capital markets ensures that income accruing to consumers is the market value of their endowments:

$$\begin{aligned} W^* &= \omega_1^* \cdot (e_{11} + e_{12}) \\ \prod{}^* &= \omega_2^* \cdot (e_{21} + e_{22}) \end{aligned} \tag{3.42}$$

Wages plus payments to capital equal *GDP* from the income perspective, thus the macroeconomic accounting property $CD^* = W^* + \Pi^*$ holds (check it by calculating all involved magnitudes). From the equilibrium values for the microeconomic variables we derive the well-known macroeconomic identities, in our example a very simple expression equating aggregate consumption with aggregate income. The lesson to be drawn here is that all familiar macroeconomic identities can be derived from basic microeconomic data, no matter how complex the economy we are dealing with is.

[4]To verify that all checks you should download the demo version of *GAMS* (http://www.gams.com) and install it in a folder of your choice. Follow the directions for doing the installation. Next save or copy the text in Appendix 2 as a text file with any name you like (i.e., `simplecge.gms` would do) and you are ready to go. Fire up *GAMS*, load the named file and run it. Make sure though that you first familiarize yourself with the way the software runs.

Given the equilibrium output levels Y^*, information on factor use by firms can be also obtained using either the conditional factor demand functions (3.39) or the technical coefficients in (3.40). This would tell us how much of each factor is used by each of the firms, helping to identify labor or capital intensive sectors.

As we can see, each agent in this economy, be it a consumer or a firm, is the recipient of an income or the origin of a payment. Consumers receive payments for their factors and pay firms for their delivery of goods. Similarly, firms receive income from their sales of goods to consumers, making in turn payments to other firms and consumers for the goods and services required as inputs in the production plans. This detailed information on income and payments can be easily assembled and compactly represented in a double entry table or matrix following the accounting convention that all payments appear in columns and all receipts appear in rows. We will classify payments (and receipts) in three categories, even though we only consider two types of agents—consumers and firms. The first category is payments to firms and these can originate only on other firms or consumers. The second category is payments to factors, labor and capital, by firms. Factors are not agents in the economy, but the income accruing to factors will be distributed to consumers according to their endowment property. The third and last category comprises payments to consumers. In the real world firms pay for the factor services they use directly to their owners, but in our representation these payments first go through the factors category to then be distributed among consumers. In this sense, factors play an intermediary role whose only justification at this point is to provide a more transparent portrayal of the income-expenditure flows.

With two consumers, two firms (and two goods, one per firm), and two factors we are able to represent all transactions in a six-by-six matrix. Using the equilibrium allocation and prices and the behavioral and technical relationships that define the model, we compute all payments and receipts and display them in Table 3.1. These types of tables are known in the literature as Social Accounting Matrices (*SAM* for short; Stone and Brown 1962; Pyatt and Round 1985; Pyatt 1988). An individual element of the six-by-six *SAM* matrix, say cell $SAM(i, j)$ occupies the i-th row and the j-th column of the *SAM*. In a *SAM* each participant category is called an ‘account’, so we have six accounts in our *SAM*. To better visualize them, we are going to re-index our six participants in such a way that firms are now labeled accounts ‘1’ and ‘2’, factors are labeled accounts ‘3’ and ‘4’, and consumers are

Table 3.1 Social accounting matrix

		1	2	3	4	5	6	Totals
1	Firm 1	20	50	0	0	15	15	100
2	Firm 2	30	25	0	0	35	10	100
3	Factor 1	40	10	0	0	0	0	50
4	Factor 2	10	15	0	0	0	0	25
5	Consumer 1	0	0	30	20	0	0	50
6	Consumer 2	0	0	20	5	0	0	25
Totals		100	100	50	25	50	25	

now seen as accounts '5' and '6'. We now need to make explicit the algebraic relationships underlying the non-zero numerical values of this *SAM* matrix, those that represent actual transactions. It can be seen they take the following form:

Payments by firms ('1', '2') to firms ('1', '2'):

$$\begin{aligned}
SAM(1,1) &= p_1^* \cdot a_{11} \cdot y_1^* \\
SAM(1,2) &= p_1^* \cdot a_{12} \cdot y_2^* \\
SAM(2,1) &= p_2^* \cdot a_{21} \cdot y_1^* \\
SAM(2,2) &= p_2^* \cdot a_{22} \cdot y_2^*
\end{aligned}$$

Firms pay other firms for the delivery of their needed intermediate inputs. For instance, *SAM*(1,2) is the amount paid in equilibrium to firm '1' by firm '2' when '2' uses the amount of '1' (i.e., $a_{12} \cdot y_2^*$) needed to generate the output of firm number '2' (i.e., y_2^*).

Consumers ('5', '6') pay to firms ('1','2') for their acquisition of final goods:

$$\begin{aligned}
SAM(1,5) &= p_1^* \cdot c_{11}(p^*, \omega^*) = p_1^* \cdot \beta_{11} \cdot \frac{\omega_1^* \cdot e_{11} + \omega_2^* \cdot e_{21}}{p_1^*} \\
&= \beta_{11} \cdot \left(\omega_1^* \cdot e_{11} + \omega_2^* \cdot e_{21}\right)
\end{aligned}$$

$$\begin{aligned}
SAM(1,6) &= p_1^* \cdot c_{12}(p^*, \omega^*) = p_1^* \cdot \beta_{12} \cdot \frac{\omega_1^* \cdot e_{12} + \omega_2^* \cdot e_{22}}{p_1^*} \\
&= \beta_{12} \cdot \left(\omega_1^* \cdot e_{12} + \omega_2^* \cdot e_{22}\right)
\end{aligned}$$

$$\begin{aligned}
SAM(2,5) &= p_2^* \cdot c_{21}(p^*, \omega^*) = p_2^* \cdot \beta_{21} \cdot \frac{\omega_1^* \cdot e_{11} + \omega_2^* \cdot e_{21}}{p_2^*} \\
&= \beta_{21} \cdot \left(\omega_1^* \cdot e_{11} + \omega_2^* \cdot e_{21}\right)
\end{aligned}$$

$$\begin{aligned}
SAM(2,6) &= p_2^* \cdot c_{22}(p^*, \omega^*) = p_2^* \cdot \beta_{22} \cdot \frac{\omega_1^* \cdot e_{12} + \omega_2^* \cdot e_{22}}{p_2^*} \\
&= \beta_{22} \cdot \left(\omega_1^* \cdot e_{12} + \omega_2^* \cdot e_{22}\right)
\end{aligned}$$

In turn, firms ('1', '2') need to use factors ('3','4') to carry out their production plans. The payments are now:

$$\begin{aligned}
SAM(3,1) &= \omega_1^* \cdot b_{11}(\omega^*) \cdot v_1 \cdot y_1^* \\
SAM(4,1) &= \omega_2^* \cdot b_{21}(\omega^*) \cdot v_1 \cdot y_1^* \\
SAM(3,2) &= \omega_1^* \cdot b_{12}(\omega^*) \cdot v_2 \cdot y_2^* \\
SAM(4,2) &= \omega_2^* \cdot b_{22}(\omega^*) \cdot v_2 \cdot y_2^*
\end{aligned}$$

For instance, if we look at entry *SAM*(3,2), we see that when firm '2' produces output at level y_2^* it needs value-added at level $v_2 \cdot y_2^*$. Applying the labor technical coefficient $b_{12}(\omega^*)$ over this amount of value-added yields required labor input for producing output y_2^*. The value of the required labor input at the equilibrium wage rate ω_1^* is therefore the payment received by the labor factor. This is the value reflected in entry *SAM*(3,2).

Finally, there are payments by factors ('3','4') to consumers ('5','6'). In fact, factors are owned by consumers who receive the income from selling their factors to firms, but we visualize the process through the two factors' accounts:

$$
\begin{aligned}
SAM(5,3) &= \omega_1^* \cdot e_{11} \\
SAM(6,3) &= \omega_1^* \cdot e_{12} \\
SAM(5,4) &= \omega_2^* \cdot e_{12} \\
SAM(6,4) &= \omega_2^* \cdot e_{22}
\end{aligned}
$$

The remaining entries of the *SAM* are set to zero since there are no transactions among the corresponding row and column entries. We can think of the *SAM* in Table 3.1 in two ways. Firstly, given the structure of our simple economy as represented by preferences and technology, the *SAM* shows the equilibrium value of all transactions given a choice of utility parameters, production coefficients, and endowments. This is the *numerical SAM*. Secondly, there is also a *symbolic SAM* that represents all possible associated equilibria for the class of economies to which the simple model belongs. In this sense, the symbolic *SAM* is a general description of the current equilibrium but also of any alternative, or counterfactual, equilibrium that could be constructed by modifying the parameter description.

The derivation of the *SAM* from the model gives rise to the following interesting question: Can we take the trip back? In other words, should we have at our disposal a *SAM* as the one in Table 3.1, could we build and specify a model which is consistent with the given data structure? This is what is known as the calibration problem, and we will turn our attention to this problem in Chap. 6.

3.6 Summary

We have seen in this chapter how to build a general equilibrium model of a simple economy with two firms and goods, two households, and two factors. Starting with a description of consumers' preferences and firms' technical possibilities, we obtained a complete characterization of the supply and demand functions for goods and services. In turn, these functions determined the equilibrium system for the economy. The equilibrium solution guarantees that all markets for goods and services clear. Given the equilibrium values for prices and output, we can obtain a whole range of economic information of an aggregate and disaggregate nature and the network of transactions among consumers, firms and factors can be represented in full detail with the use of a Social Accounting Matrix (*SAM*). The aggregate magnitudes are constructed from the values of the microeconomic variables and they satisfy the conventions of the National Income and Product Accounts. They also provide an additional valuable source of information for the analyst interested in studying the aggregate responses induced by changes taking place at the most basic level—the individual level.

The availability of micro and macro information is one of the definite advantages of numerical general equilibrium over partial equilibrium analysis and macro modeling.

3.7 Questions and Exercises

1. Work the derivation of the demand functions for Cobb–Douglas consumers (easy).
2. Do the same for the Constant Elasticity of Substitution (*CES*) utility functions (medium). Consult Varian (1992) for a formal definition of *CES* utility and production functions.
3. Work out the derivation of the value-added price index for Cobb–Douglas factor aggregators (easy).
4. Do the same for CES value-added aggregators (medium).
5. Suppose that the production technology allows for Cobb–Douglas substitution among intermediate inputs as well as among primary factors. Further assume that value-added and the composite intermediate good combine in fixed proportions to generate output. Obtain the cost function and show how to derive the matrix A of variable input–output coefficients (hard. Hint: use Shephard's lemma).
6. Do the same as in question 5 above for Cobb–Douglas substitution between value-added and the composite intermediate good (hard).
7. For the economy of this chapter show that Walras' law takes indeed the form:

$$p \cdot [CD(p,\omega) - (I - A) \cdot Y] + \omega \cdot \left[Z(\omega, y) - S(p,\omega)\right] = 0$$

In explicit summation notation this vector expression would be:

$$\sum_{i=1}^{N} p_i \cdot \left(c_{ih}(p,\omega) + \sum_{j=1}^{N} a_{ji} \cdot y_i - y_i \right) + \sum_{k=1}^{K} \omega_k \cdot \left(\sum_{j=1}^{N} b_{kj}(\omega) \cdot v_j \cdot y_j + \sum_{h=1}^{H} x_{kh}(p,\omega) - \sum_{h=1}^{H} e_{kh} \right) = 0$$

(medium-hard. Hint: Write down the households' budget constraints, make explicit that this is a private ownership economy, even if profits are zero because of the *CRS* assumption, and *presto*. . .with a little bit of algebraic manipulation and patience).

Appendix 1: The Detailed Equations of the General Equilibrium Example

1. Equilibrium in the markets for the two goods:

Good 1:

$$c_{11}(p,\omega) + c_{12}(p,\omega) + a_{11} \cdot y_1 + a_{12} \cdot y_2 = y_1$$

Notice the left-hand side includes demand for good 1 by households $h = 1, 2$ for final use and demand for good 1 by firms $j = 1, 2$ for intermediate use. The right-hand side is supply of good 1.

Good 2:

$$c_{21}(p,\omega) + c_{22}(p,\omega) + a_{21} \cdot y_1 + a_{22} \cdot y_2 = y_2$$

Households' demands are of the Cobb–Douglas variety:

$$c_{ih} = \beta_{ih} \cdot \frac{\omega_1 \cdot e_{1h} + \omega_2 \cdot e_{2h}}{p_i} \quad \text{with } i = 1,2 \text{ and } h = 1,2$$

2. Equilibrium in the markets for the two factors*:*

Factor 1:

$$b_{11}(\omega) \cdot v_1 \cdot y_1 + b_{12}(\omega) \cdot v_2 \cdot y_2 = e_{11} + e_{12}$$

The left-hand side is demand for factor $k = 1$ by firms $j = 1, 2$. The right-hand side is the supply of factor $k = 1$ by households $h = 1, 2$.

Factor 2:

$$b_{21}(\omega) \cdot v_1 \cdot y_1 + b_{22}(\omega) \cdot v_2 \cdot y_2 = e_{21} + e_{22}$$

The technical coefficients $b_{kj}(\omega)$ are derived from the Cobb–Douglas production function:

$$b_{1j}(\omega) = \alpha_{1j} \cdot \left(\frac{\omega_2}{\omega_1}\right)^{\alpha_{2j}}$$
$$b_{2j}(\omega) = \alpha_{2j} \cdot \left(\frac{\omega_1}{\omega_2}\right)^{\alpha_{1j}}$$

3. Zero profit condition for the two firms:

Firm 1:

$$p_1 = pva_1(\omega) \cdot v_1 + p_1 \cdot a_{11} + p_2 \cdot a_{21}$$

Firm 2:

$$p_2 = pva_2(\omega) \cdot v_2 + p_1 \cdot a_{12} + p_2 \cdot a_{22}$$

Here the price index for optimal value-added used in sectors $j = 1, 2$ takes once again the Cobb–Douglas form:

$$pva_j(\omega) = C_j(\omega, 1) = \omega_1^{\alpha_{1j}} \cdot \omega_2^{\alpha_{2j}}$$

Some recommendations are now in order. First, make sure that in the above equations you identify all the coefficients that are numerical data to the problem (i.e. the *vj* values). In fact, write a list of all the coefficients and parameters along with their numerical values. Second, make sure too that you match each of the equations to their numerical counterparts that appear in the main text. Finally, the system contains six equations and six unknowns (p_1, p_2, y_1, y_2, ω_1, ω_2) but remember Walras' law. One of the equations is redundant and can be eliminated, and then we would need to fix from outside one of the unknowns (i.e. *the numé raire*) to solve the system. Now check that the numerical values provided in Sect. 3.5 do indeed solve the system of equations.

Appendix 2: GAMS Code for Solving the Simple General Equilibrium Model

```
$TITLE SIMPLE GENERAL EQUILIBRIUM MODEL: CHAPTER 3
OPTION DECIMALS=3;
OPTION NLP=CONOPT;
SET I goods /1*2/;
SET K factors /1*2/;
SET H households /1*2/;
ALIAS (J,I);

TABLE E(K,H) endowments
          1        2
1        30       20
2        20        5;

TABLE BETA(I,H) CD utility coefficients
          1        2
1        0.3      0.6
2        0.7      0.4;
```

```
TABLE A(I,J) input-output coefficients
          1        2
1        0.2      0.5
2        0.3      0.25;

TABLE ALPHA(K,I) production function coefficients
          1        2
1        0.8      0.4
2        0.2      0.6;

PARAMETER V(I) value-added coefficients
/1       0.5
 2       0.25/;

VARIABLES
Z        maximizing dummy
P(I)     prices for goods
W(K)     prices for factors
Y(I)     total output
PVA(I)   price of value-added
B(K,I)   flexible factor coefficients
C(I,H)   individual demand for final consumption
CD(I)    aggregate demand for final consumption
X(K,I)   firms factor demand
XD(K)    aggregate factor demand;

EQUATIONS
VAPRICE(i)       price index for value added
PRICES(I)        price formation
DEMAND(I)        total demand for goods
HOUSDEM(I,H)     households demand for goods
LAB(I)           variable coefficient for labor
CAP(I)           variable coefficient for capital
ZDFAC(K,I)       firms demand for factors
ZFACDEM(K)       total demand for factors
EQGOODS(I)       equilibrium for goods
EQFACTORS(K)     equilibrium for factors,
MAXIMAND         auxiliary objective function;

VAPRICE(I)..     PVA(I) =E= PROD(K, W(K)**ALPHA(K,I)) ;
PRICES(I)..      P(I) =E= PVA(I)*V(I)+SUM(J,P(J)*A(J,I));
DEMAND(I)..      CD(I) =E= SUM(H, C(I,H));
HOUSDEM(I,H)..   C(I,H) =E= BETA(I,H)*SUM(K, W(K)*E(K,H))/P(I);
LAB(I)..         B('1',I) =E= ALPHA('1',I)*(W('2')/W('1'))**ALPHA('2',I) ;
CAP(I)..         B('2',I) =E= ALPHA('2',I)*(W('1')/W('2'))**ALPHA('1',I) ;
```

```
ZDFAC(K,I)..       X(K,I) =E= B(K,I)*V(I)*Y(I);
ZFACDEM(K)..       XD(K) =E= SUM(I, X(K,I));
EQGOODS(I)..       Y(I)  =E= CD(I) + SUM(J, A(I,J)*Y(J));
EQFACTORS(K)..     XD(K) =E= SUM(H, E(K,H));
MAXIMAND..         Z =E= 1;

MODEL SIMPLECGE /ALL/;

SCALAR LB lower bound /1E-4/;
P.LO(I)=LB; Y.LO(I)=LB; W.LO(K)=LB; PVA.LO(I)=LB; C.LO(I,H)=LB; B.LO(K,I)=LB;
X.LO(K,I)=LB;
W.FX('1') = 1;

SOLVE SIMPLECGE MAXIMIZING Z USING NLP;

DISPLAY P.L, Y.L, W.L, XD.L, CD.L;
```

A General Equilibrium Model with a Government Sector

4

The simple model outlined in the previous chapter is now going to be adapted to incorporate a government sector. In the world of model-building, it becomes imperative to simplify the full range of economic activities undertaken by the government if we want to keep the model under development at a tractable level. For this reason we will consider the government taking two basic types of decisions, the first one regarding the level and composition of taxation, the second one dealing with its expenditure and transfers program. Taxes will affect consumers and producers plans by modifying the prices they face—via indirect taxes—and their disposable income—both through indirect and direct taxes. On the other hand, government spending will modify final demand faced by firms either through its direct purchase of goods and services or by the induced changes in consumption demand resulting from transferring income to households. In an economy with fixed resources, however, the activities of the government have an unmistakably re-distributive flavor. Income is extracted from the private sector and then poured back in a different mix. Whether these activities promote welfare, or not, is an issue that numerical general equilibrium can suitably address.

We will start our analysis of government activities by introducing an indirect tax on output whose collections are fully returned to consumers in lump-sum form. Next we will consider the modifications in the basic model ensuing from a tax on the use of factors and an income tax. Finally, we will address the question of the government spending more (or less) than its tax collections.

4.1 An Indirect Tax on Output with Lump-Sum Transfers

The government can obtain income from the private sector by taxing transactions among firms. A tax on output can take several formats. It can be a tax on the number of units being traded, and then it is called a *unit* or *excise* tax. Or it can be a tax on the value of transactions and then be known as an *ad valorem* tax. One way or another, the tax acts as an additional cost for the firm's commodity and prices will

M.A. Cardenete et al., *Applied General Equilibrium*, Springer Texts in Business and Economics, DOI 10.1007/978-3-662-54893-6_4

reflect the increased production costs. Since *ad valorem* output taxes seem to be more prevalent than unit taxes we will focus our attention on them. We will assume that the amount of output taxes that the government collects is given back in full to households in lump-sum form. This simple, balanced budget specification, will allow us to put aside, for the time being, issues related to savings, investment, and the government deficit. We will turn to these questions further along the chapter.

Let τ_j be the *ad valorem* output tax rate on the output of firm j in the simple model and let now p denote the gross-of-tax vector of commodity prices. Because of the presence of the *ad valorem* indirect tax, average costac_j for firm $j = 1,2,\ldots,N$ will now take the form:

$$ac_j = \left(1 + \tau_j\right) \cdot \left(v_j \cdot \sum_{k=1}^{K} \omega_k \cdot b_{kj}(\omega) + \sum_{i=1}^{N} p_i \cdot a_{ij} \right) \tag{4.1}$$

Notice that if $\tau_j = 0$ then expression (4.1) reverts to the standard average cost situation outlined in Chap. 3. Commodity prices once again will satisfy $p_j = ac_j$. The amount collected by the government thanks to the output tax will be denoted T. This quantity is returned to the hands of consumers in lump-sum form according to weights δ_h that add up to unity. The budget constraint of consumer h becomes:

$$\sum_{i=1}^{N} p_i \cdot c_{ih} = \sum_{k=1}^{K} \omega_k \cdot e_{kh} + \delta_h \cdot T \tag{4.2}$$

where once again only goods generate utility for the consumers. Solving the utility maximization problem yields consumption demand by h, which will depend on p and ω, as before, but also on the value of the received lump-sum payback $\delta_h \cdot T$. This value cannot be, however, determined independently of the prices faced by firms and their level of output. We therefore have a simultaneity problem not unlike the simultaneity problem in any general equilibrium model. Consumers, for instance, need to know prices of goods and factors in order to formulate their consumption demands. But their demands will determine what prices they will eventually face. The answer to this problem is to model consumption demand parametrically dependent on prices and then solve the simultaneous system of equilibrium equations for prices. Likewise, consumers need to know the value of the lump-sum transfer previous to formulating their demand, but again it is their consumption demand that will affect output levels, prices and henceforth tax collections T. The same solution outlined for prices can be used here and we can make demands depend parametrically on government transfers T. A new unknown requires a new independent equation to keep the solvability of the equilibrium system. In this case we introduce a government revenue function that makes explicit its source of income:

$$R(p,\omega,Y)=\sum_{j=1}^{N}\tau_j\cdot\left(v_j\cdot\sum_{k=1}^{K}\omega_k\cdot b_{kj}(\omega)+\sum_{i=1}^{N}p_i\cdot a_{ij}\right)\cdot y_j \tag{4.3}$$

An equilibrium with output taxes will now be characterized by a vector Y^* of output levels, a set of prices (p^*,ω^*), and a level of tax collections T^* such that:

$$\begin{array}{ll}\text{(i)} & \left(Y^*=TD(p^*,\omega^*,T^*,Y^*)=CD(p^*,\omega^*,T^*)+A\cdot Y^*\right.\\ \text{(ii)} & S(p^*,\omega^*,T^*)=Z(\omega^*,Y^*)\\ \text{(iii)} & p^*=(pva(\omega^*)\cdot V^*+p^*\cdot A)\cdot\Gamma\\ \text{(iv)} & R(p^*,\omega^*,Y^*;\tau)=T^*\end{array} \tag{4.4}$$

where Γ is a diagonal matrix with entries $1+\tau_j$. The symbolism "; τ" that we now use in (iv) indicates that the vector of tax rates τ has a direct role in the configuration of the expression but its purpose here is merely informational. The role of τ is apparent in (iii) where the tax rates appear explicitly in the matrix Γ. The tax rates τ do not have, however, a direct role in (i) or (ii), hence the omission.[1] Their impact is felt through the changes in prices and output levels. Conditions (i)–(iii) above extend those of in (3.27). Condition (iv) ensures that transfers equal total government revenue. Notice that in (i) total demand *TD* depends on the lump-sum transfers *T* via consumption demand *CD*. The supply of factors *S* may also be a function of *T* depending on whether or not factors are utility producing. If consumers do not derive utility from the use of factors, then all of their endowments are inelastically supplied and the dependence on *T* may be omitted. The demand for factors do not depend on *T*, however, since the optimal mix of factors depends only on relative factor prices.[2]

The present equilibrium system comprises, in order of appearance in (4.4), $N+K+N+1$ equations and the same number of unknowns—N levels of output, K factor prices, N commodity prices, and the level of tax collections T. Households' demand functions are now homogeneous of degree zero in prices (p,ω) and transfers T, thus only relative prices matter. Additionally, the following version of Walras' law is satisfied for any price vector (p,ω) and transfers T:

$$p\cdot[CD(p,\omega)-(I-A)\cdot y]+\omega\cdot\left[Z(\omega,y)-S\big(p,\omega\big)\right]=T \tag{4.5}$$

Therefore the equilibrium system (4.4) can at most determine $N+K-1$ of its prices and an additional condition is required to fix the price level. As before, any normalization will do. The dimension of the problem can also be reduced along the same lines as in Chap. 3. Skipping over the details, we can write the reduced form expressions for commodity prices and output levels as follows:

[1]Commodity and factor prices will in fact be affected by the tax on output but not in a direct way that modifies the structure of the firms' and households' optimization problem.

[2]The effect of transfers to consumers will be felt through the output levels Y, but not directly.

$$\begin{aligned} p(\omega) &= pva(\omega) \cdot V \cdot \Gamma \cdot (I - A \cdot \Gamma)^{-1} \\ Y(\omega, T) &= (I - A)^{-1} \cdot CD(p(\omega), \omega, T) \end{aligned} \tag{4.6}$$

Substituting in condition (iii) in expression (4.4), we reduce the equilibrium conditions for the markets for factors to K equations with $K + 1$ unknowns, i.e. K factors prices and T. If we combine (iii) with the reduced form for the government revenue function, we have a system with $K + 1$ equations and $K + 1$ unknowns:

$$\begin{aligned} S(p(\omega), \omega, T) &= Z(\omega, Y(\omega)) \\ R(p(\omega), \omega, Y(\omega)) &= T \end{aligned} \tag{4.7}$$

In fact, by Walras' law only K of the equations are independent, and we need to use a normalization equation to exogenously fix a unit of measurement for prices. System (4.7) has a much smaller dimension than the expanded system described in (4.4), a definite advantage for some computational techniques.

4.1.1 The Micro Effects of Output Taxes: An Example

We now retake the same simple model of Sect. 3.4 to evaluate the general equilibrium effects of an indirect tax on output. We will assume to begin with a common *ad valorem* output tax τ on both sectors and lump-sum transfers to consumers that follow a linear structure according to non-negative weights $\delta = (\delta_1, \delta_2, .., \delta_H)$ such that $\sum_{h=1}^{H} \delta_h = 1$. The tax rate τ and weights δ define the policy vector of the government and the question arises as to whether an equilibrium will exist for any possible configuration of the policy parameters. Given a level of tax collections T, the weights δ are innocuous as far as the existence of equilibrium is concerned, since their only purpose is to allocate T among consumers. Not so with the tax rate τ. Notice from (4.6) that productivity of the input–output matrix A is not enough to guarantee that commodity prices p will be non-negative for any given set of factor prices ω. What is necessary here is that the product matrix $A \cdot \Gamma$ is productive, but this condition will be violated if we choose the tax rate τ to be high enough.[3] We will avoid this problem here by keeping the tax rates within reasonable values. This is in fact advice that applies when considering any real world policy analysis. "Reasonable" is however a very ill defined concept but practice and the nature of the problem at hand usually provide enough insight to select an admissible range of values.

Suppose the government decides on a tax rate τ equal to 10% while considering three possible redistribution scenarios. The first one allocates all tax proceedings to consumer 1, the second one divides tax collections equally among consumers, whereas the third one transfers all government income to consumer 2. We solve

[3]Since we will be raising both the lower and upper bound for the relevant non-negative eigenvalue.

the present tax model and report in Table 4.1 the equilibrium solution for prices and output levels for the three scenarios as well as some derived equilibrium magnitudes on welfare, resource allocation and tax collections. From right to left, each column corresponds to a smaller participation of consumer 1 in total government transfers. The results in Table 4.1 will be useful in two accounts. On the one hand, we will use them as an example of how to read specific simulation results, and on the other hand they will provide the background for a discussion of aspects of a somewhat broader scope that are common to numerical general equilibrium analysis.

It would be useful, in order to appraise the effects of the output tax, to have some reference against which to compare the three alternative policy scenarios, and for this purpose the equilibrium solution prior to the introduction of the policy is in fact a very good candidate. This equilibrium is usually referred to as the *baseline* or benchmark equilibrium. In contrast, the equilibrium solutions for the policy scenarios are termed counterfactual equilibrium states or, more simply, *simulations* since they refer to equilibrium states that would be presumably attained should those changes be introduced. A meaningful comparison between the baseline and any counterfactual equilibrium requires some cautionary words to help prevent misinterpretations of the data. First, prices are *always* expressed in terms of the chosen *numéraire*, or reference value unit. In our case it is the price of labor ($\omega_1 = 1$). As discussed earlier in Chap. 2, general equilibrium analysis cannot

Table 4.1 General equilibrium effects of an output tax

	$\delta_1 = 1$	$\delta_1 = 0.5$	$\delta_1 = 0$
Prices			
p_1	1.277	1.269	1.261
p_2	1.350	1.339	1.329
ω_1	1.000	1.000	1.000
ω_2	1.007	0.985	0.964
Output			
y_1	0.995	1.010	1.025
y_2	1.009	0.979	0.950
Utility change in %			
Δu_1	11.60	−6.67	−24.66
Δu_2	−23.31	13.24	49.19
Factor allocation			
Sector 1			
x_{11}	39.867	40.297	40.712
x_{21}	9.901	10.226	10.554
Sector 2			
x_{12}	10.133	9.703	9.288
x_{22}	15.099	14.774	14.466
Tax receipts			
R	23.939	23.579	23.228

Tax rate $\tau = 10\%$ all sectors. Redistribution parameter $\delta_1 = 1; 0.5; 0$

provide absolute price levels, only relative prices are computable, and if we are able to obtain numerical values for all prices is only by selecting an arbitrary unit of measurement. When comparing simulation prices with benchmark prices, and remember that in our earlier example prices were all unitary, the variations will always be relative to the *numéraire*. One could be tempted to claim, if we read column 1 of Table 4.1, as an example, that the price of the good produced by the first sector has gone up by 27.7% since originally it was 1.000 and now equals 1.277. This would be a misleading interpretation, however, since the model does not tell us by how much we can expect the price of labor to change following the policy vector corresponding to column 1. All we can claim is that in the base situation one unit of good 1 could buy one unit of labor whereas in the counterfactual equilibrium one unit of good 1 would buy 1.277 units of labor. Alternatively, to buy one unit of labor we would only need now 1/1.277 units of good 1. In other words, labor has become cheaper relative to good 1.

The interpretation of prices is also subjected to whether they are gross or net of tax prices. The formulation adopted in expression (4.1) corresponds to prices inclusive of taxes, but equally valid would have been to express prices net of taxes. The choice is up to the modeler but the interpretation should be coherent with the convention adopted. In the US price labels are net of taxes prices and sales taxes are added at the cash register. In Europe, however, price labels usually include taxes. The formulation makes of course no difference regarding the general equilibrium effects of a tax on output. In some cases, within the same model some prices (commodity prices) are expressed as gross of tax whereas some prices (factor prices) are not.

For the same indirect tax rate we observe that commodity prices become more expensive relative to labor in all cases but less so as government transfers shift from consumer 1 (the 'rich' one in terms on baseline nominal income; see the *SAM* in Table 3.1) to consumer 2 (the 'poor' one). Similarly, the price of capital increases relative to the price of labor when all transfers are passed onto the 'rich' consumer but this is altogether reversed when transfers are fully allocated to the 'poor' consumer.

We show the equilibrium output levels in index form by dividing the counterfactual values by their corresponding baseline values. This convention simplifies the presentation of results and makes it easier to compute the percent variations in output levels. Accordingly, benchmark levels of output are set equal to one. For instance, the output level of sector 1 falls by 0.5% and that of sector 2 goes up by 0.9% as a result of the policy vector for column 1 of Table 4.1. As transfers are shifted towards consumer 2, however, there is also a reversal in the change in activity levels: the output of the first sector increases and that of the second one decreases. Here is a significant message to recall: distributive policies are not neutral as far as output composition is concerned.

Even though prices and output levels are basically all that is needed to have a complete characterization of the equilibrium solution, a few additional magnitudes are nonetheless reported in Table 4.1. They provide additional useful information of the effects of the output tax. Their values are computed using the equilibrium values for prices and output within the utility, factors' demand and revenue functions.

The welfare change shows the distributional effects that the alternative policies may have on consumers' utility. The allocation of labor and capital by sector shows how firms adjust their demand for factors when they face changing relative factor prices and output levels. Finally, the tax revenue row displays government revenue under the three regimes.

The welfare variations move in the expected direction. Consumer 1's utility increases by 11.6% when all government revenue is transferred to this consumer type. When the transfer amounts to only 50% of tax collections, however, this consumer's utility falls by 6.67%. The income transfer from the government is not enough to compensate for the reduced real income that follows from the increase in commodity prices relative to factor prices. Similar but symmetric considerations apply to consumer 2.

An interesting observation arises from the factor allocation results. In column 1 we see that capital become more expensive relative to labor, $\omega_2 = 1.007$ versus $\omega_1 = 1$ (the opposite is true when we refer to column 3: $\omega_2 = 0.964$ versus $\omega_1 = 1$). A quick appraisal of this price change might lead us to (wrongly) conclude that firms would substitute labor for capital. In fact, *less* labor and capital is used in sector 1 and *more* of both factors are used in sector 2. What is the catch here, if any? The answer is that even though there *is* factor substitution, there also is an output effect that is strong enough to offset the substitution effect. This is a typical example of the advantages, and dangers, of using general equilibrium analysis. It tells us much more than partial equilibrium ever could, but it also demands a bit more ingenuity to properly explain and interpret the results. From a partial equilibrium viewpoint a tax on output would disregard the effects on factor prices, or on output levels, but even if it conceded that an effect does take place, it would not be capable of offering estimates. Fortunately, general equilibrium can and does give us an answer.

The last row of Table 4.1 offers another interesting observation that quite frequently is dismissed or ignored by some government authorities. Total tax collections depend not only on the tax rate τ but also in the level of prices p and ω, and output Y. As the transfer parameter δ_1 decreases, both commodity prices and the price of capital fall relative to the *numéraire*. On the other hand, since both productive sectors have roughly the same economic size (see again the *SAM* in Table 3.1), the increase in the gross output level of sector 1 is not strong enough to compensate for the fall in sector 2's gross output. As a result, tax collections fall when δ_1 decreases. The transfer policy adopted by the government is therefore not neutral regarding government income, even for a given fixed indirect tax rate. Certainly, the variation in tax revenues is not substantial in the example but the message regarding the unforeseen, and sometimes bluntly ignored, effects of policy decisions is quite relevant: indirect tax revenues depend on tax rates but also on the value of transactions, which will be certainly affected by the presence of the tax.

4.1.2 The Aggregate Effects of the Output Tax

The report on the resource allocation effects is usually accompanied by a description of major macroeconomic indicators. These indicators are constructed from the basic microeconomic results by aggregation. Table 4.2 offers a macro summary for the same three scenarios of Table 4.1. Figures are in *numéraire* units and percentage shares are used to pinpoint any composition shifts in the functional distribution of income between labor and capital.

The immediate apparent effect of the output tax is to raise nominal Gross Domestic Product (*GDP*) at market prices in all three of the policy scenarios. *GDP* at factor cost (excluding indirect taxes), on the other hand, rises in *numéraire* units when all transfers are allocated to consumer 1 but declines as the participation of consumer 1 in government transfers decreases. Since there is no source of final demand other than private consumption, *GDP* at market prices coincides with total consumption. We also observe that *GDP* is equal to total payments to primary factors plus indirect tax collections; the upholding of this identity confirms that the underlying general equilibrium model is correctly accounting for the physical and value flows between producers, consumers and the public sector. This check on the ability of the model to generate the macroeconomic identities that we know have to hold is very helpful to detect any errors in the specification of the model or any omissions in the accounting of flows.

There is a clear danger of misinterpreting the aggregate data generated by a general equilibrium model that the reader should be made aware of. In our example the benchmark data corresponds to the solution of the general equilibrium model for a given set of structural parameters (preferences and technology) in the absence of a government, or if we wish with a very inactive government as far as fiscal and expenditure policies are concerned. In actual applications, however, the benchmark data corresponds to an observed data set for a given period. Benchmark *GDP*, or for that matter benchmark private consumption and any other macro magnitudes, are measured in current monetary units. In contrast, the macro values that we compute from the simulation runs of the model are expressed in *numéraire* units. If we double the *numéraire*, we double the nominal value of all the macroeconomic variables without in fact this change of units making any difference from a microeconomic perspective. Neither relative prices nor output levels will be

Table 4.2 Macroeconomic indicators in equilibrium

	Benchmark	$\delta_1 = 1$	$\delta_1 = 0.5$	$\delta_1 = 0$
GDP (income)	75.00	99.11	98.21	97.34
Wages (% net income)	50.00 (66.66%)	50.00 (66.52%)	50.00 (67.00%)	50.00 (67.47%)
Capital income (% net income)	25.00 (33.3%)	25.17 (33.48%)	24.63 (33.00%)	24.11 (33.53%)
Indirect taxes	0.00	23.94	23.58	23.23
GDP (spending)	75.00	99.11	98.21	97.34

Tax rate $\tau = 10\%$ all sectors. Redistribution parameter $\delta_1 = 1$; 0.5; 0

affected, as we well know from general equilibrium theory. To bypass this inconvenience we can express macro variables as percentages of *GDP*, thus partly eliminating the problem since these ratios do not depend on the chosen *numéraire*. This option provides us with information on the changing composition of aggregate output, if any, as a result of a tax or other policy related change. If in addition we want to obtain an appraisal of volume effects, the use of some quantity index is then required. A common procedure is to use a Laspeyres index to measure volume changes in real *GDP* or in any of the other macro magnitudes.

In our example, the patient reader can verify that the Laspeyres index for real *GDP* remains roughly unchanged in all the simulations at the benchmark level. Conventional wisdom (by which we again mean partial equilibrium analysis) states, however, that an output tax should have a contracting effect on net output. The explanation for this seemingly surprising result can be traced to the market for factors, where both labor and capital are inelastically supplied and full employment is assumed, and the fact that the government returns to the private sector all tax collections. Even though relative factor prices do change, and firms modify their optimal factor mix accordingly, the full employment condition ensures that labor and capital remain fully employed at the new equilibrium. This keeps consumers' income from falling due to unused or unsold resources. On the other hand, the expected fall in real income induced by the tax is also offset, on average, by the lump-sum transfers that consumers receive from the government. Opposite effects tend to cancel out and overall there are no substantial effects on the real aggregate net output level. Unlike this nil impact in aggregate net output, there is a striking effect in sectoral net outputs. The reader can again verify as an exercise that the level of net output of sector 1, once all intermediate requirements have been discounted, goes down by 2.79% whereas that of sector 2 goes up by 1.86% for policy scenario $\delta_1 = 1$. Results are even more heightened under the policy $\delta_1 = 0$. Under the apparently calm waters of the macro world, there are strong resource allocation pulls modifying consumption patterns, sectoral gross and net output, and utility levels. This level of informational detail cannot be hoped to be obtained by using standard macroeconomic models that deal only with aggregate variables and disregard resource allocations shifts. Applied general equilibrium analysis, on the other hand, is perhaps the most suitable tool so far devised for this purpose, providing sector specific information on prices and output, with added informational bonuses on welfare, income distribution and macro magnitudes.

The simulation results we have obtained are nonetheless conditioned by the assumptions, behavioral and structural, that we are making, especially by the constraint that in equilibrium all resources should be used. Results could be markedly different if, for instance, as a result of the output tax some of the available resources could remain idle. This would follow from the reduction in consumers' income which in turn would give rise to a drop in final demand, therefore gross output, and would therefore lower the firms' needs for factors. Unemployment is a very persistent malaise of modern economies, much too important to be neglected, but unfortunately most applied general equilibrium exercises have somehow put this crucial factor aside. It is true that general equilibrium analysis is the kingdom of market clearing but this should not be an excuse for exiling the problem into oblivion. There are ways, perhaps not fully satisfactory, of dealing with unused resources and the analyst should consider the trade-off between purity and compliance with the basic tenets of micro-theory and empirical and policy relevance. We will come back to this issue in the next chapter.

4.2 Factor and Income Taxes

Factors of production can also be subjected to indirect taxation. The cost of using a unit of factor k if its use is taxed at the *ad valorem* rate t_k would rise to $\omega_k \cdot (1 + t_k)$. A typical example of a factor tax is the payroll or Social Security tax on the use of labor by firms. In general terms, for K primary factors and tax rates t_k, average cost for sector j becomes:

$$ac_j = v_j \cdot \sum_{k=1}^{K} \omega_k \cdot (1 + t_k) \cdot b_{kj}(\omega; t) + \sum_{i=1}^{N} p_i \cdot a_{ij} \tag{4.8}$$

where $b_{kj}(\omega; t)$ indicates now that the variable coefficients depend also on the levied factor tax rates. If the factor tax affects only 'labor,' then tax rates for 'capital use' would be zero in (4.8). We prefer to leave (4.8) as a general expression with factor taxes regardless of whether or not a specific factor is taxed at a non-zero rate. Cost covering prices will of course satisfy, once again, $p_j = ac_j$.

Turning to the modeling of income taxes, a simple approach consists in assuming that a percentage m_h of consumer's h income is due to be paid to the government.[4] The budget constraint for consumer h would now be:

$$\sum_{i=1}^{N} p_i \cdot c_{ih} = (1 - m_h) \cdot \left(\sum_{k=1}^{K} \omega_k \cdot e_{kh} + \delta_h \cdot T \right) \tag{4.9}$$

where for simplicity we assume once again that only the consumption of goods, c_{ih}, is utility producing. Notice that *all* income accruing to consumer h is considered to be taxable, both from the sales of factors and even from the government lump-sum transfer. This may not be true for all kinds of government transfers in specific applications but again for simplicity we will not address this issue here. The revenue of the government now includes two sources of income, factor taxes paid by firms and income taxes paid by consumers, with revenue function:

[4]Modeling income taxes can be a bit more complicated that modeling output or factor taxes. Unlike the latter, which usually takes the simple form of rates, the former may contemplate, depending on the specific tax laws of each country, various deductions and/or some non-taxable income ranges. For simplicity of exposition, we assume a simple personalized tax schedule with no deductions and identical average and marginal rates.

$$R(p, \omega, Y, T; m, t) = \sum_{j=1}^{N} v_j \cdot y_j \sum_{k=1}^{K} t_k \cdot \omega_k \cdot b_{kj}(\omega; t) + \sum_{h=1}^{H} m_h \cdot \left(\sum_{k=1}^{K} \omega_k \cdot e_{kh} + \delta_h \cdot T \right) \tag{4.10}$$

In expression (4.10) we explicitly show the dependence of total revenue R on the tax policy vectors of income tax rates m and factor taxes t. Notice also that, unlike the revenue function in (4.3), transfers T are now an argument in (4.10). The reason is that total revenue cannot be computed independently of the level of transfers accruing to consumers, much in the same way as consumers demand for goods needs to be made dependent on T.

The new system of equilibrium equations incorporating the income and factor taxes is summarized as follows:

$$\begin{aligned}
&\text{(i) } Y^* = TD(p^*, \omega^*, T^*; m) \\
&\text{(ii) } S(p^*, \omega^*, T^*; m) = Z(\omega^*, Y^*; t) \\
&\text{(iii) } p^* = pva(\omega^*; t) \cdot V + p^* \cdot A \\
&\text{(iv) } R(p^*, \omega^*, Y^*, T^*; m, t) = T^*
\end{aligned} \tag{4.11}$$

As in (4.4), we make explicit the role of each tax in each of the blocks of equations. The income tax rates m modify the budget constraint of consumers thus having a direct impact on consumption demand for goods and services [right hand side of (i)] and supply of factors (left hand side of ii). Factor taxes modify the optimization problem of firms and hence their appearance in the demand for factors by firms [right hand side of (ii)] and the price equation in (iii) where $pva(\omega; t)$ represents now the value-added price index, inclusive of factor taxes.

4.2.1 The Effects of Factor and Income Taxes: Another Example

The simple model is now solved under three scenarios. In the first one the government levies a 10% tax on the use of labor, whereas in the second one the government levies a 20% income tax on both consumers. The last simulation is a bit special; we look for the labor tax rate that is revenue neutral with regard to the 20% income tax rate. By revenue neutral we mean here the factor tax rate that substitutes the income tax but yields the same level of tax collections in *numéraire* units. As before, all government income is returned to consumers in lump-sum form. For the present simulations the distributive parameter is taken to be $\delta_1 = 0.5$, that is, tax collections are equally shared by both consumers. Table 4.3 shows the results for prices, an index of gross sectoral output, an index of net sectoral output—which coincides with final consumption demand in our case—welfare, and tax revenues. As in Table 4.1, all prices for goods and factors pick up the general equilibrium effects of taxes, and they are shown inclusive of all taxes. Hence they are prices

Table 4.3 General equilibrium effects of factor and income taxes

	$t_{\text{lab}} = 10\%$	$m_h = 20\%$	$t_{\text{lab}} = 37.42\%$
Prices			
p_1	1.100	0.998	1.373
p_2	1.100	0.998	1.373
ω_1	1.100	1.000	1.374
ω_2	1.099	0.994	1.370
Output			
y_1	1.001	1.004	1.002
y_2	0.998	0.992	0.995
Net output			
$y_1 - \sum_j a_{1j} \cdot y_j$	1.005	1.025	1.014
$y_2 - \sum_j a_{2j} \cdot y_j$	0.997	0.983	0.991
Utility change in %			
Δu_1	−0.92	−5.03	−2.74
Δu_2	1.83	10.05	5.49
Tax receipts			
R	5	18.713	18.173

Redistribution parameter $\delta_1 = 0.5$

faced by buyers. Since the *numéraire* is once again the net price of labor, the reported gross price of labor collects the impact of the factor tax, when applicable.

Both the payroll tax and the income tax have little effect on relative prices. The changes in output and welfare are therefore mainly due to income effects. The egalitarian distribution of transfers is all *but* egalitarian in real income terms. Consumer 1 is worse off in all three cases. The comparison between columns 2 and 3 shows that revenue neutral tax instruments are not necessarily welfare neutral, a fact commonly forgotten by government authorities. In our example, a tax on labor of about 37.4% collects the same amount of *numéraire* units as a 20% income tax, but the impact on real income levels, as measured by utility, is different for both consumers. For Consumer 1 both fiscal scenarios are detrimental but the labor tax affects less his welfare than the income tax. The opposite occurs for Consumer 2. These observations point out the need of exploring the effects of different fiscal packages in the light of their welfare and distributional effects. Finally, even though real *GDP* could be seen to remain essentially constant, there are small but non negligible reallocative output effects at the sectoral level, which are more clearly seen when we look at an index of net output, i.e., sectoral consumption in this simple case.

4.3 Savings, Investment, and the Government Budget Constraint

4.3.1 Government Activities with a Balanced Budget

Our formulation of the government expenditure policy has been so far very simple. All income accruing to the government coffers as taxes is simply returned in lump-sum form to consumers and, for a given set of tax instruments, the government's only decision is how to distribute its intake. In the simple model, the budget of the government is always balanced. There are, however, numerous ways in which government activities may depart from this simple formulation.

The government may provide a public good free of charge to all private parties, or may produce and sell goods and services exactly in the same way as any other productive sector, or may allocate part of its budget for public works and investment. Transfers to the private sector may include transfers to households but also transfers to private or public firms in the form of subsidies and tax breaks. Transfers may be in lump-sum form—like welfare payments—or be indexed to the evolution of some economic indicator—like unemployment benefits. Transfers can also be classified as discretionary or belonging to entitlement programs. On top of all this variety of activities, the government may decide to follow a non-balanced budget policy, and spend less or, more likely, more than its total accrued income.

As long as the government budget remains balanced, almost all kinds of expenditure patterns can be easily accommodated within the current structure of the simple model that we have been using all along with only minor modifications. For instance, the production of a marketable or quasi-marketable commodity can be incorporated in a straightforward way by including an additional production activity; we will not devote more time to this possibility here. The provision of public services by the government requires the use of primary factors and intermediate goods. Again, this activity can be included in the productive technology by specifying a new production function for this publicly produced good. From now on, and whenever necessary, we will reserve the index N for this new activity.

As for transfers, whether fixed or variable, they need to be carefully specified at the consumers' budget constraint level. For the time being, and until unemployment and an investment good are incorporated in the model, however, we will have to refrain from using categories like unemployment benefits or public investment.

To simplify matters, let us consolidate all government outlays in two broad types, the first one being total transfers to the private sector—denoted now by $T(p, \omega, Y)$ to indicate that they are made dependent on endogenous variables—the second one being the level of provision of the public good, E. The balanced budget constraint implies:

$$R(p, \omega, Y; \mathfrak{I}) = T(p, \omega, Y) + p_N \cdot E \tag{4.12}$$

where $\mathfrak{I}$ is the combination of available tax instruments, i.e. $\mathfrak{I} = (\tau, t, m)$ in our case, and p_N is the cost covering price of the good for public consumption. Notice that total tax revenue R, transfers T and the price p_N are all endogenous magnitudes. Their value is the result of the overall working of the economy. The balanced budget policy embedded in (4.12) therefore implies that the level of provision of E can only be determined residually; in other words, the government cannot have under its control both the deficit and the level of public consumption. Whatever amount that is left after the indexed transfers T are deducted from the endogenous tax revenue R is what can be devoted to public consumption. If the budget is balanced, then E is determined so as to satisfy or 'close' the constraint in (4.12).

If the government wishes to modify E from its closure value, then the budget will not, except by chance, be balanced and a deficit or a surplus will ensue.

This seems to negate the ability of the government to decide its spending patterns but needs not be so. If we now distinguish a fixed, or non-indexed, level of transfers $\bar{T}$ in the government budget constraint, some room for discretionary spending becomes possible, as long as total discretionary spending is compatible with the balanced budget restriction:

$$R(p,\omega,Y;\mathfrak{I}) = \bar{T} + T^{e}(p,\omega,Y) + p_N \cdot E \tag{4.13}$$

In this expression, $T^{e}(p,\omega,Y)$ would now indicate variable or economy-dependent transfers. Since R, T^{e}, and p_N are endogenous, any decision on the level of E will automatically determine $\bar{T}$, and vice versa. But again, the balanced budget constraint plus the endogenous nature of R, T^{e} and the price p_N limits the scope and level of the discretionary spending programs of the public sector.

4.3.2 Running a Non-balanced Budget

When the government makes decisions on the level, and possibly composition, of E the endogenous nature of R and T breaks the compatibility between income and spending that we embodied expressions (4.12) and (4.13) with. Total spending will not necessarily match total income accruing to the government. To reenact the matching we need to introduce a new variable D with the property:

$$R(p,\omega,Y;\mathfrak{I}) = T(p,\omega,Y) + p_N \cdot E + D \tag{4.14}$$

From a mathematical viewpoint D is a free slack variable that allows the equality between left-hand-side and right-hand-side to hold. From the viewpoint of an economist D is the balancing entry between income and spending. When $D > 0$, and therefore total government income is greater that total outlays, we say the government is running a surplus or that public savings are positive. When $D < 0$, on the contrary, the government is running a deficit.

We have a degree of freedom in our approach to modeling the spending policies of the government. If the level of spending in E is given, then (4.14) implies that the surplus or deficit will be endogenously determined in the model. Alternatively, if the government follows a balanced budget policy, then the aggregate level of spending will be an endogenous variable consistent with (4.14) for $D = 0$. In fact, nothing prevents D from being non-zero and the actual modeling options are two: either the expenditure pattern is exogenously given and the surplus or deficit are endogenous, or else, the level of D can be chosen and then the model generates the endogenous value for spending consistent with the selected size for D; in other words, spending fluctuates with the level of government income allowing for a gap. The balanced budget approach is just one in the continuum of possible budget policies.

There is more than meets the eye, however. Any non-zero value for D involves a fundamental change in the structure of the simple model, and the most likely value for D is bound to be non-zero when E is a decision variable of the government. Indeed, the well-known basic macroeconomic identity between investment and savings states:

$$\mathbf{I} = S_v + D - F \tag{4.15}$$

where $\mathbf{I}$ is the value of aggregate investment, S_v is private savings, D is the balance for government income and spending, and F is the balance of trade. The simple model corresponds to a closed economy with no private savings or investment (i.e. $\mathbf{I} = S_v = \mathrm{F} = 0$). Thus any non-zero value for D is incompatible with (4.15) given the current structure of the model. In consequence, if we want to accommodate for the possibility of an unbalanced budget, and keep the internal consistency of the model representation of the economy, we need to explicitly incorporate investment within the model.

4.3.3 Investment and Savings

The standard neoclassical general equilibrium model is static in nature. Investment is, on the other hand, an essentially dynamic phenomenon. The marriage between statics and dynamics is not an easy one. Yet, if we want to move beyond the restricted setting of the textbook examples and make general equilibrium models play a useful role in the policy analysis of contemporary issues, investment must somehow be blended into the modeling.

An option is to model investment as an additional production activity whose output is capital for tomorrow. This activity is characterized by an N dimensional vector $a_I = (a_{I1}, \ldots, a_{Ij}, \ldots, a_{IN})$ of technical coefficients showing the input requirements of the different commodities needed to obtain one unit of the investment good. Under Constant Returns to Scale and investment level equal to λ_I, the contribution of the investment activity to the final demand for good $j = 1, 2, \ldots, N$ is measured by:

$$INV_j = \lambda_I \cdot a_{Ij} \tag{4.16}$$

where INV_j is, in physical terms, investment final demand for commodity j. For the given investment technology a_I, aggregate investment $\mathbf{I}$ in the economy is therefore given by:

$$\begin{aligned}\mathbf{I} &= \sum_{j=1}^{N} p_j \cdot INV_j = \sum_{j=1}^{N} p_j \cdot \lambda_I \cdot a_{Ij} = \lambda_I \cdot \sum_{j=1}^{N} p_j \cdot a_{Ij} = \lambda_I \cdot p_{N+1} \\ &= \mathbf{I}\left(p_{N+1}, \lambda_I\right)\end{aligned} \tag{4.17}$$

where p_{N+1} is a price index for the investment good. Notice that we make explicit the dependence of aggregate investment on prices and physical investment level by explicitly writing $\mathbf{I}(p_{N+1}, \lambda_I)$.

In our closed model, investment can only be financed by private or public savings. Public savings corresponds to D in (4.14), which can be either positive or negative (i.e. government borrowing). Savings from households can be included in the model by allowing consumers to demand a new good called 'consumption tomorrow' which is in fact the same good being produced by the new investment activity. We have used $N+1$ as the index for this additional activity (and good). Assuming once again that none of the K primary factors produce utility to consumers, the utility maximization problem for individual h is now:

$$\text{Max } u_h(c_h) \text{ subject to } \sum_{i=1}^{N+1} p_i \cdot c_{ih} = (1 - m_h) \cdot \left(\sum_{k=1}^{K} \omega_k \cdot e_{kh} + \delta_h \cdot T \right) \quad (4.18)$$

with c_h being now a $N+1$ dimensional vector and where $c_{N+1,h}$ is savings demand by household h. Notice that consumer h pays a fraction m_h of her income to the government but receives a lump-sum fraction δ_h of the transfers $T = T(p, \omega, Y)$ that the government hands out. Consumption demand, in the usual sense, is given by the first N entries of the individual demand function $c_h(p, \omega)$. Total demand for tomorrow's consumption, or private savings, is obtained adding up the $N+1$ entries in the demand functions of all H households with total private savings given by the value of these demands:

$$S_v(p, \omega) = p_{N+1} \cdot \sum_{h=1}^{H} c_{N+1,h}(p, \omega) \quad (4.19)$$

We can now write identity (4.15) to take into account the structure of investment and savings as outlined in (4.17) and (4.19):

$$\mathbf{I}\left(p_{N+1}, \lambda_I\right) = S_v(p, \omega) + D - F \quad (4.20)$$

Notice that the right-hand side of (4.20) is known once prices are known, since savings S_v are endogenous and D is either endogenous or fixed—depending on the selected closure rule for the government sector. The F magnitude, which appears for completeness, would be zero since there is yet no external sector in the economy. Given the investment technology represented by a_I, the left-hand side depends only on the price of the investment good p_{N+1}—an endogenous variable—and the aggregate activity level for investment λ_I. Thus for the identity to hold within the model and obtain the matching between aggregate investment $\mathbf{I}$ and private and public savings, S_v plus D, the variable λ_I must adjust accordingly. This rule is known as the macroeconomic 'closure' of the model and its purpose is to guarantee the consistency between micro variables governed by resource allocation and aggregate macro magnitudes. All that is said about investment with this formulation is that investment is savings driven; this is clearly too simplistic a 'theory' of investment but given the static nature of applied general equilibrium models this is usually enough for most purposes. The aspect that is worth emphasizing here is that, regardless of the level of sophistication of the determinants of investment, the closure rule between savings and investment must nonetheless hold since it represents a fundamental economic identity.

4.3.4 The Equilibrium System

The equilibrium system undergoes some modifications when the activities of the government become more complex in the way described above and the savings-investment phenomenon is incorporated. The complete set of equations for a given set of tax instruments $\mathfrak{J} = (\tau, t, m)$ can be described now as follows:

$$\begin{aligned}
&\text{(i)}\ Y = TD\big(p, \omega, p_{N+1}, Y, \lambda_I, E; \mathfrak{J}\big)\\
&\text{(ii)}\ S(p, \omega; \mathfrak{J}) = Z(\omega, Y; \mathfrak{J})\\
&\text{(iii)}\ p = (pva(\omega; \mathfrak{J}) \cdot V + p \cdot A) \cdot \Gamma\\
&\text{(iv)}\ R(p, \omega, Y; \mathfrak{J}) - T(p, \omega, Y; \mathfrak{J}) = p_N \cdot E + D\\
&\text{(v)}\ \mathbf{I}\big(p_{N+1}, \lambda_I\big) = S_v(p, \omega; \mathfrak{J}) + D\\
&\text{(vi)}\ p_{N+1} = p \cdot a_I
\end{aligned} \tag{4.21}$$

In the new system total final demand for the N commodities now includes government consumption E and physical investment $\lambda_I \cdot a_I$ in addition to intermediate and consumption demand; recall that we have chosen the index N to represent the publicly provided good, so we let now E' be a column vector of zeros except for the N-*th* entry that has a value equal to E; then total demand for goods is given by:

$$TD\big(p, \omega, p_{N+1}, Y, \lambda_I, E; \mathfrak{J}\big) = A \cdot Y + CD(p, \omega; \mathfrak{J}) + \lambda_I \cdot a_I + E' \tag{4.22}$$

Condition (iv) in (4.21) corresponds to what we previously called the government revenue function but may now be more properly termed the government budget function, with the left-hand side being the government's net endogenous income.

The inclusion of the investment good requires a new quantity and a new price equation to determine the activity level λ_I and the price p_{N+1} of the good and this is accomplished by introducing conditions (v)—the closure rule that equates supply of and demand for the investment good—and (vi) that determines the price of investment as an index price based on the technical coefficients of the investment activity.

The system (4.21) comprises $2N + K + 3$ equations: N from (i), K from (ii), N from (iii), and one from each (iv), (v) and (vi). The number of unknowns is, however, $2N + K + 4$: N activity levels Y for commodities, N prices p for commodities, K prices for factors, the price p_{N+1} and activity level λ_I for investment, and the level of public consumption E and budget surplus or deficit D. The counting of equations and unknowns shows again that it is not possible for the government to simultaneously control E and D. We can reduce the number of variables to be determined to $2N + K + 3$ by exogenously fixing either E or D. Which one is chosen is arbitrary as far as modeling is concerned but in policy applications the selection will depends upon the specific global policy guidelines that the government is following. Given a combination of tax instruments $\mathfrak{I}$ and expenditure level E (or, alternatively, deficit level D), an equilibrium for this economy will be a set of commodity prices p^*, factor prices ω^*, activity levels for commodities Y^*, price of investment p^*_{N+1}, activity level for investment λ^*_I, and deficit D^* (alternatively, E^*) such that all of the equations in (4.21) are satisfied.

4.3.5 Taxes, Expenditure and the Deficit: A Final Example

We will now adapt the structure of the simple model to account for a government sector and an investment good. The government will collect a tax on output τ whose proceedings may be used to pay, partly or totally, for the provision of a public good. Of the two sectors in the model, the second one may now be identified with the sector providing public services. The input–output matrix and the valued-added production functions will retain the same parameter values. The introduction of the investment good forces us, however, to make some changes in the description of the utility functions. Households will now have preferences defined over consumption of the two commodities and savings; hence the utility functions will have to be expanded to introduce the new good. The new utility representations will again be assumed to be of the Cobb–Douglas variety; more specifically:

$$\begin{aligned} u_1(c_1, c_1, c_3) &= c_{11}^{0.3} \cdot c_{21}^{0.6} \cdot c_{31}^{0.1} \\ u_2(c_2, c_2, c_3) &= c_{12}^{0.48} \cdot c_{22}^{0.32} \cdot c_{32}^{0.20} \end{aligned} \tag{4.23}$$

where c_{3h} is now demand for savings by agent h, or 'consumption tomorrow.' The investment activity is described by the vector of coefficients:

$$a_I = (0.3, 0.7) \tag{4.24}$$

Recall from the equilibrium system in (4.21) that there are $2N + K + 3$ variables to be determined in equilibrium. In our $N = K = 2$ example, this means we need to determine a total of 9 values, once the level of public demand E is given. For the present parameter description, the reader can verify as an exercise that the equilibrium solution with no government activities, neither taxation nor purchase of goods, is given by:

Table 4.4 Effects of raising taxes under a fixed budget

	$\tau = 5\%$	$\tau = 12.43\%$
Prices		
p_1	1.133	1.370
p_2	1.167	1.470
p_{2+1}	1.157	1.000
ω_1	1.000	1.000
ω_2	1.023	1.440
Normalized activity levels		
y_1	0.984	0.976
y_2	1.023	1.047
λ	0.410	0.440
Government indicators		
D	−6.151	−6.151
E	10.000	15.000
Welfare change (%)		
Δu_1	−7.92	−18.35
Δu_2	−2.94	−6.38

Distribution parameters: $\rho = 0.50$ $\theta_1 = \theta_2 = 0.50$

$$\begin{aligned} &p_1^*, = p_2^* = \omega_1^* = \omega_2^* = p_3^* = 1 \\ &y_1^* = y_2^* = 100, \lambda_1^* = 10, D^* = 0 \end{aligned} \tag{4.25}$$

We now contemplate two alternative tax/expenditure policy scenarios. In the first one, the government decides to purchase $E = 10$ units of the publicly provided good. The financing of this purchase requires the use of some of the tax collections that will be generated. We will assume, to this effect, that the government will reserve a percentage ρ of the accrued collections. In this simulation we will assume $\tau = 5\%$ and $\rho = 0.50$, i.e., half of the collections of the 5% output tax is returned to households and half is devoted to the financing of the public demand for goods. In the second scenario, the government wishes to increase its level of demand to $E = 15$ units while at the same time maintaining the deficit constant at the previous level of $D = -6.151$. This is only possible if the output tax rate increases. We use the model to compute endogenously the appropriate tax rate. It turns out the budget deficit neutral tax rate is $\tau = 12.43\%$. Table 4.4 shows the numerical results; as before, we report the normalized activity levels for output and investment instead of their absolute values. We leave the interpretation of the results as an exercise, in any case, to the willing reader.

4.4 Summary

The basic model outlined in Chap. 3 has been extended to account for the activities of the government and the savings/investment good. Governments collect, as a general rule, three broad types of taxes—indirect taxes on output, indirect taxes on factor use paid by firms, and direct taxes on income paid by households. The collected tax revenues are then used to finance transfers to the private sector of the economy and the purchase of a public good. The distribution of the collected tax funds between transfers and expenditure is of course a policy decision.

Conceivably, other taxes could have been incorporated at this basic level of analysis; these might include, for instance, social security taxes paid by employees, excise taxes on consumption, a value-added tax, or even negative taxes such as subsidies. Any particular tax instrument could also contemplate a range of sector or household specific rates if so desired.

The model is, however, still incomplete and improvable on at least two major and several minor counts. On the first level, we are modeling an economy with no external sector, a drastic limitation as far as real policy applications are concerned; the full employment of factors is also a restrictive setting since it cushions the income or output effects that would otherwise ensue from any policy change. On the second level, the representation of technology, preferences and budget constraints is amenable to further and richer refinements. Substitution among factors could be extended to the class of Constant Elasticity of Substitution (*CES*) functions of which our Cobb–Douglas assumption is a very particular type. The same type of consideration applies to the representation of preferences where *CES* or Linear Expenditure System (*LES*) functions are commonly used in practice. Additionally, the households' budget constraint could include and distinguish distinct types of government transfers with different fiscal treatment, some of them fixed in real terms, some endogenous to the model.

We have also discussed at length some of the simulation results, our aim being to provide a guide to the reader as to the appropriate way of interpreting the results. A literal, straight reading of the numerical simulations could prove to be misleading or, even worse, plainly wrong and some warnings have been issued to this effect. The matter of proper interpretation goes of course much further than this. Here we simply refer to the syntactic interpretation and not to the methodological problem of interpreting the simulation results of a static—hence timeless—model in the context of a real world—hence essentially dynamic—economy.

A distinguishing trait of numerical general equilibrium that we have so far prudently omitted is that of computation. How is indeed a solution for an equilibrium system like those in this chapter computed?. We are well aware that the production of numbers falls short of being magic at this point. There is no magician hat, however. To satisfy the curious reader we list in Appendices 1 and 2 the *GAMS*[5] codes that have been used for all the computations reported in Tables 4.1, 4.2, 4.3 and 4.4. All the results can be reproduced in a regular desktop computer loaded with the *GAMS* programming facilities.

[5]General Algebraic Modeling System (*GAMS*) is a flexible and powerful high level mathematical optimization language that can be adapted to solving non-linear systems of equations. See Brooke et al. (1988).

4.5 Questions and Exercises

1. Write the price equation when p is not inclusive of output taxes (easy).
2. Use the information in Tables 4.1 and 4.2 to compute the Laspeyres index for Gross Domestic Product in all three policy scenarios (easy).
3. Verify that in scenario 1 of Table 4.1 the level of net output in sector 1 drops by 2.79% while that of sector 2 goes up by 1.86% (easy).
4. Use the information in Table 4.1 on equilibrium prices and output for scenario 1 to build the Social Accounting Matrix for the counterfactual economy (medium).
5. Suppose you have had the patience to construct the *SAMs* for the three scenarios in Table 4.1. Can you meaningfully compare those SAMs? (*medium*).
6. Why in the equilibrium system (4.21) there does not explicitly appear a variable T—lump-sum transfers—in the same way as in expression (4.4)? (easy).
7. Refer to Table 4.4. Use the equilibrium values reported there to calculate *GDP* from the expenditure and income viewpoints. Check that the condition "savings = investment" holds. Finally, is there a crowding out effect caused by the government purchases of the public good? (easy).

Appendix 1: GAMS Code for the CGE Model Generating Tables 4.1, 4.2 and 4.3

```
$TITLE SIMPLE GENERAL EQUILIBRIUM MODEL: CHAPTER 4, Tables 1,2,3.
* The model contemplates three types of taxes: output, labor, and income taxes.
* Users can select valued for policy parameters.
OPTION DECIMALS=3;
OPTION NLP=CONOPT;
SET I goods /1*2/; SET K factors /1*2/; SET H households /1*2/;
ALIAS (J,I);

TABLE E(K,H) endowments
          1         2
1        30        20
2        20         5;

TABLE BETA(I,H) Cobb-Douglas utility coefficients
          1         2
1        0.3       0.6
2        0.7       0.4;

TABLE A(I,J) input-output coefficients
          1         2
1        0.2       0.5
2        0.3       0.25;
```

```
TABLE ALPHA(K,I) Cobb-Douglas production function coefficients
          1        2
1        0.8      0.4
2        0.2      0.6;

PARAMETER V(I) value-added coefficients
/1       0.5
 2       0.25/;

PARAMETER
TAU(I)    output tax rates
M(H)              income tax
T(K)              factor tax
DEL(H)    lump sum shares;
TAU(I)=0; M(H)=0; T(K) = 0; DEL(H)=0;

VARIABLES
P(I)           prices for goods
W(K)           prices for factors
WN(K)          net prices for factors
Y(I)           total output
PVA(I)         price of value-added
B(K,I)         flexible factor coefficients
C(I,H)         individual demand for final consumption
CD(I)          aggregate demand for final consumption
X(K,I)         firms factor demand
XD(K)          aggregate factor demand
TC             total tax collections
OT             output tax collections
FT             factor tax collections
MT             income tax collections
Z              maximizing dummy;

EQUATIONS
VAPRICE(I)     price index for value added
PRICES(I)      price formation
FACPRICES(K)   net and gross factor prices
DEMAND(I)      total demand for goods
HOUSDEM(I,H)   households demand for goods
LAB(I)         variable coefficient for labor
CAP(I)         variable coefficient for capital
ZDFAC(K,I)     firms demand for factors
ZFACDEM(K)     total demand for factors
GOVERNM        government budget constraint
INCOMETAX      income tax collections
```

```
FACTORTAX        factor tax collections
OUTPUTTAX        output tax collections
EQGOODS(I)       equilibrium for goods
EQFACTORS(K)     equilibrium for factors
MAXIMAND         aux objective function;

VAPRICE(I)..     PVA(I) =E= PROD(K, W(K)**ALPHA(K,I)) ;
PRICES(I)..      P(I) =E= (1+TAU(I))*(PVA(I)*V(I)+SUM(J,P(J)*A(J,I)));
FACPRICES(K)..   W(K) =E= WN(K)*(1+T(K)) ;
DEMAND(I)..      CD(I) =E= SUM(H, C(I,H));
HOUSDEM(I,H)..   C(I,H)=E=(1-M(H))*BETA(I,H)*(DEL(H)*TC+SUM(K, WN(K)*E(K,H)))/P(I);
LAB(I)..         B('1',I) =E= ALPHA('1',I)*(W('2')/W('1'))**ALPHA('2',I) ;
CAP(I)..         B('2',I) =E= ALPHA('2',I)*(W('1')/W('2'))**ALPHA('1',I) ;
ZDFAC(K,I)..     X(K,I) =E= B(K,I)*V(I)*Y(I);
ZFACDEM(K)..     XD(K) =E= SUM(I, X(K,I));
GOVERNM..        TC =E= OT + FT+ MT ;
INCOMETAX..      MT =E= SUM(H, M(H)*(DEL(H)*TC+SUM(K, WN(K)*E(K,H))) );
FACTORTAX..      FT =E= SUM( (I,K), T(K)*WN(K)*B(K,I)*V(I)*Y(I) );
OUTPUTTAX..      OT =E= SUM(I, TAU(I)*Y(I)*(PVA(I)*V(I)+SUM(J,P(J)*A(J,I)))) ;
EQGOODS(I)..     Y(I)=E= CD(I) + SUM(J, A(I,J)*Y(J));
EQFACTORS(K)..   XD(K)=E= SUM(H, E(K,H));
MAXIMAND..       Z =E= 1;

MODEL SIMPLECGE  /ALL/;

SCALAR LB lowerbound /1E-4/;
P.LO(I)=LB; Y.LO(I)=LB; W.LO(K)=LB; PVA.LO(I)=LB; C.LO(I,H)=LB;
B.LO(K,I)=LB; X.LO(K,I)=LB;
Wn.FX('1') = 1;

SOLVE SIMPLECGE MAXIMIZING Z USING NLP ;

* Save benchmark results
PARAMETER
Y0(I)            Benchmark gross output of i
NY0(I)           Benchmark net output of i
PC0(I)           Benchmark consumption of i
U0(H)            Benchmark utility of h;

Y0(I)  = Y.L(I);
NY0(I) = Y.L(I)-SUM(J, A(I,J)*Y.L(J));
PC0(I) = SUM(H, C.L(I,H));
U0(H)  = PROD(I, C.L(I,H)**BETA(I,H));
```

```
* Choose Fiscal Policies for Tables 4.1, 4.2 and 4.3
TAU(I)= 0.10;
*T('1') = 0.10;
*T('1') = 0.374260;
*M(H) = 0.20;
* Choose Redistribution parameter for same Tables.
DEL('1')= 0.5; DEL('2')=1-DEL('1');

*Solve model under policy
SOLVE SIMPLECGE MAXIMIZING Z USING NLP ;

*Write simulation results
PARAMETER
U(H)            simutility
DU(H)           utility changes
WAG             wages
KAP             capital income
PC(I)           simconsumption of good i
PRC             private consumption
GDPI            GDP-income
GDPE            GDP-expenditure
NY(I)           net output
DNY(I)          change or index for net output of i
DY(I)           change or index for gross output of i ;

U(H)  = PROD(I, C.L(I,H)**BETA(I,H));
DU(H) = (U(H)/U0(H)-1)*100;
WAG   = WN.L('1')*XD.L('1'); KAP = WN.L('2')*XD.L('2');
PC(I) = SUM(H, C.L(I,H));
PRC   = SUM(I, P.L(I)*PC(I));
GDPI  = WAG+KAP+TC.L;
GDPE  = PRC;
NY(I) = Y.L(I)-SUM(J, A(I,J)*Y.L(J));

* Output indexation:
DNY(I)= NY(I)/NY0(I) ;
DY(I) = Y.L(I)/Y0(I);

DISPLAY "RESULTS";
DISPLAY DEL, TAU, T, M;
DISPLAY P.L, W.L, DY, DNY, DU, X.L, TC.L;
DISPLAY GDPI, WAG, KAP, TC.L, GDPE;
```

Appendix 2: GAMS Code for the CGE Model Generating Table 4.4

```
$TITLE SIMPLE GENERAL EQUILIBRIUM MODEL: CHAPTER 4: Table 4
OPTION DECIMALS=3;
OPTION NLP=CONOPT;
SET IT goods /1*2, S/; SET K factors /1*2/; SET H households /1*2/;
SET I(IT) /1*2/;
ALIAS (J,I);

TABLE E(K,H) households endowments
          1        2
1        30       20
2        20        5;

TABLE BETA(IT,H) Cobb-Douglas utility coefficients
          1        2
1        0.30     0.48
2        0.60     0.32
S        0.10     0.20;

TABLE A(I,J) input-output coefficients
          1        2
1        0.20     0.50
2        0.30     0.25;

TABLE ALPHA(K,I) production function coefficients
           1        2
1        0.80     0.40
2        0.20     0.60;

PARAMETER V(I) value-added coefficients
/1       0.50
 2       0.25/;

PARAMETER INV(I) investment activity
/1       0.30
 2       0.70/ ;

PARAMETER DG(I) government demand
/1       0
 2       0/ ;
```

```
PARAMETER
TAU(I)           output tax
M(H)             income tax
T(K)             factor tax
DEL(H)           lumpsum shares
RO               percentage of transfers to households;
TAU(I)=0; M(H)=0; T(K)=0; DEL(H)=0; RO=0;

VARIABLES
P(I)            prices for goods
W(K)            prices for factors
WN(K)           net prices for factors
Y(I)            total output
PVA(I)          price of value-added
B(K,I)          flexible factor coefficients
C(IT,H)         individual demand for final consumption and savings
CD(I)           aggregate demand for final consumption
X(K,I)          firms factor demand
XD(K)           aggregate factor demand
TR              transfers to households
TC              total tax collections
OT              output tax collections
FT              factor tax collections
MT              income tax collections
NIV             investment level
PINV            investment price
DEF             government deficit
GD              government expenditure
Z               maximizing dummy  ;

EQUATIONS
VAPRICE(I)      price index for value added
PRICES(I)       price formation for goods
PRICEINV        price of investment
FACPRICES(K)    net and gross factor prices
DEMAND(I)       total demand for goods
HOUSDEM(I,H)    households demand for goods
SAVPRIV(H)      savings by households
LAB(I)          variable coefficient for labor
CAP(I)          variable coefficient for capital
ZDFAC(K,I)      firms demand for factors
ZFACDEM(K)      total demand for factors
GOVINCOME       government income
GOVTRANS        government transfers
GOVSAV          savings by government
```

```
GOVDEM          government demand
INCOMETAX       income tax collections
FACTORTAX       factor tax collections
OUTPUTTAX       output tax collections
EQGOODS(I)      equilibrium for goods
EQFACTORS(K)    equilibrium for factors
SAVINV          macro closure
MAXIMAND        aux objective function;

VAPRICE(I)..    PVA(I) =E= PROD(K, W(K)**ALPHA(K,I)) ;
PRICES(I)..     P(I) =E= (1+TAU(I))*(PVA(I)*V(I)+SUM(J,P(J)*A(J,I)));
PRICEINV..      PINV =E= SUM(I, P(I)*INV(I)) ;
FACPRICES(K)..  W(K) =E= WN(K)*(1+T(K)) ;
DEMAND(I)..     CD(I) =E= SUM(H, C(I,H));
HOUSDEM(I,H)..  C(I,H)=E=(1-M(H))*BETA(I,H)*(DEL(H)*TR+SUM(K, WN(K)*E(K,H)))/P(I);
SAVPRIV(H)..    C('S',H)=E=(1-M(H))*BETA('S',H)*(DEL(H)*TR+SUM(K,
WN(K)*E(K,H)))/PINV;
LAB (I)..       B('1',I) =E= ALPHA('1',I)*(W('2')/W('1'))**ALPHA('2',I) ;
CAP(I)..        B('2',I) =E= ALPHA('2',I)*(W('1')/W('2'))**ALPHA('1',I) ;
ZDFAC(K,I)..    X(K,I) =E= B(K,I)*V(I)*Y(I);
ZFACDEM(K)..    XD(K) =E= SUM(I, X(K,I));
GOVINCOME..     TC =E= OT+FT+MT ;
GOVTRANS..      TR =E= RO*TC ;
GOVSAV..        DEF =E= TC-TR-GD;
GOVDEM..        GD =E= SUM(I, P(I)*DG(I));
INCOMETAX..     MT =E= SUM(H, M(H)*(DEL(H)*TR+SUM(K, WN(K)*E(K,H))) );
FACTORTAX..     FT =E= SUM((I,K),  T(K)*WN(K)*X(K,I) );
OUTPUTTAX..     OT =E= SUM(I, TAU(I)*Y(I)*(PVA(I)*V(I)+SUM(J,P(J)*A(J,I)))) ;
EQGOODS(I)..    Y(I) =E= NIV*INV(I) + DG(I)+ CD(I) + SUM(J, A(I,J)*Y(J));
EQFACTORS(K)..  XD(K )=E= SUM(H, E(K,H));
SAVINV..        SUM(I, NIV*INV(I)*P(I)) =E= SUM(H, PINV*C('S',H)) + DEF;
MAXIMAND..      Z =E= 1;

MODEL SIMPLECGE /ALL/;

SCALAR LB lowerbound /1E-4/;
P.LO(I)=LB; PVA.LO(I)=LB; W.LO(K)=LB; WN.LO(K)=LB ; PINV.LO=LB;
Y.LO(I)=LB; X.LO(K,I)=LB; XD.LO(K)=LB;
C.LO(I,H)=LB; CD.LO(I)=LB; B.LO(K,I)=LB;
TR.LO=0; TC.LO=0; OT.LO=0; FT.LO=0; MT.LO=0; GD.LO=0; NIV.LO=0;
WN.FX('1') = 1;

SOLVE SIMPLECGE MAXIMIZING Z USING NLP ;
```

```
PARAMETER
Y0(I)                 Benchmark gross output of i
NY0(I)                Benchmark net output of i
PC0(I)                Benchmark consumption of i
U0(H)                 Benchmark utility of h
NIV0                  Benchmark investment level;

Y0(I)   = Y.L(I) ;
NY0(I)  = Y.L(I)-SUM(J, A(I,J)*Y.L(J));
PC0(I)  = SUM(H, C.L(I,H));
U0(H)   = PROD(IT, C.L(IT,H)**BETA(IT,H));
NIV0    = NIV.L;

* Fiscal Policy
TAU(I) = 0.05;
T('1') = 0.0;
T('2') = 0.0;
M(H) = 0.0;
DEL('1')=0.5; DEL('2')=1-DEL('1');
RO = 0.50 ;
DG('1') = DG('1')+ 0    ;
DG('2') = DG('2')+ 10 ;

*Solve under policy
SOLVE SIMPLECGE MAXIMIZING Z USING NLP ;

*Write simulation results
PARAMETER
U(H)                simutility
DU(H)               utility changes
WAG                 wages
KAP                 capital income
PC(I)               simconsumption of good i
PRC                 private consumption
GDPI                GDP-income
GDPE                GDP-expenditure
FBK                 Gross capital formation
ITAX                Indirect taxation
SAV                 Savings by households
PUBC                Public consumtion
NY(I)               net output
DNY(I)              index for net output of i
DY(I)               index for gross output
DINV                index for investment;
```

```
U(H)   = PROD(IT, C.L(IT,H)**BETA(IT,H));
DU(H)  = (U(H)/U0(H)-1)*100;
WAG    = WN.L('1')*XD.L('1'); KAP = WN.L('2')*XD.L('2');
PC(I)  = SUM(H, C.L(I,H));
PRC    = SUM(I, P.L(I)*PC(I));
ITAX   = OT.L + FT.L;
GDPI   = WAG+KAP+ITAX;
FBK    = SUM(I, NIV.L*INV(I)*P.L(I)) ;
SAV    = SUM(H, PINV.L*C.L('S',H)) ;
PUBC   = SUM(I, P.L(I)*DG(I));
GDPE   = PUBC + PRC+ FBK;
NY(I)  = Y.L(I)-SUM(J, A(I,J)*Y.L(J));

* Output indexation
DNY(I)= NY(I)/NY0(I);
DY(I)  = Y.L(I)/Y0(I);
DINV   = NIV.L/NIV0;

DISPLAY "RESULTS";
DISPLAY RO, DEL, TAU;
DISPLAY P.L, PINV.L, W.L, DY, DNY, DINV, DEF.L, DG, DU;
DISPLAY FBK, PRC, PUBC, GDPI, GDPE, WAG, KAP, ITAX;
```

5 Additions to the Model: External Sector, Labor Market, Consumption Technology and Welfare Evaluation

The basic model we have thus far developed offers a comprehensive view of the resource allocation process in a closed economy with or without a government sector. These settings, valuable as they are as learning devices, are nonetheless restricted as far as actual policy applications are concerned. We need to further elaborate the model structure and we will do so next by introducing an external sector—hence opening the economy to trading partners—and by allowing some domestic resources, most notably labor, to remain partly unused. These are two phenomena that cannot be ignored in any economy-wide modeling exercise that wants to be capable of handling real-world issues. Finally, we will discuss a limitation of the present version of the model which is more of an empirical and data related nature and has to do with the fact that the goods demanded by households may not correspond to the goods produced by firms.

5.1 The External Sector

The extension of the model to account for foreign trade is less than straightforward and a variety of possibilities is available to the modeler. In fact, many general equilibrium economists consider this field to be composed of, broadly speaking, two classes of models—'fiscal' and 'trade' models, depending upon whether the modeling effort is laid on the activities of the government and the distortionary effect of taxes (Ballard et al. 1985; Shoven and Whalley 1992; Kehoe et al. 2005) or on the specification of the external sector (Dervis et al. 1982; Francois and Reinert 1997). We feel, however, than this is a somewhat artificial divide since applied general equilibrium models of both 'types' share in more than they actually differ. The differences, if any, are of emphasis rather than of substance (Shoven and Whalley 1984). Any full-fledged operational 'fiscal' model needs a trade specification, even if it is very simple, and any specifically built 'trade' model needs a tax subsystem for completeness. The subsidiary aspects may rest at a different level of

M.A. Cardenete et al., *Applied General Equilibrium*, Springer Texts in Business and Economics, DOI 10.1007/978-3-662-54893-6_5

complexity than the primary modeling concern, but they need to be incorporated to provide the applied model with all the necessary feed-back components and an appropriate representation of all the underlying structural traits. Both 'approaches' emphasize the determinant role of markets and the price system in the resource allocation process, and this is what makes them markedly different from other general equilibrium approaches such as macroeconomic modeling.

It is well known that the predictions of trade theory are not always backed by the facts, particularly in regard with specialization, and when facts tend to contradict our stylized view of the world we expect that an assortment of modeling attempts at reconciliation will appear. As a general rule, countries do not seem to specialize in commodity production and most goods are simultaneously imported and exported (cross hauling) in a given economy.[1] To incorporate this empirical fact in applied general equilibrium analysis two modeling options are commonly used. The first one defines for each good a demand for imports by domestic agents and a demand for exports of the domestic good by foreign agents. For all practical purposes, the domestic and foreign good are indistinguishable and cross hauling is unaccounted for. The second option recognizes that domestic and foreign goods are close but not identical substitutes. If foreign and domestic goods are competitive, then domestic firms can, depending on relative prices, smoothly substitute one for another in their production activities. Thus there will be a domestic demand for the external competitive good and, likewise, the foreign sector—an aggregation of the rest of the world—will demand the domestic good for their production processes, giving rise to exports. This is the so-called Armington (1969) assumption that we will adopt in what follows. Under the first option we can treat imports as an additional intermediate input in production (Kehoe and Serra-Puche 1983) or we can define net import demand functions (Ballard et al. 1985, in their standard version of the US model).

5.1.1 The Armington Assumption

The treatment we use combines the Armington assumption with the activity analysis tradition. There is an activity that produces a good for exports that we will call 'trade'. The producing technology for this good is represented by a vector of technical coefficients $a_x = (a_{x1}, \ldots, a_{xj}, \ldots, a_{xN})$. At the base level this vector a_x yields an observed amount E_x of the trade good, but the activity can be operated at different levels that we represent by λ_x. Let us use the index $N + 2$ to identify this new activity. Then total exports of good j are given by:

[1] True enough; the level of aggregation for commodities plays an important role here. The higher the digit disaggregation, the more likely is to find that commodities are imported but not exported, or the other way around.

$$EXP_j = \lambda_x \cdot a_{xj} \cdot E_x \tag{5.1}$$

whereas the value of aggregate exports is obtained as:

$$X = \sum_{j=1}^{N} p_j \cdot EXP_j = \lambda_x \cdot E_x \cdot \sum_{j=1}^{N} p_j \cdot a_{xj} = \lambda_x \cdot E_x \cdot p_{N+2} = X(p_{N+2}, \lambda_x) \tag{5.2}$$

Here p_{N+2} is a price index for the trade good. Notice the formal similarity of (5.2) with the level of aggregate investment as determined in expression (4.17). Simultaneously, the output y_j of each production sector is the result of combining two imperfectly substitute inputs, namely, domestic output Q_j and imports M_j of the trade good according to the production function:

$$y_j = \Phi_j(Q_j, M_j) \tag{5.3}$$

For a given domestic price p_j, price p_{N+2} for the trade good, and output level y_j there is an optimal mixture of Q_j and M_j resulting from solving the cost minimization problem:

$$\text{Min } p_j \cdot Q_j + p_{N+2} \cdot M_j \tag{5.4}$$

$$\text{subject to}: \quad y_j = \Phi_j(Q_j, M_j)$$

The solution to (5.4) depends parametrically on prices and output and this yield the conditional demands for domestic input and imports:

$$Q_j = Q_j\left(p_j, p_{N+2}, y_j\right) \tag{5.5}$$

$$M_j = M_j\left(p_j, p_{N+2}, y_j\right)$$

with changes in the relative price (p_j/p_{N+2}) giving rise to different demand schedules for Q_j and M_j, conditional on y_j. The aggregator function Φ_j can take the familiar Cobb–Douglas or *CES* formats. The input value Q_j that enters the production function (5.3) is in fact the output produced by the domestic firms. The technology to produce Q_j corresponds exactly to the description of the production function for y_j in the previous Chapters, that is, intermediate inputs and value-added combine in fixed proportions whereas valued-added is the composite factor resulting from combining primary factors, labor and capital.

Total imports can be expressed as:

$$M = \sum_{j=1}^{N} p_{N+2} \cdot M_j\left(p_j, p_{N+2}, y_j\right) = M(p, p_{N+2}, Y) \tag{5.6}$$

Finally, the balance of trade is given by:

$$F = X\left(p_{N+2}, \lambda_x\right) - M\left(p, p_{N+2}, Y\right) \tag{5.7}$$

The 'savings = investment' identity now reads:

$$\mathbf{I} = S + D - F \tag{5.8}$$

where the amount $-F$ can be interpreted as foreign savings and S and D represent, as before, the level of private and public savings.

5.1.2 Equilibrium and Model Closure

We now turn again to describe and discuss the characteristics of the new equilibrium system once the external sector has been added. With the same notational conventions as before and the use of the new definitions we have

$$\begin{array}{ll} \text{(i)} & Y = TD\left(p, \omega, p_{N+1}, p_{N+2}, Y, \lambda_I, \lambda_x, E; \mathfrak{J}\right) \\ \text{(ii)} & S(p, \omega; \mathfrak{J}) = Z(\omega, Y; \mathfrak{J}) \\ \text{(iii)} & p = (pva(\omega; \mathfrak{J}) \cdot V + p \cdot A) \cdot \Gamma \\ \text{(iv)} & R(p, \omega, Y; \mathfrak{J}) - T(p, \omega, Y; \mathfrak{J}) = p_N \cdot E + D \\ \text{(v)} & \mathbf{I}\left(p_{N+1}, \lambda_I\right) = S_v(p, \omega; \mathfrak{J}) + D - F \\ \text{(vi)} & F = X\left(p_{N+2}, \lambda_x\right) - M\left(p, p_{N+2}, Y\right) \\ \text{(vii)} & p_{N+1} = p \cdot a_I \\ \text{(viii)} & p_{N+2} = p \cdot a_x \end{array} \tag{5.9}$$

Quickly enumerated, the differences in regard to the previous version (4.21) of the equilibrium system are as follows: Equation (i) now incorporates final demand for exports (i.e. through the exports activity level λ_x); equation (v) now allows for foreign savings, i.e. $-F$; Eqs. (vi) and (viii) are new and they introduce the definition of the trade balance and the price equation for the trade good, respectively.

System (5.9) contains $2N + K + 5$ equations: N from (i), K from (ii), N from (iii). and one from the rest of Eqs. (iv)–(viii). The number of variables is $2N + K + 7$: N activity levels Y, N prices p, K prices ω, two prices and two activity levels for the investment and trade activities, plus the trade balance F, the budget deficit D and the level of government consumption E. We therefore have two degrees of freedom in the new equilibrium system and two variables have to be exogenously fixed to close the model. From an economic perspective, this means that there are four possible different closure rules. We can choose—as we did before—D or E to be endogenous, and this corresponds to whether the government policy gives priority to decisions on the level of public expenditure or to manage the size of the deficit. In addition, we can make the trade deficit F endogenous, hence the level of exports λ_x is fixed, or else make λ_x endogenous and then the trade deficit F remains fixed. Notice the formal similarities with the closure of the government sector. In economic terms, however, making the trade deficit endogenous and the level of exports exogenous seems to be a more appropriate option in the majority of cases. After all, we are in no way modeling the demand decisions of the foreign agent, although it is perfectly possible to modify in an *ad hoc* manner the exports technology.

5.1.3 Tariffs

The introduction of the external sector is important in two counts, the first one being the methodological need for a more comprehensive picture of the economy, the second one having to do with government intervention. Any government policies affecting relative prices and output levels will have an effect on the level and composition—domestic and foreign—of total supply but the most impact will be felt when the policy affects directly the relative price of domestic output *vis-a-vis* imports. This is the case of tariffs or taxes on imports.

Let us suppose that any domestic firm importing foreign goods pays an *ad valorem* tariff rate r_j. The cost minimization problem to derive the Armington demands becomes now:

$$\text{Min } p_j \cdot Q_j + p_{N+2} \cdot \left(1 + r_j\right) \cdot M_j \qquad (5.10)$$

$$\text{subject to } y_j = \Phi_j\left(Q_j, M_j\right)$$

and the vector $r = (r_1, \ldots, r_N)$ of tariff rates becomes an additional and powerful instrument in the hands of the government to manage the trade flows. The Armington demands for domestic goods and imports depend now upon the gross-of-tariff price $p_{N+2} \cdot (1 + r_j)$ and this yields a particularly well suited specification for studying trade liberalization effects.

5.2 The Labor Market

We have seen in Chap. 4 how government intervention in the economy using taxes and expenditure programs affects the allocation of resources. Fiscal policies have an effect on relative prices and firms and households adjust, accordingly, their production plans and consumption patterns. The overall effects that the basic model was able to capture were restricted, however, by the simplistic representation of the supply of factors. Both labor and capital were inelastically supplied, regardless of their market prices. It can be argued that this assumption is somewhat acceptable for capital, since this factor is not commonly thought of being utility producing, but the assumption is less defensible for the labor market. Consumers, or so we are told in our micro-theory classes, allocate their endowment of time between work and leisure in response to price changes. If the supply of labor, or any factor for that matter, is price dependent, then policy changes would indeed affect the level of transactions in the labor market and, in turn, an effect would be felt in the aggregate output level.

There are numerous empirical studies of the labor market estimating supply elasticities but general equilibrium modeling requires for its numerical implementation detailed data on leisure demand by households, information which is usually unavailable. Any observable unused labor endowment at the household level can either be 'optimistically' viewed as leisure or 'pessimistically' as unemployment and, which is which, is definitely a difficult empirical question. But even if we could come up with price sensitive labor supply functions we would still be imposing the market clearing conditions. To skip over this problem we will use a less elegant route that does not have the informational requirements of fully specified labor supply functions but allows for the introduction of unemployed labor resources.

Let us restrict the basic model to consider only two primary factors—labor ($k = 1$) and capital ($k = 2$). If both factors are inelastically supplied and we impose the market clearing condition, then factor prices will adjust to keep them fully utilized. For labor not to be fully employed, therefore, its price must be determined somewhere else. Suppose that the price of labor is such that allows consumers to buy a normalized basket $\gamma = (\gamma_1, \ldots, \gamma_j, \ldots, \gamma_N)$ of goods:

$$\omega_1 = \sum_{j=1}^{N} p_j \cdot \gamma_j = p \cdot \gamma \text{ with } \sum_{j=1}^{N} \gamma_j = 1 \text{ and } \gamma_j > 0 \tag{5.11}$$

where $p \cdot \gamma$ can be interpreted as a consumers' price index. In other words, the purchasing power of the real wage $\omega_1/p \cdot \gamma$ is made constant and equal to one. By adding this real wage condition in the definition of equilibrium we free the market clearing condition for labor from happening and labor demand may differ from the available pool of labor, giving rise to unemployed labor. Figure 5.1 shows a schematic representation of the labor market with and without the real wage condition.

The function $z_1(\omega, Y)$ represents demand for labor. At market clearing equilibrium values ω^o and Y^o, the whole endowment of labor is employed, i.e. $e_1 = z_1(\omega^o, Y^o), \omega_1^o$ being the equilibrium price of labor. When we fix the price of labor in terms of the price index $p \cdot \gamma$, in contrast, demand for labor at the alternative equilibrium values ω^1 and Y^1 may not be enough to hire all of the labor supplied by consumers. In this case, unemployment arises and is measured by $e_1 - z_1(\omega^1, Y^1)$ in Fig. 5.1. The first kind of market behavior describes a situation where the wage rate adjusts so as to keep all labor employed; in the second one, we may say that the unemployment rate u, defined as the percentage of unemployed labor over total labor endowment, adjusts so as to keep constant the purchasing power of the real wage. For short, we will refer to them as the 'flexible' wages (where demand and supply rule the labor market) and 'rigid' wages (where exogenous fixing of wages take place) scenarios, respectively.

The alternative formulations of the labor market are, in a sense, polar cases since we exclude by construction any feedback effects between the real wage and the unemployment rate. Rigidities in the real wage arise when workers or unions negotiate the real wage level but, in practice, the desired level for the real wage is

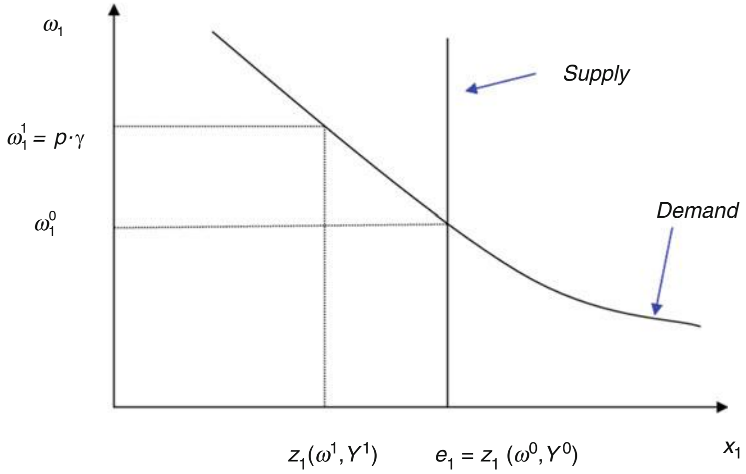

Fig. 5.1 Labor market representation

not set independently of the economic environment. A high unemployment rate may work very efficiently to reduce workers' claims for increasing real wage rates. Alternatively, we might rationalize a union's behavior by assuming some sort of preference relation between the real wage and the unemployment rate. From the union's perspective, the higher the real wage rate and the lower the unemployment rate, the better, but at the same time they are willing to accept a trade-off between them. The idea behind the proposed formulation is that of a wage curve (Oswald 1982; Blanchflower and Oswald 1994).

We explicitly introduce this trade-off in a very simple way by assuming that the real wage is sensitive to the unemployment rate so that in equilibrium the following condition is taken to hold:

$$\omega_1 = p \cdot \gamma \cdot (k \cdot (1 - u))^{1/\beta} \tag{5.12}$$

In this expression β is a non-negative parameter that measures the sensitivity of the real wage to the unemployment rate and k is a constant. Notice that when β is large, i.e. $\beta \to \infty$, the nominal wage rate approaches the price index $p \cdot \gamma$ and we obtain the 'rigid' wages case outlined above in expression (5.11). For small values of β the price of labor reacts increasingly stronger to any changes in u, no matter how small, with the limit case, i.e. $\beta \to 0$, corresponding to the 'flexible' wages case. This can be perhaps more easily seen by computing the elasticity ε of the real wage with respect to the unemployment rate. From expression (5.12) we define the real wage ϖ in terms of the purchasing power of the basket γ:

$$\varpi = \frac{\omega_1}{p \cdot \boldsymbol{\gamma}} = k \cdot (1 - u)^{1/\beta}$$

From here we take the derivative with respect to the unemployment rate:

$$\frac{d\overline{\omega}}{du} = k \cdot \frac{1}{\beta} \cdot (1-u)^{(1-\beta)/\beta}$$

We now introduce the elasticity ε of the real wage with respect to unemployment:

$$\varepsilon = \frac{d\overline{\omega}}{du} \cdot \frac{u}{\overline{\omega}} = k \cdot \frac{1}{\beta} \cdot (1-u)^{(1-\beta)/\beta} \cdot \frac{u}{\overline{\omega}} = k \cdot \frac{1}{\beta} \cdot (1-u)^{(1-\beta)/\beta} \cdot \frac{u}{k \cdot (1-u)^{1/\beta}} \tag{5.13}$$

Rearranging terms and simplifying we obtain:

$$\varepsilon = -\frac{1}{\beta} \cdot \frac{u}{1-u} \tag{5.14}$$

We see that when $\varepsilon \to 0$ then $\beta \to \infty$ (rigid wages), and when $\varepsilon \to \infty$ then $\beta \to 0$ (flexible wages). To give but an example, for unemployment rate $u = 0.06$ and $\beta = 3$, the elasticity takes the value $\varepsilon = -0.0213$. For this elasticity value, a 1% point increase in the unemployment rate (that is, from 6% to 7%) would reduce the real wage rate by 0.35%, approximately. Indeed, from (5.13) and (5.14) we can write:

$$\frac{d\overline{\omega}}{\overline{\omega}} = \varepsilon \cdot \frac{du}{u} = \left(-\frac{1}{\beta} \cdot \frac{u}{1-u}\right) \cdot \frac{du}{u} = -\frac{1}{\beta} \cdot \frac{du}{1-u}$$

and the numerical value follows when using the elasticity value ($\varepsilon = -0.0213$), the reference unemployment rate ($u = 0.06$) and the unemployment rate change ($du = 0.01$). For an appreciation of the role of the real wage condition, we present in Table 5.1 the values for the elasticity and the induced real wage response to 1% changes in the unemployment rate for a 'low', 'medium' and 'high' value of the parameter β. Notice that for large values of β, the response of the real wage becomes increasingly small, as predicted by the real wage rule.

5.3 The Consumption Technology

It is often the case in empirical applications that data on commodities demanded by consumers do not correspond to data on commodities produced by firms.[2] To account for this distinction we introduce a somewhat artificial consumption technology specifying the input requirements of production goods necessary to obtain a unit of a given consumption good. Suppose we have N 'producer' goods and

[2]In fact this is a common restriction imposed by data. Production activities are classified using the input–output conventions whereas disaggregate data on household consumption follows the usually different budget and expenditure survey conventions.

Table 5.1 Sensitivity of the real wage to unemployment

Elasticities	Unemployment rates		
β	3%	6%	9%
0.5	−0.0619	−0.1277	−0.1978
3	−0.0103	−0.0213	−0.0330
100	−0.0003	−0.0006	−0.0010
Real wage response to 1 point percent increase in unemployment			
Elasticities	Unemployment rates		
β	3%	6%	9%
0.5	−2.0619	−2.1277	−2.1978
3	−0.3436	−0.3546	−0.3663
100	−0.0103	−0.0106	0.0110

M 'consumption' goods and, further, suppose that no primary factors are used in the production of the consumption goods. Let b_{ij} denote the amount of producer good i, $(i = 1, 2, \ldots, N)$, needed to obtain one unit of consumption good j, $(j = 1, 2, \ldots, M)$. Under Constant Returns to Scale we can represent the consumption technology by a $N \times M$ matrix B that we will call the transition or conversion matrix. With the conversion matrix B we can transform any $M \times 1$ vector X of consumption commodities into an $N \times 1$ vector X' of production commodities simply by

$$X' = B \cdot X \tag{5.15}$$

Similarly, we can also distinguish now production prices from consumption prices. Given the fixed coefficients technology B, any $1 \times N$ vector of production prices p is converted into a $1 \times M$ vector q of consumption prices by

$$q = p \cdot B \tag{5.16}$$

The consumer's optimization problem (3.1) is straightforwardly redefined. The argument of a representative households utility function is now a M dimensional vector of consumption goods, whereas the budget constraint is defined, for a given vector of factor endowments, by the consumption price vector q and the factors price vector ω. The individual demand function can be written as $c_h(q, \omega)$, with market demand $CD(q, \omega)$ being as usual the aggregation of all consumers' individuals demands. Final consumption demand for production goods is obtained using (5.15) as $B \cdot CD(q, \omega)$. Notice that all actual production takes place only in the production sectors and the matrix B is just a convenient mapping for reconciling different commodity aggregations. If no such a distinction existed, B would reduce to the identity matrix and we would be back to any of the versions of the standard general equilibrium system.

5.4 Welfare Evaluation and Measures

In general, the main objective for developing an applied general equilibrium model is the evaluation of the potential socio-economic effects that may arise from the implementation of a specific policy. There are very many possible policies that the government may implement and a few explicit examples are in order. For instance we may wish to consider the effects of changing the structure of the tax system, or variations in the level and composition of public spending in investment and consumption, or modifications in the terms of trade through an increase/decrease in import tariffs. As we have seen in the specific examples that have been introduced in previous chapters, the effects of economic policies are widespread. New policies alter the state of the economy through their impact on resource allocation. All markets and all agents end up being somehow affected and thus a natural question to ask is whether the economy-wide effects that follow from these policies are beneficial or detrimental to society. Even more, since different policies can be designed to achieve a given policy goal, we need ways to compare among alternative policies in order to suggest which one should be chosen.

5.4.1 Measuring the Effects of a Fiscal Policy

Suppose, as an example, that the government wants (or needs) to reduce the public deficit. This goal can be achieved either by reducing public spending or by increasing public revenues (or a combination thereof). The reader will no doubt think of many possible implementations of these goals. Should the government reduce public expenditures in a proportioned way among all sectors or should it focus the reduction in a subset of them? Should the government put the emphasis in reducing public investment or public consumption? In terms of taxation, should the government raise indirect taxes such as the value-added tax or raise income taxes? Or perhaps should the government first eliminate subsidies and tax breaks to specific groups of agents before raising any tax rates to the whole population? Clearly, policy making is not an easy task.

Facing this complex situation, the question is how to determine which policy is better, and why so. This is a straightforward question that requires some kind of an answer from us economists. We believe two types of issues are relevant. The first has to do with the efficiency effects that follow from implementing a policy. Public finance theory shows that enacting a tax, or raising the current rate of an existing tax, will always have a detrimental effect on real output and real income. This loss is for the economy as a whole and even if the tax receipts are completely reinvested, the loss will always happen. This type of loss we refer to as an efficiency loss since the performance of the economy, after the tax, is worse than it was before. The second issue has to do with distribution since, even if the economy is overall in worse shape, perhaps some agents will be better off.

Let us recall the simulation results from Table 4.1, in particular the middle simulation where after the passing of a 10% output tax, all accrued tax revenues are equally and fully returned to the two agents. Recall too that before this policy was considered, equilibrium prices for the two goods were 1 and equilibrium activity levels were 1 as well. Let us denote these initial values by $p^0 = \left(p_1^0, p_2^0\right) = (1, 1)$ and $y^0 = \left(y_1^0, y_2^0\right) = (1, 1)$. Once the policy is implemented, the counterfactual results show that prices for goods and output levels would be:

$$p^1 = \left(p_1^1, p_2^1\right) = (1.269,\ \ 1.339)$$

$$y^1 = \left(y_1^1, y_2^1\right) = (1.010,\ \ 0.979)$$

The effect on real output of this policy can be measure using a quantity index. If we use Laspeyres (which uses original prices as reference) or Paasche (which uses final prices) quantity indices to measure the effects on real output we would find:

$$I_L = \frac{\sum_i p_i^0 \cdot y_i^1}{\sum_i p_i^0 \cdot y_i^0} = \frac{1 \cdot 1.010 + 1 \cdot 0.979}{1 \cdot 1 + 1 \cdot 1} = 0.9945$$

$$I_P = \frac{\sum_i p_i^1 \cdot y_i^1}{\sum_i p_i^1 \cdot y_i^0} = \frac{1.269 \cdot 1.010 + 1.339 \cdot 0.979}{1.269 \cdot 1 + 1.339 \cdot 1} = 0.9941$$

In the first case the measured loss in real output is -0.55% whereas in the second is -0.59%, but we detect a real loss with both indices. This loss is the efficiency loss caused by the distortion upon the equilibrium created by the tax policy. Notice however that consumer 1 is worse off (a utility loss of 6.67%) whereas consumer 2 is better off (a utility gain of 13.24%). These are the distributional effects of the tax policy. Do we need to assert that both are important in the evaluation of policies?

It is common to refer to these different effects as welfare effects. True enough, if the policy moves the economy to a state where less overall output is available, the economy is certainly worse off in terms of overall welfare since total output has contracted. At the same time, individual welfare, as measured by the utilities, indicate that there are losers (consumer 1) and winners (consumer 2) as a result of the policy. It is important to keep this distinction in mind when assessing the welfare effects of policies. In addition, it is also relevant to be aware of the variety of alternative welfare measures and have sense in selecting the appropriate one for the problem under analysis. Later in this section we will present additional welfare indicators that could be used in the evaluation of policies involving taxation, such as the average and marginal excess burden.

Besides the commented distinction between efficiency and distributional issues, in practical terms economists also like to use impact indicators borrowed from macro and microeconomics. Typical macroeconomic indicators quite often used are changes in gross domestic product, *GDP*, or in total unemployment levels. These are aggregate indicators that measure global, rather than specific or sectoral, economic performance and they are attractive because they summarize a conglomerate of effects in single interpretative figures which are easier to report and understand. Microeconomic indicators, in contrast, are less popular with the public and not so universally used. Relative prices, agents' utilities, output composition, terms of trade, etc. are profoundly relevant measures but they are not well understood by the public at large. Nonetheless, these measures are the bread and butter of applied general equilibrium.

We should however be careful with the use of macroeconomic indicators. *GDP* is usually expressed in nominal terms but changes in nominal *GDP* are not really informative. Nominal *GDP* depends on the selected unit of value, or *numéraire*. A change in the numéraire will change the expression of *GDP*. But what we really need to measure are real, rather than nominal, changes in *GDP*. Let us go back to the results of Sect. 4.1.2, in particular to the same simulation we just commented above, i.e. an output tax of 10% with all tax revenues returned equally to both consumers (50% each). Table 5.2 summarizes the results of changing the numéraire from the wage rate (as it was reported in Table 4.1) to the price of capital. Before the policy, net and gross *GDP* coincide since there is no tax system. The first observation is that nominal *GDP* increases under any unit of value but the nominal figures are different. In other words, we always need to know which reference value unit is being used. To be absolutely correct in our assessments, we need to refer any observed changes in a value magnitude to the chosen unit of reference.

A second, but ultimately wrong, observation would be to assert that *GDP* has risen, from 75 units to 98.21 (or 99.68) units. The comparison would be wrong on two accounts. First we would be comparing the initial net *GDP* with final gross *GDP*. We should use final net of taxes *GDP*. Secondly, even if we compare net *GDP*, the numéraire is still around affecting the expression of the observed measures. When we use the wage rate as numéraire, net *GDP* would be $98.21 - 23.58 = 74.63$ (less than 75) whereas when we use the price of capital net *GDP* would be $99.68 - 23.93 = 75.75$ (more than 75). So is *GDP* going up or down? To be absolutely precise all we can say is that nominal *GDP* in terms of the wage rate goes down and nominal *GDP* in terms of the price of capital goes up. What is going on beneath these numbers is that the relative price between labor and capital is always $\omega_1/\omega_2 = 1/0.985 = 1.015/1 = 1.015$. Changing the numéraire from $\omega_1 = 1$ to $\omega_2 = 1$ just re-scales values everywhere by this factor, without affecting any other equilibrium magnitudes at all. This is a well-known property of competitive equilibrium.

Clearly we need to move beyond nominal expressions and measure changes in real terms, that is, independently of the unit of value being used. When we do so and compute real *GDP* we observe in Table 5.2 that it falls. If we measure the real change with a Laspeyres index, the fall is -0.013% while with a Paasche index the fall is a bit larger and reaches -0.147%. This is the same kind of distortionary effect caused by taxation that we discussed above when measuring the impact on gross output. No matter how we estimate the change in real terms there is always a loss in real *GDP*, an efficiency loss predicted by economic theory. Notice too that the selection of numéraire is irrelevant for the calculations. In the Laspeyres case, it is so because we use the vector of initial prices (labeled as p^0) in both sides of the ratio. For the Paasche index, we use final prices (labeled as p^1) in both sides of the

Table 5.2 Macroeconomic indicators and the influence of the numéraire

		Policy effects to the equilibrium	
	Benchmark	Numéraire: Wage $\omega_1 = 1$	Numéraire: Price of Capital $\omega_2 = 1$
GDP (nominal)	75.00	98.21	99.68
Wages	50.00	50.00	50.75
(% net income)	(66.66%)	(67.00%)	(67.00%)
Capital income	25.00	24.63	25.00
(% net income)	(33.33%)	(33.00%)	(33.00%)
Output tax	0.00	23.58	23.93
(% gross income)	(0.00%)	(24.01%)	(24.01%)
Real *GDP* (at p^0)	75.00	74.99	74.99
Real *GDP* change(%)	–	−0.013%	−0.013%
Real *GDP* (at p^1)	98.33	98.18	98.18
Real *GDP* change(%)	–	−0.147%	−0.147%

Output tax rate $\tau = 10\%$ in all sectors. Redistribution parameter $\delta_1 = 0.5$

ratio and any change in the selected numéraire would be neutral since it would at the same time re-dimension numerator and denominator. We leave the verification of the details of these calculations to the reader as an exercise.

In addition to these technical issues regarding the measurement of changes in *GDP*, there is the conceptual fact that *GDP* itself may not be the best of indicators. Recall, for instance, that GDP does not include non-market activities that are beneficial to society as elements in production, as is leisure, or that it includes as production a set of activities that have in fact detrimental effects to society such as the consumption of fuel fossils and the corresponding pollutant emissions. We will not dwell any more on these social accounting considerations here since they are beyond the scope of this book.

Another issue regarding the numéraire, of a deep theoretical nature, has to do with the presence of market power. It turns out that when market power is present in at least one of the markets, then the choice of numéraire may matter for the values that real magnitudes take in equilibrium. Real results under a non-competitive general equilibrium set-up, as Ginsburgh (1994) explains, may be quite sensitive to the choice of unit. It is bad news when something apparently innocuous, as it is the selection of the unit of measurement, may end up affecting the object being measured. Once again, this problem is beyond the scope of this book but we feel it is a relevant and worth-retaining piece of information. In fact, most economists seem to be unaware of this circumstance.

5.4.2 Individual Welfare Issues

We now move to discuss welfare indicators at the individual level. From a pure theoretical perspective, changes in utility levels are the most basic indicators for individual welfare change. If an agent's utility goes up (down) in percentage terms between two equilibrium states, that agent is better (worse) off. If the change in utility is the result of enacting a new policy, then that policy turns out to be beneficial (detrimental) for the said agent. However, microeconomic theory teaches us that utilities do not have a cardinal interpretation; they only order different consumption bundles. The higher in the order, the better the consumption bundle form the agent's perspective, but no quantitative meaning can be ascribed to the level of utility. Even when we sometimes use "utils" as units of utility, we know that they cannot be interpreted in any physical way. These units are just a pedagogical convenience that we eliminate as soon as we can in the learning of economics. Let us take a simple example. Consider the very simple utility function $u(c) = c$ with $c > 0$ being a real number. The consumption of c units of bread yields c *utils* of satisfaction. So if the consumer goes from having $c_0 = 1$ units of bread to $c_1 = 2$ units, the utility of his consumption would increase a 100%. But remember from your microeconomic theory that any monotone increasing transformation of a utility function is also a utility function that represents exactly the same preference relation. From here $w = (u)^2$ is also a valid representation of the same consumer's preferences. But now the same consumptions would give utilities $w(c_0) = 1$ and $w(c_1) = 4$. Nothing has really changed in real terms for this consumer but the percentage increase in utility would be now 300%! Clearly raw "utils" are not very useful for sensibly measuring individual welfare. The sign of the percentage change do tell us something meaningful about welfare (whether goes up or down) but the numerical values do not. We need something better that raw utility changes.

Luckily economic theory provides a few indicators of individual welfare that have sensible meaning.[3] The two most common and accepted measures are the compensating and equivalent variations (*CV* and *EV*, respectively). They provide value figures that have a quick interpretation in terms of income. We will take a given consumer and examine the effects of a policy on his welfare. Let us use the following notation. The initial level of prices and income that this consumer is facing will be denoted by the pair (p^0, m^0). The consumer faces the pair (p^0, m^0) and solves the utility maximization problem to derive his optimal (Marshallian) demand bundle $c^0 = c(p^0, m^0)$ and with this bundle he achieves a utility level of $u^0 = u(c(p^0, m^0))$. A policy shock triggers now a change in equilibrium. The new vector of prices is p^1 and this consumer's income is now m^1. When facing this new pair, the consumer's optimal solution would now be given by the demand bundle $c^1 = c(p^1, m^1)$ with a utility level of $u^1 = u(c(p^1, m^1))$. If it is the case that $u^1 - u^0 < 0$, the consumer is worse off since his utility has fallen down. Let us take this case for the sake of the following argument.

[3] We follow Varian (1992) for these developments.

The compensating variation CV is defined as the level of additional income needed to compensate this consumer for the loss in utility caused by the price and income changes. With this additional income, the consumer would be back to his initial level of utility but under the new set of prices. Since utility is monotone increasing in income, this means we need to increase the new level of income m^1 exactly by CV so as to take the consumer to his initial level of utility:

$$u\left(c\left(p^1, m^1 + CV\right) = u\left(c\left(p^0, m^0\right)\right)\right. \tag{5.17}$$

Since CV is an extra income that the consumer does not actually have, the value CV is a measure of the actual utility loss due to the policy. In other words, if the consumer had to be compensated with CV units of extra income to regain utility, then the welfare loss he is suffering would be measured by its negative value $–CV$.

In contrast, the equivalent variation EV is defined as the level of income that should be detracted from the initial income at the initial prices, (p^0, m^0 – EV), so that the consumer would attain the same utility facing (p^0, m^0 – EV) as facing (p^1, m^1) after the adjustment.

$$u\left(c\left(p^0, m^0 - EV\right) = u\left(c\left(p^1, m^1\right)\right.\right. \tag{5.18}$$

Under the welfare loss $u_1 - u_0 < 0$ that we are assuming in this example, the sign detracted income $–EV$ measures the welfare loss. Obviously all the arguments should be adequately reversed if the situation would be the opposite, that is, the consumer is better off in the new equilibrium state, i.e. $u^1 – u^0 > 0$. We leave these details as an exercise for the reader.

It is relevant to remark two significant aspects in expressions (5.17) and (5.18). The first one is that they associate welfare changes to income changes between the two compared equilibrium states. The second one is that these measures are independent of the chosen utility representation. Take as before the monotone increasing transformation $w = (u)^2$ and it is obvious that nothing substantial would change in these expressions apart from the formulaic representation of utilities. The equations to calculate CV and EV would not be affected.

These two definitional expressions, however, are not very convenient in order to effectively compute CV and EV. It is easier if we use the expenditure function that derives from the expenditure minimization problem. In this problem a consumer chooses the optimal bundle $c = (c_1, c_2,\ldots, c_n)$ that guarantees a given level of utility with the minimum possible cost:

$$\begin{aligned} &Min \ \sum_{i=1}^{n} p_i \cdot c_i \\ &\text{subject to } \ u(c_1, c_2, .., c_N) \geq u \end{aligned} \tag{5.19}$$

The specific solution to problem (5.19) depends on the price vector p and the utility target u. The functions $c_i(p, u)$ that pick up this dependence are known in the literature as Hicksian or compensated demand functions. The expenditure function gives the minimal cost to achieve utility target u and is defined by:

$$e(p, u) = \sum_{i=1}^{n} p_i \cdot c_i(p, u) \tag{5.20}$$

This function has exactly the same mathematical properties as the cost functions for the production side of the economy that we have being using since Chap. 3. We are now going to rephrase the concepts of CV and EV using the expenditure function defined in (5.20). Remember that CV is income and that expenditure is nothing but the use of income. CV is the extra income that would move the consumer from the utility level u^1 under the pair (p^1, m^1) [i.e. and with an expenditure at that point of $e(p^1, u^1)$] back to the utility level u^0 under the new prices p^0 [i.e. with an expenditure of $e(p^1, u^0)$]. The difference in expenditures is nothing but the compensating variation:

$$CV = e\left(p^1, u^1\right) - e\left(p^1, u^0\right) \tag{5.21}$$

Similarly, the equivalent variation is how much less income from the initial point, which has utility u^0 and expenditure $e(p^0, u^0)$, would give the same final utility u^1 but at the initial prices p^0, in this case with an expenditure level of $e(p^0, u^1)$. From here:

$$EV = e\left(p^0, u^1\right) - e\left(p^0, u^0\right) \tag{5.22}$$

We are now, almost, in a position to greatly simplify welfare calculations. In most of the applied general equilibrium work the utility functions are assumed to be homogeneous of degree 1. When the utility function has this characteristic, then the following mathematical property can be seen to hold true:

$$e(p, u) = e(p, 1) \cdot u \tag{5.23}$$

Notice that property (5.23) for the expenditure function is analogous to property (3.10) that we used for the cost function. We can separate total expenditure in two multiplicative components, namely, the expenditure associated to achieving a utility level of 1 and the utility level itself. Take expression (5.21) and rewrite using (5.23) to obtain:

$$CV = e\left(p^1, u^1\right) - e\left(p^1, u^0\right) = e\left(p^1, 1\right) \cdot \left(u^1 - u^0\right) = e\left(p^1, 1\right) \cdot u^1 \cdot \frac{u^1 - u^0}{u^1}$$

But expenditure is income and what the consumer spends to achieve utility u^1 at prices p^1 is his income m^1. Thus from $e(p^1,1) \cdot u^1 = e(p^1, u^1) = m^1$ we finally conclude:

$$CV = m^1 \cdot \frac{u^1 - u^0}{u^1} \tag{5.24}$$

This gives us a very simple way to calculate the compensating variation. All we need is the final level of income and the two utility levels in both equilibrium states. With a similar derivation (that we omit here) we would find that the equivalent variation is given by:

$$EV = m^0 \cdot \frac{u^1 - u^0}{u^0} \tag{5.25}$$

In both expressions the initial and final utility levels depend only on the consumption bundles of the agent which do not depend on how relative prices are normalized with some numéraire. From expression (5.24) we can conclude, however, that under general equilibrium the income level m^1 does depend on the choice of numéraire. Indeed, the initial endowment vector of the agent will be valued in terms of the market prices for factors which in turn depend on the numéraire.

In conclusion, the compensating *CV* variation is not neutral regarding the numéraire whereas the equivalent variation *EV* is neutral since m^0 only depends on initial prices. This is a little known fact that has been disregarded in applied general equilibrium analysis where the *CV* measure has been profusely used without any reference to its direct dependence on the selected final expression of prices. If we want to use this welfare measure in general equilibrium we need to 'compensate' the *CV* and we can do so normalizing it with a price index built using the same numéraire. If *CPI* is a consumers' price index, the normalized and correct welfare measure would be *CV*/*CPI*. Any changes due to the numéraire will now cancel out.

Some calculations will be useful to fix these ideas. We return to the same simulation commented above and taken in Table 5.2. We now use the description of the economy, as outlined in Chap. 4, and with some small modifications in the computer code of Appendix 1 we would obtain initial and final utility levels and income levels for the two consumers. The super-index indicates the equilibrium state (0 and 1) whereas the sub-index identifies the consumers (1 and 2). The price index is calculated as a simple average of final prices.

We can see in Table 5.3 how the choice of numéraire is irrelevant except when we arrive to the *CV*. The fact that disposable income depends on the unit of measurement explains the result. Once we normalize the *CV* values by the *CPI* all checks well. The message is that in a general equilibrium setting you cannot use the raw *CV* numbers from the analytical formula; you need to correct for the bias that the numéraire creates.

Table 5.3 Welfare Indicators and the Influence of the Numéraire

	New equilibrium effects	
	Numéraire: Wage $\omega_1 = 1$	Numéraire: Price of capital $\omega_2 = 1$
Initial utility levels		
u_1^0	27.144	27.144
u_2^0	12.754	12.754
Final utility levels		
u_1^1	25.333	25.334
u_2^1	14.443	14.444
Utility change in %		
Δu_1	−6.672	−6.672
Δu_2	13.242	13.242
Equivalent variation		
EV_1	−3.335	−3.335
EV_2	3.311	3.311
Compensating variation		
CV_1	−4.396	−4.462
CV_2	4.294	4.358
Consumers price index	1.304	1.324
CV (normalized by CPI)		
CV_1/CPI	−3.370	−3.370
CV_2/CPI	3.292	3.292

Output Tax Rate $\tau = 10\%$ all sectors. Redistribution Parameter $\delta_1 = 0.5$

5.4.3 Welfare Aggregation

The results of the numerical example show two things regarding the implementation of the fiscal policy. First, output goes down whether we measure it with total gross output or with *GDP*. This is the result of the distortionary effect caused by taxation. The second observation is that individual welfare for agent 1 goes down whereas it goes up for agent 2, no matter how we measure it. We might be tempted to add up the percentage utility changes as an indicator of total welfare change. In this case, we would have $\Delta u_1 + \Delta u_2 > 0$. But we already know we cannot do that. Remember that utilities are merely ordinal and we can transform their numerical expression using any monotone transformation. If instead of the utility function u_1 we use $w_1 = (u_1)^3$ the equilibrium would be unaffected but now the percentage utility change for agent 1 would turn out be $\Delta w_1 = -18.710$ with a result opposite in sign $\Delta w_1 + \Delta u_2 < 0$!. Clearly, the addition of utilities makes no-sense as a collective welfare indicator.

What about adding up CV and EV across all agents?. They are, after all, income measures and income is not an ordinal magnitude. We can add income and the summation result is at least meaningful in terms of income itself. In the example of Table 5.3, for instance, we would find that $CV_1 + CV_2 < 0$ and $EV_1 + EV_2 < 0$. In aggregate terms the collective welfare would go down, under both measures. This result agrees with the previous observation that aggregate output also goes down because of taxation. Overall there is less output in the economy and no matter how this smaller amount is distributed the end result is that the economy is, because of the distortion generated by the output tax, worse off in terms of aggregate welfare. However, a note of caution should be stresses at this point. Even though both agents share the same structure of the utility function and they are homothetic, these agents present different utility patterns since they have different utility weights for the goods that they consume. Therefore the summation of *EV* and *CV* across agents may not be too sound a practice. Boadway (1974), for instance, explains the difficulties that may arise from the aggregation of these welfare measures. Despite some reservations it is usual to use their aggregation as a proxy for welfare changes at the level of the whole economy (Shoven and Whalley 1984).

There is another issue related to aggregation in applied general equilibrium and it has to do with the familiar assumption of using a single representative consumer. In an economy with a single consumer, this individual's welfare is unambiguously also the economy's welfare, at least in private terms. In building an empirical model we find often that is difficult, or sometimes even impossible, to collect information on the individual level. In many cases we are only able to observe aggregate magnitudes such as total households' consumption and income. In these cases the only option is to implement the presence of a unique private consumer in the model as a representative of the variety of consumers in the real economy. The question is to what extent this substitution procedure is theoretically consistent.

It is reasonable to assume that such an aggregation cannot be straightforward and valid for all possible utility functions. This is indeed the case. Gorman (1953) stated the conditions for this aggregation to be valid. The use of a representative consumer means that the sum of all the individuals' Marshallian demands can be substituted by a unique aggregate demand function. Let us recall expression (3.20) which gives the vector of aggregate consumption demand *CD* for the economy as the sum of the individual demand functions c_h for the H consumers:

$$CD(p,\omega) = \sum_{h=1}^{H} c_h(p,\omega) \tag{5.26}$$

Let us make the following modification in the individual demand functions by making explicit the income level m_h as a function of factor prices' and initial endowments:

$$c_h(p,\omega) = c_h(p,m_h) = c_h\left(p, \sum_{k=1}^{K} \omega_k \cdot e_{kh}\right) \tag{5.27}$$

Aggregate demand can therefore be seen as a function:

$$CD(p, m_1, m_2, \ldots, m_H) = \sum_{h=1}^{H} c_h(p, m_h) \tag{5.28}$$

Under a unique representative consumer the economy's demand function should be equal to the sum of the individual demand functions:

$$CD\left(p, \sum_{h=1}^{H} m_h\right) = \sum_{h=1}^{H} c_h(p, m_h) \tag{5.29}$$

If condition (5.29) is fulfilled then aggregation to a single representative consumer is correct. Gorman's aggregation theorem sets the necessary and sufficient conditions for the condition to hold. The requirements have to do with the structure of the indirect utility functions. These are the functions that give maximal utility levels for any configuration of prices and income levels:

$$v_h(p, m_h) = \underset{c_h}{\mathrm{Max}} \left\{ u(c_h) / \sum_{i=1}^{N} p_i \cdot c_{ih} = m_h \right\} \tag{5.30}$$

According to Gorman's theorem aggregation to a representative consumer as in (5.29) is possible if and only if all the individual indirect utility functions satisfy the condition:

$$v_h(p, m_h) = a_h(p) + b(p) \cdot m_h \tag{5.31}$$

In this case, aggregation is possible and there is a representative consumer with indirect utility function:

$$v(p, m) = a(p) + b(p) \cdot m \tag{5.32}$$

with $a(p) = \sum_h a_h(p)$ and $m = \sum_h m_h$. Gorman's conditions are quite restrictive but there are two cases when they are satisfied. One is when all utility functions are identical and homothetic. The second case is when individual utility functions are quasi-linear. Summarizing, when aggregation is possible changes in the level of welfare of the representative consumer, measured using either *CV* and/or *EV*, will indicate the effects of a change in the equilibrium state induced by the implementation of a new policy. The representative consumer's welfare will reflect those of the group of identical consumers that is being represented. Changes in utility levels, when appropriately measured, have the advantage that are connected to preferences and changes in some non-market activities, i.e. leisure, can be easily captured and include as an argument in the utility function. We cannot do the same with changes in real *GDP*, as we commented before.

5.4.4 The Excess Burden of Taxation

We have seen in Sect. 5.4.1 how an output tax affects the decisions of firms and consumers. The tax changes relative prices, affect their income levels and the agents react accordingly to the new situation. The presence of the tax influences households' consumption patterns as well as the quantity of commodities being produced the mix of factors used in their production. We have also seen that total output and GDP fall after the tax. This is the so-called distortionary effect, or efficiency cost, of the tax and it implies a loss for society as a whole. The tax aims at raising income for the government so that it can finance its public expenditure and social programs. But even when all the collected tax income is returned to consumers, overall there is less output and less income around. The transfer of income from private agents to the government implies that in the process some income is lost and not collected as tax revenue.

The costs linked to these distortions are known in the literature as the excess burden (*EB*) of taxation, or deadweight loss. More specifically, the *EB* is defined as the amount of income lost in excess of what the government collects that is then transferred as a lump-sum transfer to society. The *EB* is usually expressed as the variation in total surplus per unit of additional tax revenue once the tax is raised or introduced:

$$EB = \frac{\Delta \text{Total Surplus}}{\Delta \text{Tax Revenues}} \tag{5.33}$$

When production takes place under constant returns to scale, firms make no pure profits and total surplus reduces to the surplus of consumers. Under this technological assumption supply functions for commodities are flat. In order to fix these ideas, in Fig. 5.2 we depict the situation in a partial equilibrium as an approximation—bear this in mind. The supply function S is flat at price p^0 reflecting a constant marginal cost. The demand function D is downward sloping reflecting the so-called "law of demand". With a tax rate of τ the price goes up to $p^1 = p^0(1 + \tau)$ and output would fall from y^0 to y^1.

The rectangular area named T is the level of tax collections with $T = p^0\, \tau\, y^1$ and ΔCS is the loss in consumers' surplus from the presence of the tax. The area $T + DW$ measures the loss in consumers' surplus from the fact that the tax raises the price from p^0 to p^1. It has two components, the tax paid by consumers T and what is known as the 'deadweight' loss, DW. Even if the government returns to consumers all of the tax collections T, there is a pure loss in welfare given by the area DW. In this partial equilibrium example, the excess burden will be $EB = DW/T$. For each unit of income collected as taxes, the economy endures an irretrievably loss of *EB* units of income, even if all the tax collections are returned to the consumers.

The variation is consumers' surplus *CS*, however, is an approximation to the exact evaluations of welfare loss that the previously introduced measures *CV* and *EV* provide. Additionally, the calculation of *CS* requires a given and fixed Marshallian demand function but in a general equilibrium setting demand functions

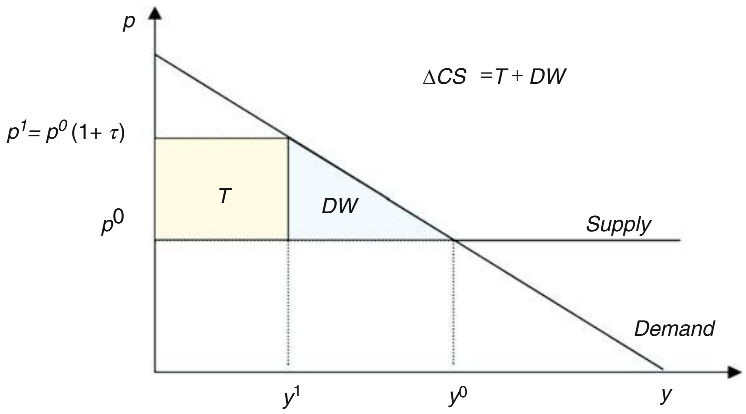

Fig. 5.2 Welfare effects of a commodity tax under partial equilibrium

will shift position because of all the interactions that take place once a tax is enacted or raised. Instead of the variation in surplus induced by the tax, we can place *EV* or *CV* in the numerator of expression (5.33). For instance, Kay (1980) recommends using *EV* as the most appropriate indicator in the numerator since approximates better the Harberger 'triangle' or deadweight loss *DW*. Other authors, in contrast, stress that there is no preference in using *CV* or *EV* (Zabalza 1982).

There are two possible ways to calculate the excess burden *EB*. Suppose we substitute a given distortionary tax with a non-distortionary lump-sum tax that collects exactly the same tax revenue. This would be neutral from the point of view of revenue for the government but the distortionary tax would generate a double impact on agents, via substitution and income effects, whereas the lump-sum tax would only affect via income effects. The bulk of the substitution effects measured in welfare units per unit of tax revenue is the average excess burden, *AEB*. The marginal excess burden, *MEB*, evaluate the effects of small changes in (5.33). We change a tax rate by a small amount, say a 1%, and measure the variation in welfare that follows from the small change. Both indicators can be used in a general equilibrium model but *MEB* is probably better since it involves small changes rather than the large discrete changes required to compute *AEB*. Small changes perturb the structure of the model less than large changes. Given the level of uncertainty on parameters, elasticities and functional relationships, that surrounds the building of large scale general equilibrium models, it is probably advisable to keep changes under control to avoid exacerbating approximation errors.

In Table 5.4 we illustrate these ideas regarding the marginal excess burden showing some empirical results borrowed from ongoing research by the authors. The reported figures update the work of Sancho (2004) using the most up-to-date database built for the Spanish economy in 2010 from official data and some newly estimated behavioral elasticities. We show the calculations using the equivalent variation *EV*. The exercise assumes a small change in a given tax rate, i.e. 1 per thousand, for the seven tax types that the model includes and then equilibrium is

Table 5.4 Marginal Excess Burden, Spain 2010

Tax type	EV	ΔTax	$MEB = EV/\Delta$Tax
Value-added tax	−0.569	0.411	−1.384
Net production tax	−0.312	0.103	−3.029
Tariffs	−0.014	0.005	−2.800
Social security-employers	−1.166	0.451	−2.585
Social security-employees	−0.391	0.381	−1.026
Social security-unemployed	−0.085	0.089	−0.955
Income tax	−0.831	0.870	−0.955
Total of all taxes	−3.857	2.503	−1.541

Thousands of millions of Euros

recomputed obtaining the welfare loss EV and the increase in tax collections measured in real terms. This adaptation is necessary to take care of the fact that final tax receipts depend not only on the final equilibrium price vector but also on the chosen numéraire. The last row of the Table reports a simulation in which all tax rates are changed at the same time and allows us to have an impression of the overall efficiency of the Spanish tax system. On average, the tax system incurs in a welfare loss of about 1.5 € on top of each additionally collected Euro. The taxes most conducive to welfare losses are net output taxes, tariffs and social security contributions by employers, in this order. The tax with the smallest induced marginal excess burden is the income tax which reflects the fact that this is the closest tax type to being lump-sum.

5.5 Summary

Going beyond the essentials of the basic general equilibrium model shows the enormous potentiality of the approach. In fact, the restrictions in building a model stem more from the limitations of data than from the limitations of economic theory. We can develop a quite rich—in terms of descriptive features—general equilibrium model as long as we are able to endow it with numerical parameters and elasticities that go hand in hand with its conceptual architecture. A good theoretical model with no empirical data to back it is just a curiosum device, whereas producing data without a model or frame of reference to incorporate it is just a waste of energy and resources. Sound economic modeling requires a balance of ideas and data. Both are required if we want to discern properties of the economy in the real world. Once a model is conceived (theory) and implemented (data), it must be capable of producing insights and results that otherwise would not be available or easy to deduce, and these results must be correctly read and interpreted. This is the ‘value-added’ of economic modeling that we must always pursue.

5.6 Questions and Exercises

1. Derive the Armington Demands for the Cobb–Douglas Aggregator

$$y_j = \mu_j \cdot Q_j^{\alpha_j} \cdot M_j{}^{\left(1-\alpha_j\right)}$$

   and the *CES* aggregator

$$y_j = \left[\left(\alpha_j \cdot Q_j\right)^{\rho} + \left(\left(1-\alpha_j\right) \cdot M_j\right)^{\rho}\right]^{\frac{1}{\rho}}$$

   if firms pay an ad-valorem tariff rate r_j (medium).
2. Obtain an expression for the government revenue function with tariffs and indicate which equations of the equilibrium system (5.9) would be affected by the tariffs (medium).
3. Interpret the unemployment/real wage condition as a supply function for labor and give its expression (medium).
4. Work out the reduced dimensionality form for the equilibrium system (5.9) (easy).
5. We know that the $j-th$ column of the input–output matrix A describes the intermediate goods technology for producing a unit of commodity j. Consider enlarging the matrix A to include the investment and trade activities and try to rewrite the equilibrium system with this new convention (hard).

6 Data Base and Model Calibration

In Chaps. 3, 4 and 5 we have developed a simple but increasingly complex general equilibrium model. Starting with the standard textbook version of a private, closed economy we showed how the different pieces of the model interact with each other to give rise to a system of equations that capture and describe market equilibrium. The addition of the government and the external sector, as well as the modification of the labor market to allow for unemployment, provided a more realistic picture of an actual economy and laid the grounds for the study of various policy issues. In each of the examples used, however, the specification of the model parameters was arbitrary, except for a choice of units that yielded convenient solution values for prices and output levels.

The equilibrium solution was used in one instance to construct a *SAM* for the economy that compiled the equilibrium value of all transactions between firms, factors, and households (Table 3.1). In fact, the *SAM* could have also been constructed and presented for the subsequent equilibrium solutions for each of the increasingly more complex equilibrium systems. In all these cases, new accounts representing the activities of the government and the external sector would have had to be introduced. The common starting point in all the examples was a description of technology, preferences, and behavior that once coupled with a parameter specification lead to a system of equations; from here, if so desired, a *SAM* could have been constructed.

An essential ingredient for any large scale applied model to be operational is the availability of parameters for its functional forms. The question arises, therefore, of how to choose or determine the model's parameters so as to have an adequate representation of the economy's underlying structure. But since a *SAM* is a depiction of the full set of transactions in the economy, the necessary information is somehow embodied in the structure of the *SAM*. All we need is to recover the parameters from the data on the *SAM*. This procedure is called calibration and in general refers to the use of an empirical *SAM* to obtain parameter values for a given model structure in such a way that the model equilibrium solution reproduces the observed *SAM* data. A good discussion of the calibration protocol versus standard statistical estimation

M.A. Cardenete et al., *Applied General Equilibrium*, Springer Texts in Business and Economics, DOI 10.1007/978-3-662-54893-6_6

can be read in Mansur and Whalley (1984). A complete account can be found in Dawkins et al. (2001). For the actual implementation of the practical calibration details in applied general equilibrium modeling see Sancho (2009).

In this chapter we will first discuss in more detail the structure of a *SAM* and some of the most common problems researchers find in assembling *SAM*s from raw data. We will next introduce the calibration techniques and show how, starting from the data on a *SAM*, parameter values can be constructed for the functional forms of the basic model.

6.1 Social Accounting Matrices

Simply stated, a *SAM* is a tabular presentation of the whole set of flows of an economy in a given moment in time. The economy in question can be national or regional, depending on the issue of interest. The purpose of building a *SAM* is to have an integrated system of accounts relating production, consumption, trade, investment and the government in a consistent, closed form. Consistency is to be understood in both the micro and the macroeconomic sense. Each agent's income-expenditure flows satisfy a budget constraint—microeconomic consistency—and all agents' flows put together satisfy the standard aggregate identities—macroeconomic consistency. This dual consistency is essential for economic analysis modeling since it allows us researchers to match actual data with an operational model whose analytical structure is based on data. This marriage of data and modeling has been made possible thanks to the availability of large scale *SAM*s following the pioneering work of Sir Richard Stone (see Stone and Brown 1962, or the Nobel lecture in Stone and Corbit 1997). Very clear applications of the *SAM* tool in policy modeling can be found in Pyatt (1988) and Thorbecke (2000), among many others.

6.1.1 A Simple Macroeconomic *SAM*

To see how the *SAM* structure fits into the national income and product accounts, consider the following aggregate accounting expressions for a very simple economy:

$$
\begin{aligned}
&(a) \quad GDP = CD + I + E + X - M \\
&(b) \quad GDP = W + K + R_1 \\
&(c) \quad W + K = CD + S_v + R_2 \\
&(d) \quad D = R_1 + R_2 - E \\
&(e) \quad F = X - M
\end{aligned}
\tag{6.1}
$$

where GDP: Gross Domestic Product, CD: Private Consumption, I: Investment, E: Government Spending, X: Exports, M: Imports, W: Payments to Labor services, K: Payments to Capital services, S_v: Private Savings, R_1: Indirect Taxes, R_2: Income

Table 6.1 Macroeconomic social accounting matrix

		(1)	(2)	(3)	(4)	(5)	(6)	(7)
(1)	Production	A	0	0	CD	I	E	X
(2)	Labor	W	0	0	0	0	0	0
(3)	Capital	K	0	0	0	0	0	0
(4)	Consumers	0	W	K	0	0	0	0
(5)	Capital account	0	0	0	S_v	0	D	$-F$
(6)	Government	R_1	0	0	R_2	0	0	0
(7)	External sector	M	0	0	0	0	0	0

Taxes, D: Budget Deficit, and F: Trade Balance. In expression (6.1) we have just the most common national accounting relationships as they are described in any standard intermediate macroeconomics textbook.

Define now accounts for the *SAM* tallying income and expenditure for 'production activities', factors such as 'labor' and 'capital', 'consumers', a 'capital account' for savings and investment, 'government', and the 'external sector'—seven accounts in total. If we use the convention that expenditures are recorded in columns and income is recorded in rows, the macro *SAM* takes the form described in Table 6.1. In this *SAM*, A stands for intermediate transactions, a magnitude that is netted out in the macroeconomic identities to avoid double accounting.

Let $\sum_i^c$ and $\sum_i^r$ denote, respectively, the sum of column i and row i of the *SAM*. Using the macroeconomic identities in (6.1), it is easily verified that $\sum_i^c = \sum_i^r$ for each of the seven columns. Indeed, $\sum_1^c = \sum_1^c$ corresponds, with the addition of intermediate inputs A, to (a) and (b) in (6.1). Identity (c) is simply $\sum_4^c = \sum_4^c$. The condition "savings = investment" is made explicit in $\sum_5^c = I = S_v + D - F = \sum_5^c$ which is just a rearrangement of (a), (b) and (c) in (6.1). For the government account, $\sum_6^c E + D = R_1 + R_2 = \sum_6^c$ is (d) and for the external sector $\sum_7^c X - F = M = \sum_7^c$ is (e).

Each account of the *SAM* can be interpreted as a budget constraint and the fact that $\sum_i^c = \sum_i^r$ means that the constraint is satisfied for each and every account. As an example, firms obtain their income by selling intermediate goods to firms, A, and by selling final goods to consumers, CD, the capital account, I, the government, E, and the external sector, X. All these transactions are recorded in the first row of the *SAM*. With their income, firms make payments to the pool of hired labor, W, and the used capital services, K. Further payments are made to the government in the form of an indirect tax, R_1, and the external sector, M, for the imported commodities used in production. These payments are recorded in the first column of the *SAM*. As another example, consumers' income, which include payments to labor W and capital services K in row 4, is fully used—see column 4—to finance consumption CD, savings S_v and income taxes R_2. Similar considerations apply to the production account. The budget constraints are satisfied for the government, external sector, and savings-investment account thanks to the balancing entries D and F. Since in the aggregate total income necessarily equals total spending, if all but one of the budget constraints is satisfied, then the latter must also be. This is the *SAM* counterpart of the familiar Walras' law in general equilibrium analysis.

6.1.2 A Detailed Microeconomic *SAM*

The macro *SAM* gives a very simple but compact representation of all aggregate flows in a base period. Its present structure can be easily adapted to provide finer detail in any of its accounts. The production account can be disaggregated to encompass a sectoral classification of activities like that of the input–output accounts. Factors can likewise be disaggregated by qualification or productivity, e.g. low, medium and high skilled labor. Consumers, or Households, can be distinguished by income or even more detailed socioeconomic characteristics: age, urban versus rural, level of education, among others. For the adopted Households' classification different consumption patterns from national or regional budget surveys can also be fitted into the *SAM* structure. In doing so, all that is needed for this purpose, should there be distinct classifications of consumption and production goods, is a transition matrix like B from Chap. 5 mapping the consumption aggregation into the production aggregation. In empirical terms this means, for example, linking the Classification of Individual Consumption by Purpose (*COICOP*) to the North American Industrial Classification System (*NAICS*) in the United Sates, to the Statistical Classification of Economics Activities (*NACE*) in the European Union, or to the Australia and New Zealand Industrial Classification System (*ANZICS*).

In the government account we can separate output, sales, factor taxes, and tariffs within the indirect tax category; income taxes may include personal income taxes and individual social security contributions, a distinction that can again be included if desired. Government spending may include expenditures in goods and services and transfers to households and firms. In essence, the only constraint in the degree of detail of an empirical *SAM* is given by the availability and quality of the data.

Table 6.2 is an extension of the simple *SAM* in Table 6.1 to accommodate for the description of the actual economic structure of an empirical *SAM* of the Spanish economy. In Table 6.2 we introduce the following new notation: B is the conversion matrix transforming demand CD for the consumption goods into demand for production goods; in the government revenue category we now distinguish R_p: indirect output taxes, R_c: indirect consumption taxes (excise and value-added

Table 6.2 Structure of the Spanish *SAM*

		(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
(1)	Production goods	A	B	0	0	0	I	E	X
(2)	Consumption goods	0	0	0	0	CD	0	0	0
(3)	Labor	W	0	0	0	0	0	0	0
(4)	Capital	K	0	0	0	0	0	0	0
(5)	Consumers	0	0	$W-R_w$	K	0	0	T	T_x
(6)	Capital account	0	0	0	0	S	0	D	$-F$
(7)	Government	R_p	R_c	R_w	0	R_h	0	0	R_x
(8)	External sector	$M+R_x$	0	0	0	0	0	0	0

taxes), R_w: payroll taxes paid by employees, R_h: income taxes, R_x: tariffs, T: government transfers to consumers, and T_x: external remittances to consumers.

The schematic structure in Table 6.2 is expanded in the numerical *SAM* of Table 6.12 in Appendix 1 with a disaggregation of the production and consumption goods accounts. In the former we include three productive sectors—Agriculture, Industry, and Services, whereas in the latter we distinguish three different types of consumption goods—Nondurables, Durables and Personal Services. The households' category comprises three types of consumers classified by age—Young, Adult, and Retired. The external sector is divided in two trading areas, Europe (*EU*) and the Rest of the World (*ROW*) with a similar distinction in tariffs. To make explicit the different tax instruments and their assignment to paying accounts, the numerical *SAM* contains several 'tax' collector accounts which then transfer their income to the government account proper.

A few aspects of the Spanish *SAM* need to be commented. First, notice that all capital income ends up in the hands of consumers and, therefore, no savings takes place at the firm level. Implicitly, all retained earnings are assigned to households as private savings. This option has to do with the static nature of the general equilibrium model and the assumption of perfect competition. Since all capital is owned by households, any retained capital income will reduce current utility levels without increasing future utility. We bypass this difficulty by assigning all income to households and let them decide their demand for future consumption.

Secondly, payments to firms by the government—subsidies—do not appear explicitly in the *SAM*. Since subsidies act as negative output taxes, they are accordingly netted out in the 'output tax' account.

Finally, unlike other *SAM*s, excise and value-added taxes on consumption are assigned to consumption goods instead of consumers. This convention allows us to have specific tax rates for goods instead of personalized consumption tax rates. Again, this is just a matter of convenience for tax policy exercises.

It is an interesting exercise to use the numerical *SAM* in Table 6.12 to verify that the national accounting identities check. Looking at expression (6.1) we will confirm that *GDP* from the expenditure and income sides coincide. For simplicity we will use the numerical identifiers of the labels describing the different agents in the economy. So "Agriculture" will be account 1, "Industry" account 2, etc. Starting from the expenditure side, private consumption in Table 6.12 is calculated from:

$$CD = \sum_{i=4}^{6} SAM(i,9) + \sum_{i=4}^{6} SAM(i,10) + \sum_{i=4}^{6} SAM(i,11) = 26228.72$$

with 4 being consumption of "Non-durables", 5 consumption of "Durables" and 6 consumption of "Services". In turn, index 9 represents the "Young" consumer, 10 the "Adult" one and 11 the "Retired" agents. Aggregate gross capital formation, public consumption and exports will be, respectively:

$$I = \sum_{i=1}^{3} SAM(i, 12) = 5811.69$$
$$E = \sum_{i=1}^{3} SAM(i, 13) = 5132.00$$
$$X = \sum_{i=1}^{3} SAM(i, 17) + \sum_{i=1}^{3} SAM(i, 19) = 3204.52 + 1984.89 = 5189.41$$

As for Imports from the European Union (account 17) and the Rest of the World (account 19) we will have:

$$M = \sum_{j=1}^{3} SAM(17, j) + \sum_{j=1}^{3} SAM(19, j) = 3550.85 + 3192.25 = 6743.10$$

With this data, *GDP* is given by:

$$\begin{aligned} GDP &= CD + I + E + X - M = \\ &= 26228.72 + 5811.69 + 5132.00 + 5189.41 - 6743.10 = 35681.72 \end{aligned}$$

We turn now to the income side of the economy. First, labor and capital income (accounts 7 and 8) are given by:

$$W = \sum_{j=1}^{3} SAM(7, j) = 16284.60$$
$$K = \sum_{j=1}^{3} SAM(8, j) = 16050.70$$

Indirect taxation includes four different types, namely, an excise tax (account 14), a value-added tax (account 15), other indirect taxes (account 16) and tariffs (accounts 18 and 20):

$$R_c = \sum_{j=4}^{6} SAM(14, j) + \sum_{j=4}^{6} SAM(15, j) = 213.25 + 1915.90 = 2129.15$$
$$R_p = \sum_{j=1}^{3} SAM(16, j) = 834.85$$
$$R_x = \sum_{j=1}^{3} SAM(18, j) + \sum_{j=1}^{3} SAM(20, j) = 251.50 + 121.90 = 373.40$$

Finally, we find *GDP* from the income side as:

$$GDP = W + K + R_c + R_p + R_x =$$
$$= 16284.60 + 16059.70 + 2129.15 + 834.85 + 373.40 = 35681.70$$

with a small difference in the last decimal regarding expenditure *GDP* due to the inevitable rounding off errors in the data.

6.1.3 Assembling a *SAM*

The compilation of *SAM*s by national statistical agencies is, much to the regret of users and researchers, still an uncommon event. As a consequence, *SAM*s are built with the help of a variety of published as well as unpublished, but available upon request, economic data, including the National Income and Product Accounts (*NIPA*), the Input–output Tables (*IOT*), budget surveys and a host of tax, demographic and socioeconomic data. A quick observation of the *SAM* in Tables 6.2 and 6.9 makes it relatively clear how these data sources fit into the integrated structure of the *SAM*. In practice, however, each data block is compiled using different estimating procedures and consistency across data sources is, more often than not, lacking. The most notorious discrepancies usually occur between the information in the budget surveys and the input–output tables or national income accounts. For instance, reported spending and income levels in the budget surveys are systematically lower than those of the *IOT* and *NIPA* sources. The discrepancies can be tracked down to different estimating methodologies, different commodity valuations, but also to a systematic downplaying of reported expenditures by overcautious consumers. Similarly, *NIPA* figures for trade usually disagree with sectoral figures; input–output value-added data do not match their aggregate *NIPA* counterpart, and so on.

To achieve overall micro and macro consistency in the *SAM*, therefore, reconciliation of the available data sources is often required. There is no well-established, standard procedure to perform this reconciliation, however, and here we will only provide some basic guidelines. When data sources provide different estimates of a given magnitude, one of them has to be chosen as pivotal, which in a sense implies accepting its estimates to be the 'true' or, more modestly, 'best' ones, and then proceed to adjust the remaining data to the chosen values. If we believe that the *NIPA* report the best aggregate figure for private consumption, we can use this estimate to adjust disaggregated households' consumption data to match the *NIPA* value. A related problem arises when researchers have access to the most recent *NIPA* figure but individual data on consumption for the same period is not yet available. Once again, a solution is to update the individual data until a matching with the aggregate value is obtained. In both cases, the procedure entails keeping the available consumption ratios while adjusting their level to achieve consistency with the selected primary data source.

This is in fact the procedure suggested, and extensively used in the development of their United States *SAM*, by Reiner and Roland-Holst (1992). They start with a complete set of current macroeconomic identities that provide the basic value levels for further disaggregation. Using input–output data on intermediate transactions, value-added and budget data, they use the implicit ratios to expand or contract the original levels to the *NIPA* levels. Each time a *SAM* vector or submatrix is adjusted, though, some discrepancies of a newer kind are generated. These are numerical discrepancies resulting from partial adjustments. As a result, row totals and column totals are likely to differ and a new mathematical adjustment, oftentimes involving the *RAS* method, is required (Bacharach 1965, 1970) to yield aggregate consistency. The *RAS* method was devised as a technique to extrapolate tabular or matrix data when newer row and column totals are known but new individual entries are not. The method adjusts older matrix entries in a way consistent with the most recent aggregate values. The end result is a new matrix whose structure inherits the old structure but respects the new aggregate data. See Sect. 6.3 ahead for a more in-depth discussion of *SAM* updating procedures.

The task of assembling *SAM*s, still mainly falling on small groups of researchers, is nonetheless being facilitated in the countries of the European Union by the mandatory publication of integrated an homogeneous national accounts and input–output tables that respond to a common standard, the European System of Accounts (*ESA*-2010). The data discrepancies, if any, are resolved at a much earlier stage of development, when very detailed survey information is available, rather than by using ex-post adjustment techniques on a given body of data. This is of tremendous help in developing a *SAM* and has been a significant step in the right direction, yet adjustments have still to be made to incorporate the budget data into the base framework laid out by the primary data sources. But as long as the development of fully integrated *SAM*s remains outside the responsibility of national statistical agencies, users with limited resources will bear the cost of assembling *SAM*s, a very tedious and time consuming task. Unfortunately, there is no unanimous accepted way of proceeding in assembling a *SAM* since, given the diversity and availability of data sources across countries, each case is a world on its own requiring its own procedures and rules. See Keuning and de Ruitjer (1988) for a detailed exposition of strategies, problems and possible solutions in building a *SAM*.

We now present a practical example to show the reader in a very simplistic but useful way how the data in an input–output table, *IOT*, can be expanded into a *SAM* using the information contained in the *NIPA*. With this objective, as in Sect. 5.5.3, we use data for the Spanish economy, in this case for the year 2010.

We split the procedure to build a SAM for Spain in 2010 in its different constituent steps. The first step would consist in obtaining a square input–output table, *I–O*, known as the Symmetric Input–Output Table (*SIOT*). If this information is not available, then we would use some technique to reconcile the flows contained in the so-called Use Table with those in the so-called Make Table (Miller and Blair 2009). Table 6.3a is a reduced version of the Spanish *SIOT* for 2010 compiled and published by the Spanish National Institute of Statistics (*INE*). However, the flows of the original *SIOT* are valued at basic prices, i.e. they include production costs plus taxes paid by corporations or firms independently of the amount sold or

Table 6.3 (a) Spanish *SIOT* at basic prices, 2010. (b) Spanish *SIOT* at market prices, 2010

(a)

	Agriculture	Industry	Services		Final demand at basic prices	Total demand at basic prices
Agriculture	1.944	26.246	2.856		20.643	51.689
Industry	11.786	415.642	113.757		506.130	1047.315
Services	3.419	129.046	323.724		764.501	1220.690
Total intermediate demand at basic prices	17.149	570.934	440.337	Total final demand and total uses at basic prices	1291.274	2319.694
Net taxes on Products	0.148	3.872	15.963		71.017	91.000
Total intermediate demand at market prices	17.297	574.806	456.300	Aggregate final demand and total demand at market prices	1362.291	2410.694
Value-added at basic prices	24.060	233.261	732.592			
Imports	10.332	239.248	31.798			
Total supply at basic prices	51.689	1047.315	1220.690			

(b)

	Agriculture	Industry	Services		Final demand at market prices	Total demand at market prices
Agriculture	1.827	27.659	2.371		33.976	65.833
Industry	12.659	427.475	127.477		468.692	1036.303
Services	2.811	119.672	326.452		859.619	1308.554
Total intermediate demand at market prices	17.297	574.806	456.300	Aggregate final demand and total demand at market prices	1362.291	2410.694
Value-added at basic prices	24.060	233.261	732.592			
Imports	10.332	239.248	31.798			
Tariffs	0.054	0.781	0.000			
Value-added tax	0.744	18.994	39.075			
Other net taxes on products	6.389	−150.394	175.355			
Total net taxes on products	7.187	−130.619	214.430	91.000	Aggregate Net Taxes on Products	
Transport and trade margins	6.957	119.609	−126.566			
Total supply at market prices	65.833	1036.303	1308.554			

Thousands of millions of Euros

produced. As a result, the original *SIOT* flows do not incorporate relevant information on indirect taxes at an industry or sectoral level, such as excise taxes, tariffs on imported products, and other ad-quantum or ad-valorem taxes levied on consumption activities. From the perspective of an applied general equilibrium modeler, it is crucial to have disaggregated information about these categories of indirect taxes. One reason is because taxes heavily influence the economy's price structure and this has to be adequately reflected. Another reason has to do with the research goal of analyzing the tax system and its possible reforms. The more itemized the information regarding indirect taxes, the more accurate will be the evaluated economy-wide impacts of the policy in question. Therefore, the second step in building a *SAM* will require as much additional information as possible on these indirect taxes and this usually calls for information from other statistical sources.

Once we have collected this additional information, the third step consists in transforming the transactions in the initial *SIOT* from basic prices to market prices. This step is optional but highly recommended because it eases the calibration of the model and the interpretation of the results. In fact, *SIOT* at market prices are rarely available. The work of Lucena and Serrano (2006) defines a sound procedure for this conversion while providing a helpful and detailed roadmap. We have followed their method to estimate the Spanish *SIOT* at market prices that is presented in Table 6.3b. From this Table we can see that the Spanish *SIOT* at market prices distinguishes three types of net indirect taxes: value-added taxes, taxes on imported commodities and the remaining net taxes on products, most of them excise taxes. In addition, when transactions are valued at market prices they must also include transport and trade margins. Although these transactions can be modeled and introduced in the general equilibrium model, in most cases they are 'redistributed' throughout the *SAM* using the *RAS* method or any similar adjustment approach (see Sect. 6.3). The practical explanation for this is that very often trade and transport margins are either incompatible with the assumptions of the model, i.e. perfect competition, or irrelevant for the typical research question.

The fourth step implies the incorporation of all transactions included in the *NIPA*. The *SIOT* gives us information about 'who' produces, 'what' and how much of that is produced, and 'how' it is technologically produced. On its part, a *SAM* incorporates information about the distribution of primary and secondary income among agents. A *SAM* compiles both the origin and destination of primary and secondary income for each agent or institutional account in the economy, i.e. households, government, corporations and the rest of the world. As stated before in this section, all this information can be taken from the *NIPA*. Table 6.4 shows total resources and total uses of income for the Households' category in 2010, as extracted from the Spanish *NIPA*.

We are reaching the fifth and final step. It consists in plugging in all the compiled information relative to income for all agents and accounts into the existing accounts of the *SIOT* at market prices. Recall that the whole goal in assembling a *SAM* is to have a description of the complete interplay of the circular flow of income. Table 6.5 illustrates a possible way for households' income sources and resources to be assigned to the different *SAM* accounts. If we look at this Table we can see that

Table 6.4 Households' Resources and Uses of Income

		Resources	Uses
Primary income	Gross labor income	542.399	
	Gross operating surplus and mixed income	163.310	
	Property income	49.542	17.955
Secondary income	Social contributions paid by households		42.972
	Social contributions paid by employers		112.833
	Direct or income taxes		79.871
	Welfare benefits	181.084	0.310
	Other transfers	46.438	52.091
	Final consumption		607.981
	Adjustment for the variation in pension funds participation	−0.178	
	Gross savings	68.582	
	Total resources = total uses	982.668	982.668

Spanish National Income and Products Accounts, 2010. Thousands of millions of Euros

Table 6.5 Assignment of households' resources and uses of income among *SAM* accounts

	Accounts paying for households' income resources	Accounts receiving households' income uses
Gross labor income	Labor account	
Gross operating surplus and mixed income	Net capital account	
Property income	Property income account	Property income account
Social contributions paid by households		Institutional accounts: government
Social contributions paid by employers		Institutional accounts: government
Direct taxes		Institutional accounts: government
Welfare benefits	Institutional accounts: government	Institutional accounts: government
Other transfers	Institutional accounts: government, households and the rest of the world	Institutional accounts: households and the rest of the world
Final consumption		Industries-sectors
Adjustment for the variation in pension funds participation	Investment account	
Gross savings	Investment account	

Households are the owners of all the labor and capital resources. Then these primary factors are hired by industries or sectors to carry out the production of goods and services and as owners of the resources, Households are compensated according to current market prices for these resources. As the reader probably knows, property income refers to the income generated (or paid) from the ownership (or rental) of natural resources, financial assets and capital equipment. If the objective of the general equilibrium analysis is, for instance, to evaluate the impact of a change in the economy's property tax, then this property income could be modeled as an additional factor of production along with labor and capital (Waters et al. 1997). For other research purposes, however, property income could be treated as 'a passive source of primary income', that is to say, dealt with differently from other primary factors, and in this case the general equilibrium model would not contemplate a specific market for this input.

Regarding resources and uses of secondary income, taxes paid by households refer to direct taxes on income, i.e. property taxes and personal income tax and labor taxes that are transferred to the government. The remaining part of secondary income relates to welfare benefits, i.e. unemployment benefits and retirement pensions, among other received transfers. Welfare benefits are mainly provided by the public sector while other transfers are usually granted by other households and the foreign sector (external remittances). Lastly, the sum of all net primary and secondary Households' income is either consumed or saved, the latter being used to finance investment activities.

6.2 Calibration

We have seen how a given model structure and a collection of parameters for the functional forms of the model give rise to a *SAM* as a depiction of an equilibrium state. In this section we take the opposite route and discuss the viability of coupling a model structure and a *SAM* to yield the necessary parameters for model implementation. The method of calibration, in contrast to statistical methods used to determine the parameters of an economic model, is essentially a deterministic procedure. Calibrating a general equilibrium model to a *SAM* means choosing parameters so as to reproduce the flows of the *SAM* as an equilibrium. To achieve this goal, the *SAM* for a given base period is used in conjunction with a chosen model structure and the restrictions implied by the assumption of utility and profit maximization followed by households and firms.

A problem in using the data in the *SAM* is that all data are expressed in currency units for the base period. Each entry of the *SAM* is a value entry and in general it is not possible to obtain separate information to distinguish the price components from the physical quantities. An ingenious way out of this problem is to redefine physical units in such a way that each unit is worth one unit of currency. This is accomplished simply by taking the observed value figures as though they were physical quantities. Of course, the new units need not have any kind of official existence and its purpose is solely one of yielding a convenient way to use the data in the *SAM*. Suppose we have an observation of a value figure V which we know is the product of price p and quantity Q, i.e. $V = p \cdot Q$. Suppose $p = 10$ €/kg and $Q = 100$ kg. so that the observed value is $V = 1000$ €. No matter what the new unit conventions for prices, p', and quantities, Q', are, it will still be true that $V = p' \cdot Q'$; now if we set $Q' = V$, then necessarily $p' = 1$ € and implicitly we are using a new unit of measurement, which in this example happens to be the decimal system unit known as *hectogram* (hg). Thus $Q' = 1000$ hg. and $p' = 1$ €/hg. Using the value data in the *SAM* for calibration is therefore akin to setting all underlying base prices equal to unity. This normalization is a real convenience which turns out to be very useful in applied modeling.

We now turn to specific examples of calibrating structural parameters for utility and production functions.

6.2.1 Calibration of Cobb–Douglas Utility Functions

Let us assume that consumer h has Cobb–Douglas preferences over the M consumption goods, that is:

$$u(c_1, c_2, \ldots, c_M) = \prod_{j=1}^{M} c_j^{\beta_j} \tag{6.2}$$

where we now omit the identifying household subindex h for notational simplicity. We can now set up the utility maximization problem for a consumer endowed with a net income level of R that faces a vector $q = (q_1, \ldots, q_j, \ldots, q_M)$ of consumption prices to obtain demand functions:

$$c_j(q, R) = \frac{\beta_j \cdot R}{q_j} \tag{6.3}$$

and from here:

$$\beta_j = \frac{q_j \cdot c_j(q, R)}{R} \tag{6.4}$$

Thus we can obtain β_j if we know net income R and the level of expenditure $q_j \cdot c_j(q, R)$ in good j. But this is information readily available from the *SAM*. For instance, for the Young consumer in the *SAM* of Table 6.12 in Appendix 1 we have:

$$\begin{aligned}
&R = \sum_k SAM(\text{‘Young’}, k) - SAM(\text{‘Government’}, \text{‘Young’}) \\
&-SAM(\text{‘Capital Acc.’}, \text{‘Young’}) \\
&q_1 \cdot c_1(q, R) = SAM(\text{‘Non-durables’}, \text{‘Young’}) \\
&q_2 \cdot c_2(q, R) = SAM(\text{‘Durables’}, \text{‘Young’})
\end{aligned}$$

and so on. For instance, the calibrated value of β_1 turns out to be:

$$\beta_1 = \frac{q_1 \cdot c_1(q, R)}{R} = \frac{169.21}{844.89 - 103.04 - 141.40} = 0.2818$$

Notice that we have opted for considering the expenditure on savings as external to the optimization problem of the consumer. The adaptation of the calibration to savings being derived from the utility maximization problem, as we explained in Chap. 4, is straightforward. The Cobb–Douglas utility indicators yield a particularly simple way for determining the utility parameters, but there is a clear modeling trade-off between their simplicity and their restrictive properties.

6.2.2 Calibration of Cobb–Douglas Production Functions

In the basic model firms produce the composite factor value-added combining two primary factors with a Constant Returns to Scale (i.e. $\alpha_1 + \alpha_2 = 1$) Cobb–Douglas technology

$$VA = \mu \cdot x_1^{\alpha_1} \cdot x_2^{\alpha_2} \tag{6.5}$$

where VA is value-added, x_1 represents labor, and x_2 represents the use of capital services, and again we omit the subindex for firms for notational simplicity. Here calibration requires determining μ, α_1 and α_2. The Constant Returns to Scale assumption implies that only two parameters, μ, and either α_1 or α_2, need to be specified.

If we set up the cost minimization problem:

$$\begin{gathered} Min\ \ \omega_1 \cdot x_1 + \omega_2 \cdot x_2 \\ \text{Subject to } VA = \mu \cdot x_1^{\alpha_1} \cdot x_2^{\alpha_2} \end{gathered} \tag{6.6}$$

the first order optimization conditions for the problem in (6.6) can be seen to imply:

$$\frac{\omega_1}{\omega_2} = \frac{\alpha_1 \cdot x_2}{\alpha_2 \cdot x_1} \tag{6.7}$$

Therefore:

$$\frac{\alpha_1}{\alpha_2} = \frac{\omega_1 \cdot x_1}{\omega_2 \cdot x_2} \tag{6.8}$$

The numerator and denominator of the right-hand side of (6.8) correspond to recorded transaction values that appear in the *SAM*, hence we can fix the ratio between α_1 and α_2 from the published data. This plus the *CRS* assumption determine the values for the alpha coefficients. To obtain the value for parameter μ, we need to use the Cobb–Douglas cost function obtained from solving problem (6.6)

$$C(\omega; VA) = \mu^{-1} \cdot \alpha_1^{-\alpha_1} \cdot \alpha_2^{-\alpha_2} \cdot \omega_1^{\alpha_1} \cdot \omega_2^{\alpha_2} \cdot VA \tag{6.9}$$

Since we are implicitly choosing units whose worth is a currency unit, in other words, we are taking $\omega_1 = \omega_2 = 1$ and $C(1; VA) = VA$, μ is easily computed as

$$\mu = \alpha_1^{-\alpha_1} \cdot \alpha_2^{-\alpha_2} \tag{6.10}$$

Notice that calibration entails the use of only two independent observations ($\omega_1 \cdot x_1$ and $\omega_2 \cdot x_2$, since $VA = \omega_1 \cdot x_1 + \omega_2 \cdot x_2$) to determine two independent parameters, say μ and α_1. The production parameters for the Cobb–Douglas value-added function for the first productive sector are as follows. First we check the appropriate values from the *SAM*:

$$\omega_1 \cdot x_1 = SAM(\text{'Labor'}, \text{'Agriculture'}) = 548.70$$
$$\omega_2 \cdot x_2 = SAM(\text{'Capital'}, \text{'Agriculture'}) = 1727.25$$

From these values we obtain the required ratio:

$$\frac{\alpha_1}{\alpha_2} = \frac{\omega_1 \cdot x_1}{\omega_2 \cdot x_2} = \frac{548.70}{1727.25} = 0.3177$$

Since $\alpha_1 + \alpha_2 = 1$ we find $\alpha_1 = 0.2411$ and $\alpha_2 = 0.7589$. Finally we obtain the value for the remaining parameter:

$$\mu = \alpha_1^{-\alpha_1} \cdot \alpha_2^{-\alpha_2} = 1.7373$$

and we have the complete production function for the composite factor that we call value-added in sector 1 ('Agriculture' in the *SAM*). Alternatively, once we have determined the α_j coefficients, we could derive the parameter μ from the definition of the production function in (6.6):

$$\mu = \frac{VA}{x_1^{\alpha_1} \cdot x_2^{\alpha_2}}$$

using the following values taken from the *SAM*: $VA = x_1 + x_2 = 548.70 + 1727.25 = 2275.95$. The reader will verify that we obtain exactly the same result.

We leave the calibration of the remaining two Cobb–Douglas production functions to the reader as an exercise.

6.2.3 Calibration of CES Production Functions

Suppose now that we wish to modify the value-added production function to encompass a wider range of technical substitution possibilities and for this we posit a Constant Elasticity of Substitution (*CES*) production function such as:

$$VA = \mu \cdot [(\alpha_1 \cdot x_1)^\rho + (\alpha_2 \cdot x_2)^\rho]^{\frac{1}{\rho}} \tag{6.11}$$

where $\rho = (\sigma - 1)/\sigma$ and σ is the elasticity of substitution between labor x_1 and capital x_2. Here, as with the Cobb–Douglas case, we can only use two independent data observations but there are now four parameters to be determined. We can reduce the degrees of freedom by noticing that (6.11) can be rewritten as:

$$VA = [(\widetilde{\alpha}_1 \cdot x_1)^\rho + (\widetilde{\alpha}_2 \cdot x_2)^\rho]^{\frac{1}{\rho}} \tag{6.12}$$

where $\widetilde{\alpha}_1 = \mu \cdot \alpha_1$ and $\widetilde{\alpha}_2 = \mu \cdot \alpha_2$. Hence, and without loss of generality, we can assume $\mu = 1$. But this still leaves three parameters and two observations and we need to select one of the parameters exogenously. The parameter ρ is the natural candidate to be fixed if there are independent econometric estimates available for the substitution elasticity.

We set up now the new cost minimization problem:

$$Min \ \ \omega_1 \cdot x_1 + \omega_2 \cdot x_2 \tag{6.13}$$

$$\text{Subject to } VA = [(\alpha_1 \cdot x_1)^\rho + (\alpha_2 \cdot x_2)^\rho]^{\frac{1}{\rho}}$$

to obtain from the first order optimization conditions:

$$\frac{\omega_1}{\omega_2} = \frac{\alpha_1 \cdot (\alpha_1 \cdot x_1)^{\rho-1}}{\alpha_2 \cdot (\alpha_2 \cdot x_2)^{\rho-1}} \tag{6.14}$$

Multiplying both sides of expression (6.14) by $(\omega_1/\omega_2)^{\rho-1}$ and rearranging we obtain:

$$\frac{\alpha_1}{\alpha_2} = \frac{\omega_1}{\omega_2} \cdot \left[\frac{\omega_2 \cdot x_1}{\omega_1 \cdot x_1}\right]^{\frac{\rho-1}{\rho}} \tag{6.15}$$

By our choice of units $\omega_1 = \omega_2 = 1$; besides, magnitudes $\omega_i \cdot x_i$ are known from the *SAM*, and we can therefore determine the ratio α_1/α_2. Further elaboration would yield the *CES* cost function:

$$C(\omega; VA) = \left[\left(\frac{\omega_1}{\alpha_1}\right)^\tau + \left(\frac{\omega_2}{\alpha_2}\right)^\tau\right]^{\frac{1}{\tau}} \cdot VA \tag{6.16}$$

where $\tau = \rho/(\rho - 1)$. By the choice of units we again have unitary factor prices, $\omega_1 = \omega_2 = 1$, and, in addition, $C(1; VA) = VA$. Expression (6.16) therefore yields:

$$\alpha_1^{-\tau} + \alpha_2^{-\tau} = 1 \tag{6.17}$$

and (6.15) and (6.17) allow us to determine the alpha parameters of the *CES* valued-added function for a given substitution elasticity σ. For the Spanish data, we now illustrate the calibration of the production parameters for the CES technology of sector 1 under a hypothetical value of the substitution elasticity of $\sigma = 0.5$. From this elasticity value, we derive the associated parameters $\rho = (\sigma - 1)/\sigma = -1$ and $\tau = \rho/(\rho - 1) = 0.5$.

We determine now the ratio implicit in expression (6.15) using the data from the *SAM* and the value of ρ:

$$\frac{\alpha_1}{\alpha_2} = \frac{\omega_1}{\omega_2} \cdot \left[\frac{\omega_2 \cdot x_2}{\omega_1 \cdot x_1}\right]^{\frac{\rho-1}{\rho}} = \frac{1}{1} \cdot \left[\frac{SAM(\text{'Capital', 'Agriculture'})}{SAM(\text{'Labor', 'Agriculture'})}\right]^{\frac{-1-1}{-1}} = \left[\frac{1727.25}{548.70}\right]^2 = 9.9092$$

Plugging this value into (6.17):

$$\alpha_1^{-\tau} + \alpha_2^{-\tau} = (9.9092 \cdot \alpha_2)^{-0.5} + (\alpha_2)^{-0.5} = 1$$

and solving for α_2 we obtain $\alpha_2 = 1.7363$ From here $\alpha_1 = 17.2050$ and the CES production function for sector 1 is fully specified.

6.2.4 Calibration in the Presence of Taxes

In the previous examples, both for the Cobb–Douglas and *CES* production functions, the calibration procedure is quite straightforward since we were conveniently omitting the fact that the magnitudes for labor and capital may be affected by taxation. What we did was to ignore the fact that the figures for *SAM*('Labor', 'Agriculture') and *SAM*('Capital', 'Agriculture') were gross of tax magnitudes. We used them as though taxes were not present. But in fact there is indirect taxation affecting the cost structure. We consider this possibility next, when we will consider the case of the government levying taxes on the use of labor and capital by firms. This is precisely the case in the Spanish *SAM*, where we can check that the 'Labor' and 'Capital' accounts make payments to the 'Government' account; in the case of the labor factor this payment correspond to the Social Security contributions paid by employers.

Let t_1 and t_2 denote the *ad valorem* tax rates on the use of labor and capital, respectively, so that the gross of tax prices for labor and capital are given by $\omega_1 \cdot (1 + t_1)$ and $\omega_2 \cdot (1 + t_2)$. For the Cobb–Douglas technology, the first order conditions for the cost minimization problem yield the equivalent expression of (6.8) above:

$$\frac{\alpha_1}{\alpha_2} = \frac{\omega_1 \cdot (1 + t_1) \cdot x_1}{\omega_2 \cdot (1 + t_2) \cdot x_2} \tag{6.18}$$

with the cost function being now

$$C(\omega; VA) = \mu^{-1} \cdot \alpha_1^{-\alpha_1} \cdot \alpha_2^{-\alpha_2} \cdot (\omega_1 \cdot (1 + t_1))^{\alpha_1} \cdot (\omega_2 \cdot (1 + t_2))^{\alpha_2} \cdot VA \qquad (6.19)$$

The numerator in (6.18) corresponds to gross of tax payments to labor, a known quantity from the *SAM* data. The denominator is also known and recorded in the *SAM*. This fixes the ratio α_1/α_2, which thanks to the *CRS* assumption determines the values of α_1 and α_2. On the other hand, calibration requires once again $\omega_1 = \omega_2 = 1$ and equating the cost of generating *VA* currency units of value-added to be, precisely, that level of value-added, i.e. $C(1; VA) = VA$.

Therefore from (6.19) we can now obtain the parameter μ as:

$$\mu = \alpha_1^{-\alpha_1} \cdot \alpha_2^{-\alpha_2} \cdot (1 + t_1)^{\alpha_1} \cdot (1 + t_2)^{\alpha_2} \qquad (6.20)$$

Notice that expression (6.20) reduces to expression (6.10) when the tax rates are zero. We now proceed to complete the calculation of all the parameters for the production function in sector 1. From (6.18) and *CRS* we will derive the coefficients α_1 and α_2:

$$\frac{\alpha_1}{\alpha_2} = \frac{\omega_1 \cdot (1 + t_1) \cdot x_1}{\omega_2 \cdot (1 + t_2) \cdot x_2} = \frac{SAM(\text{'Labor', 'Agriculture'})}{SAM(\text{'Capital', 'Agriculture'})} = \frac{548.70}{1727.25}$$

From here we again find $\alpha_1 = 0.2411$ and $\alpha_2 = 0.7589$. To obtain μ we need first to calculate the tax rates t_1 and t_2. These rates can be directly obtained from the *SAM* since the *SAM* contains data on the payments of factors to the government. The indirect tax rate on labor, for instance, can be easily calculated since we know total payments to labor and labor tax collections:

$$\begin{aligned} t_1 &= \frac{SAM(\text{'Government', 'Labor'})}{SAM(\text{'Labor', 'Agriculture'}) + \text{SAM}(\text{'Labor', 'Industry'}) + \text{SAM}(\text{'Labor', 'Services'})} \\ &= \frac{3994.34}{548.70 + 6051.82 + 9093.90} = 0.3250 \end{aligned}$$

A similar reading of the data for capital in the *SAM* would allow to obtain the second tax rate $t_2 = 0.0307$. Using all these values in expression (6.20) we find the value of the scale parameter $\mu = 1.9024$. Now, how can we be confident that the calibration is correct? The answer is surprisingly easy. If all has gone well the production function should exactly reproduce the data in the *SAM*. Let us check it is indeed so. The first thing is to record the figure for total value-added in 'Agriculture'. It is given by:

$$VA = SAM(\text{'Labor', 'Agriculture'}) + SAM(\text{'Capital', 'Agriculture'}) = 2275.95$$

The second step is to calculate value-added from the calibrated production function. For this we need the actual values for labor and capital as inputs, i.e. x_1 and x_2. For the case of the labor input, we know $\omega_1 \cdot (1 + t_1) \cdot x_1 = SAM$ ('Labor', 'Agriculture') = 548.70 and from here:

$$x_1 = \frac{SAM(\text{'Labor'}, \text{'Agriculture'})}{\omega_1 \cdot (1 + t_1)} = 414.1131$$

since we know the numerator, the tax rate for labor t_1 and the choice of units condition $\omega_1 = 1$ implicit in the calibration. Do the same for capital to find $x_2 = 1675.8187$. Plug now everything into the value-added production function:

$$VA = \mu \cdot x_1^{\alpha_1} \cdot x_2^{\alpha_2} = (1.9024) \cdot (414.1131)^{0.2411} \cdot (1675.8187)^{0.7589} = 2275.95$$

All checks, as promised!.

Turning to the *CES* production function, the first order conditions for cost minimization would now yield:

$$\frac{\alpha_1}{\alpha_2} = \frac{\omega_1 \cdot (1 + t_1)}{\omega_2 \cdot (1 + t_2)} \cdot \left[\frac{\omega_2 \cdot (1 + t_2) \cdot x_2}{\omega_1 \cdot (1 + t_1) \cdot x_1}\right]^{\frac{\rho - 1}{\rho}} \tag{6.21}$$

with the *CES* cost function being:

$$C(\omega; VA) = \left[\left(\frac{\omega_1 \cdot (1 + t_1)}{\alpha_1}\right)^{\tau} + \left(\frac{\omega_2 \cdot (1 + t_2)}{\alpha_2}\right)^{\tau}\right]^{\frac{1}{\tau}} \cdot VA \tag{6.22}$$

Set again $\omega_1 = \omega_2 = 1$, $C(1; VA) = VA$ and use the calibrated tax rates t_1 and t_2 to obtain from (6.21) and (6.22) the sought values for α_1 and α_2. The reader should verify that these values turn out to be $\alpha_1 = 22.7967$ and $\alpha_2 = 1.7895$. Armed with these *CES* coefficients and plugging them into the value-added *CES* production function (6.12) with labor and capital input values of $x_1 = 414.1131$ and $x_2 = 1675.8187$ yields exactly the recorded value-added for the 'Agriculture' sector in the *SAM*, i.e. $VA = 2275.95$, as it should be expected if the calibration has been correctly implemented.

6.2.5 Calibration of the Fixed Coefficients Technology

The specification of the production side of the economy is not complete until the input–output and conversion matrices A and B are determined. For each productive sector j total gross output y_j is given by the row or column sum total, a figure inclusive of foreign output. To compute the input–output matrix we need to net out imports from total output to obtain domestic output Q_j, since the input–output coefficients show intermediate requirement per unit of domestically produced output. Let $SAM(i, j)$ denote the value in row i and column j of the *SAM*. Using this notation, total domestic output, that is, the difference between total output and gross of tariffs imports is given by:

$$Q_j = y_j - M_j - R_{xj} = \sum_{i=1}^{20} SAM(i,j) - \sum_{k=17}^{20} SAM(k,j) \tag{6.23}$$

for the three productive sectors $j = 1, 2, 3$. Therefore the matrix A of input–output coefficients is obtained as:

$$A = \{a_{ij}\} = \left\{ \frac{SAM(i,j)}{Q_j} \right\} \tag{6.24}$$

for $i,j = l, 2, 3$. As an example, the coefficient a_{12} has the value:

$$a_{12} = \frac{SAM(1,2)}{Q_2} = \frac{2203.51}{37410.96 - 3216.67 - 249.20 - 2552.62 - 116.84} = 0.0705$$

The conversion matrix B is in turn given by:

$$B = \{b_{ij}\} = \left\{ \frac{SAM(i,j)}{\sum_{k=1}^{3} SAM(k,j) + SAM(14,j) + SAM(15,j)} \right\} \tag{6.25}$$

for indices $i = 1, 2, 3$ and $j = 4, 5, 6$. The denominator includes the tax cost (value-added tax and excise taxes) associated to the purchase of each of the consumption goods. As an example, we show the numerical value for coefficient b_{14}:

$$\begin{aligned} b_{14} &= \frac{SAM(1,4)}{SAM(1,4) + SAM(2,4) + SAM(3,4) + SAM(14,4) + SAM(15,4)} \\ &= \frac{993.28}{8063.85} = 0.1232 \end{aligned}$$

6.2.6 Calibration of Tax Rates

The structural utility and technology parameters need to be complemented with all the relevant fiscal parameters that appear in the utility maximization and cost minimization problems of households and firms. These parameters are crucial in determining the optimal consumption and production plans and constitute the basic policy vector for government tax intervention in the economy. All tax rates are computed as average effective tax rates based on actual tax collections by the government. At the calibrated tax rates total government revenue, as computed using the revenue function, must be equal to actual tax collections. Any existing tax evasion is considered to be neutral, that is, uniformly distributed among the different goods, services, and households.

We use the following notation for the different tax instruments computed from the recorded *SAM* tax data, with all sub-indices referring now to the position of the respective accounts in the empirical *SAM* of Table 6.12 in Appendix 1.

τ_j: Output tax rate on good $j(j=1,2,3)$
t_k: Factor tax rate on factor use $k(k=7,8)$
e_i: Excise tax rate on consumption good $i(i=4,5,6)$
vat_i: Value-added tax rate on consumption good i $(i=4,5,6)$
r_j^E: Tariff rate on European imports of good $j(j=1,2,3)$
r_j^R: Tariff rate on Rest of World imports of good $j(j=1,2,3)$
m_h: Personal income tax rate on household $h(h=9,10,11)$

We indicate first how to compute output tax rates:

$$\tau_j = \frac{SAM(16,j)}{Q_j - SAM(16,j)}(j=1,2,3) \tag{6.26}$$

with the numerator being taxes paid by firms in the production sector j and the denominator being domestic output, as calculated from expression (6.23), net of output taxes. We assume the two recorded factors, labor and capital, are homogeneous and taxed at a common sectoral rate given by:

$$t_k = \frac{SAM(13,k)}{\sum_{j=1}^{3} SAM(k,j) - SAM(13,k)} \tag{6.27}$$

the numerator being total labor tax collections and the denominator payments to labor net of labor use taxes. Recall that we already used the *SAM* data in the previous section to calculate these two rates.

There are two taxes on consumption, an excise tax on selected goods (such as alcoholic beverages and oil products within the non-durables account) and a general value-added tax which is applied at the end of the transaction. Both are levied as *ad valorem* tax rates and are cumulative. Hence we first need to net out all consumption taxes from gross of tax consumption expenditures to obtain, say, the excise tax rates and then net out only value-added tax collections to compute the value-added tax rates. Excise rates are therefore given by:

$$e_i = \frac{SAM(14,i)}{\sum_{j=1}^{3} SAM(j,i)}(i=4,5,6) \tag{6.28}$$

whereas value-added tax rates are computed as:

$$vat_i = \frac{SAM(15,i)}{\sum_{j=1}^{3} SAM(j,i) + SAM(14,i)} (i = 4,5,6) \tag{6.29}$$

Let us do the calculations for non-durables ($i = 4$) using the reported *SAM* data. The excise tax rate is calculated from:

$$e_4 = \frac{SAM(14,4)}{\sum_{j=1}^{3} SAM(j,4)} = \frac{73.05}{993.28 + 3837.61 + 21854.42} = 0.0098$$

or 0.98%. The value-added tax, in turn, is given by:

$$vat_4 = \frac{SAM(15,4)}{\sum_{j=1}^{3} SAM(j,i) + SAM(14,4)} = \frac{574.49}{7416.31 + 73.05} = 0.0767$$

To make sure that the tax calibration has been performed rightfully we will check that tax collections are correctly recomputed when using the calibrated rates. Total sales of non-durables before any consumption taxes are applied amounts to 7416.31. When the excise tax rate e_4 is applied over this magnitude we recover total excise tax collections, i.e. $73.05 = e_4 \cdot 7416.13$. Next we apply the derived value-added tax rate vat_4 over the sum of the net of taxes sale plus the excise tax, i.e. $574.49 = vat_4 \cdot (7416.31 + 73.05)$. Everything checks out fine.

Finally, tariff rates for the two trading partners, Europe and the Rest of the World are computed as the ratio of tariff collections to net of tariffs imports:

$$\begin{aligned} r_j^E &= \frac{SAM(18,j)}{SAM(17,j)} (j = 1,2,3) \\ r_j^R &= \frac{SAM(20,j)}{SAM(19,j)} (j = 1,2,3) \end{aligned} \tag{6.30}$$

The computation of the income tax rates can be a little trickier depending on what income sources are subject to taxation. In Spain, for instance, government transfers to consumers in the form of unemployment benefits have been tax free. In addition, social security contributions paid by employees are tax deductible. The computation of income tax rates should take into account these specifics so as to carefully define the taxable base. Since the level of disaggregation of the micro *SAM* does not offer detailed information on these matters, we will assume for simplicity that all income accruing to consumers is taxable and make no distinction between average and marginal rates. With these simplifying assumptions, we have:

$$m_h = \frac{SAM(13,h)}{\sum_{j=1}^{20} SAM(h,j) - SAM(13,h)} (h = 9,10,11) \tag{6.31}$$

Take the young consumer ($h = 9$). His total income tax payment to the government is reported as $SAM(13, 9) = 103.04$, whereas his income from all sources (sale of primary factors, transfers from the government and abroad) totals $\sum_{j=1}^{3} SAM(9, j) = 844.89$. The calibrated income tax rate for the young household is now quickly computed:

$$m_9 = \frac{103.04}{844.89 - 103.04} = 0.1389$$

Notice that all the computations related to calibration described above can be extended in a natural and systematic way to any higher level of disaggregation of the *SAM*.

6.3 Updating a *SAM*

In an ideal set-up we would have the availability of a *SAM* yearly. But what is ideal is not always realistic or cost effective. Assembling a *SAM* is costly and time consuming even for official statistical agencies. Data collection and processing is expensive and an excessive temporal lag in the production and publication of official data is quite often an unavoidable reality. In practical terms we should be happy if statistical agencies would produce their input–output tables, which are one of the core elements of a *SAM*, with short time delays. But unfortunately this is not the case and years end up passing before a new input–output data set is made available. Since up-to-date data is a critical element for the implementation of credible policy evaluations, researchers face a real challenge when such current data turns out to be absent. A way out of this conundrum is badly needed.

6.3.1 An Updating Procedure Based on Information Theory

When base data for assembling an up-to-date *SAM* is not available, we have the recourse of using updating procedures that permit the projection of a given known *SAM* for period t to an estimated *SAM* for period $t + 1$. Suppose that we know the "true" *SAM* for an initial period $t = 0$ and that we need, for modeling purposes, a new *SAM* for period $t = 1$. Let us make the following notational conventions and simplifications. Any given *SAM* will be denoted, in generic terms, by a $n \times n$ square matrix $\mathbf{A}$. The given "true" *SAM* at $t = 0$ will be denoted by $\mathbf{A}^0$. In period $t = 1$ there is also a "true" *SAM*, represented by $\mathbf{A}^1$, but unfortunately that *SAM* happens to be unobservable. The most we can do is to estimate a *SAM* for $t = 1$ using the known initial data $\mathbf{A}^0$ and whatever partial information pertaining to $t = 1$ is available. We will refer to this estimate of a *SAM* by $\hat{\mathbf{A}}^1$. The analytical question is how to perform the projection of the matrix $\mathbf{A}^0$ into the new matrix $\hat{\mathbf{A}}^1$.

For the initial *SAM* $\mathbf{A}^0$ we calculate the sum of the elements of all its rows and columns:

$$\sum_{j=1}^{n} a_{ij}^0 = z_i^0 \quad \text{for all rows } i = 1, 2, \ldots, n$$
$$\sum_{i=1}^{n} a_{ij}^0 = z_j^0 \quad \text{for all columns } j = 1, 2, \ldots, n$$

Each account in a *SAM* must satisfy the balance between total receipts (i.e. rows) and total outlays (i.e. columns), thus $i = j$ implies $z_i^0 = z_j^0$. The vector $z^0 = \left(z_1^0, z_2^0, \ldots, z_n^0\right)$ is called the vector of marginals for the *SAM* $\mathbf{A}^0$.

Suppose we do not know the true new *SAM* $\mathbf{A}^1$ but that we do know its vector of marginal values z^1. Can we use this partial information to obtain from $\mathbf{A}^0$ an estimate $\hat{\mathbf{A}}^1$ of the true but unknown and unobservable $\mathbf{A}^1$? The error in using $\hat{\mathbf{A}}^1$ instead of $\mathbf{A}^1$ can be measured by a distance (or loss) function $d\left(\mathbf{A}^1, \hat{\mathbf{A}}^1\right)$ such that the smaller the distance between them, the smaller the error in the estimation. But $\mathbf{A}^1$ is unknown and all the information we have is $\mathbf{A}^0$ and z^1. An option is to use $\mathbf{A}^0$ instead of $\mathbf{A}^1$ in the distance function and estimate $\hat{\mathbf{A}}^1$ in such a way that their distance is minimized subject to the new information z^1 being incorporated into the estimation procedure. Consider thus the following problem: find matrix $\hat{\mathbf{A}}^1 = \left(\hat{a}_{ij}^1\right)$ such that it solves:

$$\begin{aligned} &\text{Min } d\left(\mathbf{A}^0, \hat{\mathbf{A}}^1\right) \\ &\text{subject to} \\ &(1) \quad \sum_{j=1}^{n} \hat{a}_{ij}^1 = z_i^1 \quad (i = 1, 2, \ldots, n) \\ &(2) \quad \sum_{i=1}^{n} \hat{a}_{ij}^1 = z_j^1 \quad (j = 1, 2, .., n) \end{aligned} \tag{6.32}$$

The two conditions (1) and (2) impose that the estimated coefficients must add up to the known true vector of marginals z^1. We still need to provide some specifics regarding the distance function d. Using information theory as a foundational basis (Shannon 1948), a function that has been used is:

$$d\left(\mathbf{A}^0, \hat{\mathbf{A}}^1\right) = \sum_{i=1}^{n} \sum_{j=1}^{n} \hat{a}_{ij}^1 \cdot \ln\left(\frac{\hat{a}_{ij}^1}{a_{ij}^0}\right) \tag{6.33}$$

The natural logarithm part in the expression (6.33) is a measure of the information loss between the estimated elements in $\hat{\mathbf{A}}^1$ and the initial or prior elements in $\mathbf{A}^0$. There are n^2 elements in each matrix, and the double summation yields the expected information loss associated to the set of n^2 estimates with regard to the prior n^2 data. The minimization problem searches the values that minimize the expected information loss (see Robinson et al. 2001, for details). In other words, the problem aims at reducing the entropy (a measure of uncertainty) between the initial *SAM* and the estimated *SAM*. A few caveats are in order and deserve additional comments.

The first one is that the problem (6.32), as stated above, is fully equivalent to the well-known bi-proportional scaling algorithm known as *RAS* (McDougall, 1999).

The second one is that besides the main conditions (1) and (2) some technical restrictions must be taken into account when performing a solution to the problem. One is the non-negativity of all entries in the optimization problem. In case the initial *SAM* contains a negative entry (a subsidy, for instance) the corresponding cell must be relocated to its symmetrical position in the matrix, i.e. from position (i, j) to position (j, i), and its sign changed. This preserves aggregate values and guarantees non-negativity throughout. After a solution is found the operation should of course be reversed. Another technical restriction has to do with the preservation of zeros in the estimated *SAM*. There are two types of zeros in a *SAM*, hypothetical and conceptual zeros. A hypothetical zero results when an input is not used in the production of an output but could in fact be used under a different productive technology. A conceptual zero corresponds to the situation when it is not possible that a non-zero entry would ever arise. For instance, in no *SAM* labor is directly involved in the production of consumption nor in the generation of imports. The updating cannot be allowed to change this type of zero cells into non-zero ones. In both cases, however, we feel that the safest course of action is for the estimated *SAM* to inherit the structure of zeros of the prior *SAM*, unless strong empirical evidence suggests otherwise. Once the objective function is specified, as in expression (6.33), the complete minimization problem would take the form:

$$
\begin{aligned}
&\text{Min } d\left(\mathbf{A}^0, \hat{\mathbf{A}}^1\right) = \sum_{i=1}^{n}\sum_{j=1}^{n} \hat{a}^{\,1}_{ij} \cdot \ln\left(\frac{\hat{a}^{\,1}_{ij}}{a^0_{ij}}\right)\\
&\text{subject to}\\
&(1) \quad \sum_{j=1}^{n} \hat{a}^{\,1}_{ij} = z^1_i \quad (i = 1, 2, \ldots, n)\\
&(2) \quad \sum_{i=1}^{n} \hat{a}^{\,1}_{ij} = z^1_j \quad (j = 1, 2, .., n)\\
&(3) \quad \hat{a}^{\,1}_{ij} \geq 0 \text{ and } a^0_{ij} > 0 \ \text{ implies } \hat{a}^{\,1}_{ij} > 0
\end{aligned}
\tag{6.34}
$$

A final comment regarding the objective function is that other functions based on alternative mathematical philosophies, different from information theory, are also possible. For instance, we could borrow a well-known least square function from statistics as the minimand in the problem:

$$
d\left(\mathbf{A}^0, \hat{\mathbf{A}}^1\right) = \sum_{i=1}^{n}\sum_{j=1}^{n}\left(\hat{a}^{\,1}_{ij} - a^0_{ij}\right)^2 \tag{6.35}
$$

We could also use a similarity relation taken from information retrieval theory (Salton and McGill 1983) such as the cosine similarity. For any given two non-negative vectors x and y consider their angle $\boldsymbol{\theta}(x, y)$. The cosine of this angle, $\cos\boldsymbol{\theta}(x, y)$, is a measure of the similarity between x and y. The following can be easily seen; first, $0 \leq \cos\boldsymbol{\theta}(x, y) \leq 1$, second, $x = y$ implies $\cos\boldsymbol{\theta}(x, y) = 1$ (i.e. maximum similarity) and, finally, that if x and y are orthogonal them $\cos\boldsymbol{\theta}(x, y) = 0$ (i.e. maximum dissimilarity). If we compare columns in the initial *SAM* with columns in the set of possible *SAM* estimates, we are in fact comparing the closeness of the cost structures (Cardenete and Sancho 2004). By using cosine similarity and selecting the closest possible columns to the initial given columns we would be maximizing the similarity between $\mathbf{A}^0$ and $\hat{\mathrm{A}}^1$. In this case the minimand function would be:

$$d\left(\mathbf{A}^0, \hat{\mathbf{A}}^1\right) = -\sum_{j=1}^{n} \cos \boldsymbol{\theta}\left(\hat{a}_{ij}^1, a_{ij}^0\right) \tag{6.36}$$

The three reported objective functions, (6.33), (6.35) and (6.36), establish a relationship between the total flows in the prior matrix $\mathbf{A}^0$ and the update candidates $\hat{\mathbf{A}}^1$. A different approach would to use unitary coefficients instead of total flows. This would entail normalizing each column of a *SAM* by its column sum and replace total flows by coefficients in the minimands. For the initial *SAM* $\mathbf{A}^0$ we would define parameter coefficients $b_{ij}^0 = a_{ij}^0/z_j^0$ were vector z^0 is the prior marginal vector, and for the estimates we would use as variables the coefficients $\hat{b}_{ij}^1 = \hat{a}_{ij}^1/z_j^1$. For example expression (6.33) would become:

$$d\left(\mathbf{A}^0, \hat{\mathbf{A}}^1\right) = \sum_{i=1}^{n} \sum_{j=1}^{n} \hat{b}_{ij}^1 \cdot \ln\left(\frac{\hat{b}_{ij}^1}{b_{ij}^0}\right) = \sum_{i=1}^{n} \sum_{j=1}^{n} \left(\hat{a}_{ij}^1/z_j^1\right) \cdot \ln\left(\frac{\hat{a}_{ij}^1/z_j^1}{a_{ij}^0/z_j^0}\right) \tag{6.37}$$

Once the optimization program would return the solution in coefficients $\hat{b}_{ij}^1$, the estimated *SAM* would be built from $\hat{a}_{ij}^1 = \hat{b}_{ij}^1 \cdot z_j^1$. It is important to remark that the *SAM* thus derived would be different form the *SAM* projected using (6.33) as minimand, as Robinson et al. (2001) very clearly show. Proximity in flows and proximity in coefficients do not yield the same results.

6.3.2 An Example of *SAM* Updating

Would it not be nice if we could play a game and know all the cards beforehand? Well, we are in a position to do just that. Remember that the updating problem would not exist if we had the initial and final true *SAM*s, $\mathbf{A}^0$ an $\mathbf{A}^1$, for the two periods of interest. When $\mathbf{A}^1$ is not available but partial information for the period is, such as the vector or marginals z^1, the procedure outline above allow us to estimate a *SAM* that is chosen to be somehow close to the true but unknown one. Let us consider two true *SAM*s, the first one being the Social Accounting Matrix in Table 3.1. This *SAM* presents in tabular form the equilibrium magnitudes for the parameters describing the simple economy of the example. We reproduce here that information as Table 6.6.

We now build a new *SAM* by modifying some of the model parameters and recalculating the equilibrium. These are the changes that we introduce. We make both consumers richer in terms of their ownership of factors. The new endowments are:

Table 6.6 True Social accounting matrix $\mathbf{A}^0$

		1	2	3	4	5	6	Totals
1	Firm 1	20	50	0	0	15	15	100
2	Firm 2	30	25	0	0	35	10	100
3	Factor 1	40	10	0	0	0	0	50
4	Factor 2	10	15	0	0	0	0	25
5	Consumer 1	0	0	30	20	0	0	50
6	Consumer 2	0	0	20	5	0	0	25
Totals		100	100	50	25	50	25	

$$e_1 = (e_{11}, e_{21}) = (30 + 12, 20)$$
$$e_2 = (e_{12}, e_{22}) = (20, 5 + 3)$$

We introduce some productivity gains in the input–output matrix that now is:

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 0.20 & 0.50 - 0.10 \\ 0.30 - 0.05 & 0.25 - 0.02 \end{pmatrix}$$

Finally, we also increase the productivity of value-added by reducing the amount needed per unit of output:

$$\begin{aligned} v_1 &= 0.5 - 0.04 \\ v_2 &= 0.25 - 0.03 \end{aligned}$$

All the remaining parameters of the economy are kept at the same levels. With this new configuration, we use the *GAMS* code in Appendix 2 in Chap. 3, replacing these changed parameters, to solve for the new equilibrium and then we proceed to recalculate the *SAM*. The data for the new *SAM* appears in Table 6.7.

Now is when things become interesting. The next step is to estimate the *SAM* $\hat{\mathbf{A}}^1$ assuming that the only available information is the prior *SAM* $\mathbf{A}^0$ (from Table 6.6) and the new vector of marginals z^1 (i.e. the vector of Totals in Table 6.7). We now use the *GAMS* code in Appendix 2 to perform the updating of the *SAM*. Table 6.8 presents the estimate for the entropy objective function (6.33).

We also present the estimates for the least squares objective (6.35) and cosine similarity (6.36) objective functions in Tables 6.9 and 6.10, respectively. The reader can then appraise, at first sight, the numerical differences that these alternative goals produce in the *SAM* estimates. Observe that in the three cases the structure of zeros in the prior *SAM* is preserved and that all flows in the three alternative updates add-up to the known new marginal vector z^1.

Despite their expected numerical differences, consequence of the different goals being assumed, at first glance the three *SAM* estimates resemble each other quite substantially. More precise statements, beyond the first impression of a casual visualization, are possible. Since we are lucky to play this game with the advantage of knowing the true *SAM*, we can perform some ex-post checking to measure the closeness between the true *SAM* and the three estimates. For this purpose, we will

Table 6.7 True Social accounting matrix $\mathbf{A}^1$

		1	2	3	4	5	6	Totals
1	Firm 1	21.00	46.81	0	0	19.59	17.59	104.99
2	Firm 2	23.60	24.20	0	0	45.70	11.73	105.23
3	Factor 1	48.31	13.69	0	0	0	0	62.00
4	Factor 2	12.08	20.53	0	0	0	0	32.61
5	Consumer 1	0	0	42.00	23.29	0	0	65.29
6	Consumer 2	0	0	20.00	9.32	0	0	29.32
Totals		104.99	105.23	62.00	32.61	65.29	29.32	

Table 6.8 Estimated Social accounting matrix $\widehat{\mathbf{A}}^1$ using entropy minimization

		1	2	3	4	5	6	Totals
1	Firm 1	18.03	48.72	0	0	20.29	17.95	104.99
2	Firm 2	25.71	23.15	0	0	45.00	11.37	105.23
3	Factor 1	48.81	13.19	0	0	0	0	62.00
4	Factor 2	12.44	20.17	0	0	0	0	32.61
5	Consumer 1	0	0	38.69	26.60	0	0	65.29
6	Consumer 2	0	0	23.31	6.01	0	0	29.32
Totals		104.99	105.23	62.00	32.61	65.29	29.32	

Table 6.9 Estimated Social accounting matrix $\widehat{\mathbf{A}}^1$ using least squares minimization

		1	2	3	4	5	6	Totals
1	Firm 1	17.59	47.65	0	0	22.62	17.13	104.99
2	Firm 2	27.65	22.71	0	0	42.67	12.19	105.23
3	Factor 1	45.97	16.03	0	0	0	0	62.00
4	Factor 2	13.78	18.84	0	0	0	0	32.61
5	Consumer 1	0	0	38.74	26.55	0	0	65.29
6	Consumer 2	0	0	23.26	6.06	0	0	29.32
Totals		104.99	105.23	62.00	32.61	65.29	29.32	

use three different proximity indicators. These are the similarity indices of Le Masné (1990) and Chenery and Watanabe (1958), both of which are defined to compare matrices, and then a standard statistic such as the coefficient of determination R^2 (the square of Pearson correlation). See Appendix 3 for the specific formulaic details of each of these proximity metrics. In Table 6.11 we present the results of comparing the true *SAM* with each of the three estimates.

For all three estimating procedures, these measures are always within the $[0, 1]$ real interval. If it happened that $\hat{\mathbf{A}}^1 = \mathbf{A}^1$ then all proximity indicators would be 1 (i.e. maximum proximity). Thus the closer an indicator is to 1 the better the proximity between the true and the estimated matrices. Being close to 1 (maximum proximity) is good and being close to 0 (minimum proximity) is bad. The quantitative results of our updating example are satisfactory with high proximity measures in all cases. It is worth noting, however, that the entropy objective function

Table 6.10 Estimated Social accounting matrix $\widehat{\mathbf{A}}^1$ using similarity maximization

		1	2	3	4	5	6	Totals
1	Firm 1	18.14	48.79	0	0	20.34	17.72	104.99
2	Firm 2	27.03	21.64	0	0	44.95	11.60	105.23
3	Factor 1	46.57	15.43	0	0	0	0	62.00
4	Factor 2	13.24	19.37	0	0	0	0	32.61
5	Consumer 1	0	0	38.55	26.74	0	0	65.29
6	Consumer 2	0	0	23.45	5.87	0	0	29.32
Totals		104.99	105.23	62.00	32.61	65.29	29.32	

Table 6.11 Summary of proximity indicators between true $\mathbf{A}^1$ and estimated $\widehat{\mathbf{A}}^1$

	Le Masné	Chenery-Watanabe	Pearson R^2
Entropy	0.969	0.937	0.992
Least squares	0.953	0.905	0.987
Cosine similarity	0.960	0.919	0.989

produces dominant indicators, even if for a small margin, for all three measures, with cosine similarity coming always in second position.

Whenever the updating of a *SAM* becomes necessary, it seems that we can rely with a certain degree of confidence in using projections obtained through an optimization problem such as the one in expression (6.32), and these projections could save us the day! In a real-world updating procedure, however, we would not know the true *SAM* and we could not actually check the proximities. Ex-ante confidence should be accompanied with additional assurances that the updating procedures also offer robust results in modeling. Along these lines, Cardenete and Sancho (2004) examine the results of introducing a policy shock in a general equilibrium model calibrated using a collection of different *SAM* databases built from projection procedures such as the ones described in this section. They conclude that results are sufficiently robust and the choice of database, provided they represent the same underlying aggregate structure, should not be an issue for obtaining good policy evaluations.

6.4 Summary

The Social Accounting Matrix is a coherent disaggregated data base offering a detailed description of the structure of the economy in a given period. The *SAM* is in itself useful as an accounting device integrating statistical information from various, and sometimes conflicting, data sources such as the national accounts, the input–output tables, and the expenditure and budget surveys. The *SAM* is, in fact, a complete microeconomic picture of the circular flow of income. The emphasis here has been laid, however, on the use of the *SAM* as the numerical background for general equilibrium models and the procedure by which the data in the *SAM* is transformed into parameters of an empirical model. This is the so-called calibration method.

Calibration uses the data in the *SAM*, the structure of the model, and the restrictions from the utility and profit maximization problems to yield a set of model parameters and coefficients with the property that, once we plug them into the equations of the model, the *SAM* data base can be exactly reproduced as an equilibrium. Depending upon the structure of the pieces of the general equilibrium model, however, the data in the *SAM* may not be sufficient to generate all parameters. This is the case for the *CES* utility or production functions, where the elasticity values need to be given exogenously. These values can be suggested by the econometric literature, if estimates exist, or more crudely fixed in an *ad-hoc* manner based on educated guesses. When elasticities or other parameters are given from outside the *SAM*, the calibration procedure adjusts the remaining parameters so as to reproduce the benchmark data as an equilibrium.

The deterministic nature of the calibration method is sometimes seen as a drawback for applied general equilibrium modeling since it offers no information on the quality or reliability of the estimated parameters and, by extension, of the simulation results obtained using those parameters. The relevant question is, however, whether an alternative estimation procedure is available. Applied general equilibrium models are in general large in size and this makes classical econometric estimation difficult or simply unfeasible due to the lack of a large enough body of data. Using available elasticity estimates along with calibration is clearly a compromise solution dictated by the constraints on information, but one that has the advantage of combining a huge amount of structural information about the economy—the *SAM*—with selected exogenous parameters that reflect the statistical regularities governing the relationship between some model variables. A second consideration refers to the robustness, or lack thereof, of model results as a consequence of the particular way these models are numerically implemented. One option is to somehow randomize these models or more specifically, some of their parameters, and recheck the variability of the derived results in comparison to those arising from the benchmark simulations (Harrison and Vinod 1992; Harrison et al. 1993). Kehoe et al. (1995) and Lima et al (2016), in contrast, study the validity of applied general equilibrium models by way of comparing model results with real world developments. After updating some inner model characteristics to capture external change, these authors conclude that these models are sufficiently robust for the evaluation of tax policy scenarios. Finally, Cardenete and Sancho (2004) shift the attention from comparing databases to comparing equilibrium simulation results and show that model robustness follows satisfactorily under differently updated *SAM* databases.

6.5 Questions and Exercises

1. Use the functional forms of the simple model of Chap. 3 and the *SAM* in Table 3.1 to calibrate the simple model. Verify that the calibrated parameters coincide with the chosen parameters for the example (medium).
2. Assume a *CES* production function with *K* factors:

$$VA = \left[\sum_{i=1}^{K} (\alpha_i \cdot x_i)^{\rho} \right]^{1/\rho}$$

Show that in calibrating this function to observed data $(VA, \omega_i \cdot x_i)$ and exogenously given ρ (or τ), the coefficients α_i can be written as (hard):

$$\alpha_i = \left[\frac{VA}{\omega_i \cdot x_i} \right]^{1/\tau}$$

3. For the same *CES* function show that if a tax t_i is levied on the use of factor i, then (medium):

$$\alpha_i = \left[\frac{VA}{\omega_i \cdot (1 + t_i) \cdot x_i} \right]^{1/\tau} \cdot (1 + t_i)$$

4. Show that in the calibration of the Cobb–Douglas production function we could have obtained the parameter μ from the expression:

$$\mu = \frac{VA}{x^{\alpha_1} \cdot x^{\alpha_2}}$$

and verify that the numerical results using this expression coincide with those reported in the chapter (easy).

5. Suppose a consumer first makes a decision on how to allocate income between a composite good called present consumption and future consumption according to a *CES* utility function. Then the consumer decides how to allocate his/her present income among different consumption goods using a *C-D* utility function. Formulate the consumer's problem and derive the demand schedules for present and future consumption (hard).
6. Use the data of the Spanish *SAM* in Table 6.12 and the expressions from question 2 above to calibrate the parameters of the *CES* Armington specification (medium).
7. Reformulate the updating problem in (6.32) using an entropy function that projects the coefficients of matrix A^0 instead of its flows a_{ij}^0. For this you first need to introduce unitary coefficients defined by $b_{ij}^0 = a_{ij}^0 / z_j^0$ and redefine the problem. Adapt the computing code in Appendix 2 and compare the results in terms of derived *SAM*s (hard).

Appendix 1: An Empirical Social Accounting Matrix

Table 6.12 An empirical social accounting matrix

Table 6.3	1. Agr	2. Ind	3. Ser	4. Nondur	5. Dur	6. Ser	7. Lab	8. Cap	9. Young	10. Adult
1. Agriculture	568.63	2203.51	160.55	993.28	4.51	5.03	0.00	0.00	0.00	0.00
2. Industry	1233.60	14011.69	4918.11	3837.61	1316.15	3729.46	0.00	0.00	0.00	0.00
3. Services	362.84	2928.96	5365.41	2585.42	4173.81	7454.28	0.00	0.00	0.00	0.00
4. Nondurables	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	169.21	6826.56
5. Durables	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	134.88	4942.32
6. Services	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	296.36	11026.49
7. Labor	548.70	6051.82	9684.08	0.00	0.00	0.00	0.00	0.00	0.00	0.00
8. Capital	1727.25	5238.56	9093.90	0.00	0.00	0.00	0.00	0.00	0.00	0.00
9. Young	0.00	0.00	0.00	0.00	0.00	0.00	379.87	325.65	0.00	0.00
10. Adult	0.00	0.00	0.00	0.00	0.00	0.00	11910.39	13953.23	0.00	0.00
11. Retired	0.00	0.00	0.00	0.00	0.00	0.00	0.00	1302.61	0.00	0.00
12. Capital account	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	141.40	4401.64
13. Government	0.00	0.00	0.00	0.00	0.00	0.00	3994.34	478.20	103.04	4288.62
14. Excise tax	0.00	0.00	0.00	73.05	0.00	140.20	0.00	0.00	0.00	0.00
15. Value added tax	0.00	0.00	0.00	574.49	376.08	965.33	0.00	0.00	0.00	0.00
16. Other taxes	−92.29	841.09	86.04	0.00	0.00	0.00	0.00	0.00	0.00	0.00
17. Europe	133.82	3216.67	200.36	0.00	0.00	0.00	0.00	0.00	0.00	0.00
18. Tarifs EU	1.77	249.20	0.53	0.00	0.00	0.00	0.00	0.00	0.00	0.00
19. Rest of world	233.79	2552.62	342.84	0.00	0.00	0.00	0.00	0.00	0.00	0.00
20. Tarifs ROW	4.00	116.84	1.06	0.00	0.00	0.00	0.00	0.00	0.00	0.00
Totals	4722.11	37410.96	29852.88	8063.86	5870.55	12294.31	16284.60	16059.70	844.89	31485.63

11. Ret	12. CapAc	13. Gov	14. Ex-tax	15. VA-tax	16. Otax	17. EU	18. T-EU	19. ROW	20. T-R0W	Totals
0.00	217.02	0.00	0.00	0.00	0.00	498.24	0.00	71.34	0.00	4722.11
0.00	4631.41	0.00	0.00	0.00	0.00	2267.20	0.00	1465.73	0.00	37410.96
0.00	963.26	5132.00	0.00	0.00	0.00	439.08	0.00	447.82	0.00	29852.88
1068.09	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	8063.86
793.36	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	5870.55
971.45	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	12294.31
0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	16284.60
0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	16059.70
0.00	0.00	99.01	0.00	0.00	0.00	33.60	0.00	6.76	0.00	844.89
0.00	0.00	4146.13	0.00	0.00	0.00	1228.74	0.00	247.14	0.00	31485.63
0.00	0.00	3322.20	0.00	0.00	0.00	61.00	0.00	12.27	0.00	4698.08
1340.14	0.00	27.30	0.00	0.00	0.00	0.00	0.00	878.20	0.00	6788.69
525.04	0.00	0.00	213.25	1915.90	834.85	0.00	251.50	0.00	121.90	12726.64
0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	213.25
0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	1915.90
0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	834.85
0.00	977.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	4527.85
0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	251.50
0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	3129.25
0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	121.90
4698.08	6788.69	12726.64	213.25	1915.90	834.85	4527.85	251.50	3129.25	121.90	

Appendix 2: GAMS Code for the Updating of a Social Accounting Matrix

```
$TITLE SAM UPDATING: CHAPTER 6 Tables 6.5,6.6,6.7
OPTION NLP=CONOPT;
SET I accounts in SAM /1*6/;
ALIAS (J,I);

TABLE A0(I,J) prior SAM
        1       2       3       4       5       6
1       20      50      0       0       15      15
2       30      25      0       0       35      10
3       40      10      0       0       0       0
4       10      15      0       0       0       0
5       0       0       30      20      0       0
6       0       0       20      5       0       0;

PARAMETER Z1(I) new marginals
/1      104.99
 2      105.23
 3       62.00
 4       32.61
 5       65.29
 6       29.32 /;

VARIABLES
D               Minimand
X(I,J)          Estimated SAM cells
SICOS(I)        Cosine similarity;

D.L = 0;
X.L(I,J) = A0(I,J); X.LO(I,J) = 0;
SICOS.L(I) = 0.5; SICOS.LO(J)=0; SICOS.UP(J) = 1;

EQUATIONS
GOAL_E         Objective function: Entropy
GOAL_L         Objective function: Least squares
GOAL_C         Objective function: Cosine similarity
ROWSUM(I)      Row sum restrictions
COLSUM(I)      Column sum restrictions
ZERO(I,J)      Zero structure
SIMCOS(I)      Column similarity;
```

```
$ONTEXT

Three possible objective functions according
 to three closeness philosophies: entropy,
 least squares and similarity. The user can select
 which of the corresponding three models will
 be solved. Solution data is
 saved in a MS Excel sheet

$OFFTEXT

GOAL_E..      D =E= SUM((I,J)$A0(I,J), X(I,J)*LOG(X(I,J)/A0(I,J)));
GOAL_L..      D =E= SUM((I,J)$A0(I,J), SQR(X(I,J)-A0(I,J))) ;
GOAL_C..      D =E= -SUM(J, SICOS(J));

ROWSUM(I)..   SUM(J, X(I,J)) =E= Z1(I);
COLSUM(I)..   SUM(J, X(J,I)) =E= Z1(I);
ZERO(I,J)..   X(I,J)$(A0(I,J) EQ 0) =E= 0;
SIMCOS(I)..   SICOS(I) =E= SUM(J$A0(J,I),X(J,I)*A0(J,I))
                                       /
              SQRT(SUM(J$A0(J,I),SQR(X(J,I)))*SUM(J$A0(J,I),SQR(A0(J,I))));

MODEL ENTROPY        /GOAL_E, ROWSUM, COLSUM, ZERO/;
MODEL LEASTSQUARES   /GOAL_L, ROWSUM, COLSUM, ZERO/;
MODEL COSINE         /GOAL_C, ROWSUM, COLSUM, ZERO, SIMCOS/;

PARAMETER SAM(I,J) to upload updating result;

SOLVE ENTROPY MINIMIZING D USING NLP;
SAM(I,J) = X.L(I,J);
$LIBINCLUDE XLDUMP SAM UPDATES.XLS ENTROPY

SOLVE LEASTSQUARES MINIMIZING D USING NLP;
SAM(I,J) = X.L(I,J);
$LIBINCLUDE XLDUMP SAM UPDATES.XLS LEASTSQUARES

SOLVE COSINE MINIMIZING D USING NLP;
SAM(I,J) = X.L(I,J);
$LIBINCLUDE XLDUMP SAM UPDATES.XLS SIMILARITY
```

Appendix 3: Proximity Indicators

Consider two square matrices $\mathbf{A}^1 = \left(a_{ij}^1\right)$ and $\hat{\mathbf{A}}^1 = \left(\hat{a}_{ij}^1\right)$ such that for all $j = 1, 2, \ldots, n$ we have:

$$\sum_{i=1}^{n} a_{ij}^1 = \sum_{i=1}^{n} a_{ji}^1 = \sum_{i=1}^{n} \hat{a}_{ij}^1 = \sum_{i=1}^{n} \hat{a}_{ji}^1 = z_j$$

Define:

$$b_{ij}^1 = a_{ij}^1/z_j, \hat{b}_{ij}^1 = \hat{a}_{ij}^1/z_j, \omega_j = z_j/\sum\nolimits_i z_i, \bar{a} = \left(1/n^2\right) \cdot \sum\nolimits_{i,j} a_{ij}^1 \text{ and } \hat{\bar{a}} = \left(1/n^2\right) \cdot \sum\nolimits_{i,j} \hat{a}_{ij}^1.$$

By the way, notice that in our example we have in fact that $\bar{a} = \hat{\bar{a}}$. Then:
Le Masné (1990) proximity index with $LM \in [0, 1]$:

$$LM = \sum_{j=1}^{n} \omega_j \cdot \left(1 - 0.5 \cdot \sum_{i=1}^{n} \left|b_{ij}^1 - \hat{b}_{ij}^1\right|\right)$$

Chenery and Watanabe (1958) proximity index with $CW \in [0, 1]$:

$$CW = \sum_{j=1}^{n} \omega_j \cdot \frac{\sum_{i=1}^{n} \left|a_{ij}^1 - \hat{a}_{ij}^1\right|}{0.5 \cdot \sum_{i=1}^{n} \left(a_{ij}^1 + \hat{a}_{ij}^1\right)}$$

Pearson correlation with $R^2 \in [0, 1]$:

$$R = \frac{\sum_{i=1}^{n} \sum_{j=1}^{n} \left(a_{ij}^1 - \bar{a}\right) \cdot \left(\hat{a}_{ij}^1 - \hat{\bar{a}}\right)}{\sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} \left(a_{ij}^1 - \bar{a}\right)^2 \cdot \sum_{i=1}^{n} \sum_{j=1}^{n} \left(\hat{a}_{ij}^1 - \hat{\bar{a}}\right)^2}}$$

7 Real-World Examples of Applied General Equilibrium Analysis

In previous chapters we have described applied general equilibrium analysis in practical terms. In this chapter we begin by presenting a succinct explanation of the history of general equilibrium and the theoretical underpinnings of this type of analysis and models. We then move on to report on actual empirical applications to give the reader a flavor of the potential of *AGE* modeling.

An *AGE* model is based on general equilibrium theory, first developed by Walras (1874a, b) over a century ago and then elaborated and improved upon by many others such as Edgeworth, Arrow, Debreu, McKenzie, Gale, Scarf and others. In 1954, Arrow and Debreu introduced the first complete and general mathematical proof of the internal consistency of the Walrasian general equilibrium model. Over the past half-century the *AGE* model approach has been refined and applied to numerous economic problems, including impact evaluations of different possible policy alternatives.

General equilibrium models, of which *AGE* models are one variant, consider all parts of the economy and thus they are able to evaluate how an external shock will ripple throughout the economy causing shifts in prices and output levels until the economy absorbs the shock and reaches a different equilibrium. The responses by productive sectors, households, and agents in general, to the shock would thus determine its impact. By comparing the initial equilibrium values with the newly reached post-shock equilibrium, *AGE* models, as simplifying representations of the underlying real economy, can therefore be used to examine and quantify impacts retrospectively but they can be also used to predict change using scenario analysis. General equilibrium theory remained a conceptual framework for nearly 100 years. As such it has decisively influenced the state of the art in economics and has provided a very fruitful framework for pushing economics forward as a science. The practical implementation of *AGE* models, however, required tools that did not become widely available until recently, namely, first mainframe and later on personal computers and the specialized software that runs in them. These tools, along with the use of empirical data to give models numerical content, have given

M.A. Cardenete et al., *Applied General Equilibrium*, Springer Texts in Business and Economics, DOI 10.1007/978-3-662-54893-6_7

economists the ability to model something as complex as an economy in realistic ways that resemble its actual operation.

7.1 Theoretical Origins

We have to go back to the marginalists or Neo-classical School (school of economics active in the third quarter of the nineteenth century) to find the historical origin of general equilibrium theory. From the theoretical basis of this school, Gossen (1854), Jevons (1871) and Walras (1874a, b)—who pioneered the use of mathematical notation—and Menger (1871)—who did not—made the first steps in the development of the theory. The most relevant author among them, and the one who can be considered the father of general equilibrium theory, in its competitive version, is Walras (1874a, b).

As we have seen in Chap. 2, general equilibrium's simplest problem lies in the analysis of exchange economies. In this type of economy, a demander's budget restriction is established by his initial resource endowment and a set of prices. The individual demand function represents the equilibrium of the individual consumer confronted with different price levels. The market demand function is obtained as the aggregation of individual functions and market equilibrium emerges when we find as set of prices for which all excess demands (i.e. aggregate demand minus aggregate endowment for each good) equals zero. This idea was already expressed by classical economists when they stated that supply should match demand; Cournot (1838), in his discussion on international money flow and Mill (1848), in his arguments on international trade, had already advanced it. Nevertheless, its expression as a set of mathematical equations is due to Walras (1874a, b).

Years later, Pareto (1909) defined a normative property of market equilibrium, i.e. efficiency, that turned out to be critical since it linked the concept of market equilibrium with the property that guarantees that such market equilibrium cannot be improved upon for the whole set of agents. In a market equilibrium allocation no agent can be made better off without making some other agent worse off. The result that any competitive allocation is Pareto efficient has been known as the First theorem of welfare economics and it has been discussed in Chap. 2 in more detail. At the same time, a Pareto efficient allocation can be obtained as a competitive equilibrium provided some additional requirements are met. This is referred to as the Second theorem of welfare economics. The first formal approach to the relationship between competitive allocations and Pareto efficiency is due to Arrow (1951).

The following step in the development of general equilibrium theory was the extension from exchange economies to economies with production. Now the supply of goods includes both the non-produced endowments plus produced goods. As in the previous case, market equilibrium would be achieved when, for a set of prices, supply of goods by producers and households matched their demand by households and producers, respectively. Originally, Walras contemplated productive sectors with simple production, i.e. each industry produced just one good. The natural generalization of this simple production proposal included the introduction of joint production, a task completed by Hicks (1939). Previous to that, Cassel (1918) had developed a model with a productive sector, understood as a set of potential linear activities. He applied a simplified Walrasian model that preserved demand functions and production coefficients but did not deduce the demand functions from the utility functions or preferences. The model was later generalized by Von Neumann (1937) to allow the articulation of production in a spatial context. Koopmans (1951), in turn, made a more complete and elaborated analysis developing a model where intermediate products were explicitly introduced.

On the other hand, an alternative model of the productive sector was developed that emphasized productive organization at the firm level rather than activities or technology. The equilibrium condition in the productive sector was that each firm maximizes its profits, obtained as the value of the input–output combination on its production possibility set, given prices for inputs and outputs. This vision of production, made explicit in a partial equilibrium context by Cournot (1838), was already implicit in the work of Marshall (1890) and Pareto (1909) and became quite explicit in a general equilibrium context in the work of Hicks (1939) and, especially, in the Arrow and Debreu (1954) version of the model. Although Wald (1936) had already pointed out the role of Walras' law and had provided a proof of the existence of equilibrium, the version of Arrow and Debreu is the one that has been identified as the first complete, general and logically consistent proof of existence in a general equilibrium model. Using advanced mathematical techniques, in particular introducing the use of fixed point theorems in economics,[1] they formally proved the existence of equilibrium with productive sectors formed by firms. Each firm faced a set of production possibilities and selected—within its set of technical possibilities—the input–output combination that maximized profits at a given set of market prices. Their version of the general equilibrium model was, in addition, the first to directly include preference relations to derive the demand side of the economy. The modern versions of the general equilibrium theorems have completed Walras' original idea and have provided the basis for the far-reaching subsequent developments in economic theory, in particular, and economics in general.

7.2 From Theory to Applications

The great leap forward from the theoretical analysis to the applied dimension took place between 1930 and 1940, when discussions arose on the feasibility of calculating Pareto efficient allocations in socialist economies, which would be susceptible of being implemented by planners (see Von Mises 1920; Robbins 1934; Lange 1936; Hayek 1940). Leontief (1941) with his input–output analysis made the

[1]Recall the elementary proof of existence for exchange economies that we presented in Chap. 2 using Brouwer's fixed point theorem.

subsequent development, perhaps the most decisive step, in the attempt to endow Walras' theory with an empirical dimension and the definite capability for using it in the analysis of the effects of economic policies. Later on, the linear and non-linear planning models of the 1950s and 1960s, based on the works of Kantorovitch (1939), Koopmans (1947) and others, were seen as an improvement of input–output techniques through the introduction of optimization and as the first attempt to develop some sort of practical, or applied, general equilibrium.

The contribution of Scarf (1973), developing a computational algorithm to actually locate fixed points that satisfy the conditions of Brouwer's fixed-point theorem, was critical for the development of applied general equilibrium models. Going beyond the existence theorems, which guarantee that an equilibrium exists but provide no clue whatsoever about its specific values, Scarf's algorithm could be used to locate and approximate, with the desired degree of numerical tolerance, the solution to the equilibrium system in economic models.

Many of the first applied general equilibrium models used Scarf's algorithm for finding their solution. Some of the present models are still based on that method, although more quickly convergent variations based on developments by Merril (1971), Eaves (1974), Kuhn and McKinnon (1975), Van der Laan and Talman (1979) and Broadie (1983) are also used. From the latter, Merril's variation is the one most often used. Newton-type methods or local linearity techniques have been as well implemented. Even though convergence is not guaranteed, these last methods can be as quick, if not more, than the former.

Another approach, implicit in the work of Harberger (1962), consisted in using a linearized equilibrium system to obtain an approximate equilibrium and, in certain cases, to improve the initial estimator through multi-stage procedures so that approximation errors are eliminated. This method was also adopted by Johansen (1974), and improved by Dixon et al. (1982), de Melo and Robinson (1980), among others, who were among the first researchers elaborating applied general equilibrium models as such.

7.3 Computational Issues

The main problems faced by applied *AGE* modelers have changed substantially over time. As emphasized by Shoven and Whalley (1984), initially there was a lot of concern about the power of computational methods to find the solution to large and non-linear systems of equations. Nowadays, after the development of efficient algorithms and their implementation in powerful software, attention has switched towards the availability of reliable data for calibration, or towards systematic sensitivity analysis to evaluate the impact of different choices for parameter values. Furthermore, in an attempt to obtain more realistic model specifications, authors have incorporated novel assumptions: imperfect competition in product and factor markets, factor mobility between different spatial locations and structural equations related to inter-temporal optimization by firms and consumers.

There is therefore plenty of specialized software that, with the supervision of modelers, completely adjusts data, calibrates models to the data set and/or to externally given parameters, calculates equilibrium points, and delivers extensive reports to the end-user. The most popular include *GEMODEL*, *GEMPACK*, *GTAP* and, especially, *GAMS*, with all their different solvers, or solution algorithms adapted to the different models' necessities (i.e. database dimensions; multiregional, dynamic or static models). Given the advances in computational techniques, there is no doubt that at present the problem is not so much how to solve and find an equilibrium but, as in other fields of economics, how to obtain the good and up-to-date data which is necessary to specify the model parameters in an *AGE* model.

At the beginning of the 1980s, World Bank researchers developed *GAMS*, i.e. General Algebraic Modeling System. *GAMS* was specifically designed for modeling linear, nonlinear and mixed integer optimization problems. The system is especially useful with large, complex problems. *GAMS* is available for use on personal computers, workstations, mainframes and supercomputers. *GAMS* allows the user to concentrate on the modeling problem by making the general setup of an optimization problem as simple as possible but in a natural and logical way as well. The system takes care of the time-consuming details of the specific machine and system software implementation and shields the end-user from concerning himself with such details. Users can modify the formulation of a model quickly and easily, can switch from one solver to another—depending on the nature of the problem—and can even convert from linear to nonlinear with little trouble. Since *GAMS* is an optimization package, the solution of a nonlinear system of equations, such as those of *AGE* models, requires some adjustment. There are several possibilities to this effect. One is to rewrite the general equilibrium system of equations as a dummy nonlinear optimization program and use "as are" any of the nonlinear solvers in *GAMS*. *GAMS* solves the nonlinear program and in doing so yields the sought solution of the system. A second option is to directly use the nonlinear solver Constrained Nonlinear System (*CNS*). *CNS* does not require the program to be in optimization format but requires a careful accounting of equations and unknowns to keep the system square. The first option is usually much more flexible since it allows the use of derived or secondary variables and equations without affecting the internal consistency account of independent variables and equations. Finally, there is *GAMS/MPSGE*, a specialized general equilibrium module developed by Rutherford (1998, 1999). There are pros and cons to any of these options but even though the learning curve is steeper for algebraic *GAMS* than for *GAMS/MPSGE*, the added flexibility and the powerful options in regular *GAMS* are worth the extra time invested. The examples in previous chapters, for instance, have all been coded using standard *GAMS*.

GAMS has a fundamental advantage, namely, it lets users concentrate on modeling. By eliminating the need to think about purely technical machine-specific problems such as address calculations, storage assignments, subroutine linkage, and input–output and flow control, *GAMS* increases the time available for conceptualizing and running models, and for analyzing and reporting their results. As a programming language *GAMS* is very, very strict. Errors of all types (grammar related, in compiling, or in execution) are reported back to users. Until errors are all processed and corrected, *GAMS* is quite inflexible and will not allow users to proceed to the actual problem solving. The need for concise and precise specification of all model ingredients instills, by hook or by crook, good modeling habits in users. The *GAMS* language is formally similar to other commonly used programming languages. Hence, anyone with some programming experience can quickly become familiar with the intricacies and possibilities of *GAMS*.

7.4 Representative Uses of AGE Analysis and Models

One of the great advantages of general equilibrium models is their capacity to explain the consequences of incorporating major changes in a particular policy parameter or in a sector's characteristic in relation to the economy as a whole. It has been customary in economics to examine the economy-wide effect of some parameter change assuming that those changes are small and using linearized approximation based on relevant elasticity estimates. If the number of sectors is small, two-sector models as those used in international trade theory, are likewise employed. However, when considering a disaggregated model and the possibility of several changes, not necessarily marginal, taking place at once, then the natural option is to resort to the construction of applied general equilibrium numeric models for the economy under scrutiny and use the built model for comparative statics exercises. These counterfactual exercises can be undertaken safely, for small or large changes, provided *AGE* models have unique solutions. As pointed out by Kehoe and Whalley (1985) this seems to be the prevalent case in actual practice.

In reviewing some of the pioneer applications of this type of modeling, we will use a working classification of the main areas where applied general equilibrium models have been used and have had their greatest impact. These real-world uses of *AGE* models are briefly described in the following Sub-sections. For the record, we do not claim that these examples give a complete listing of all the applications and their authors—regrettably some omissions are inevitable—but we hope the reader will get enough of the flavor of what is possible and what has actually been done. See Table 7.1 in the Appendix.

7.4.1 Fiscal Policies

In the taxation area, from the first two-sector models of Harberger (1962) and Shoven and Whalley (1977) researchers have moved to modeling on much larger scales, like Piggot and Whalley (1977) did for Great Britain; Ballard et al. (1985) for the United States; Kehoe and Serra-Puche (1983) for Mexico; Keller (1980) for Holland, and Piggot (1980) for Australia, among others. More recently, Rutherford and Light (2002) evaluate the cost of raising additional government revenue from different tax sources in Colombia, Cardenete and Sancho (2003) used a *AGE* model to analyze the effects of the income tax reform in Spain at the regional level, Mabugu (2005) undertook a similar analysis for South Africa with a dynamic *AGE* model, while Ferreira (2007) examined poverty and income distribution issues following a tax reform.

Recently, Yusuf et al. (2008) have analyzed various aspects of fiscal policies in Indonesia and Xiao and Wittwer (2009) have used a dynamic *AGE* model of China with a financial module and sectoral detail to examine the real and nominal impacts of several policies, from using a nominal exchange rate appreciation alone to using fiscal policy alone, or a combined fiscal and monetary package, to redress China's external imbalance. The fiscal policy area is probably the area where this type of economic modeling has been more widely adapted and developed.

More recently, André et al. (2010) have studied the possible marriage of *AGE* modeling with Multi-Criteria Decision Making (*MCDM*) and have explored its potential for a better assessment of the design and implications of fiscal policies. They show how this area of fiscal policy analysis can indeed benefit from the merging of the joint capabilities of *AGE* modeling and the *MCDM* paradigm. The use of this new joint approach to fiscal policies creates new modeling challenges, but also allows for more realistic formulations and more pragmatic solutions to the design of public policies. This is especially so when more complex different fiscal policies—from the point of view of both income and/or expenditure—are considered simultaneously.

7.4.2 Trade Policies

The use of general equilibrium to study trade policies has revolved around the issue of protectionism and its consequences on an economy's efficiency and welfare levels. At the risk of oversimplification, trade models can be classified into two main groups. On the one hand, we find the small economy models whose main characteristic is full price endogeneity. On the other hand, we find large economy models that incorporate the assumption of price exogeneity in traded goods.

We can mention, among others, the global general equilibrium model developed by Deardorff and Stern (1986) which was used to evaluate policy options in the negotiation rounds at the *GATT* meetings. Dixon et al. (1982) built a large scale model for Australia, which has been widely used by government agencies to evaluate various trade policies concerning that country. There is also the group of models developed by the World Bank for different countries (Dervis et al. 1982). These models have provided key information for government authorities of borrowing countries, and have been useful in the assessment of different trade liberalization options for various developing countries. De Melo (1978) presents the core structure of the simulation models used for trade policy analysis. The economy-wide simulations that these models are able to produce have been instrumental for quantifying the tradeoffs implicit in the policy packages that the World Bank discusses with interested parties.

Piergiorgio (2000) studies the long-term effects of trade policy initiatives implemented by the European Union (*EU*) in the last decade, i.e. the European

Agreements, the Euro-Mediterranean Agreements and the Customs Union with Turkey. Vaittinen (2004), on his part, studies trade and integration at three levels: regional, national and global. The impacts of trade and integration have been evaluated in three different cases: Finland's accession to the *EU*, global trade liberalization set off by the World Trade Organization negotiations and the enlargement of the *EU* with new members from the former socialist central European countries. In all these cases, *AGE* models have been used as the preferred tool of analysis.

Narayanan et al. (2009) have developed a model that captures international trade, domestic consumption and output, using Constant Elasticity of Transformation (*CET*) and Constant Elasticity of Substitution (*CES*) structures, under market clearing conditions and price linkages, all nested within the standard *GTAP* model. Standardi (2010), for his part, has built a global *AGE* trade model at *NUTS*1[2] level (sub national level) for the *EU*15 regions. The focus is on the production side. The model is used to assess production reallocation across sectors in each *NUTS*1 regions after the adoption of an agricultural tariff liberalization policy. Likewise, it can also be used to simulate other trade policy reform according to the special objective of the researcher. The model is parsimonious in terms of data at the *NUTS*1 level. Unskilled and skilled labor inputs are the source of the heterogeneity across the *NUTS*1 regions. A stylized model is built in order to interpret the results. A sensitivity analysis on trade policy results according to two different degrees of skilled/unskilled labor mobility is conducted.

7.4.3 Stabilization Policies

The adverse external shocks experienced by most developed countries from the beginning of the 1980s, with falling exports, foreign trade losses, high interest rates and debt increments due to the *US* dollar appreciation, led, together with the decrease of trade bank benefits, to drastic adjustments. Subsequent adjustment programs were designed mostly separately by the *IMF* and the World Bank. These programs were characterized by emphasizing measure on both the demand side, when reducing short-term depressions, and on the supply side, to pursue greater efficiency through structural adjustments. The two components of the strategy (stabilization and structural adjustment) were not casually separated, partly due to the dimension of the required adjustments.

Macroeconomic models and standard general equilibrium models have proved inappropriate to analyze these problems. The high level of aggregation in macro models makes it difficult to use them for examining resource allocation shifts between different sectors and agents. On the other hand, in standard general

[2]The Nomenclature of Territorial Units for Statistics (*NUTS*) is instrumental in European Union's Structural Fund delivery mechanisms. The current *NUTS* classification valid from January 1, 2008 until December 31, 2011 lists 97 regions at *NUTS*1.

equilibrium models money is neutral and it only affects relative prices via *numé raire*. There is no theoretically satisfying way to study inflation, nominal wage rigidity or exchange rate nominal policies with traditional general equilibrium tools. For this reason, some economists have developed so-called "general equilibrium financial models". These models try to integrate money and financial assets into the multi-sector and multi-account structure of general equilibrium models. Despite these efforts, however, no consensus has yet emerged on how to introduce money and financial assets into general equilibrium models. Authors like Lewis (1994), who studied the case of Turkey, and Fargeix and Sadoulet (1994) for Ecuador, have contributed to this area of study. In turn, Seung-Rae (2004) goes a step forward suggesting the merger of optimal control models with dynamic general equilibrium models. This paper demonstrates the usefulness of *AGE* techniques in control theory applications and provides a practical guideline for policymakers in this relatively new field. Issues such as uncertainty, short-term quantity adjustment processes, and sector-specific political preferences are taken into account in exploring the optimal time paths of adjustments in an economy where the government adheres to those explicit policy goals. These provisional results highlight the importance of the structures of political preferences and uncertainty when performing optimal stabilization policy exercises. More recently, and with an empirical perspective for the Spanish economy, Alvarez (2010) has presented an *AGE* evaluation of a set of distinct government policies aimed at stabilization.

7.4.4 Environmental Analysis

Facing the reality of a progressive environmental degradation, in direct consonance with a worldwide increase in the demand for energy goods and energy related goods, has positioned the environment in the forefront of societies' worries. Over the past two decades *AGE* analysis has also been intensively used to studying the impacts of different policies relating to the environment. We will distinguish three such types of policies and classify the discussion according to them: water policies, CO_2 emission control policies and climate change policies.

In relation to water policies, Decaluwe et al. (1999) and Thabet et al. (1999) analyze the impact and efficiency of water prices setting. Seung et al. (1998) study the welfare gains of transferring water from agricultural to recreational uses in the Walker River Basin. Seung et al. (2000) combine a dynamic *AGE* model with a recreation demand model to analyze the temporal effects of water reallocation in Churchill County (Nevada). Diao and Roe (2000) analyze the consequences of a protectionist agricultural policy in Morocco and show how the liberalization of agricultural markets creates the necessary conditions for the implementation of efficient water pricing, particularly through the possibility of a market for water in the rural sector. Goodman (2000) shows how temporary water exchanges provide a lower cost option than the building up of new dams or the enlargement of the existing water storage facilities. In Gómez et al. (2004) we can find a nice summary of the use of several *AGE* models to study water issues. Given the important economic functions that water performs as an input, environmental engineers have been working to integrate traditional environmental modeling methods with the *AGE* approach. The journal Ecological Economics recently devoted an entire issue to integrated hydro-economic modeling emphasizing the role of *AGE* models [see in particular Van Heerden et al. (2008), Strzepek et al. (2008), Brouwer et al. (2008)].

In the second block of applications focusing on CO_2 emissions, many applied general equilibrium models have been successfully implemented in the last few years. Some of the applications of this type of analysis can be found in André et al. (2005) where an environmental tax reform is examined using regional data. An extensive sensibility analysis of the role played by substitution elasticities in CO_2 emission levels is presented in Sancho (2010) using an *AGE* model with price sensitive energy coefficients. While O'Ryan et al. (2005) analyze a range of environmental taxes in Chile, Schafer and Jacoby (2005) extend the analysis to transport accounts using a detailed specification of the different technologies available. Böhringer et al. (2001) focus in the role of efficiency gains in climate policies designed for the mitigations of multi-greenhouse emissions. Finally, Bergman (2005) and Turner et al. (2009) examine changes in environmental trade balances adopting an *AGE* modeling approach.

AGE models have also been recently used for the analysis of issues related to climate change. Kremers et al. (2002) review and compare different *AGE* models used in the evaluation of environmental policies designed to mitigate climate change. The effects of capital mobility in a global world model with trade are explored by Springer (2003). Nijkamp et al. (2005) use an environmental version of the *GTAP* model to study the impact of some international redress policies on climate change. The role of environmental taxation, structural change and market imperfections on climate change is studied in Böhringer et al. (2001). Roson et al. (2007) use a world *AGE* model to capture the effects on climate change of worldwide energy demand also under different market power scenarios. Finally, and most recently, the incorporation of dynamics in the *AGE* approach to examine climate change policies has been modeled in Eboli et al. (2009). A nice multi country set of studies, which use a common *AGE* model, regarding the trade-offs between growth and the promotion of free trade with the environment can be found in Beghin et al. (2002).

7.5 General Equilibrium and Multicriteria Programming

Traditional economic analysis builds on the principle that agents are considered to be rational. This idea is typically rendered as the assumption that they set out to optimize some objective function subject to some constraints. In this way, most economic problems can be expressed as particular cases of mathematical optimization. Recall how in Chap. 3 consumers were assumed to maximize their utility subject to their budget constraints, and firms were assumed to maximize their profits subject to technology and market environment constraints. This approach is appealing for at least two reasons. On the one hand, it provides a sound and logical way to think about decision making from a conceptual point of view. It establishes that economic agents pursue goals that they want to achieve at the highest possible levels but it also establishes that their actions are going to be restricted by market situations beyond their direct control. On the other hand, by resorting to mathematical optimization theory, it provides economic theory with a powerful and consistent analytical tool suitably matched to the way agents behave.

Following this trend, the design of public policies has also been traditionally envisioned as an optimization problem. Governments have goals too that they want to achieve and they are also restricted in their actions. Once a target is set, the next step consists in finding a tool to accomplish it. We call policies to these tools. Since not all possible policies are equally effective, the idea of choosing the best policy to attain a given goal becomes relevant. To this effect, finding an optimal policy has become a traditional economic problem. In practical terms, this means using an objective function that represents the values the government wants to promote and link it to a set of actions. The problem thus set up is then a typical optimization problem. We should choose the value of instruments that yield the best result for the realization of the sought goals. Typically, the problem reduces to the maximization of the utility function of a representative consumer as a proxy for welfare.

In real-world policy making, the government faces a certain degree of complexity in the sense that society's interests are multi-faceted. It is common to pursue different objectives which usually conflict with each other. In this light the design of public policies can be understood as a decision problem with several policy goals or objectives. From a purely economic viewpoint, an active employment policy is likely to exert upward pressure on prices; raising labor taxes is bad for employment but helps in sustaining the pension funds; expansive expenditure policies could be positive for economic activity but negative for the public budget and the incurred debt, and so on. In summary, it is not generally possible to find a policy that is beneficial for all the objectives at the same time.

n the 1970s some authors recognized that policy making was a multicriteria issue and made some attempts to connect multicriteria techniques with econometric models to devise policy recommendations (see Spivey and Tamura 1970; Wallenius et al. 1978; Zeleny and Cochrane 1973). Nevertheless, this branch of work was not very influential in economics. One likely reason for this lack of impact is the Nobel Prize winner Robert Lucas' well-known critique of the use of estimated (reduced-form) econometric models to predict the effect of macroeconomic policies. The main idea is that the estimated equations of a reduced form econometric model reflect a combination of the behavior of economic agents as well as the prevailing policy framework. As a consequence, those equations are not valid for predicting the effects of a policy that is different from the one that was being implemented when the model was estimated. Therefore, if one wants to evaluate the effects of alternative policy settings, it is not suitable to use a reduced form model. What is needed is a structural model specifying behavior functions for all the agents, and guaranteeing that the policy variables are clearly separated from the underlying fundamentals of economics (i.e. the technological structure and the agents' preferences).

Applied general equilibrium models are, without a doubt, structural models of the economy. Hence they offer a proper platform to integrate the multicriteria approach in the evaluation of economic policies. For the presentation of this integration we will rely on André et al. (2010). Figure 7.1 outlines the basics of this methodological research approach to public policy design. The government sets desirable objectives such as production increases, price stabilization, improvement in employment and the like and it has instruments such as different taxes and different tax rates, expenditure plans in public consumption and public investment, etc. But notice that these instruments can only perform their effects when going through the different agents that compose the economy and induce changes in the equilibrium. It is therefore the behavioral response of agents to these policies and their translation into new equilibrium values that will make said policies successful, or not.

To wit, the implementation of this multicriteria approach requires the following elements:

1. The identification of relevant policy objectives as measured by specific economic indicators. They can be micro or macro indicators, depending on the problem under analysis.
2. The determination of the policy instruments and the feasible range for those instruments.

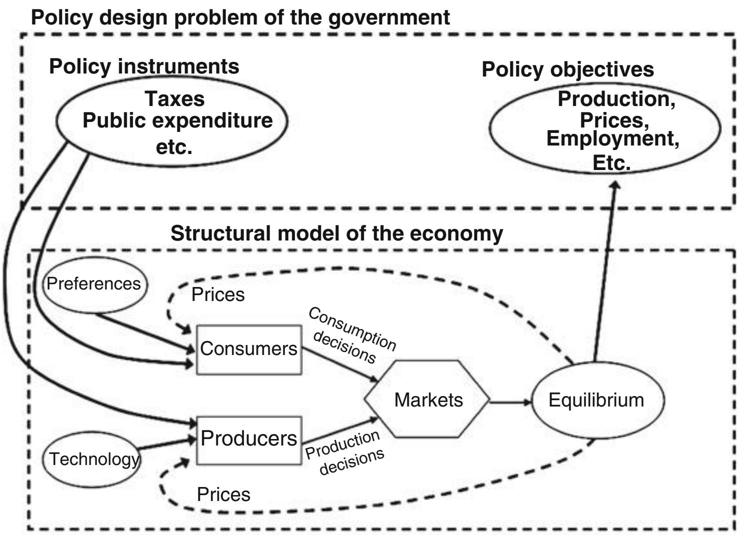

Fig. 7.1 Outline of the methodological proposal. Source: André et al. (2010)

3. The availability of a structural model that includes behavior functions for all economic agents and from which it is possible to calculate the equilibrium state and the levels of the policy objectives as a function of the policy instruments.
4. A reliable and consistent database in order to find the parameter values of the model by some estimation or calibration procedure.
5. Some multicriteria technique to be applied in order to handle the decision problem.

7.5.1 An Empirical Example

We now illustrate the potential of this integrated methodology, and for this we use some of the results of André and Cardenete (2009). In summary, the example uses multi-objective programming to identify efficient policies in terms of two indicators related to output and prices. Under any circumstances, more output is always a desirable goal. Similarly, price stability is also considered a good economic trait. Small price changes are thus preferable to large changes. The instruments are taxes and public expenditure, the two essential ingredients in fiscal policy. Using this technique it is possible to derive the so-called efficiency frontier and then compare it to real data from the Spanish economy. The comparison suggests that the observed situation is strictly above the frontier and, therefore, displays some degree of inefficiency. Let us see a few details.

A first piece of useful information is the so-called payoff matrix. The two chosen indicators are output gains and price changes, measured in percentage terms, which we denote respectively by γ and π. Any changes in output are explained exclusively by resource (re)allocation, resulting from the adoption of a policy. Price changes, in turn, are referred to the price of labor which is used as numéraire in the equilibrium model. Let us now consider the following question: if the government only cared for output improvements and price changes were ignored, what level of gains could be achieved using the available instruments and making them work through the structural model? In that case, what would be the resulting effect on prices? In actual policy making, however, the freedom of the government to change the level of its instruments is commonly restricted. Because of the grinding effects of the political machinery, big policy changes are hard to negotiate and difficult to implement. Assume then that sectoral expenditure in goods and services may change at most $\pm 20\%$ and assume too that tax rates may change but also within the same $\pm 20\%$. Take now a global neutrality assumption so that any individual changes must be such that total expenditure and total tax revenues remain constant. Given these two types of constraints regarding the policy instruments, the applied general equilibrium model is solved over the set of possible changes.

In a first solve, the model finds the values for the set of policy instruments that give rise to the highest possible output gains regardless of the price changes. The

Table 7.2 Payoff matrix with a lower bound for price changes

	γ: Output gains	π: Price changes
Max γ (unrestricted)	3.62	6.59
Max γ (subject to π bound)	1.57	0.50

All values in %
Source: André and Cardenete (2009)

values are $\gamma = 3.62\%$ for output change and $\pi = 6.59\%$ for price changes relative to the chosen model numéraire. In a second solve, the emphasis is shifted to minimize the effect of policy on prices without considering the effects on output. In this case, the model finds $\pi = -6.76\%$ and $\gamma = -9.69\%$. A fall in labor-measured prices of this magnitude is not a good thing, and even more so when it turns out that implies a huge output loss. These numbers are really bad as far as economic indicators go. Remember the proviso that small changes in prices are better than large changes. So we could fix an ad-hoc level of price changes that would seem reasonable, given the usual policy considerations. Let us assume that level to be $\pi = 0.50\%$, and let us ask the model again about the best mix of instruments that guarantee this level of price change. The price restricted solution yields now an output change of $\gamma = 1.57\%$.

The payoff matrix with a lower bound for price changes is reproduced in Table 7.2. The bold numbers represent the ideal values for each indicator when the other indicator is disregarded.

The data in Table 7.2 gives us two efficient points in the output-prices space, determined by the ideal values, namely (1.57, 0.50) and (3.62, 6.59). It is possible to complete the set of efficient points between them, at least approximately. The method involves constructing a grid of the feasible values of π, i.e. [0.50 to 6.59]. The number of points in the grid depends on how accurate one wishes the analysis to be. In this case, $n = 10$ values appear to be sufficient to provide a good approximation of the efficient set. Let π_n denote one specific value of π in the grid. For each of these values, we then solve the problem:

$$\begin{aligned} &\text{Max } \gamma \\ &\text{subject to}: \quad (1)\ \pi \leq \pi_n, \\ &\qquad\qquad\qquad (2)\ \text{all the equations of the general equilibrium model.} \end{aligned}$$

Figure 7.2 maps these calculations for a structural model of the Spanish economy in 1995, showing as well the ideal and anti-ideal solutions. We refer to the polygonal connecting all the efficient solutions as the *efficiency frontier*. Any combination above this frontier is considered as inefficient, as it entails either a higher price change for the same output gain (if compared with its vertical projection onto the frontier) or a lower output gain for the same change in prices (when compared to the horizontal counterpart in the frontier). All the combinations below the frontier are infeasible.

The slope of the efficient frontier can be understood as the policy trade-off or opportunity cost among objectives. Clearly, this slope is always positive, but it is

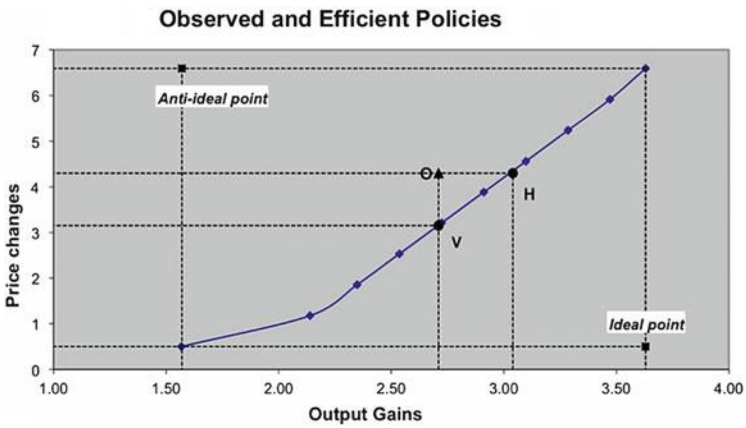

Fig. 7.2 Projecting the observed policy on the efficient frontier. Source: André and Cardenete (2009)

not constant in absolute terms. Indeed, the frontier can be roughly divided in two parts: the bottom segment (with low values for both indicators), and the top segment (with high values). The bottom segment slopes less than the top segment. This means that, if the output change rate is high, it takes larger increments in prices to attain additional points of output improvement. Alternatively, for small price changes, it would be more costly in terms of lost output to achieve additional reductions than otherwise.

In Fig. 7.2, point *O* shows the observed data in output and price changes in Spain in 1995: $\gamma = 2.71\%$, $\pi = 4.3\%$ (Source: *INE*, Spanish Statistical Institute). Since this point lies strictly above the frontier, a first impression conclusion is that policies enacted by the government are somewhat inefficient in terms of the selected objectives. Note that point *H* ("horizontal projection") provides the same change in prices with a strictly higher output gain (specifically, $\gamma_H = 3.02$, $\pi_H = 4.3$), whereas point *V* ("vertical projection") provides the same output gain with a strictly lower change in prices ($\gamma_V = 2.71$, $\pi_V = 3.15$). Of course, reality is usually quite more complex than any model can capture and the observed data is the end result of a myriad of factors beyond the result of the policies proper. Observed real-world indicators and the model counterparts will not be measuring exactly the same magnitudes. Hence some caution in this comparison and its interpretation is called for.

Despite these warnings for caution, it is nonetheless interesting to determine in which direction(s) the (fiscal) policies could be reformulated to improve the results in terms of efficiency. For this, we solve two optimization problems. The first maximizes γ subject to $\pi \leq \pi_H = 4.3$ (observed price changes). The second minimizes π subject to $\gamma \geq \gamma_V = 2.71$ (observed output change). Solving these formulations is equivalent to projecting point *O* onto *H* and *V*, respectively.

The results of these exercises are shown in Table 7.3. The column headed "Observed" shows the values of the policy instruments (public expenditure and five different tax types with sectoral rates) under observed conditions (resulting from calibration). The columns headed "Point *H*" and "Point *V*" present the changes that should be applied in order to move from the observed situation (*O*) to *H* and *V*,

Table 7.3 Values of policy instruments (observed and projected)

			Point H		Point V	
	Sector	Observed[a]	Value[a]	Change rate[b]	Value[a]	Change rate[b]
Public expenditure	5	3295	3954	20.00	3954	20.00
	6	119	143	20.00	143	20.00
	9	80,362	79,679	−0.85	79,679	−0.85
Value added tax	1	0.65	0.52	−20.0	0.52	−20.0
	2	1.30	1.04	−20.0	1.04	−20.0
	3	3.29	2.63	−20.0	2.63	−20.0
	4	2.28	1.82	−20.0	1.82	−20.0
	5	1.02	1.22	20.0	1.22	20.0
	6	1.42	1.71	20.0	1.71	20.0
	7	1.89	2.26	19.5	1.86	−1.7
	8	1.70	2.04	20.0	2.04	20.0
	9	3.61	2.89	−20.0	2.89	−20.0
Social security contributions by employers	1	11.17	8.94	−20.0	8.94	−20.0
	2	39.64	31.72	−20.0	31.72	−20.0
	3	36.22	28.98	−20.0	28.98	−20.0
	4	27.28	21.83	−20.0	21.83	−20.0
	5	32.33	32.73	1.2	29.57	−8.5
	6	28.52	34.23	20.0	34.23	20.0
	7	25.58	28.05	9.6	26.70	4.4
	8	23.28	27.94	20.0	27.94	20.0
	9	26.60	27.44	3.2	24.84	−6.6
Tariffs	1	0.15	0.15	0.0	0.15	0.0
	2	0.11	0.11	0.0	0.11	0.0
	4	0.57	0.56	−1.75	0.57	0.0
	5	0.56	0.66	17.85	0.56	0.0
	6	1.62	1.62	0.0	1.59	−2.2
	7	0.89	0.89	0.0	0.89	0.0
Income tax		10.29	10.75	4.5	11.47	11.5
Social security contribution by employees		6.50	5.17	−20.0	5.17	−20.5

Units: [a]Million Euros for public expenditure, and average percentage rate for taxes; [b]Percentage rate of change with respect to the observed value.
Source: André and Cardenete (2009)

respectively. In each case, the column labelled “Value” displays the value of each instrument, whereas the column named “Change” displays the rate of change with respect to the observed situation.

7.6 General Equilibrium and Dynamics

When we review the applied general equilibrium literature we find that most of the models developed over the years are static models. What this means is that these models analyze the state of the economy in a single period. When we examine the effects that different types of shocks—policy induced or not—exert on the equilibrium state, we do it using the technique known as comparative statics. Under this technique we compare two equilibrium states, before and after the considered shock takes place. However, in some empirical applications it may be interesting, or perhaps even needed, to glimpse the path over time that some key endogenous variables follow. For this we need a dynamic or multi-period model that concatenates these variables in a sequence of equilibrium states, one for each period being considered. With a dynamic model we are able to generate equilibrium trajectories.

Dynamic general equilibrium models provide another powerful tool for analyzing the impact of different policies in more or less complex representations of real-world economies. The wide array of theoretical models available today offers different possibilities in the design of a dynamic general equilibrium model with an empirical basis. Consumers are sometimes taken as infinitely lived and sometimes they have a finite span that overlaps with agents from a different generation. Sometimes there is just one consumer that represents the whole of the economy’s consumption side and sometimes we encounter heterogeneous consumers. Likewise with the production side, going from a single production function to multisectoral settings. The field of dynamic general equilibrium still has plenty of room for progress, for example with regard to expectations formation, learning mechanisms, uncertainty and the treatment of misspecified models. See Cardenete and Delgado (2015) for a more complete explanation of these issues.

All these dynamic models incorporate elements of growth through the changes that take place over time in the capital stock. The most common specification in the literature of dynamic general equilibrium adopts as starting point the Ramsey (1928) growth model with its infinitely lived consumer. The Ramsey model behaves differently depending on whether or not the solution picks up a stationary steady state. In a steady state key endogenous variables (capital, investment, output, etc.) all grow at the same rate. This initial model of Ramsey was later improved by Cass (1965) and Koopmans (1965). But it was not until the seventies, with the work of Scarf (1973) that dynamic general equilibrium models were strengthened and, like static models, their numerical solution could finally be computed. It was Johansen (1974) who, in a very simple way, developed the first model able to represent the economy of Norway in a dynamic setting. Another of the pioneers in using this dynamic approach was Harberger (1962), who examined the distortionary impact of a tax on capital in a model with two sectors.

More recently other time-dependent approaches have been incorporated to the growing pool of applied general equilibrium models. Under the heading of dynamic and stochastic general equilibrium (*DSGE*) we find models that study how the economy evolves over time (this is the dynamic part) while being subject to random shocks such as technological change, foreign price fluctuations, or macroeconomic policy intervention (this is the stochastic part). Under the overlapping generations (*OLG*) heading we have equilibrium models where agents live for a finite period of time that partially share with another generation of agents. Each generation lives in periods with schooling, work and finally retirement providing the natural framework to study resource allocation over time and across generations.

Since the nineties, these dynamic applied general equilibrium models have become popular for the analysis of numerous policy issues in various disciplines such as foreign trade policy, price control and optimal taxation, and changes related to climate policies. A few illustrative examples of applications follow but you should be aware that these examples are just a very small sample within the whole range of interesting possibilities. Thus, Jorgenson and Wilcoxen (1990) use a steady state dynamic model to explore the economic impact of government regulation in the United States. Vennemo (1997) uses a Ramsey type dynamic model to examine the effects of a reduced labor productivity induced by capital depreciation. Rasmussen (2001) extends the Ramsey model to capture endogenous technological progress while Gerlagh and Van der Zwaan (2003) use a similar approach to tackle multiple technologies. The contribution by Dissou et al. (2002) is noteworthy for the introduction of monopolistic competition in a Ramsey type model for Canada. As for climate and environmental issues, Hazilla and Kopp (1990) estimate the social cost of quality regulations mandated by clear air and clean water legislation; Blitzer et al. (1994) built a dynamic general equilibrium model for Egypt to analyze carbon emission restrictions in that country; Boyd and Ibarrarán (2009) quantify cross sectoral vulnerability, adaptation and mitigation policies, and climate change to measure the impact of drought on agriculture, forestry and hydroelectric sectors; finally, Bye (2000) discusses environmental tax reform and the possibilities of a double dividend with a dynamic Ramsey type model of Norway.

All applied general equilibrium models aim at providing insights regarding the way the economy works while at the same time attaching numerical estimates to those insights that allow us to quantify their role and weight. But this is only possible when we are able to calibrate the model to data. Hence good quality data is indispensable for the successful implementation of an applied general equilibrium model. However, the more complex the model, the more demanding its data requirements is. Adding layers of modeling complexity, if indeed it happens to be necessary at all, has an informational cost in terms of data. Unfortunately, the available economic data for model implementation is limited, even in the current state of expanding digital information. Static models require a staggering amount of economy-wide data for their implementation, and dynamic models require not less. A nice but complex modeling structure with a poor database must be put side by side with what an also nice but simpler model for which good data exists is capable to offer. The tradeoff between economic modeling architecture and economic data is, and will be for the foreseeable future, inevitable. This is the art of modeling that all researchers must learn if they want to survive with success in their field.

7.7 Summary

Despite the brevity of the exposition and the inevitably omission of many valuable scholarly references, we hope the reader will be by now sufficiently aware of the very broad range of problems that *AGE* models can tackle. The proof of their success is the ever increasing flow of new models and new applications and the corresponding technical reports published on academic journals. The adoption by government agencies of these models is also an indirect proof of their perceived usefulness.

Building these applied models is not always easy though. The issues to be examined have to be clearly defined; the context of analysis and the appropriate characteristics of the underlying economy have to be understood; adequate and up-to-date good data have to be available; flexible enough computer software has to be mastered and a proper translation of the economic model into computer code has to be checked and rechecked for overall consistency; last but not least, computer results have to be correctly interpreted within the logic of the specific economic model and general economic theory, a task which is not always easy but is critical for the credibility of these models. As an eminent researcher (Whalley 1985) once pointed out, experts in this field have to master different competences: in economics, in data acquisitions and manipulation, in computer programming and, of course, they had better have some good communication skills. As we hinted in the Introduction our goal in writing this book has been to provide readers with a road map and some basic expertise tips on how to proceed if readers become interested and motivated enough in pursuing this exciting *AGE* road.

Appendix

Table 7.1 Summary of specific AGE contributions

Type of analysis	Author/s	Area of application
Fiscal policy analysis	Harberger (1962)	2 × 2 × 2 model
	Shoven and Whalley (1977)	2 × 2 × 2 model
	Piggott and Whalley (1977)	UK
	Keller (1980)	Netherlands
	Piggott (1980)	Australia
	Kehoe and Serra-Puche (1983)	Mexico
	Ballard et al. (1985)	USA
	Rutherford and Light (2002)	Colombia
	Cardenete and Sancho (2003)	Spain
	Mabugu (2005)	South Africa
	Ferreira (2007)	Brazil
	Yusuf et al. (2008)	Indonesia
	Xiao and Wittwer (2009)	China
	André et al. (2010)	Spain
Trade policy analysis	De Melo (1978)	Developing countries
	De Melo and Robinson (1980)	Developing countries
	Dixon et al. (1982)	GATT
	Dervis et al. (1982)	Developing countries
	Deardorff and Stern (1986)	GATT
	Piergiorgio (2000)	EU-Mediterranean
	Vaittinen (2004)	Finland
	Narayanan et al. (2009)	World Bank
	Standardi (2010)	EU-15
Stabilization policy analysis	Lewis (1994)	Turkey
	Fargeix and Sadoulet (1994)	Ecuador
	Seung-Rae (2004)	Dynamic AGE
	Alvarez (2010)	Spain
Environmental analysis	Seung et al. (1998)	Water
	Decaluwe et al. (1999)	
	Thabet et al. (1999)	
	Seung et al. (2000)	
	Diao and Roe (2000)	
	Goodman (2000)	
	Gómez et al. (2004)	
	Van Heerden et al. (2008)	
	Strzepek et al. (2008)	
	Brouwer et al. (2008)	

(continued)

Table 7.1 (continued)

Type of analysis	Author/s	Area of application
	Böhringer et al. (2001)	CO_2 emissions
	O'Ryan et al. (2005)	
	Schafer and Jacoby (2005)	
	Bergman (2005)	
	André et al. (2005)	
	Turner et al. (2009)	
	Sancho (2010)	
	Beghin et al. (2002)	Climate change
	Kremers et al. (2002)	
	Springer (2003)	
	Nijkamp et al. (2005)	
	Böhringer et al. (2006)	
	Roson et al. (2007)	
	Eboli et al. (2009)	

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M.A. Cardenete et al., *Applied General Equilibrium*, Springer Texts in Business and Economics, DOI 10.1007/978-3-662-54893-6