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Understanding Auctions

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Foreword

To the bidder in a traditional ascending auction-the sort conducted by major auction houses like Sothebys and Christies and online by eBay-deciding how to bid might seem like a simple affair. If a bidder is interested in just one particular item and knows the maximum price he or she is willing to pay, then bidding strategy can be pretty simple: buy the item if the price does not rise above that maximum.

Looking deeper, though, matters quickly get complicated for both the bidder and the auctioneer. Suppose, for example, that the bidder wishes to buy a particular type of wine and there are multiple lots of the wine offered in the auction. How high should she bid for the first lot? If she bids up to her maximum for the first lot, she may miss out on a bargain on a later lot. Or, suppose that the seller has the option not to sell and instead to keep the item for a later auction. Should he set a high reserve, even if doing so might discourage some participants? Should the reserve be public or secret? Or suppose that a bidder for antiques is bidding against a dealer who is expert about resale values. How should the bidder account for the *winner's curse*, which is the risk that she wins mostly inferior pieces on which better-informed bidders choose not to compete?

The more one looks at auctions, the more apparent it becomes that there is much to study. Lots are fixed in most auctions, but bidders in Dutch flower auctions are permitted to decide what quantities they want to take. Why is that? Why are Treasury bills almost always sold by sealed bids, while cattle are mostly sold in a series of ascending auctions? In auctioning the assets of bankrupt businesses, the auctioneer can offer the assets individually, or accept only *entirety bids* for all the assets together, or take both kinds of bids. How does it decide which to do? What strategies are used to bid for rights to the radio spectrum used for mobile services, when a business can use licenses at different frequencies to support alternative business plans?

Until now, the best answers scholars have offered to these sorts of questions have been expressed mainly in the abstract language of mathematics, putting them out of the reach of the average reader. This book fills an important gap by making the main ideas and findings of auction research accessible. It accomplishes that by doing two things other books about the economics of auctions haven't: First, it uses ordinary language and avoids unnecessary jargon, while still defining the terms needed for a precise exposition of the key ideas and findings. It introduces ideas and terms in the context of real-world problems to engage the readers' intuition and bring ideas to life. Second, the book is filled with quantitative examples that show readers how and why applications of its ideas are important.

Auction researchers have applied a variety of methods, including theoretical, experimental, empirical, and computational ones. In a theoretical analysis of auctions, the researcher may represent any particular auction as a game played according to very precise rules and then apply game theory to make predictions about what bidders will (or should) do and how results are likely to vary with changes in the rules. Experimental researchers may simulate auctions in an economics laboratory, usually with undergraduate students acting as the bidders, or may attempt to conduct experimental tests "in the field." For the latter, the researcher might offer to sell randomly selected items on different auction websites that use different rules. The bidding data can then be analyzed to compare how bidders behave with different rules and how this may lead auction outcomes to vary. In traditional empirical research, the researcher does not design and conduct experiments, but instead analyzes data from auctions run by others. One can find such studies in recent years using data from auctions from eBay, from wholesale autos, from fish, Treasury bills, radio spectrum, timber, and more. Many of these studies have contributed new insights about how auctions in particular industries work in practice. When auctions are introduced into a new setting, it can be unclear which lessons from past auctions are likely to apply. Then, computations to simulate the performance of a given set of rules in the new setting can sometimes be of great value.

The sheer volume of auction research can seem daunting, but ***Understanding Auctions*** is an accessible introduction to and summary of the most important parts, written in engaging language by two scholars who know their subject.

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Paul Milgrom
Shirley and Leonard Ely

Preface

The term *auction* covers a wide range of market mechanisms that are used to exchange products and services by determining who receives an item and how much is paid for it. Although there are different auction designs, all of them share two properties. First, they are universal, that is, they can be used to sell (or buy) anything. Second, auctions are anonymous, which means that the identities of the participants do not affect the outcome of the auction (see [41]).

The aim of this book is to simplify the task for readers who want to approach the vast and unknown world of auctions. Traditionally, the word auction evokes the classic art auction in which participants increase the price by placing their bids. More recently, the Internet, with numerous websites dedicated to auction sales, has made them more common. However, these auctions account for only a small fraction.

This manual introduces the reader to auctions, to raise awareness of the importance of the sector and the diversity of auction models that exist. We offer a detailed description of how the main auction models work, which allows interested readers to understand them step by step. To accomplish this, in each chapter the theoretical analysis is accompanied by practical examples. The early chapters begin with explanations of the basic terms of auction literature and the simplest models by which an item may be offered. Subsequently, we analyze more sophisticated auctions that offer multiple items. Note that it is important to read this manual sequentially; if it is read out of order, certain chapters will be difficult to understand. The chapters included in the book are as follows:

- **Chapter 1: Auction basics**
  In this chapter, we briefly review the historical evolution of auctions. Then, the chapter focuses on important concepts such as: value, bid, price, bidding strategies, reserve price, and opening bid.
- **Chapter 2: Standard single-unit auctions**
  The second chapter is devoted to the study of the four standard auction models of one item: ascending (English), descending (Dutch), first-price sealed-bid, and second-price sealed-bid auctions. We also analyze the equivalence among these four models and discuss the implications of the Revenue Equivalence Theorem.

- **Chapter 3: Other single-unit auctions**
  In addition to the four standard auction models included in Chap. 2, there are other options for awarding one item. For example, the seller can combine the properties of different models, set different pricing rules, different ending rules, or use hybrid auctions. These alternatives are studied in the third chapter, where we also explain the difference between optimal auctions and efficient auctions.
- **Chapter 4: Assigning multiple homogeneous items in a single auction**
  The fourth chapter introduces environments in which multiple items are auctioned. In this section, we consider only those cases in which identical items are offered in a single auction. We study both sealed-bid auctions (uniform price, discriminatory price, and Vickrey auctions) and dynamic auctions (English, Dutch, and Ausubel), as well as other hybrid processes.
- **Chapter 5: Sequential and simultaneous auctions**
  In this chapter we study multi-unit auctions for allocating homogeneous and heterogeneous items. We discuss situations in which items are not allocated in the same auction but sold at sequential or simultaneous auctions.
- **Chapter 6: Double auctions**
  In this chapter we introduce the concept of the double auction, which involves multiple buyers submitting bids to be matched with the multiple asks submitted by sellers.
- **Chapter 7: Introduction to combinatorial auctions**
  In the seventh chapter, we introduce combinatorial auctions, in which a seller offers multiple related items in a single auction, and bidders can bid on any combination of items (packages) as well as on individual items. These auctions are particularly relevant when substitutes and complements items are involved. This chapter also explains the winner determination problem (WDP), that is, how to calculate the winners of a combinatorial auction, as well as the implications of choosing a particular pricing rule.
- **Chapter 8: Combinatorial auction models**
  Like any other auction, a combinatorial auction involves setting variables such as bidding format, bid increments, and pricing rules. This chapter explains some of the principal designs of both sealed-bid and iterative combinatorial auctions, such as the ascending proxy auction, the clock-proxy auction, and the combinatorial clock auction.
- **Chapter 9: Online auctions**
  In recent years, Internet auctions have had strong growth in many markets. In this chapter, we focus on reviewing the main models of online auctions, and we examine the strategic behavior of both buyers and sellers.

Those readers who are interested in exploring the world of auctions can continue their study with different manuals. Krishna [41] is a root reference regarding the theory of auctions. The author does an excellent job explaining the main concepts and theoretical developments of auctions, both for single-unit and multiple-unit environments. Klemperer [39], consisting of two volumes, is a compilation of the main articles on the subject in their original form. The same author later published a manual that explains the basics of auction theory with reference to previous research, complementing the theoretical analysis with practical analysis of the radio spectrum licenses auctions held by many European governments (see [40]). In the same year Milgrom [52] published an excellent synthesis of the contributions to the field. The book also combines theoretical analysis with practical problems. For those readers interested in double auctions, the manual by Friedman and Rust [31] is indispensable. To go deeper in combinatorial auctions, the book by Shoham et al. [20] is an excellent guide.

The manuals mentioned herein are an important foundation for the further study of auctions. However, this field is characterized by continuous development. Therefore, to maintain up-to-date knowledge of the sector, it is essential to keep abreast of the latest contributions.

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April 2014

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The world of auctions is much more extensive and complex than it may initially seem. The common understanding is based on the concept of classic ascending auctions, which are relatively simple. However, auctions have undergone a remarkable evolution, and they are becoming more widespread. This book is intended as an introductory guide to auctions that will cover fundamental concepts, from the simple to the complex. This project would not have been possible without the participation of several friends and collaborators, who we thank sincerely for their contributions. First, we express our thanks to Professor Francisco Mochón, who, with his extensive experience as a writer, made numerous contributions to the organization and writing. We also thank Ana Ancochea for her impartial and valuable editorial work. To Juan Francisco Pagés, Fernando Savirón, Daniel Franco, and Nicole Antoinette Büttner, we extend our thanks for generous corrections, comments, and contributions that have been key to creating a much more didactic and illustrative text. We thank Professor Pedro Isasi and Jose Luis Gómez, original members from our team, for their valuable and unconditional support. Thanks to Silvia Console Battilana and Auctionomics for giving us the opportunity of approaching real high stakes auctions and accepting us as members of their team. Finally, a very special thanks to Professor Paul Milgrom, auctions undisputed expert, for his ability to make the complex simple. With love and dedication, he has become our master.

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Contents

Acronyms

CA	Combinatorial auctions
CCA	Combinatorial clock auction
CPC	Cost per click
CDA	Continuous double auction
EP	Eligibility points
HRB	Highest-rejected-bid
ICA	Iterative combinatorial auction
LAB	Lowest-accepted-bid
PWYB	Pay what you bid
SMR	Simultaneous multiple round
SAA	Simultaneous ascending auction
VCG	Vickrey–Clarke–Groves
WDP	Winner determination problem

1 Auction Basics

1.1 Introduction

An **auction** is a market mechanism, operating under specific rules, that determines to whom one or more items will be awarded and at what price.[1] Auctions are specially important in environments in which it is difficult to set a market price because they are able to provide answers to questions such as: How much is a Picasso worth? For how much can we sell a coin from the fifteenth century? How much will someone be willing to pay for a movie poster from *Breakfast at Tiffany's*? How much will a company pay to obtain radio spectrum licenses or for pollution rights? In an environment of uncertainty, in which selecting a price is a complicated decision, auctions are the best way to buy and sell.

In this introductory chapter, we first review the historical importance of auctions. We then explain key concepts, such as private, common and interdependent values, winner's curse, reserve price, income of the bidder or seller's revenue. These concepts are essential to understand the mechanisms discussed in subsequent chapters.

1.2 A Brief Overview of the Evolution of Auctions

The first known historical reference to auctions was made by the Greek historian Herodotus of Halicarnassus, who recorded that the Babylonians auctioned women of marriageable age. Certain societies also used auctions to sell slaves. The Romans

[1] Sellers can use auctions to determine the selling price and the winners (highest bid) of an auction, **forward auction**. However, buyers can also use auctions to buy items from the seller who makes the lowest offer, **reverse auction**, such as those conducted by governments (**procurement auctions**). This manual, unless otherwise noted, discusses different models of auction mechanisms in which the item is awarded to the participant who made the highest bid; in other words, we focus on forward auctions.

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held auctions to distribute the spoils of war, which were auctioned under a spear stuck in the ground. From this tradition comes the term that, even today, in Italian, is used to refer to auctions: *asta*, which means spear. Referring to this same practice, according to the Spanish Royal Academy, the word auction comes from the Latin: *sub hasta*, meaning under the spear. The English term *auction* is also derived from the Latin *augeo* (increase) which is indicative of the pricing rule used in traditional auction, ascending auction (see Chap. 2). With the fall of the Roman Empire and until the seventeenth century, the importance of auctions decreased, primarily because of limited population concentrations and the poor circulation of currency. In the seventeenth century, auction houses appeared, but it was not until the second half of the eighteenth century that they began to specialize by market. Art and antiques is the market with the strongest tradition in auctions, for which the main auction houses are, even today, Christies, based in London, and Sotheby's, based in New York. Shubik [75] and Cassady [16] are two good references that explore the history of auctions.

In recent years, auctions have observed strong growth, fueled largely by the advance of the Internet. The Internet has emerged as the ideal channel to connect supply and demand for any item, eliminating barriers (both temporal and geographic) and reducing transaction costs. The best example of the web's strong penetration into the field of auctions has been the proliferation of companies dedicated exclusively to conducting online auctions. The paradigmatic example of a company that has relied on the Internet to auction all types of items is *eBay*.

With the development of auction theory and experimental research, more sophisticated auction designs have appeared that promote an efficient allocation and generate competitive revenues for the seller. This has encouraged many markets to turn to auctions as a method for allocating items, including: fish, flowers, slots at airports, industrial equipment, cars, real estate, among others.

Moreover, governments, aware of the advantages of auctions, have decided to use these mechanisms to allocate and set selling prices in environments of uncertainty. Some of the public goods that have already been allocated through auctions include radio spectrum licenses, CO_2, emission rights, bonds, bus lines, electricity and wood. Governments are currently the economic operators that most often resort to auctions, not just to sell public assets but also to buy items and services from suppliers offering the lowest price (**procurement auctions**).

In addition to their historical significance, auctions serve as a barometer of market prices for items that are exchanged, specially when it is difficult to determine the final price. In this sense, auction prices can serve as a reference for both buyers and sellers for future transactions.

1.3 Values, Bids, and Prices

To understand the bidders' behavior in an auction, it is essential to clearly comprehend the differences among value, bid, and selling price. The **value**, v_i is the maximum price that bidder i is willing to pay for an item. The **bid**, b_i is the offer that bidder i submits for an item. The three basic strategies that a bidder can follow are:

- **Sincere bidding**: the bidder makes a bid equal to his valuation: $v_i = b_i$.
- **Underbidding**: the bidder makes a bid below his valuation: $v_i > b_i$.
- **Overbidding**: the bidder makes a bid above his valuation: $v_i < b_i$.

The highest bid made by any bidder is denoted by b^*, which represents the winning bid.[2] The **selling price**, p^* is the final price that the bidder actually pays for the item awarded. The final price to pay will depend on the **pricing rule** that the seller establishes. As explained in the following chapters, bid and price to pay may not be equal. Only under the **first-price** rule the bid submitted by the winner is equal to the selling price $p^* = b^*$, see Chap. 2.

The main problem facing the seller in an auction is the inherent uncertainty about the bidders' values. If the seller were to know these values, he could directly establish a fixed price equal to or very close to the highest and obtain the maximum revenue. Because the seller does not have this information, the auction is used to set the selling price. However, there are situations in which bidders do not have a clear value of the items and only have an estimate thereof. The auction theory identifies the following bidders' types or preference structures:

- **Private value**: before the auction, each bidder knows his personal value of the item and does not know the values of the other bidders. Nevertheless, even if he knew these values, this information would not change his private value. Private values are strongly influenced by personal and emotional aspects because the items are purchased for personal use. Furthermore, a bidder has a private value only if there is no resale market in which he can obtain profits. In the examples throughout this book, we always assume that bidders have private values unless otherwise indicated.

  To better understand the concept of private value, consider the following example. A basketball enthusiast has a private value of 100 euros for a Michael Jordan shirt. However, a person who does not like basketball has a private value of zero euros. Even before the auction, if the first bidder learned of the other's value, this information would not change his value. The fan would continue valuing it at 100 euros, while the other participant would not be interested in the shirt.
- **Interdependent value**: before the auction, each bidder only knows their personal estimate of the item's value, but valuations of his rivals may affect his own value. Therefore, if the bidder observes signals about his rivals' estimates, this information can affect his value.

[2] If there are two bidders with the same highest bid $b_i^* = b_j^*$, the seller must establish a rule to solve the tie and award the item.

Continuing the example above, for the person who is not interested in basketball and whose private value is zero, he could change his value if he had the option to resell the shirt. If he knew how much other bidders were willing to pay for the item, he would surely increase his bid to buy the item, resell it, and obtain profits. In this case, the bidder would have an interdependent value.

- **Common value**: common value is the extreme case of interdependent value in which, before the auction, each individual has his own value of an item based on his estimates, but after the auction and having complete information regarding the item being offered, all of the bidders have the same value.

  For example, imagine a farmer who auctions a truck full of oranges. The price of a kilo of oranges is one euro, but before the auction, bidders do not know exactly how many kilos the truck contains. Bidders will make their bids according to their estimates of the number of kilos, so that each participant will have a different estimation and value. However, at the end of the auction, when they learn the number of kilos of oranges that were in the truck, all bidders will have the same value. In this example, the participants had a common value; even though the participants had different values before the auction because of their different estimates, the value of the item became the same when uncertainty is eliminated.

Milgrom and Weber [54] presented a model in which they established different degrees of correlation (interdependence) between the participants' values, including the two extreme cases (private values and common values). According to this model, the actual value of the item to bidder i relies on his own value estimates or signals (X_i), on some real variables that affect the item's value (S) and on the signals obtained from his rivals ($\{X_k\}_{k \neq i}$).[3]

1.4 The Winner's Curse

The uncertainty generated by auctioning items for which participants have common values can have a strong effect on the outcome of an auction. In these situations, the **winner's curse** is specially relevant. This term refers to the notion that the bidder who wins the item probably has not gotten a good deal; if nobody else was willing to pay as much, he probably overestimated. These negative effects will be more

[3] In a single-unit auction with n bidders, $X = (X_1, \ldots, X_n)$ is the vector of the real-valued informational variables (or value estimates or signals) observed by bidder i. $S = (S_1, \ldots, S_m)$ is a vector of additional real-valued variables which affects the value of the item. Bidder's i value of the item is denoted by $V_i = u_i(S, X)$. Furthermore, these authors assumed that $u_i(S, X) = u(S, X_i, \{X_k\}_{k \neq i})$. Therefore, S affects bidders' values in the same way and each bidder's value is affected by his rivals' signals. If bidders have private values $V_i = X_i$. In the case of common values $V_i = S_1$.

meaningful the larger the difference between the second highest bid and the winning bid because a large difference implies a significant deviation in the estimate of the winner bidder.

To study the winner's curse, laboratory experiments have been conducted, as described by Thaler [77]. For example, professors from Boston University auctioned a box of paperclips. Each paperclip cost four cents, and students had to estimate the total value of the box and place their bids. It is obvious that the students had common values as before the auction, no one knew the exact value because they did not know how many paperclips there were but once the auction was over, the value would be equal for all. The experiment was repeated in different classes, and the result was that, on average, the winner paid 10.01 dollars for the box of paperclips that was worth eight dollars. In this case, the winner was affected by the winner's curse because he had overestimated the value of the item and incurred losses by buying it.

The winner's curse has a direct effect on the bidding strategies as bidders tend to respond to the risk of overpaying by decreasing their bids with respect to their values. As will be shown in the following chapters, selecting the appropriate auction model can reduce this effect.

Among the studies of the winner's curse, the work of Bulow and Klemperer [14] and Kagel and Levin [36] should be highlighted.

1.5 Setting a Reserve Price

One of the risks that every seller faces in an auction is that the selling price might not be attractive to him. To avoid this situation, the seller can set a **reserve price**, p^{r} that is, the minimum price for which he is willing to sell the item. Thus, if the final price is below the reserve price, the seller is not obliged to make the sale. The main problem in setting a reserve price is to determine what the price should be, that is not a trivial decision. Studies such as those by Engelbrecht-Wiggans [27] and Levin and Smith[43] analyze the optimal reserve price in different auction models.

When setting a reserve price, the seller also has to decide whether the reserve price will be revealed (public) or kept secret (private). specially in Internet auctions, there is an open debate about the effects of public and private reserve prices (see [37, 80]). If the seller sets a private reserve price, bidders could know that there is a reserve price but never know the amount, or they may not know that there is a reserve price at all. However, if the seller sets a public reserve price, it will match the **starting bid**, p^{s}.[4] In other words, the starting bid will be the reserve price, $p^{\mathrm{s}} = p^{\mathrm{r}}$.

To better understand the difference between a public and private reserve price, consider the following example. A person wants to sell an item but not for less than

[4] Also known as **opening bid** or **minimum bid**.

200 euros. In this situation, the seller has two options. The first is to auction the item by setting a low starting bid (less than 200 euros) and setting a minimum sale price of 200 euros, which will be unknown to the bidders (secret reserve price). The second option is to set a starting bid of 200 euros. In this case, the starting price matches with the reserve price, which is to say that the reserve price is public.

One of the advantages of a secret reserve price is that a low starting bid can attract the attention of potential buyers, who are then hooked on the auction and end up buying the item for an amount greater than the reserve price. However, there are buyers who think that using a secret reserve price and starting the auction with a very low price wastes their time, and they may choose not to participate.

Sellers may have another alternative to avoid selling the item for a low price without setting a reserve price. They have the option to do **shill bids**[5] that is, to bid on their own items through multiple identities, family members or friends to make the auction more exciting, artificially increasing the price and avoiding a final price that is lower than desired.

1.6 Income of the Bidder and Seller's Revenue

After the auction, bidder i wins the item if his bid is higher than the bid placed by any other bidder k ($b_i > \max_{k \neq i} b_k$), that is, if he makes the highest bid of the auction $b_i = b^*$. In single-unit auctions, the **income of the bidder** i is equal to his value of the item obtained:

$$\Gamma_i^* = v_i, \tag{1.1}$$

and the **surplus of the bidder** i is equal to the difference between income and cost:

$$\Pi_i^* = \Gamma_i^* - p^*, \tag{1.2}$$

where p^* is equal to the price to be paid by the winner.

If the bid placed by bidder i is lower than a bid made by another bidder k ($b_i < \max_{k \neq i} b_k$), bidder i does not submit the highest bid $b_i < b^*$, he does not win the item, and his income and surplus are both zero.

To understand these concepts, consider the following example. An auctioneer offers an item and bidder i's value is equal to $v_i = 50$ euros. Let's assume that bidder i submits the highest bid equal to $b_i = b^* = 40$ euros, he wins the item, and the selling price is equal to his bid.[6] The income of the bidder is equal to his value

[5] Shill bid, also known as **bid padding**, is prohibited in most auctions.

[6] As will be shown in Chap. 2, the auctioneer can set different pricing rules, so that the selling price does not always match the highest bid.

$\Gamma_i^* = v_i = 50$ euros and his surplus equal to $\Pi_i^* = \Gamma_i^* - p^* = 50 - 40 = 10$ euros. Obviously, the greater the difference between the value and the selling price, the greater will be the surplus for the buyer.

The **seller's revenue** in a single-unit auction is equal to the price paid by the winning bidder[7]:

$$R^* = p^*. \tag{1.3}$$

In the example above, the seller's revenue is equal to $R^* = p^* = 40$ euros.

Before the auction, bidders must face the difficult decision of determining which bid to place, that is, setting their bidding strategy. This decision involves a trade-off between the probability of winning the item and the potential profits. A higher bid increases the odds of winning the item but decreases the potential profits. In contrast, a low bid decreases the chance of winning but allows for higher profits. What, then, is the best bidding strategy that a bidder could follow given this uncertainty?

This decision is conditioned by the attitude toward risk of each bidder. From a theoretical point of view, a bidder may be: **risk averse**, **risk loving**, or **risk neutral**. Faced with this dilemma, an individual with risk aversion will prefer to make a higher bid to increase the chances of winning the item, even if it means reducing the surplus. In contrast, a risk lover will choose to place a low bid to seek greater potential profits. Finally, assuming that bidder i is risk neutral,[8] the selected bidding strategy is one that maximizes his **expected surplus**, $E[\Pi_i^*]$.

For a better understanding consider an auction in which the winner pays an amount equal to his bid ($b^* = p^*$). *Before* the auction bidder i does not know whether he will win the item or not. Therefore, the **expected payment** for bidder i, $E[p_i^*]$ is equal to the probability of winning the item with the bid made multiplied by the price to pay:

$$E[p_i^*] = \text{Prob[winning]} b_i. \tag{1.4}$$

This uncertainty also affects the bidder's surplus. Before the auction the expected surplus is equal to the difference between expected income and expected payment. In this example (first-price auction) it is calculated as:

$$E[\Pi_i^*] = \text{Prob[winning]}(v_i - b_i). \tag{1.5}$$

A risk-neutral bidder will estimate the expected surplus obtained with different bids and will finally submit the bid that maximizes his expected surplus.

[7] In multi-unit auctions the seller's revenue is the sum of the prices paid by all winning bidders, as will be shown in later chapters.

[8] From this point forward, bidders are assumed to be risk neutral unless otherwise indicated.

As discussed in the following chapters, according to the pricing rule selected, the winning bid might not match the final price ($b^* \neq p^*$), which will affect expected surplus and bidding strategies.

1.7 Variables Used in This Chapter

In this chapter, we use the following variables:

- $I = (1, 2, \ldots, N)$: Bidders.
- v_i: Value of bidder i.
- b_i: Bid of bidder i.
- b^*: Highest bid made by any bidder (winning bid).
- p^s: Starting bid (minimum bid).
- p^r: Reserve price. If the reserve price is public, it holds that $p^r = p^s$.
- p^*: Selling price. Only if the auctioneer sets the first-price rule the selling price matches the highest bid, $p^* = b^*$.
- Γ_i^*: Income of bidder i from the acquired item. In single-unit auctions, it holds that $\Gamma_i^* = v_i$.
- Π_i^*: Surplus of bidder i from the acquired item.
- R^*: Seller's revenue. In single-unit auctions, it holds that $R^* = p^*$. Only if the seller sets the first-price rule it holds that this matches the highest bid $R^* = p^* = b^*$.
- $E[p_i^*]$: Bidder i expected payment (before the auction).
- $E[\Pi_i^*]$: Bidder i expected surplus (before the auction).

1.8 Exercises

1. In a single-unit auction the seller has set a reserve price of $p^r = 70$ euros and according to the rules of the auction, the winning bidder will have to pay an amount equal to his bid. The winning bidder has a value equal to $v_i = 100$ euros and has made a bid of $b_i = 90$ euros. Find:
   (a) The seller's revenue.
   (b) The surplus of the winning bidder.
   (c) The seller's revenue, if he had established a reserve price of $p^r = 95$ euros.
2. A single-unit auction is conducted in which the final price to pay will match the highest bid made (first-price rule), and the seller sets a reserve price of $p^r = 100$ euros. There are two bidders. The first one wants to win the item for personal reasons, has a private value of $v_1 = 200$ euros, and decides to make a bid of $b_1 = 180$ euros. The second bidder is only interested in winning the item for resale, and his bid is equal to $b_2 = 100$ euros. Find:
   (a) Who is the winner of the auction?
   (b) What price will the winner pay and what surplus will he make?
   (c) If the second bidder would have known his rival's bid before the auction, how would that affect his bidding strategy?

3. In a single-unit auction with first-price rule there is a risk-neutral bidder with a private value of $v_i = 20$ euros. Point out his best strategy in the following cases:
   (a) With strategy A: the bidder makes a bid of $b_i = 10$ euros and has a chance of winning the item of Prob[winning] $= 0.5$; with strategy B: the bidder makes a bid of $b_i = 15$ euros and has a chance of winning the item of Prob[winning] $= 1$.
   (b) With strategy A: the bidder makes a bid of $b_i = 10$ euros and has a chance of winning the item of Prob[winning] $= 0.5$; with strategy C: the bidder makes a bid of $b_i = 12$ euros and has a chance of winning the item of Prob[winning] $= 0.9$.
   (c) If the bidder follows strategy C and happens to be the winner, what is his surplus? What is the seller's revenue?

1.9 Solutions to Exercises

1. In this example we obtain the following results:
   (a) As $b^* = p^* = 90 > 70 = p^{\mathrm{r}}$, the item will definitely be sold. The seller's revenue is equal to $R^* = p^* = 90$ euros.
   (b) The surplus of the winning bidder is: $\Pi_i^* = \Gamma_i^* - p^* = 100 - 90 = 10$ euros.
   (c) By increasing the reserve price, the seller's revenue becomes zero. Because the highest bid is below the reserve price ($p^* = 90 < 95 = p^{\mathrm{r}}$) and the seller is not obliged to make the sale.
2. In this exercise the first bidder has a private value (he wants to buy the item for personal reasons), and the second one has an interdependent value (he wants to buy the item for resale).
   (a) The first bidder makes the highest bid and wins the auction: $b^* = b_1 = 180 > b_2 = 100$. In this case, setting the reserve price does not affect the outcome of the auction because the final price exceeds the reserve price: $p^* = 180 > p^{\mathrm{r}} = 100$.
   (b) With the first-price rule, the final price matches the highest bid: $p^* = b_1 = 180$ euros. Thus, the winner's surplus is equal to $\Pi_1^* = \Gamma_1^* - p^* = 200 - 180 = 20$ euros.
   (c) The value of the second bidder is affected by the value of the first one (interdependent values), so he would have tend to increase his bid.
3. A risk neutral bidder will follow the strategy that maximizes his expected surplus $E[\Pi_i^*] = \text{Prob[winning]}(v_i - b_i)$.
   (a) With strategy A: $E[\Pi_i^*] = 0.5(20 - 10) = 5$ euros, and with strategy B: $E[\Pi_i^*] = 1(20 - 15) = 5$ euros. Therefore, the bidder will be indifferent between these two strategies.

(b) With strategy A: $E[\Pi_i^*] = 0.5(20 - 10) = 5$ euros, and with strategy C: $E[\Pi_i^*] = 0.9(20 - 12) = 7.2$ euros. Therefore, the bidder will follow strategy C because it yields higher expected surplus.
(c) If the bidder bids according to strategy C ($b_i = 12$) and becomes the winner, the revenue of the seller would be: $p^* = b_i = R^* = 12$ euros. The bidder's surplus would be equal to $\Pi_i^* = \Gamma_i^* - p^* = 20 - 12 = 8$ euros.

2 Standard Single-Unit Auctions

2.1 Introduction

This chapter discusses the standard single-unit auctions: ascending (English), descending (Dutch), first-price sealed-bid, and second-price sealed-bid. Advantages, disadvantages, and distinguishing features of each format are described. Furthermore, equivalences among them are mentioned. The final section is devoted to the Revenue Equivalence Theorem, which states that, under certain assumptions, the four auction formats yield the same expected revenue to the seller. The implications of relaxing each assumption are also shown.

2.2 Key Factors in Auction Design

When designing a single-unit auction, the auctioneer has to decide the parameters that set the rules. The first one is the **bidding rule**. We can distinguish two types of auctions according to the bidding rules:

- **Dynamic auctions**: auctions that allow bidders to make multiple bids and where some information is shown during the auction. The bids are called **open bids** and can be made in two ways:
  - Bids at discrete rounds: bidders submit their bids in multiple iterative rounds, **iterative auctions** or **multi-round auctions**.
  - Continuous bidding: bidders bid on a continuous basis, as occurs in **clock auctions**.[1]
- **Single-round auctions**: bidders can submit only one bid in a single round, known as **sealed-bid**. These auctions are also called **sealed-bid auctions**.

[1] Theoretical clock auctions have continuous bidding. In practice, this auction model is also usually implemented with discrete clock rounds to facilitate the auction process.

A. Mochón, Y. Sáez, *Understanding Auctions*, Springer Texts in Business and Economics, DOI 10.1007/978-3-319-08813-6_2

The main advantage of dynamic auctions is that during the course of the auction, a process of **price discovery** occurs. Bidders gain information about the valuations of their rivals that reduces uncertainty and thereby the effect of the **winner's curse**. However, this bidding rule also has some drawbacks. First, it can deter the weakest bidders (those with the lowest valuations) from participating in the auction. It can also make the auction too long and complex, thereby increasing the costs of the auction. In contrast, sealed-bid auctions are quick, easy to implement and reduce the risk of **collusion** among bidders. The main disadvantage of single-round auctions is that they may cut down the income of the seller if the bidders make low bids in an effort to avoid the winner's curse.

A second parameter that must be set is the **pricing rule**, that is, how to determine the amount that the winning bidder should pay. We distinguish between:

- **First-price**: the winning bidder pays the amount of his bid, which is the highest bid of the auction: $p^* = b^*$. This pricing rule is also called **pay-what-you bid** (PWYB).
- **Second-price**: the winning bidder pays an amount that is equal to the second highest bid for the awarded item.

The first-price rule is easy to understand and implement. However, placing a bid equal to the value implies zero profit, which may encourage participants to decrease their bids. This strategy will probably involve an inefficient allocation of items because the bidder with the highest value will probably not be the same as the highest bidder.[2] Furthermore, this pricing rule is more vulnerable to the winner's curse. Instead, the use of the second-price rule encourages bidders to bid according to their true values because with this strategy, they still derive positive profits, which tend to generate an efficient allocation of items and reduce the negative effects of the winner's curse.

2.3 Standard Single-Unit Auction Models

Combining the parameters described in the previous section, we define four basic single-unit auction models.

2.3.1 Ascending-Bid Auction

In an **ascending-bid auction** or **English auction** the seller sets a starting price (that is relatively low) and increases in small increments until there is only one bidder

[2] See **efficient auction** in Chap. 3.

Table 2.1 Ascending-bid auction

Round (t)	Round price (p^t)	b_1^t	b_2^t	b_3^t
$t = 0$	$p^0 = 900$	900	900	900
$t = 1$	$p^1 = 1000$	0	1000	1000
$t = 2$	$p^2 = 1100$	0	0	1100
	Values v_i	$v_1 = 1000$	$v_2 = 1100$	$v_3 = 1800$

interested in buying the item. The bidder who holds the highest bid is the one who is finally awarded the item.

There are different ways to implement an ascending-bid auction:

- *Bidders increase the price:* in this model, bidders send their bids in different rounds, indicating the price they are willing to pay for the item. The auction ends when a bidder submits a bid that exceeds any other. This is the best-known auction format as has been traditionally used by art and antiques auction houses. This version of the ascending auction is also very frequent in electronic auctions.
- *The auctioneer increases the price:* the seller increases the price from round to round, so that bidders can decide whether to bid at the current round or not.
- *Price increases continuously:* the price rises steadily and the bidders decide when to stop bidding. This auction model is called **ascending clock auction**.[3]

For a better understanding, we analyze the following example, see summary in Table 2.1. There is one single item and three bidders with the following private values: $v_1 = 1000$ euros, $v_2 = 1100$ euros, and $v_3 = 1800$ euros. We assume bidders will bid sincerely and will keep bidding until the price in round t is equal to their values ($p^t = v_i$). The seller sets the starting price at $p^s = 900$ euros (which equals the reserve price) and a price increment between rounds of 100 euros. At the starting price in round $t = 0$, the three bidders bid: $b_1^0 = b_2^0 = b_3^0 = 900$ euros. For all bidders this price is lower than their values ($p^s < v_i$). In the next round ($t = 1$) the price is $p^1 = 1000$ euros and the first bidder drops out ($b_1^1 = 0$). At this price, he is indifferent between winning or losing (the price is equal to his value $p^1 = v_1 = 1000$ euros and he would obtain zero profit). The same happens to the second participant in round $t = 2$ ($p^2 = v_2 = 1100$ euros), so bidder three is the only one remaining. The auction ends and the third bidder is the winner, the final price is $p^* = 1100$ euros.

It is important to highlight that in an ascending auction, the winner pays an amount equal to the second highest bid plus the bid increment. In this example, the second highest bid has been made by the second participant ($b_2^1 = 1000$ euros) in round $t = 1$ and the increment is 100 euros. The surplus of the winner

[3]The **Japanese auction** is an ascending-bid clock auction that starts with a low price for all bidders to bid. Then, the price increases continuously, and bidders leave the auction when they are no longer interested in the item. Bidders are not allowed to bid again after dropping out.

Table 2.2 Descending-bid auction

Round (t)	Round price (p^t)	b_1^t	b_2^t	b_3^t
$t = 0$	$p^0 = 2000$	0	0	0
$t = 1$	$p^1 = 1900$	0	0	0
$t = 2$	$p^2 = 1800$	0	0	1800
	Values v_i	$v_1 = 1000$	$v_2 = 1100$	$v_3 = 1800$

is $\Pi_3^* = \Gamma_3^* - p^* = 1800 - 1100 = 700$ euros and the seller's revenue is $R^* = p^* = 1100$ euros.

The ascending-bid auction is a **dynamic auction**, in which the **second-price** rule applies. An advantage of this auction model is that it is simple to implement and is easily understood by all bidders. It also reduces the effect of the **winner's curse** because bidders are informed about their rivals' values as a result of the **price discovery** as rounds go on. This issue encourages bidders to raise their bids, and increase the seller's revenue. This is specially relevant in environments where bidders have **interdependent values** or **common values**, where the winner's curse may have a significant effect. Another advantage is that the auction may require a reasonable amount of time, which contributes to generate a good atmosphere and promotes participation.

2.3.2 Descending-Bid Auction

The **descending-bid auction** or **Dutch auction**[4] starts with a relatively high price, which decreases until a bidder submits a bid.

To understand how a decreasing auction works, we can refer to the example of the previous section. We have three bidders with the same private values bidding sincerely: $b_i = v_i$ (see Table 2.2). The starting price is $p^s = 2000$ euros and the price decrements 100 euros. The starting price is higher than the values of all bidders, so none of them place bids, and price drops to $p^1 = 1900$ euros. Still, there are no bids in $t = 1$ but in $t = 2$, the price drops to $p^2 = 1800$ euros, which equals the value of the third bidder, so he bids and the auction ends ($v_3 = 1800 = b_3^2$). This bidder wins the item, pays $p^2 = p^* = 1800$ euros and obtains a surplus of $\Pi_3^* = \Gamma_3^* - p^* = 1800 - 1800 = 0$ euros. As we will explain, in descending auctions, bidders always wait to submit bids below their valuations to seek positive profits.

The descending auction is a **dynamic auction** in which the **first-price** rule applies, the winning bidder pays an amount equal to the highest bid. This format is often used to sell perishable products such as fish, flowers, or tobacco. It is used

[4] The descending auction is also known as the Dutch auction because it was the method used in the flower market in the Netherlands.

Table 2.3 Sealed-bid auction

Round (t)	b_1	b_2	b_3
$t = 0$	1000	1100	1800
Values v_i	$v_1 = 1000$	$v_2 = 1100$	$v_3 = 1800$

to eliminate unwanted stocks that may be more expensive to keep than to sell off at a low price.

2.3.3 First-Price Sealed-Bid Auction

In a **first-price sealed-bid auction** all bidders simultaneously submit their bids in a single round without the possibility of subsequent amendment. When all bids are received, the seller determines the winner (the bidder who made the highest bid), and the winner pays an amount equal to his bid.

Continuing with the previous example, Table 2.3 shows bids that the bidders would submit if they bid sincerely ($b_i = v_i$). The winner is the third bidder because he makes the highest bid, and the final price is $p^* = 1800$ euros. With this bidding strategy he obtains zero profits $\Pi_3^* = \Gamma_3^* - p^* = 1800 - 1800 = 0$ euros, just as in the **descending-bid auction** with the same strategy. Therefore, in this auction model, the bidders also tend to bid below their values because a bid equal to the value implies zero surplus.

2.3.4 Second-Price Sealed-Bid Auction

In a **second-price sealed-bid auction**, all bidders simultaneously submit a single bid in one round. The winner is the participant who makes the highest bid, but the price he pays is equal to the second highest bid (**second-price** rule).

If we continue with the previous example and bidders keep their bidding strategy ($b_i = v_i$), the seller receives the bids shown in Table 2.3. The third bidder is again the winner, but the price he pays is equal to the second highest bid, $p^* = 1100$ euros. His surplus is $\Pi_3^* = \Gamma_3^* - p^* = 1800 - 1100 = 700$ euros, that is equal to the surplus made on the **ascending-bid auction** with all bidders doing sincere bidding.

This auction model is also known as the **Vickrey auction** for single-unit because it was Vickrey [79] who demonstrated that bidders' **dominant strategy**[5] in this auction format is **sincere bidding**s ($b_i = v_i$).[6] If bidders follow this strategy, the bidder with the highest value will submit the highest bid and will win the item.

[5]Strategy that does at least as well as any other strategy for one bidder, no matter how that bidder's opponents may play.

[6]Included in the appendix at the end of this chapter, we show that submitting sincere bids is the dominant strategy in a Vickrey auction.

Hence, an efficient allocation of items is achieved, and therefore the second-price sealed-bid auction is an **efficient auction**.[7]

The second-price sealed-bid auction is an example of the **Vickrey–Clarke–Groves mechanism** (**VCG mechanism**).[8] A VCG mechanism for allocating items states that each bidder has to pay an amount equal to the opportunity cost of the awarded item. For a single-unit auction, this corresponds to the second-price sealed-bid auction because the opportunity cost of the awarded item is equal to the second highest bid. By setting this pricing rule, the VCG mechanism is **incentive compatible**, that is, the dominant strategy of all bidders is to make sincere bids.

2.4 Bidding Strategies: Equivalences Among the Standard Single-Unit Auctions

In the previous section, we studied the four standard single-unit auctions, assuming that bidders bid sincerely ($b_i = v_i$). However, is this the bidding strategy that all bidders will actually follow?

If the **first-price** rule applies, the winner will pay an amount equal to his bid. Thus, by submitting sincere bids, he will achieve zero surplus. This pricing rule applies both in descending and first-price sealed-bid auctions. Therefore, in these auction models, bidders will never follow the sincere strategy and instead will bid below their valuations ($b_i < v_i$) in an attempt to achieve a positive surplus.[9] But, how much below his value will the participant bid? This is the decision faced by bidders, and it implies a trade-off between potential surplus and the probability of winning.

In a descending auction, bidders have to decide at what price to bid as the seller decreases the price. While the auction is active and the price is going down, each bidder only knows that no other opponent is willing to pay the current amount. However, the only information he receives is provided when one bidder finally bids. At this point the auction ends, and there is no chance to react. Therefore, despite being a dynamic auction, it is strategically equivalent to a first-price sealed-bid auction.[10] In other words, with the same bidding strategy, the bidder will obtain the same surplus in both auction models (which will obviously depend on the rivals' bids). Making a specific bid in a first-price sealed-bid auction is equivalent to submitting the same bid in a descending auction (if the auction is still open).

In these auction models, it is frequently observed that the greater the number of bidders (more competition), the higher the optimal bids are, and the closer they are

[7]The notion of efficient auction is discussed in Chap. 3.

[8]See Vickrey [79], Clarke [17] and Groves [35].

[9]This behavior tends to yield an inefficient allocation of items, as the bidder with the highest value is not necessarily the winner.

[10]Two games are strategically equivalent if for every strategy in one game, a player has a strategy in the other game, which yields the same outcomes.

to the bidders' values. Likewise, the optimal bid of a **risk averse** bidder tends to be higher than if the bidder is **risk neutral** or **risk loving**.

A similar analysis can be performed for **second-price** auctions (ascending and second-price sealed-bid auction), but restricted to the case in which the bidders have **private values** and the information received during the ascending auction does not affect their values nor bidding strategies. In an ascending auction, any bidder will continue bidding, provided that the price is less than his value, and he will never bid above his valuation (which would yield negative surplus). Therefore, the optimal strategy in an ascending auction is to make sincere bids, $b_i = v_i$, as it is in a second-price sealed-bid auction. Therefore, the two auction models are strategically equivalent, but only if bidders have private valuations.[11]

Krishna [41] determines the **Bayesian Nash equilibrium** for the first and second-price auction, showing the equilibrium bidding strategies.[12]

2.5 Revenue Equivalence Theorem

In the previous section we have observed that bidding strategies differ from one auction model to another. If the **first-price** rule applies (descending and first-price sealed-bid auctions), bidders tend to bid below their valuations ($b_i < v_i$). Nevertheless, in **second-price** auctions (ascending and second-price sealed-bid auctions), bidders tend to submit sincere bids ($b_i = v_i$). Thus, among these four standard formats, which will the seller choose to maximize his revenues?

According to the **Revenue Equivalence Theorem**,[13] under certain assumptions, the four standard single-unit auctions yield the same expected revenue to the seller. These assumptions are the following:

- *Risk neutral bidders*: a risk neutral bidder plays the strategy that maximizes his expected surplus (expected income minus expected payment).

[11] In the case where the bidders have **interdependent values**, information obtained during the course of an ascending auction can influence their bidding strategy. Therefore, this auction model would not be strategically equivalent to a second-price sealed-bid auction in which there is no such information available.

[12] The **Nash equilibrium** is the set of strategies, one for each player of the game, such that each player's strategy is a best response to the others' strategies. The Bayesian Nash Equilibrium is an extension of the Nash equilibrium to incomplete information games. In the equilibrium, each player plays a best response to the strategies of the other players (evaluated after a player learns his private information but before he learns his rival's private information). Each player maximizes his expected utility given his private information, the joint distribution of others' private information, and the strategies of the other players. Private information is drawn from a common joint distribution and beliefs about strategies are consistent.

[13] Many studies have been conducted on the Revenue Equivalence Theorem. We can highlight the first study presented by Vickrey [79] and subsequent generalizations made by Myerson [56] and Riley and Samuelson [65], among others.

- *Bidders with independent-private-values*: bidder i has an **independent-private-value** when his private valuation is independent of his rivals' values. Values of different bidders are independently distributed.
- *Symmetric bidders*: if the values of all bidders are distributed according to the same distribution function F, then they are **symmetric bidders**.
- *No budget constraints*: bids are not constrained by a budget cap.

This theorem has been studied by authors, such as Milgrom and Weber [54], Maskin and Riley [45], McAffe and McMillan [47], and Krishna [41], who also discuss the implications of the relaxation of the assumptions. Some of these implications are summarized below.

- **Risk averse bidders**

  A risk averse bidder will follow a strategy that maximizes his expected utility.[14] These bidders tend to submit higher bids in order to increase their chances of winning, although this behavior decreases the potential surplus. If the bidders are risk averse but the other assumptions hold, *the first-price sealed-bid auction generates higher expected revenue for the seller than the ascending auction or the second-price sealed-bid auction*.

  The bidding strategy of a risk averse bidder, both in an ascending and second-price sealed-bid auction, remains unchanged even under the assumption of risk neutrality. In other words, continue bidding until the price equals the valuation (ascending auction) or make a bid equal to the valuation (second-price sealed-bid auction). However, in both Dutch and first-price sealed-bid auction, bidders tend to submit bids below their valuations. Determining how much to bid below their valuations will be affected by the bidder's attitude toward risk. In fact, if the participants are risk averse, they will tend to bid closer to their valuations (although always lower) than if they are risk neutral, thus increasing their chances of winning the item. This strategy involves an increase in the expected revenue for the seller.
- **Bidders with interdependent values and affiliated signals**

  If the bidders have interdependent values, the signals received by a bidder about the estimation of their rivals affect his own valuation. These signals may be independent, or they may be correlated. If the signals are positively correlated (increasing one value also tends to increase the other), this is referred to as **affiliation** or **affiliated signals**.

  If the assumptions of the Revenue Equivalence Theorem hold, but the bidders have interdependent values and affiliated signals, then the **winner's curse** has a strong impact on the bidding strategies. In auctions where the bidders do not receive any information about the valuations of their rivals, the bidders tend to

[14] To examine risk averse bidders, it is common to use the von-Neumann–Morgenstern utility function that satisfies $u(0) = 0$, $u' > 0$ and $u'' < 0$. Maskin and Riley [44] analyze optimal auctions and risk averse bidders.

decrease their bids to avoid the winner's curse. However, in an ascending auction, the bidders gain information about their rivals as the auction progresses, thereby reducing the risk of the winner's curse and allowing the bidders to make higher bids. In this context, *the expected revenue for the seller in an ascending auction is greater than or equal to that obtained in a second-price sealed-bid auction, which in turn is greater than or equal to that obtained in a first-price sealed-bid auction*.

- **Asymmetric bidders**

  Asymmetric bidders do not have their values distributed according to same distribution function.[15] Having two or more types of bidders means that there are asymmetries. If this occurs but the other assumptions of the model hold, the four standard single-unit auctions do not generate the same expected revenue for the seller. Nevertheless, *in this case, we cannot determine which model generates higher expected revenue for the seller.*
- **Participants with budget constraints**

  If bidders are budget constraints, *the seller's expected revenue in a first-price sealed-bid auction is higher than in a second-price sealed-bid auction*.

2.6 Variables Used in This Chapter

In this chapter, we use the following variables:

- $I = (1, 2, \ldots, N)$: Bidders.
- v_i: Value of bidder i.
- b_i: Bid of bidder i in a sealed-bid auction (single-round).
- b_i^t: Bid of bidder i in round t of a dynamic auction.
- b^*: Highest bid made by any bidder (winning bid).
- p^{s}: Starting bid (minimum bid).
- p^{r}: Reserve price. If the reserve price is public, it holds that $p^{\mathrm{r}} = p^{\mathrm{s}}$.
- p^t: Price in round t of a dynamic auction.
- p^*: Selling price. Only if the auctioneer sets the first-price rule the selling price matches the highest bid $p^* = b^*$.

[15]The following is an example of an auction with asymmetric bidders. A government wants to award a license for road construction, and there are two types of companies. There are strong companies that have been in the market for a long time and whose valuations have a common distribution function F_1, and there are weak companies, start-ups, with another distribution function F_2 for their valuations. That the valuations of all strong companies belong to the distribution function F_1 does not mean that they all have the same value, but it does mean that they all have a valuation with the same range and described by the same function. The distribution, for example, could be a normal distribution between 0 and 100. Similarly, weaker companies will have different valuations from each other but will belong to the same normal function F_2 with a range, for example, of 50–100. In this example, we consider two types of bidders (strong and weak companies), but there can be m types, and $m \leq n$, where n is the number of bidders.

- Γ_i^*: Income of bidder i. In single-unit auctions, it holds that $\Gamma_i^* = v_i$.
- Π_i^*: Surplus of bidder i.
- R^*: Seller's revenue. In single-unit auctions, it holds that $R^* = p^*$. Furthermore, only if the seller sets the first-price rule it holds that this matches the highest bid $R^* = p^* = b^*$.

2.7 Exercises

1. Assume an ascending auction with three bidders with the following private valuations: $v_1 = 500$ euros, $v_2 = 450$ euros, and $v_3 = 440$ euros. All bidders decide to continue bidding if the price is lower than their values. The starting price (reserve price) is $p^s = p^r = 430$ euros. The bid increment is 10 euros. Determine the following:
   (a) The winning bidder.
   (b) The price to be paid by the winning bidder.
   (c) Surplus of each bidder.
   (d) Seller's revenue.
2. Using the data of the previous exercise, analyze the result if we conduct a descending auction in which bidders bid for only 90 % of their valuations; the starting price is equal to $p^s = 460$ euros, and the price decrement is 10 euros. Determine the following:
   (a) The winning bidder.
   (b) The price to be paid by the winning bidder.
   (c) Surplus of each bidder.
   (d) Seller's revenue.
3. Using the data and bidding strategies given in exercise two, assume a first-price sealed-bid auction model. Determine the following:
   (a) The winning bidder.
   (b) The price to be paid by the winning bidder.
   (c) Surplus of each bidder.
   (d) Seller's revenue.
4. Using the same data from the ascending auction proposed in the first exercise, now consider a second-price sealed-bid auction in which all bidders submit sincere bids. Determine the following:
   (a) The winning bidder.
   (b) The price to be paid by the winning bidder.
   (c) Surplus of each bidder.
   (d) Seller's revenue.

2.8 Solutions to Exercises

1. Using the data for the exercise, the evolution of the ascending auction in which bidders bid until the price equals their valuations is shown in Table 2.4.

Table 2.4 Ascending auction

Round (t)	Round price (p^t)	b_1^t	b_2^t	b_3^t
$t = 0$	$p^0 = 430$	430	430	430
$t = 1$	$p^1 = 440$	440	440	0
$t = 2$	$p^2 = 450$	450	0	0
	Values v_i	$v_1 = 500$	$v_2 = 450$	$v_3 = 440$

Table 2.5 Descending auction

Round (t)	Round price (p^t)	b_1^t	b_2^t	b_3^t
$t = 0$	$p^0 = 460$	0	0	0
$t = 1$	$p^1 = 450$	450	0	0
	Values v_i	$v_1 = 500$	$v_2 = 450$	$v_3 = 440$

Therefore:

(a) The winner is the first bidder because he makes the highest bid, $b^* = b_1^2 = 450$ euros, which equals the second highest bid plus the bid increment.
(b) The winner pays a price of $p^* = 450$ euros.
(c) The first bidder's surplus is equal to $\Pi_1^* = \Gamma_1^* - p^* = 500 - 450 = 50$ euros. The other two bidders do not have any surplus because they do not win the item.
(d) The seller's revenue is equal to $R^* = p^* = 450$ euros.

2. In a descending auction, the winner pays an amount equal to his bid, and therefore it does not make sense to submit a bid equal to the value, which would yield a zero surplus. Bidders follow the strategy of bidding up to 90 % of their valuations. That is, they intend to submit the following bids: $b_1 = 450$ euros, $b_2 = 405$ euros, and $b_3 = 396$ euros. Table 2.5 shows the evolution of the auction, in which only the first player makes his bid in round $t = 1$. At that round the auction ends.

   The final outcome is as follows:
   (a) The winner is the first bidder, $b^* = b_1^1 = 450$ euros.
   (b) The price paid is $p^* = 450$ euros.
   (c) The first bidder's surplus is $\Pi_1^* = \Gamma_1^* - p^* = 500 - 450 = 50$ euros and zero for the second and third.
   (d) The seller's revenue is equal to $R^* = p^* = 450$ euros.

3. As in a descending auction, in a first-price sealed-bid auction, the winning bidder pays his bid. We assume that the bidders strategy is to submit bids for 90 % of the value: $b_1 = 450$ euros, $b_2 = 405$, and $b_3 = 396$ euros. The outcome of the auction is as follows.
   (a) The winner is the first bidder, $b^* = b_1 = 450$ euros.
   (b) The final price is $p^* = 450$ euros.
   (c) The first bidder's surplus is $\Pi_1^* = \Gamma_1^* - p^* = 500 - 450 = 50$ euros. Zero for the second and third.
   (d) The seller's revenue is equal to $R^* = p^* = 450$ euros.

If we assume that bidders use the same bidding strategy in a descending and a first-price sealed-bid auction, both auctions yield the same outcome. In other words, they are equivalent.

4. In a second-price sealed-bid auction, the dominant strategy is to bid according to the true value ($b_i = v_i$). The submitted bids simultaneously in a single round are: $b_1 = 500$ euros, $b_2 = 450$, and $b_3 = 440$ euros. Results are the following:
   (a) The winner is the first bidder, $b^* = b_1 = 500$ euros.
   (b) The final price is $p^* = 450$ euros, which is the second highest bid.
   (c) The first bidder's surplus is $\Pi_1^* = \Gamma_1^* - p^* = 500 - 450 = 50$ euros. Zero for the second and third.
   (d) The seller's revenue is equal to $R^* = p^* = 450$ euros.

   Assuming private values and same bidding strategies, we obtain the same results in an ascending and a second-price sealed-bid auction.

   In the four examples summarized in these exercises, it holds that the seller receives the same revenue, regardless of the auction model. According to the Revenue Equivalence Theorem, if the four basic assumptions hold, the four auction models generate the same expected revenue to the seller.

Appendix

We present, in a simple way, the **dominant strategy** in a **second-price sealed-bid auction** or **Vickrey auction** for a single-unit (see [26]).

Bidder i, with value v_i, submits the bid b_i and his surplus is Π_i^*. Let $b_{\max k}$ be the highest bid made by all of the other bidders, $k \neq i$, p^* the final price and b^* the winning bid.

Overbidding

If bidder i submits a bid higher than his value, $b_i > v_i$, the following situations may occur:

- *The highest bid of the other bidders is below the value of bidder i ($b_{\max k} < v_i$).*
  The bid submitted by bidder i is higher than his valuation ($b_i > v_i$), and is also the highest bid of the auction ($b^* = b_i > b_{\max k}$), so bidder i wins the item, and pays $p^* = b_{\max k}$. The surplus he obtains is $\Pi_i^* = v_i - b_{\max k} > 0$. If, under these circumstances, bidder i would have made a sincere bid ($b_i = v_i$), he would have obtained the same surplus.
- *The highest bid of the other bidders is higher than the value of bidder i but below his bid ($v_i < b_{\max k} < b_i$).*
  Bidder i wins the items because his bid is higher than his rivals' bids, and the price he pays is equal to the second highest bid ($p^* = b_{\max k}$). However, because $b_{\max k} > v_i$, the price is higher than his valuation, and therefore he incurs a loss: $\Pi_i^* = v_i - b_{\max k} < 0$. Under these assumptions, if the bidder would have made

a sincere bid ($b_i = v_i$), he would have been better off because although he would not have won the item, he would not have incurred a loss.

- *The highest bid of the other bidders is higher than bidder i's bid ($b_{\max k} > b_i$).* Bidder i does not win the item, but sincere bidding would have yielded the same outcome. Either way, he has no surplus or losses.

If bidder i would have bid sincerely ($b_i = v_i$) instead of overbidding ($b_i > v_i$), he would not have been worse off and could have even been better off.

Underbidding

If bidder i submits a bid below his value, $b_i < v_i$, the following situations may occur:

- *The highest bid of the other bidders is below the bidder i's bid ($b_{\max k} < b_i$).* Bidder i has made the highest bid ($b^* = b_i$), wins the item and pays $p^* = b_{\max k}$. His surplus is $\Pi_i^* = v_i - b_{\max k} > 0$. If, under these circumstances, bidder i would have made a sincere bid ($b_i = v_i$), he would have obtained the same surplus.
- *The highest bid of the other bidders is below bidder i's value but higher than the bid that he submits ($b_i < b_{\max k} < v_i$).* Bidder i does not win the item and has no surplus nor losses. However, if he would have bid according to his true value ($b_i = v_i$), he would have won the item and obtained a surplus equal to $\Pi_i^* = v_i - b_{\max k} > 0$.
- *The highest bid of the other bidders is higher than bidder i's value ($b_{\max k} > v_i$).* Under this assumption, bidder i does not win the item. Bidding sincerely would have resulted in the same outcome for bidder i.

If bidder i would have bid sincerely ($b_i = v_i$) instead of underbidding ($b_i < v_i$), he would not have been worse off and could have even been better off.

Sincere bidding

The analysis for both strategies (overbidding and underbidding) shows that, in this auction model, bidder i's outcome is never worst by bidding sincerely and could have even been better.

A **dominant strategy equilibrium** is a refinement of a Nash equilibrium (or a Bayesian Nash equilibrium for games of incomplete information). In the equilibrium each player's strategy is a best response, regardless of the strategies of the other players. Behavior in dominant strategy equilibria is robust to uncertainty about rivals' strategies and private information. With private values, *bidding one's value is a dominant strategy equilibrium in the Vickrey auction*.

Other Single-Unit Auctions

3

3.1 Introduction

In the previous chapter, we studied the four standard single-unit auctions. However, the seller has multiple options when designing an auction. In this chapter, we examine other single-unit models, such as hybrid auctions (Anglo-Dutch and Dutch–English auctions), auctions with different pricing rules (third-price or average price), auctions with different closing rules, and all-pay auctions. The chapter ends with the definition of optimal and efficient auctions.

3.2 Anglo-Dutch Auction

The **Anglo-Dutch auction** for one item is a hybrid auction that combines a dynamic phase with a sealed-bid phase, see [38]. The first phase is an ascending-bid auction in which the seller sets an opening bid that increases until there are only two bidders. The second phase is a first-price sealed-bid auction. The two bidders who were active in the last round of the previous phase each place a final sealed-bid that cannot be lower than the bid entered in the last round of the first phase. The winner is the bidder who placed the highest sealed-bid and pays an amount equal to his bid.

The following example illustrates an Anglo-Dutch auction. A seller offers one unit for which there are four interested bidders with the following private values: $v_1 = 500$ euros, $v_2 = 1000$ euros, $v_3 = 600$ euros, and $v_4 = 1200$ euros. The seller sets a reserve price for Phase I of $p^{\mathrm{I,s}} = 400$ euros and a bid increment of 100 euros. Assuming that the bidders continue bidding as long as the price is lower than their values, the first phase unfolds as follows (see Table 3.1). All bidders bid at the reserve price ($t = 0$). In round $t = 1$, the price is equal to the first bidder's value $p^1 = v_1 = 500$ euros; thus, he stops bidding. In round $t = 2$, the price equals the value of the third bidder $p^2 = v_3 = 600$ euros, so he also stops bidding. Phase I ends in this round and only the second and fourth bidder continue bidding

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Table 3.1 Anglo-Dutch auction

Phase I:	Ascending				
Round (t)	Price ($p^{\mathrm{I},t}$)	$b_1^{\mathrm{I},t}$	$b_2^{\mathrm{I},t}$	$b_3^{\mathrm{I},t}$	$b_4^{\mathrm{I},t}$
$t = 0$	$p^{\mathrm{I},0} = 400$	400	400	400	400
$t = 1$	$p^{\mathrm{I},1} = 500$	0	500	500	500
$t = 2$	$p^{\mathrm{I},2} = 600$	0	600	0	600
Phase II:	Sealed-bid	b_1^{II}	b_2^{II}	b_3^{II}	b_4^{II}
Bid		0	900	0	1080
	v_i	$v_1 = 500$	$v_2 = 1000$	$v_3 = 600$	$v_4 = 1200$

in Phase II. The minimum bid in this second phase is $p^{\mathrm{II,s}} = 600$ euros, which was the price in the last round of the ascending phase (Phase I). If we assume that these bidders bid 90 % of their private values, the final sealed-bids are $b_2^{\mathrm{II}} = 900$ euros and $b_4^{\mathrm{II}} = 1080$ euros. The fourth bidder is the winner and pays $p^* = 1080$ euros for the item. The surplus of this bidder is $\Pi_4^* = 1200 - 1080 = 120$ euros, and the seller's revenue is $R^* = p^* = 1080$ euros.

This auction mechanism aims to combine the advantages of dynamic auctions with those of single-round auctions. In the first phase, bidders receive information about their rivals' values, reducing the **winner's curse** effect. With this process it is also more likely to award the item to the bidder who most values it. However, the second sealed-bid phase favors the participation of the bidders and reduces the risk of **collusion** between them.

3.3 Dutch–English Auction

The **Dutch–English auction** for one item is also a two-stage hybrid auction that proceeds as follows. The first phase corresponds to a Dutch auction in which the seller starts the auction with a relatively high price that decreases until a bidder bids. This price is set as the opening bid of the second phase, which is an ascending or English auction.

A seller offers one item with an opening bid in round $t = 0$ of Phase I of $p^{\mathrm{I,s}} = 700$ euros and in which a decrement of 100 euros is set. There are three bidders with the following private values: $v_1 = 600$ euros, $v_2 = 800$ euros, and $v_3 = 400$ euros. We assume that bidders will not bid in Phase I until the price is 50 % of their values. In the first round, no bidder bids, so the price decreases to $p^{\mathrm{I},1} = 600$ euros, a price for which there are again no bids. The same happens in round $t = 2$. Finally, in the next round $p^{\mathrm{I},3} = 400$ euros, the second bidder bids, thus ending Phase I and triggering Phase II, for which the bid increment is set to 40 euros.

Assuming that bidders will bid up to 80 % of their private values in Phase II means that the first bidder will bid up to 480 euros and the second one up to 640 euros. The third bidder will not submit bids in this phase as the starting price equals his value. The opening bid of the ascending auction is the closing price of

Table 3.2 Dutch–English auction

Phase I: descending				
Round (t)	Price ($p^{\mathrm{I},t}$)	$b_1^{\mathrm{I},t}$	$b_2^{\mathrm{I},t}$	$b_3^{\mathrm{I},t}$
$t = 0$	$p^{\mathrm{I},0} = 700$	0	0	0
$t = 1$	$p^{\mathrm{I},1} = 600$	0	0	0
$t = 2$	$p^{\mathrm{I},2} = 500$	0	0	0
$t = 3$	$p^{\mathrm{I},3} = 400$	0	400	0
Phase II: ascending				
Round (t)	Price ($p^{\mathrm{II},t}$)	$b_1^{\mathrm{II},t}$	$b_2^{\mathrm{II},t}$	$b_3^{\mathrm{II},t}$
$t = 0$	$p^{\mathrm{II},0} = 400$	400	400	0
$t = 1$	$p^{\mathrm{II},1} = 440$	440	440	0
$t = 2$	$p^{\mathrm{II},2} = 480$	0	480	0
	v_i	$v_1 = 600$	$v_2 = 800$	$v_3 = 400$

the previous phase, $p^{\mathrm{II,s}} = 400$ euros. At this price, both first and second bidder bid. They continue bidding in round $t = 1$ of Phase II at price $p^{\mathrm{II},1} = 440$ euros. Finally, in the next round ($p^{\mathrm{II},2} = 480$ euros), only the second bidder continues bidding, meaning that he is the winner. The price that the second bidder pays for the item, which is also the seller's revenue, is equal to $R^* = p^* = 480$ euros. The surplus of the winner is equal to $\Pi_2^* = 800 - 480 = 320$ euros. A summary of the auction is shown in Table 3.2.

3.4 Auctions with Different Pricing Rules

In a single-unit auction the winning bidder is the one with the highest bid b^*. However, depending on the **pricing rule** established by the seller, the price to be paid can be different from this bid. In the previous chapter, we studied the main pricing rules: first-price and second-price. However, the sellers can set other rules such as the third-price or the average of the bids places.

If the seller chooses the **third-price** rule, the winner pays an amount equal to the third highest bid. For example, in a sealed-bid auction, the seller receives the following bids: $b_1 = 50$ euros, $b_2 = 300$ euros, $b_3 = 100$ euros, and $b_4 = 150$ euros. The second bidder is the winner ($b^* = b_2 = 300$ euros), but the price to be paid is $p^* = 100$ euros.

Another pricing rule that can be established is one in which the winner pays the **average of the bids** placed. This pricing rule may be of interest in situations similar to that described below. Two people own a house and want to stop sharing the property, but both want to keep it. One way to settle who remains and at what cost is to hold a sealed-bid auction in which the winner pays the average of the two bids. For example, if the bidders make the bids $b_1 = 200000$ euros and $b_2 = 150000$ euros, the first bidder wins ($b_1 = b^*$), but the price paid is $p^* = 175000$ euros, which is the average of the two bids. With this pricing rule, the bidder who does not win the item is compensated by receiving more than he was willing to pay for his half of the property.

3.5 Auctions with Different Closing Rules

In the dynamic auctions considered so far, it was always assumed that the auction ended when no bidder was interested in further increasing the current price. However, the seller can set a fixed moment in time at which the auction ends. These auctions are called **deadline auction** and are most frequently used on the Internet. Sites such as *eBay* use deadline auctions that end at a specific time without the possibility of an extension (*hard close*). There are also other sites with deadline auctions that automatically extend the closing time when new bids are submitted.

In deadline auctions, the winner is the bidder who makes the highest bid prior to the deadline, so bidders adapt their bidding strategies to this rule. The auction literature describes **last minute bidding** (or **sniping**) as a strategy in which bidders wait until the end of the auction (even when facing the possibility of not being able to bid) to send their bids. We highlight the work of Ockenfels and Roth [58, 59], who studied the behavior of bidders in deadline auctions and detected the use of this strategy to avoid price wars, the transmission of information to their rivals, and the loss of opportunities to bid on similar auctions being conducted simultaneously. This strategy is used primarily by **risk loving** bidders, who prefer to risk the inability to bid but aim to obtain higher surplus.

Other auction models set the end of the auction with a certain degree of randomness. For example, it was common in the Middle Ages to use **candle auctions**, which are auctions that end when the candle burns to its end.

3.6 All-Pay Auctions

In an **all-pay auction**, all bidders, including winners and losers, must pay an amount equal to the bid placed ($b_i = p_i^*$), and the winner is the bidder who submits the highest bid, b^*.[1] In this auction model, the seller receives a revenue equal to the sum of the bids made by all bidders:

$$R^* = \Sigma_{i=1}^{N} b_i = \Sigma_{i=1}^{N} p_i^* . \tag{3.1}$$

Although this auction format is rarely used in practice, it can be useful to analyze and understand many social situations. Baye et al. [11] analyzed political processes in which lobbyists make payments to politicians through campaign contributions. The purpose of these up-front payments is to be favored if the political group is

[1] This auction model corresponds to a first-price auction in which everyone pays.

Table 3.3 Dollar auction

Bids = payments	$b_1 = p_1^*$	$b_2 = p_2^*$	$b_3 = p_3^*$	$b_4 = p_4^*$
	0.10	0.30	0.35	0.40
Highest bid				b^*
				0.40
Seller's revenue: $R^* = 1.15$				

elected. Therefore, individuals who make such contributions before the election are competing in an auction in which everyone pays, as payments are not refunded after the election.

A particular example of this type of auction is the **dollar auction** first presented by Shubik [74] and later analyzed by others, such as O'neill [61]. In this auction, a seller offers a dollar that is awarded to the bidder who submits the highest bid, but all bidders must pay their placed bids. Suppose that you perform a dollar auction, and the seller receives the bids listed in Table 3.3. In this example, the fourth bidder wins by placing the highest bid $b^* = b_4 = 0.4$ USD and thus obtains a dollar minus his bid, 0.6 USD. However, the other bidders must pay their offers, so the seller receives a revenue of 1.15 USD. In this example, the seller obtains a profit of 0.15 USD.

Many experiments have been done using the dollar auction with ascending rounds, and the following behavior has been observed. In the beginning, the bidders make small bids because they believe that they have little to lose. However, as the price approaches one dollar, only a few bidders remain active (often only two). At this point, the bidders who continue to bid are keen on winning the dollar because, otherwise, they lose all that they have already offered. Surprisingly, these strategies often result in the dollar going for a price exceeding a dollar. In fact, it is common that after the bids have exceeded one dollar, the bidders fail to act rationally and are influenced by the social perception that others have of them; thus, they continue bidding to avoid being perceived as losers. On some occasions, this behavior was so intense that the final auction price reached five or six dollars.

The dollar auction is used to explain situations of escalating conflict in which the objectives change according to the auction's progress (the conflict). At first, the bidders want to win, then they want to lose as little as possible, and finally they simply do not want their rivals to win and they try to prove that they are stronger, which leads them to engage in irrational behavior. Often, governments are involved in conflicts whose intensity increases progressively, where de-escalation also becomes progressively more difficult and rational behavior ceases.[2]

[2] An example might be the race of many governments to invest in nuclear weapons. Although it would be more rational to reach an agreement that no one has nuclear weapons, some governments invest in them if other countries have done so.

3.6.1 War of Attrition

A special type of all-pay auction is the **war of attrition**.[3] According to this model, all losing bidders pay the bid that they have made, and the winner pays the second highest bid.[4]

As all-pay auctions, the war of attrition is commonly used to model conflicts between companies or animals. Maynard Smith [46] devotes a chapter of his book to explore this auction and creates a model in which two animals compete for the same territory. The loser is the animal that first withdraws from the competition, and both pay for the time and any damage caused by the conflict. The authors Bishop, Cannings, and Maynard Smith [12] have also discussed conflicts between animals using this model. Transferring this conflict to the business environment, Fudenberg and Tirole [32] developed a duopoly model in which two companies are competing for market share.

3.7 Optimal Auctions and Efficient Auctions

Upon reviewing the previous sections and chapters, it is clear that many auction models can be used to award an item. The design of an auction involves deciding on the bidding format, the pricing rule, the closing rule, etc. The choice of a specific auction model is conditioned by the objectives to be achieved. Usually, sellers have two main goals: revenue maximization and efficient allocation of the items. According to which of the objectives is accomplished, the auction may be categorized as optimal or efficient.

- **Optimal auction**
  Optimal auctions are those that *maximize the expected revenue for the seller*. According to the **Revenue Equivalence Theorem**, if bidders have private values and under certain assumptions,[5] the four standard single-unit auctions yield the same expected revenue to the seller, so all would be considered optimal mechanisms. However, as discussed in Chap. 2, when these assumptions are relaxed, the expected revenues are not equal. The first work on optimal auctions was presented by Myerson [56]. This topic has since been developed by many authors, among which we can highlight the contributions of Riley and Samuelson [65], Cremer and McLean [21], and Bulow and Klemperer [13], as well as the study developed by Engelbrecht-Wiggans [28] on multi-unit auctions.

[3] Krishna and Morgan [42] compared the expected revenue for an all-pay auction and a war of attrition, assuming that the bidders have interdependent values.

[4] This auction model corresponds to a second-price auction in which everyone pays.

[5] Under the assumptions mentioned in Chap. 2: risk neutral bidders, bidders with independent-private-values, symmetric bidders and no budget constraints.

- **Efficient auction**
  An auction is efficient if, after the auction, *the items are awarded to the bidders who most value them*. **Incentive compatible** auctions, i.e., auctions in which the dominant strategy of all bidders is to submit sincere bids, will be efficient because the bidder with the highest valuation will make the highest bid and therefore win the item. The **second-price sealed-bid auction** satisfies this property and is therefore an efficient auction. Conversely, bidders tend to bid below their values in first-price auctions, so the bidder who placed the highest bid might not have the highest valuation; thus, the final allocation will probably not be efficient.

It is quite common for the two goals to be conflicting: revenue maximization versus efficient allocation. If the seller tries to maximize the expected revenue, he often fails to award the items to the bidders who value them most or even does not award all of them.[6]

3.8 Variables Used in This Chapter

In this chapter, we use the following variables:

- $I = (1, 2, \ldots, N)$: Bidders.
- v_i: Value of bidder i.
- b_i: Bid of bidder i in a sealed-bid auction (single-round).
- b_i^{II}: Bid of bidder i in Phase II of a hybrid auction whose Phase II corresponds to a sealed-bid auction (single-round).
- $b_i^{\mathrm{I},t}$: Bid of bidder i in round t of Phase I of a hybrid auction whose Phase I corresponds to a dynamic auction.
- $b_i^{\mathrm{II},t}$: Bid of bidder i in round t of Phase II of a hybrid auction whose Phase II corresponds to a dynamic auction.
- b^*: Highest bid made by any bidder (winning bid).
- $p^{\mathrm{I},t}$: Price in round t of Phase I of a hybrid auction whose Phase I corresponds to a dynamic auction.
- $p^{\mathrm{II},t}$: Price in round t of Phase II of a hybrid auction whose Phase II corresponds to a dynamic auction
- $p^{\mathrm{I,s}}$: Starting bid (minimum bid) of Phase I of a hybrid auction.
- $p^{\mathrm{II,s}}$: Starting bid (minimum bid) of Phase II of a hybrid auction.
- p^*: Selling price. Only if the auctioneer sets the first-price rule the selling price matches the highest bid $p^* = b^*$.

[6] Ausubel and Cramton [6] show that if there is an efficient resale market, an efficient auction can also maximize the seller's expected revenue.

- Γ_i^*: Income of bidder i. In single-unit auctions, it holds that $\Gamma_i^* = v_i$.
- Π_i^*: Surplus of bidder i.
- R^*: Seller's revenue. In single-unit auctions, it holds that $R^* = p^*$. Furthermore, only if the seller sets the first-price rule it holds that this matches the highest bid $R^* = p^* = b^*$.

3.9 Exercises

1. An Anglo-Dutch auction is held with the following features: four bidders with private values of $v_1 = 50$ euros, $v_2 = 50$ euros, $v_3 = 90$ euros, and $v_4 = 80$ euros; a starting bid of $p^s = 40$ euros; and a bid increment of 10 euros. Find:
   (a) The bidders who reach Phase II of the auction assuming that all bidders bid while the price is below their private valuation.
   (b) The minimum bid in Phase II.
2. Using data from the previous exercise, find the winning bidder, his surplus, and the seller's revenue after Phase II if:
   (a) Both bidders place sincere bids, $b_i^{\mathrm{II}} = v_i$.
   (b) The fourth bidder places a sincere bid, but the third bids $b_3^{\mathrm{II}} = 85$ euros.
   (c) The fourth bidder places a sincere bid, but the third bids $b_3^{\mathrm{II}} = 70$ euros.
3. In a sealed-bid auction, three bidders place the following bids: $b_1 = 50$ euros, $b_2 = 100$ euros, and $b_3 = 150$ euros. Calculate the winning bidder, the price paid, and the seller's revenue for the following pricing rules:
   (a) The first-price rule.
   (b) The second-price rule.
   (c) The third-price rule.
   (d) The average of the bids rule.
4. Two bidders have the following private values $v_1 = 100$ euros and $v_2 = 80$ euros. We assume that the second bidder always places sincere bids: $b_2 = v_2 = 80$ euros.
   (a) Compute the auction outcome if the seller holds a second-price sealed-bid auction and the first bidder places a bid equal to his value ($b_1 = v_1$).
   (b) Compute the auction outcome if the seller holds a first-price sealed-bid auction and the first bidder is risk averse, and therefore, submits a bid just below his value: $b_1 = 90$ euros.
   (c) Compute the auction outcome if the seller holds a first-price sealed-bid auction and the first bidder is risk loving, and therefore, submits a bid substantially below his value: $b_1 = 50$ euros.
   (d) For each case, comment on whether the objectives of profit maximization (optimal auction) and efficient allocation of items (efficient auction) have been achieved.

3.10 Solutions to Exercises

1. Table 3.4 shows Phase I of the Anglo-Dutch auction. At the starting price, all bidders participate. In round $t = 1$, the first and the second bidders stop bidding because the price reaches their valuations.
   (a) The third and fourth bidders are allowed to bid in Phase II.
   (b) The minimum bid in Phase II is 50 euros.
2. The final outcome will depend on the bids placed by the active bidders in Phase II.
   (a) Table 3.5 shows the submitted bids assuming that the bidders bid according to their true valuations. The winner is the third bidder, and the price paid, which is also the seller's revenue, is equal to $p^* = R^* = 90$ euros. The buyer's surplus is $\Pi_3^* = 90 - 90 = 0$ euros.
   (b) If the third bidder wants to obtain a positive surplus, he should bid below his value. By bidding $b_3^{\mathrm{II}} = 85$ euros, he still wins the auction and pays $p^* = R^* = 85$ euros, which is also the seller's revenue. His surplus is equal to $\Pi_3^* = 90 - 85 = 5$ euros (see Table 3.6).
   (c) The third bidder can also submit an even lower bid to try to increase his surplus. However, if he bids $b_3^{\mathrm{II}} = 70$ euros, he is no longer the winner, and instead, the fourth bidder gets the item ($b^* = b_4^{\mathrm{II}}$), see Table 3.7. The new winner pays $p^* = R^* = 80$ euros. The surplus of the buyer is $\Pi_4^* = 80 - 80 = 0$ euros, because he placed a sincere bid.

Table 3.4 Anglo-Dutch auction (exercise 1)

Phase I: ascending					
Round (t)	Round price (p^t)	b_1^t	b_2^t	b_3^t	b_4^t
$t = 0$	$p^s = 40$	40	40	40	40
$t = 1$	$p^1 = 50$	0	0	50	50
	v_i	$v_1 = 50$	$v_2 = 50$	$v_3 = 90$	$v_4 = 80$

Table 3.5 Anglo-Dutch auction (exercise 2.a)

Phase II: sealed-bid				
	b_1^{II}	b_2^{II}	b_3^{II}	b_4^{II}
bid	0	0	90	80
v_i	$v_1 = 50$	$v_2 = 50$	$v_3 = 90$	$v_4 = 80$

Table 3.6 Anglo-Dutch auction (exercise 2.b)

Phase II: sealed-bid				
	b_1^{II}	b_2^{II}	b_3^{II}	b_4^{II}
bid	0	0	85	80

Table 3.7 Anglo-Dutch auction (exercise 2.c)

Phase II: sealed-bid				
	b_1^{II}	b_2^{II}	b_3^{II}	b_4^{II}
bid	0	0	70	80

Table 3.8 Optimal auction and efficient allocation

Pricing rule	Allocation	Seller's revenue	b_1	b_2
Second-price	Efficient	80	100	80
First-price	Efficient	90	90	80
First-price	Inefficient	50	50	80

3. The winner is always the third bidder, as he submitted the highest bid: $b^* = b_3 = 150$ euros. However, the seller's revenue (final price) will depend on the pricing rule selected.
   (a) According to the first-price rule: $R^* = p^* = 150$ euros.
   (b) According to the second-price rule: $R^* = p^* = 100$ euros.
   (c) According to the third-price rule: $R^* = p^* = 50$ euros.
   (d) According to the average of the bids rule: $R^* = p^* = \frac{50+100+150}{3} = 100$ euros.
4. According to the auction model and the submitted bids, the following outcomes are obtained.
   (a) *Second-price sealed-bid auction*: If $b_1 = 100$ euros and $b_2 = 80$ euros, the first bidder is the winner. The amount to be paid is equal to the second highest bid placed, $p^* = R^* = 80$ euros, which is the revenue of the seller, and the buyer obtains a surplus of $\Pi_1^* = \Gamma_1^* - p^* = 100 - 80 = 20$ euros.
   (b) *First-price sealed-bid auction with a risk-averse first bidder:* If $b_1 = 90$ euros and $b_2 = 80$ euros, the first bidder is the winner. His surplus is equal to $\Pi_1^* = \Gamma_1^* - p^* = 100 - 90 = 10$ euros. The seller's revenue is equal to $R^* = p^* = 90$ euros.
   (c) *First-price sealed-bid auction with a risk-loving first bidder:* If $b_1 = 50$ euros and $b_2 = 80$ euros, the first bidder is no longer the winner and $b^* = b_2$. The surplus of the second bidder is equal to $\Pi_2^* = 0$, because he has submitted a sincere bid. The revenue of the seller is $R^* = p^* = 80$ euros.
   (d) Comments:

   Second-price sealed-bid auction: the dominant strategy of all bidders is to bid sincerely. With this strategy, the auction yields an efficient allocation (efficient auction). In this example, the first bidder has the highest valuation and won the item (Section a).

   First-price sealed-bid auction: the bidders will always tend to bid below their values to achieve a positive surplus (because sincere bidding results in zero surplus if the first-price rule is applied). Therefore, depending on the submitted bids, the result can be efficient or inefficient. If the bidders are risk-averse, they will tend to raise their bids closer to the values, which generates more revenue for the seller (Section b). However, if bidders are risk-loving, they will tend to reduce their bids further below their values, which will reduce the revenue of the seller (Section c). Table 3.8 shows the outcomes.

Assigning Multiple Homogeneous Items in a Single Auction

4

4.1 Introduction

In previous chapters, the main single-unit auction models were discussed. However, in many instances auctions are used to award multiple related units. For example, if an olive oil factory wants to sell part of its stock, it can choose to conduct an auction. The items to be auctioned may be homogeneous (oil bottles of the same size and acidity) or heterogeneous (different sizes and acidities) and may be awarded in a single auction or in different auctions. This chapter presents situations in which multiple identical items (homogeneous) are available in the same auction. These auctions can be dynamic, sealed-bid (single-round), or hybrid processes. We also assume that the items to be auctioned are **substitutes**,[1] so the marginal value of the second item for bidder i cannot exceed the marginal value of the first item: $v_{i,1} \geq v_{i,2}$, where $v_{i,1}$ is bidder i's value for the first item.

4.2 Multi-unit Dynamic Auctions of Homogeneous Items

In a multi-unit auction the seller will have to choose between a dynamic or a sealed-bid process. If the first option is chosen, the main models are the ascending auction, the descending auction, and the Ausubel auction.

[1]In multi-unit auctions, items can be substitutes or **complements**. If the items are substitutes, the value of winning the second unit is less than the value of winning the first. Items are complements (**synergies**) if the value of winning the second unit is higher than that of winning the first, if the first item has already been purchased. In Chap. 6, we will explain these concepts in more detail.

A. Mochón, Y. Sáez, *Understanding Auctions*, Springer Texts in Business and Economics, DOI 10.1007/978-3-319-08813-6_4

4.2.1 Multi-unit Ascending Auction of Homogeneous Items

In a **multi-unit ascending auction**, also called **multi-unit English auction** a seller offers M identical items ($J = (1, 2, \ldots, M)$) to N bidders ($I = (1, 2, \ldots, N)$). The seller sets a minimum bid (p^s) and each bidder i indicates which quantity q_i^0 is willing to buy at that price (round $t = 0$). At the end of each round, the seller calculates the aggregate demand ($Q^t = \Sigma_{i=1}^N q_i^t$). If it is greater than the supply ($Q^t > M$), the seller raises the price in the next round, and the bidders submit new bids (q_i^{t+1}). As the price increases, the bidders' demand decreases, and the auction ends when the number of items demanded does not exceed the supply ($Q^T \leq M$), where T is the last round of the auction. All winners pay the price of the last round per acquired item ($p^* = p^T$). Bidder i's payment is:

$$P_i^* = p^* q_i^*, \tag{4.1}$$

where q_i^* is the number of items won by bidder i. The seller's revenue is equal to the payment made by all winning bidders:

$$R^* = \Sigma_{i \in W} P_i^*, \tag{4.2}$$

where W is the set of winning bidders. The values of bidder i are represented by the vector $V_i = (v_{i,1}, v_{i,2}, \ldots v_{i,M})$, where $v_{i,j}$ is bidder i's marginal value for item j.[2] Bidder i's income for having won q_i^* items is equal to the sum of the private values of the items won:

$$\Gamma_i^* = \Sigma_{j=1}^{q_i^*} v_{i,j}. \tag{4.3}$$

The surplus of bidder i is equal to the difference between the income and the total payment:

$$\Pi_i^* = \Gamma_i^* - P_i^*. \tag{4.4}$$

To better understand this model, consider an auction of seven identical items ($M = 7$) with an opening bid per item of $p^s = 2$ euros and a bid increment of one euro. The first bidder's valuation vector is $V_1 = (12, 10, 8, 6, 4, 0, 0)$, the second's vector is $V_2 = (6, 5, 4, 3, 2, 0, 0)$, and the third's vector is $V_3 = (6, 4, 3, 2, 1, 0, 0)$. We assume that all bidders bid on items whose prices are below their marginal valuation ($p^t < v_{i,j}$). In round $t = 0$ bidders submit the following demands: $q_1^0 = 5$ items, $q_2^0 = 4$ items, and $q_3^0 = 3$ items. The aggregate demand in this round is $\Sigma_{i=1}^N q_i^0 = Q^0 = 12$ items, which exceeds supply ($M = 7$); thus, the seller raises the price. In the following round ($t = 1$), the price rises to $p^1 = 3$ euros, and the bidders place the following bids: $q_1^1 = 5$ items, $q_2^1 = 3$ items, and $q_3^1 = 2$ items.

[2] In this chapter, we assume substitute items: decreasing marginal values, $v_{i,1} \geq v_{i,2} \geq \cdots \geq v_{i,M}$.

Table 4.1 Rounds in a multi-unit ascending auction of homogeneous items

Round (t)	Price (p^t)	q_1^t	q_2^t	q_3^t	Q^t
$t = 0$	$p^0 = 2$	5	4	3	12
$t = 1$	$p^1 = 3$	5	3	2	10
$t = 2$	$p^2 = 4$	4	2	1	7
Values	$v_{i,j}$	$v_{1,j}$	$v_{2,j}$	$v_{3,j}$	
	$j = 1$	12	6	6	
	$j = 2$	10	5	4	
	$j = 3$	8	4	3	
	$j = 4$	6	3	2	
	$j = 5$	4	2	1	
	$j = 6$	0	0	0	
	$j = 7$	0	0	0	

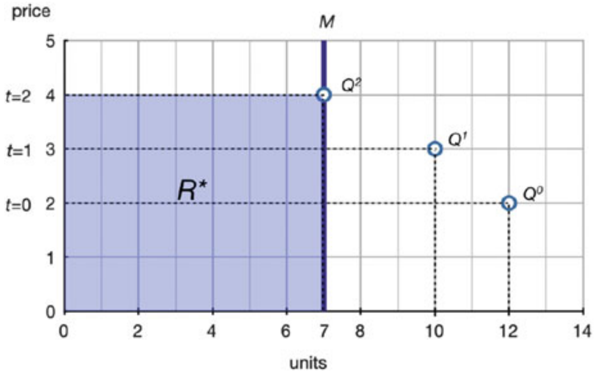


Fig. 4.1 Multi-unit ascending auction of homogeneous items

As total demand continues to exceed supply ($Q^1 = 10 > 7 = M$), the seller raises the price again. In round $t = 2$, the price is $p^2 = 4$ euros, and the bidders place the following bids: $q_1^2 = 4$ items, $q_2^2 = 2$ items, and $q_3^2 = 1$ item. In this round, demand equals supply ($M = Q^2 = 7$ items), and the auction ends. The auction is summarized in Table 4.1.

The auction yields the following outcome. The first bidder wins $q_1^* = 4$ items and pays $P_1^* = 4 \times 4 = 16$ euros, the second bidder wins $q_2^* = 2$ items for $P_2^* = 2 \times 4 = 8$ euros, and the third bidder obtains $q_3^* = 1$ item at a price of $P_3^* = 1 \times 4 = 4$ euros. The price per item is the same for all bidders, $p^* = 4$ euros. Each bidder's income is equal to the values for the items acquired: $\Gamma_1^* = \Sigma_{j=1}^{4} v_{1,j} = 36$ euros, $\Gamma_2^* = \Sigma_{j=1}^{2} v_{2,j} = 11$ euros, and $\Gamma_3^* = v_{3,1} = 6$ euros; thus, the surpluses obtained are $\Pi_1^* = \Gamma_1^* - P_1^* = 36 - 16 = 20$ euros, $\Pi_2^* = \Gamma_2^* - P_2^* = 11 - 8 = 3$ euros, and $\Pi_3^* = \Gamma_3^* - P_3^* = 6 - 4 = 2$ euros. Finally, the seller gets a revenue equal to the sum of the payments from all bidders, $R^* = P_1^* + P_2^* + P_3^* = 16 + 8 + 4 = 28$ euros.

Figure 4.1 shows the aggregate demand in each round which is greater than supply ($Q^t > M$) in rounds $t = 0$ and $t = 1$, so the price increases. However, demand equals supply in round $t = 2$, and the auction ends. The final price is $p^* = 4$ euros per item. The seller obtains a revenue equal to the shaded area $R^* = 7 \times 4 = 28$ euros.

It may happen that as the price increases, the aggregate demand decreases below the quantity supplied ($Q^T < M$). In order to avoid having unsold items, the seller can set a **rationing rule** to allocate the items. The appendix included at the end of the chapter explains one of the most common rationing rules: the proportional rationing rule.

4.2.2 Multi-unit Descending Auction of Homogeneous Items

In a **multi-unit descending auction**, also known as a **multi-unit Dutch auction**, a seller offers M identical items ($J = (1, 2, \ldots, M)$) to N bidders ($I = (1, 2, \ldots, N)$). The auction begins with a relatively high starting price per unit (p^s) set by the seller. This price decreases successively in each round, and when a bidder is interested in purchasing an item at the price in round t (p^t), a bid is placed. At this point, the bidder automatically wins the item. The total number of items assigned in round t to the bidders who have bid is equal to:

$$Q^{t*} = \Sigma_{i=1}^{N} q_i^{t*}, \tag{4.5}$$

where q_i^{t*} is the number of items won by bidder i in round t. The total number of items won from round 0 to round l by all bidders is equal to $\Sigma_{t=0}^{l} Q^{t*}$. The seller continues decreasing the price per item while the items assigned until round l is less than the total supply $\Sigma_{t=0}^{l} Q^{t*} < M$. The auction ends at round T when no items remain $\Sigma_{t=0}^{T} Q^{t*} = M$.[3] Bidder i's acquired items at the end of the auction is equal to the items acquired in all rounds:

$$q_i^* = \Sigma_{t=0}^{T} q_i^{t*}. \tag{4.6}$$

In contrast to the multi-unit ascending auction, each bidder pays a different amount for the items obtained in this model. For each item, they pay the standing price of the round in which each item is won. The final payment made by bidder i is equal to the sum of the items acquired in each round multiplied by the prices in the corresponding rounds (p^t):

$$P_i^* = \Sigma_{t=0}^{T} p^t q_i^{t*}. \tag{4.7}$$

[3]The aggregate demand in round T could exceed the remaining supply, $\Sigma_{t=0}^{T} Q^{t*} > M$. To solve this tie, the seller can set different rules to allocate the items among the bidders that have submitted offers in this last round. There are many options such as: allocating the items to those who bid first or in a random way.

The income for bidder i is equal to the sum of the values of the items won at the end of the auction:

$$\Gamma_i^* = \Sigma_{j=1}^{q_i^*} v_{i,j}, \tag{4.8}$$

and the surplus is the difference between the income and the total payment:

$$\Pi_i^* = \Gamma_i^* - P_i^*. \tag{4.9}$$

Finally, the seller's revenue is equal to the sum of the payments made by all winning bidders $i \in W$:

$$R^* = \Sigma_{i \in W} P_i^*. \tag{4.10}$$

The following example describes a multi-unit descending auction of six identical items in which there are three bidders. The valuation vectors of the three bidders are $V_1 = (250, 200, 150, 100, 50, 25)$ for the first bidder, $V_2 = (200, 150, 125, 100, 75, 50)$ for the second, and $V_3 = (150, 140, 130, 120, 110, 100)$ for the third. To simplify we assume that bidders bid on any item for which the price equals the marginal value: sincere bidding ($p^t = v_{i,j}$).[4]

The starting price per item is $p^s = 300$ euros, a price at which nobody bids. The bid decrement is 50 euros, so the price in the following round is $p^1 = 250$ euros. The first bidder bids for one item ($q_1^{1*} = 1$) at 250 euros. In the next round ($p^2 = 200$ euros), the first and the second bidder bid on an item, $q_1^{2*} = 1$, and $q_2^{2*} = 1$, which they acquire at the price set in this round. The total number of items assigned until round $t = 2$ is less than the supply ($\Sigma_{t=0}^{2} Q^{2*} = 3 < 6 = M$), and thus, the price continues to decrease. The auction concludes in the third round when the price reaches $p^3 = 150$ euros and all three bidders bid on an item: $q_1^{3*} = 1$, $q_2^{3*} = 1$, and $q_3^{3*} = 1$. Table 4.2 summarizes the auction.

The first bidder wins 3 items ($q_1^* = 3$), i.e., one in round $t = 1$, another one in round $t = 2$, and a third in round $t = 3$; and his total payment is $P_1^* = (1 \times 250) + (1 \times 200) + (1 \times 150) = 600$ euros. The second bidder wins two items ($q_2^* = 2$): one in round $t = 2$ and another in $t = 3$. For these items he pays $P_2^* = (1 \times 200) + (1 \times 150) = 350$ euros. Finally, the third bidder only wins one item ($q_3^* = 1$) and pays $P_3^* = (1 \times 150) = 150$ euros. Each bidder's income is equal to the sum of the values of the items acquired: $\Gamma_1^* = \Sigma_{j=1}^{3} v_{1,j} = 250 + 200 + 150 = 600$ euros for the first bidder, $\Gamma_2^* = \Sigma_{j=1}^{2} v_{2,j} = 200 + 150 = 350$ euros for the second bidder, and $\Gamma_3^* = v_{3,1} = 150$ euros for the third bidder. In this example, because all bidders placed sincere bids, none of them have a positive surplus: $\Pi_1^* = \Gamma_1^* - P_1^* = 0$

[4] In multi-unit descending auction bidders will not bid sincerely because this strategy implies zero surplus. Bidders tend to wait until the price is less than the valuation to obtain a positive surplus: underbidding ($p^t < v_{i,j}$).

Table 4.2 Rounds in a Multi-unit descending auction of homogeneous items

Round (t)	Round price (p^t)	q_1^{t*}	q_2^{t*}	q_3^{t*}	$\Sigma_{t=0}^{l} Q^{t*}$
$t = 0$	$p^0 = 300$	0	0	0	0
$t = 1$	$p^1 = 250$	1	0	0	1
$t = 2$	$p^2 = 200$	1	1	0	3
$t = 3$	$p^3 = 150$	1	1	1	6
Values	$v_{i,j}$	$v_{1,j}$	$v_{2,j}$	$v_{3,j}$	
	$j = 1$	250	200	150	
	$j = 2$	200	150	140	
	$j = 3$	150	125	130	
	$j = 4$	100	100	120	
	$j = 5$	50	75	110	
	$j = 6$	25	50	100	

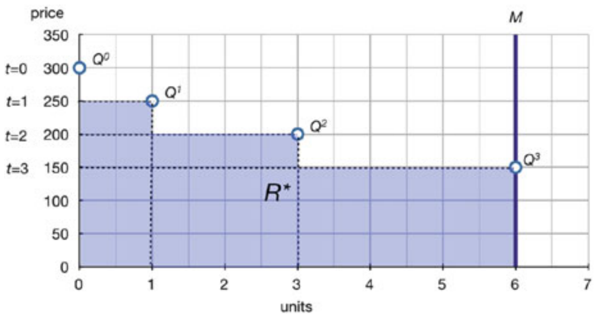


Fig. 4.2 Multi-unit descending auction of homogeneous items

euros, $\Pi_2^* = \Gamma_2^* - P_2^* = 0$ euros, and $\Pi_3^* = \Gamma_3^* - P_3^* = 0$ euros. Finally, the revenue obtained by the seller after allocating the six items is equal to the amount paid by all of the winning bidders: $R^* = P_1^* + P_2^* + P_3^* = 1100$ euros.

Figure 4.2 illustrates the aggregate demand in each round. At the starting price there are no bids. In round $t = 1$, a bidder demands and wins one item at that round's price $p^1 = 250$ euros. In round $t = 2$, bidders demand two items at a price of $p^2 = 200$ euros. The last three items are allocated in round $t = 3$ at a price of $p^3 = 150$ euros. In this auction model, each bidder pays a different price for the items won. The seller's revenue is equal to the shaded area $R^* = (1 \times 250) + (2 \times 200) + (3 \times 150) = 1100$ euros.

4.2.3 The Ausubel Auction

The **Ausubel auction**, presented by Ausubel [4], is a particular mechanism of multi-unit ascending auctions. The seller sets a starting bid per item (p^s) in round $t = 0$. In subsequent rounds, the price increases, and the bidders indicate the number of items that they are willing to buy (q_i^t). With the bids submitted in round t, the seller calculates the aggregate demand $Q^t = \Sigma_{i=1}^{N} q_i^t$ and increases the price if the demand is greater than the supply ($Q^t > M$). Bidders indicate demand at the updated price of the next round, which can never be greater than their demand in the previous rounds.[5] The auction ends in round T when the aggregate demand of all bidders does not exceed the supply ($Q^T \leq M$).[6]

The difference between this auction and the multi-unit ascending auction lies in the price that bidder i must pay for items acquired at the end of the auction q_i^*. The Ausubel auction states that each bidder has to pay the price for each item in the round in which the item was obtained (*clinched*). A bidder obtains an item in round t when the aggregate demand of the other bidder's $k \neq i$ is less than the supply ($\Sigma_{k \neq i} q_k^t < M$) but the total demand of all bidders is greater than the supply ($Q^t > M$).[7] At this point, it holds that the residual supply of bidder i is greater than zero ($M_i^t > 0$). Bidder i's residual supply is calculated as the total supply minus the demand of the rivals:

$$M_i^t = M - \Sigma_{k \neq i} q_k^t. \tag{4.11}$$

As bidders cannot increase their bids in future rounds, if bidder i's residual supply is greater than zero ($M_i^t > 0$), this implies that he has already guaranteed to win an item (clinched). The number of items that bidder i has clinched until round t (C_i^t) is equal to the maximum value between zero and the residual supply of bidder i in that round, calculated as follows:

$$C_i^t = \max\{0, M_i^t\}. \tag{4.12}$$

The number of clinched items in round t is equal to:

$$c_i^t = C_i^t - C_i^{t-1}, \tag{4.13}$$

Therefore, we can also calculate the total number of items clinched by bidder i until round t as the sum of the items assured in each round:

$$C_i^t = \Sigma_{t=0}^T c_i^t. \tag{4.14}$$

When the auction ends, the number of items awarded to bidder i is equal to $q_i^* = C_i^T$. For each item allocated to bidder i, he will pay the corresponding price

[5] According to the rules of the auction, when the price increases, bidders are required to maintain or decrease their bid, but not increase it. Furthermore, bidders who decide to stop bidding in a round cannot bid again in later rounds.

[6] It could happen that at a certain increase of the price, the aggregate demand decreases below the supply ($Q^T < M$). To avoid unsold items, the seller can set a rationing rule to allocate all items, see Appendix at the end of this chapter.

[7] This procedure sequentially implements the **Vickrey rule**, which establishes that each bidder pays an amount equal to the opportunity cost of the item won.

Table 4.3 Rounds in an Ausubel auction

Round (t)	Round price (p^t)	q_1^t	q_2^t	q_3^t	Q^t
$t = 0$	$p^0 = 100$	3	4	4	11
$t = 1$	$p^1 = 150$	2	3	4	9
$t = 2$	$p^2 = 200$	1	2	3	6
$t = 3$	$p^3 = 250$	1	2	2	5
$t = 4$	$p^4 = 300$	1	1	2	4
Values	$v_{i,j}$	$v_{1,j}$	$v_{2,j}$	$v_{3,j}$	
	$j = 1$	350	350	400	
	$j = 2$	200	300	350	
	$j = 3$	150	200	250	
	$j = 4$	50	150	200	

to the round in which he clinched the item. Bidder i's payment for the acquired items is equal to:

$$P_i^* = \Sigma_{t=0}^{T} p^t c_i^t. \tag{4.15}$$

The income for bidder i is equal to the sum of the values of the acquired items:

$$\Gamma_i^* = \Sigma_{j=1}^{q_i^*} v_{i,j}, \tag{4.16}$$

and the surplus is difference between the income and the payment:

$$\Pi_i^* = \Gamma_i^* - P_i^*. \tag{4.17}$$

The seller's revenue is the sum of the payments made by all winning bidders, $i \in W$:

$$R^* = \Sigma_{i \in W} P_i^*. \tag{4.18}$$

To depict the step-by-step operation of this auction, let us review the following example. A seller holds an Ausubel auction to award four identical items ($M = 4$) to three bidders with the following valuation vectors: $V_1 = (350, 200, 150, 50)$ for the first bidder, $V_2 = (350, 300, 200, 150)$ for the second bidder, and $V_3 = (400, 350, 250, 200)$ for the third bidder. The seller sets $p^s = 100$ euros and a bid increment of 50 euros. If we assume that the bidders bid while the price is below their marginal value, the auction develops as shown in Table 4.3. To determine how many items are allocated to each bidder, the Ausubel auction works like a multi-unit ascending auction. In this example, the auction ends in round $t = 4$. The first and second bidder win one item $q_1^* = q_2^* = 1$, and the third obtains two items $q_3^* = 2$.

The difference between a multi-unit ascending auction and an Ausubel auction is how to calculate the final payments. Table 4.4 shows the residual supply of

Table 4.4 Residual supply and clinched units

Round (t)	M_1^t	M_2^t	M_3^t	C_1^t	C_2^t	C_3^t
$t = 0$	-4	-3	-3	0	0	0
$t = 1$	-3	-2	-1	0	0	0
$t = 2$	-1	0	1	0	0	1
$t = 3$	0	1	1	0	1	1
$t = 4$	1	1	2	1	1	2

each bidder (M_i^t) and the accumulated clinched items per round (C_i^t), which is the maximum of 0 and the residual supply.

At the starting price, the aggregate demanded is greater than the supply ($Q^0 = 11 > 4 = M$). In this round, the residual supply of all bidders is less than zero ($M_1^0 = -4$, $M_2^0 = -3$, and $M_3^0 = -3$), so no one clinched an item ($C_1^0 = C_2^0 = C_3^0 = 0$). In the next round, the seller increases the price per item, and the bidders place new bids (equal to or lower than the previous round). Still, no bidder clinches an item. In round $t = 2$, the total demand is greater than the supply ($Q^2 = 6 > 4 = M$), but the demand of all bidders except the third is less than the total supply ($\Sigma_{k \neq 3} q_k^2 = 3 < 4 = M$). In this round, the residual supply of the third bidder is greater than zero, i.e., $M_3^2 = 1$. The third bidder thus clinches the first unit ($C_3^2 = 1$) at the price of the current round ($p^2 = 200$ euros). In the next round, the total demand is again greater than the total supply, and the second bidder clinches one item. In this round, the demand of the second bidder's rivals is lower than the total supply ($\Sigma_{k \neq 2} q_k^3 = 3 < 4 = M$), and his residual supply is $M_2^3 = 1$; therefore, the second bidder clinches one item $C_2^3 = 1$. For this item, the second bidder pays $p^3 = 250$ euros. The auction ends in round $t = 4$, in which the total demand equals the supply ($Q^4 = 4 = M$). In this round, the first bidder wins his first item ($M_1^4 = 1$, thus clinching one item $C_1^4 = 1$) and pays $p^4 = 300$ euros. The third bidder wins his second item in this round ($M_3^4 = 2$) and has accumulated two items $C_3^4 = 2$ (the first item clinched in round $t = 2$). For this second item, the third bidder pays $p^4 = 300$ euros.

The final outcome of the auction is the following. The first bidder wins one item ($q_1^* = 1$) and pays $P_1^* = 1 \times 300 = 300$ euros. The second bidder also wins one item ($q_2^* = 1$) and pays $P_2^* = 1 \times 250 = 250$ euros. Finally, the third bidder wins two items ($q_3^* = 2$) and pays $P_3^* = (1 \times 200) + (1 \times 300) = 500$ euros. The surplus obtained by the bidders are equal to $\Pi_1^* = \Gamma_1^* - P_1^* = 350 - 300 = 50$ euros for the first bidder, $\Pi_2^* = \Gamma_2^* - P_2^* = 350 - 250 = 100$ euros for the second bidder, and $\Pi_3^* = \Gamma_3^* - P_3^* = (400 + 350) - 500 = 250$ euros for the third bidder. The seller's revenue is equal to $R^* = P_1^* + P_2^* + P_3^* = 300 + 250 + 500 = 1050$ euros. Figure 4.3 illustrates the aggregate demand per round. As it shows, the auction ends in round $t = 4$ when the total demand is equal to the supply $Q^4 = M$.

In this auction model, a graph can be used to calculate the amount paid by each bidder for the acquired items. Let us focus on the third bidder as a reference point. Figure 4.4 shows his bids in each round (q_3^t) and the number of accumulated clinched items per round (C_3^t). The bidder clinches an item in round t when the

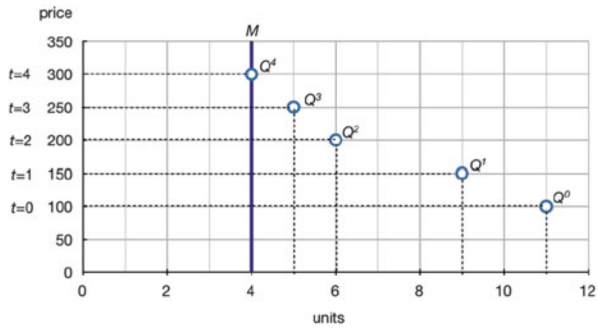


Fig. 4.3 Ausubel auction of homogeneous items

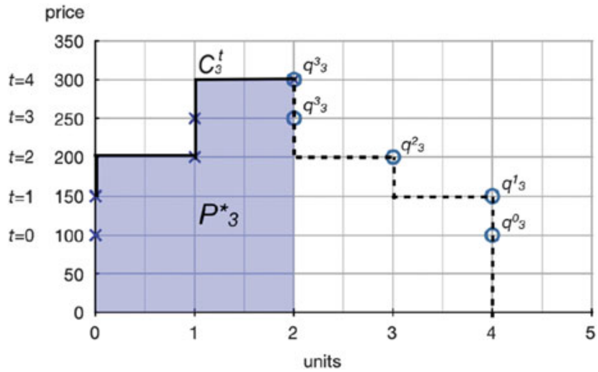


Fig. 4.4 Demand and clinched items per round for bidder three in an Ausubel auction

residual supply is greater than zero and must pay the price at that round. In this example, his final payment is equal to the shaded area ($P_3^* = (1\times200)+(1\times300) =$ 500 euros), i.e., the area below the clinched items (C_3^t) until the intersection with his demand (q_3^t). To calculate the seller's revenue, this analysis would have to be performed for all winning bidders and add the shaded areas.

The main advantage of this auction model is that with private values and diminishing marginal values, **sincere bidding** by every bidder constitutes a perfect equilibrium (*ex post*), which implies an efficient allocation of the items (**efficient auction**).[8]

4.3 Multi-unit Sealed-Bid Auction of Homogeneous Items

Instead of using dynamic auctions, the seller may choose to conduct a sealed-bid (single-round) auction. In a **multi-unit sealed-bid auction**, the seller offers M homogeneous items and bidders are asked to submit M bids. Each bidder submits a bid vector indicating the price that he is willing to pay for each unit.

[8]See demonstration in [4].

$B_i = (b_{i,1}, b_{i,2}, \ldots, b_{i,M})$ is the vector of bidder i, where $b_{i,j}$ indicates how much bidder i is willing to pay for item j.[9] We can consider a bid vector as an inverse demand function which can thus be inverted to obtain bidder i's demand function (d_i). Given a price p, bidder i demands j units for which the bids placed are equal to or higher than the price $p \leq b_{i,j}$. We obtain the demand function via the following equation:

$$d_i(p) \equiv \max\{j : p \leq b_{i,j}\}. \quad (4.19)$$

For example, in an auction of three identical items, bidder i places the following bids: $B_i = (b_{i,1}, b_{i,2}, b_{i,3}) = (6, 4, 2)$. To obtain his demand function we have to find the number of items he will be interested in at each price. If the unit price were $p = 6$ euros, the bidder would demand one item. If the unit price were $p = 4$ euros, the bidder would demand two items. Finally, if the price were $p = 2$ euros, the bidder would be interested in acquiring three items, as all bids are greater than or equal to two euros.

In this auction model, the seller receives a total of $N \times M$ bids because each of the N bidders places M bids, one for each item. The seller lists the bids in decreasing order such that M items are awarded to the M highest bids. The winning bidders are those who have made the M highest bids.[10]

In an auction with two bidders and three items, the following bids are submitted: $B_1 = (6, 4, 2)$ and $B_2 = (5, 3, 1)$, Table 4.5. The seller lists all bids in descending order $(6, 5, 4, 3, 2, 1)$, and the aggregate demand is obtained. Figure 4.5 shows the aggregate demand of the bidders and its intersection with the fixed supply: $M = 3$. The winning bids, which correspond to the M highest bids, are those that fall to the left of the supply line. In this example, the winning bids are $(b^*_{1,1}, b^*_{2,1}, b^*_{1,2}) = (6, 5, 4)$. The first bidder wins two items and the second one wins one. The losing bids are to the right of the supply.[11]

Table 4.5 Bids in a multi-unit sealed-bid auction of homogeneous items

	$b_{1,j}$	$b_{2,j}$
$j = 1$	6	5
$j = 2$	4	3
$j = 3$	2	1

[9] We have assumed that the marginal values are decreasing (substitute items). Therefore, the bid vector must satisfy the following condition: $b_{i,1} \geq b_{i,2} \geq \cdots \geq b_{i,M}$.

[10] To calculate the winners, the aggregate demand function must first be obtained by horizontally adding the N individual demand functions of the bidders. The supply will then determine the winner bidders.

[11] If there is a tie among bidders, the seller can set a tie-breaking rule such as: by order of submission and randomly.

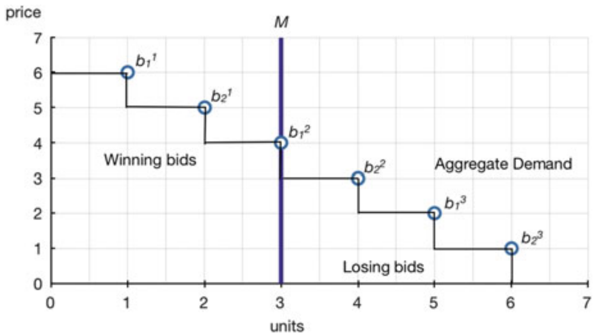


Fig. 4.5 Multi-unit sealed-bid auction of homogeneous items

With this mechanism, the seller solves the allocation problem. The price paid by each winner will depend on the pricing rule selected. The main options are the discriminatory auction, uniform-price auction, and Vickrey auction.[12]

4.3.1 Discriminatory Auction

In a **discriminatory auction**, also called **pay-your-bid auction**, the winning bidders pay an amount equal to the sum of the bids placed on the items acquired. If bidder i has q_i^* winning bids (among the M highest bids) and obtains q_i^* items, his total payment is equal to:

$$P_i^* = \Sigma_{j=1}^{q_i^*} b_{i,j}^*, \tag{4.20}$$

where $b_{i,j}^*$ represents the winning bid submitted by bidder i for item j. The surplus of bidder i is equal to the difference between the income and the cost:

$$\Pi_i^* = \Gamma_i^* - P_i^*. \tag{4.21}$$

Bidder i's income is equal to the sum of the values of the items acquired:

$$\Gamma_i^* = \Sigma_{j=1}^{q_i^*} v_{i,j}. \tag{4.22}$$

Finally, the seller's revenue is equal to the sum of the payments done by all of the winning bidders:

$$R^* = \Sigma_{i \in W} P_i^*. \tag{4.23}$$

[12]The specific auction chosen affects the bidding strategy of the bidders, which makes it difficult to determine the model that generates the greatest revenue for the seller.

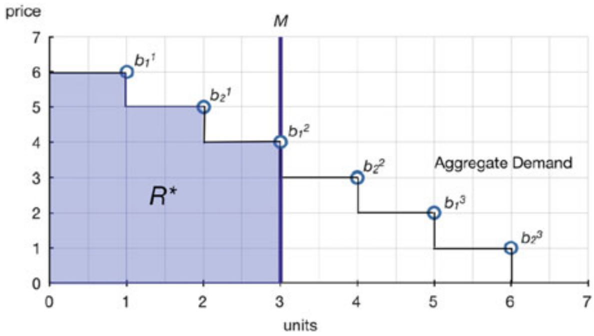


Fig. 4.6 Discriminatory auction

Continuing with the example of the previous section, whose bids are contained in Table 4.5, the amount that the first bidder pays for the winning items is $P_1^* = b_{1,1}^* + b_{1,2}^* = 6 + 4 = 10$ euros. The amount that the second bidder pays for the winning item is $P_2^* = b_{2,1}^* = 5$ euros. The total revenue that the seller obtains is $R^* = P_1^* + P_2^* = 15$ euros. As shown in Fig. 4.6, the seller's revenue is equal to the shaded area below the winning bids: $R^* = (1 \times 6) + (1 \times 5) + (1 \times 4) = 15$ euros.

The discriminatory auction is an extension of the **first-price sealed-bid auction** when there is a supply of multiple identical items.

4.3.2 Uniform-Price Auction

In an **uniform-price auction**, all bidders pay the same price for the acquired items (p^*), the market-clearing price at which demand equals supply. Demand equals supply at any value between the *highest-rejected-bid* (*HRB*) and the *lowest-accepted-bid* (*LAB*).[13] In this book, we will use the HRB to fix the clearing price.

If bidder i wins q_i^* items, his payment is equal to:

$$P_i^* = p^* \times q_i^*. \tag{4.24}$$

The income of bidder i is equal to the sum of the values of the q_i^* items obtained:

$$\Gamma_i^* = \Sigma_{j=1}^{q_i^*} v_{i,j}, \tag{4.25}$$

and his surplus is equal to the difference between income and payment:

$$\Pi_i^* = \Gamma_i^* - P_i^*. \tag{4.26}$$

[13] Cramton and Sujarittanonta [19] analyzed the effect of choosing the HRB or the LAB in a multi-unit ascending auction.

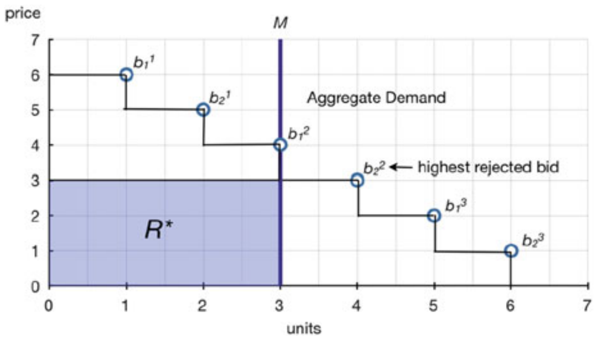


Fig. 4.7 Uniform-price auction

The seller's revenue is equal to the payments made by the winning bidders:

$$R^* = \Sigma_{i \in W} P_i^*, \tag{4.27}$$

where W is the group of winning bidders.

Continuing with the data of the previous exercise contained in Table 4.5, the bid made by the second bidder for the second item is the HRB, $b_{2,2} = 3$ euros, the market-clearing price at which the demand equals supply ($p^* = 3$ euros). Thus, the first bidder who wins two items pays $P_1^* = 3 \times 2 = 6$ euros. The second pays $P_2^* = 3 \times 1 = 3$ euros. The seller's revenue is the sum of both payments: $R^* = P_1^* + P_2^* = 9$ euros, represented by the shaded area in Fig. 4.7 ($R^* = 3 \times 3 = 9$ euros).[14]

4.3.3 Multi-unit Vickrey Auction of Homogeneous Items

A **Vickrey auction** of M identical items is a sealed-bid auction (single-round) in which the winning bidders pay the opportunity cost for the items obtained. If bidder i wins q_i^* units, he will pay the q_i^* highest rejected bids of the other bidders (the q_i^* highest bids not including his own).[15]

To compute this payment, we first find the *vector of competing bids* facing bidder i, $C_{-i} = (c_{-i,1}, c_{-i,2}, \ldots, c_{-i,M})$ which is the M-vector of the highest bids submitted by his rivals. We denote $c_{-i,1}$ as the highest bid of the others, $c_{-i,2}$ is the second highest, and so on. This vector is obtained by rearranging the $(N-1) \times M$

[14]If the price were calculated according to the LAB, it would be equal to $p = 4$ euros ($b_{1,2} = 4$); therefore, the first bidder would have paid $P_1^* = 4 \times 2 = 8$ euros for the two items and the second bidder $P_2^* = 4 \times 1 = 4$ euros. In this case, the seller's revenue would have been $R^* = 4 \times 3 = 12$ euros.

[15]The Vickrey auction is a **VCG mechanism**, in which each winning bidder pays an amount equal to the opportunity cost of the acquired item.

bids of all bidders except i in decreasing order and selecting the M highest bids. These would be the M winning bids if bidder i would have not placed a bid.

Then we have to determine whether the highest bid of bidder i exceeds the lowest bid included in C_{-i}, i.e., if $b_{i,1} > c_{-i,M}$. In the event that this condition is met, bidder i wins an item and pays an amount equal to $c_{-i,M}$, which is the highest rejected bid submitted by the others. For bidder i to win the second item, his second highest bid must exceed the second lowest competing bid ($b_{i,2} > c_{-i,M-1}$). If so, bidder i pays $c_{-i,M-1}$ for the second item. To calculate the number of items that bidder i wins, the process is continued until the jth highest bid placed by bidder i is lower than the jth lowest competing bid.

According to the Vickrey rule, bidder i pays the q_i^* highest rejected bids placed by the rivals for the q_i^* items obtained. Bidder i's total payment is calculated as follows:

$$P_i^* = \Sigma_{j=1}^{q_i^*} c_{-i,M-q_i^*+j}. \tag{4.28}$$

The income of bidder i is equal to the sum of the values of the q_i^* items won:

$$\Gamma_i^* = \Sigma_{j=1}^{q_i^*} v_{i,j}, \tag{4.29}$$

and the surplus is the difference between the income and the cost:

$$\Pi_i^* = \Gamma_i^* - P_i^*. \tag{4.30}$$

The seller obtains a revenue equal to the sum of payments done by the winning bidders:

$$R^* = \Sigma_{i \in W} P_i^*. \tag{4.31}$$

Consider the following example in which there are three bidders and three items to be auctioned. Suppose that each bidder submits the following bids: $B_1 = (10, 8, 6)$, $B_2 = (12, 7, 5)$, and $B_3 = (15, 13, 9)$. Table 4.6 depicts bids done by all bidders in decreasing order.

Table 4.7 shows the vector of competing bids facing each bidder and the auction outcome. To determine if the first bidder wins an item, first we obtain his vector

Table 4.6 Bids in decreasing order

Bids	15	13	12	10	9	8	7	6	5
Bidders	3	3	2	1	3	1	2	1	2

Table 4.7 Multi-unit Vickrey auction of homogeneous items

Bidders	$b_{i,1}$	$b_{i,2}$	$b_{i,3}$	$c_{-i,1}$	$c_{-i,2}$	$c_{-i,3}$	q_i^*	P_i^*
$i = 1$	10	8	6	15	13	12	0	0
$i = 2$	12	7	5	15	13	10	1	10
$i = 3$	15	13	9	12	10	8	2	18

of competing bids, the highest three bids placed by the others: $C_{-1} = (15, 13, 12)$. Next, we check if the highest bid made by the first bidder is greater than the lowest bid of the rivals, that is, if $b_{1,1} > c_{-1,3}$ holds. This condition does not hold because $b_{1,1} = 10 < 12 = c_{-1,3}$, so the first bidder does not win an item ($q_1^* = 0$).

We perform the same process for the second bidder, whose vector of competing bids is equal to $C_{-2} = (15, 13, 10)$. This bidder wins an item because his highest bid is greater than the lowest bid submitted by his rivals, $b_{2,1} = 12 > 10 = c_{-2,3}$. For this first item, the bidder pays an amount equal to the highest rejected bid made by the others: 10 euros. This bidder earns no more items, as $b_{2,2} = 7 < 13 = c_{-2,2}$. Finally, the vector of competing bids for the third bidder is $C_{-3} = (12, 10, 8)$. This bidder wins two items because his two highest bids are greater than the two lowest bids included in his vector of competing bid: $b_{3,1} = 15 > 8 = c_{-3,3}$, and $b_{3,2} = 13 > 10 = c_{-3,2}$. This bidder pays 8 euros for the first item and 10 euros for the second one. The final outcome is summarized as follows. The first bidder does not win any items, the second bidder wins one item and pays $P_2^* = 10$ euros, and the third bidder acquires two items for $P_3^* = 8 + 10 = 18$ euros. The seller's revenue is $R^* = 10 + 18 = 28$ euros.

The **second-price sealed-bid auction** is a particular case of the Vickrey auction in which only one item ($M = 1$) is auctioned.

4.4 Bidding Strategies and Equivalences of Multi-unit Auction Formats

When a seller offers M homogeneous items, the decision of the auction model is not trivial. Bidders will have different strategies depending on the type of auction chosen, thereby affecting the final result. In a multi-unit descending auction and in a discriminatory auction, in which winning bidders pay an amount equal to the bids placed, the bidders will always tend to bid below their private valuations to obtain a surplus. In contrast, in a multi-unit ascending auction, bidder i will not stop bidding on item j as long as the unit price is below the valuation ($v_{i,j}$). Finally, in an Ausubel auction and in a Vickrey auction, the bidders have incentives to submit sincere bids because the final price paid depends only on the bids made by the rivals.

In the specific case in which the bidders have **private values**, the following equivalences hold: the descending auction is equivalent to the discriminatory auction, the ascending auction is equivalent to the uniform price auction, and the Ausubel auction is equivalent to the Vickrey auction. To further explore the behavior of bidders in these models see [41].

4.5 Multi-unit Anglo-Dutch Auction of Homogeneous Items

The multi-unit **Anglo-Dutch auction**, developed by Klemperer [38], is a hybrid auction in two phases. The first phase is a multi-unit ascending auction and the second one is a sealed-bid auction. This two-stage auction was used to allocate 3G spectrum licenses in Europe.

Phase I is an ascending auction in which M items are available. The seller sets $p^{\mathrm{I,s}}$ as the starting price per unit, and the bidders indicate which quantities they are willing to buy, $q_i^{\mathrm{I},0}$. At the end of each round, the seller obtains the aggregate demand $Q^{\mathrm{I},t} = \Sigma_{i=1}^{N} q_i^{\mathrm{I},t}$ and incrementally increases the price whenever the supply is greater than the demand. This phase ends when the demand is greater than the supply by only one unit, $Q^{\mathrm{I},t} = M + 1$.[16] Bidders who were active in the last round of the ascending phase have the option to bid in Phase II, which is a one round, sealed-bid auction in which the bidders indicate the price that they are willing to pay for the items they bid on in the previous phase ($B_i^{\mathrm{II}} = (b_{i,1}^{\mathrm{II}}, b_{i,2}^{\mathrm{II}}, \ldots, b_{i,j}^{\mathrm{II}})$). The bids of this second phase cannot be less than the price per item established in the last round of the previous phase; thus, the opening bid of Phase II ($p^{\mathrm{II,s}}$) is equal to the closing price of Phase I. The bidders with the M highest bids are the winners.

Consider the following example in which three items are auctioned to six bidders and each bidder is only allowed to win one. The opening bid per item of Phase I is $p^{\mathrm{I,s}} = 5$ euros, a price at which all bidders demand one unit. In the following round, the price is $p^{\mathrm{I},1} = 10$ euros, and the first bidder stops bidding. In round $t = 2$, the price increases to $p^{\mathrm{I},2} = 15$ euros, and the second bidder stops bidding. In this round, the Phase I ends because the demand is greater than the supply by one unit ($M + 1 = 4 = Q^{\mathrm{I},2}$). Only bidders three, four, five, and six, who were active in the final round of the ascending phase, participate in Phase II. These bidders submit their final sealed-bid in one round. The minimum bid in Phase II is 15 euros per item, which is the closing price of the previous phase. The active bidders submit the following bids: $b_3^{\mathrm{II}} = 50$ euros, $b_4^{\mathrm{II}} = 15$ euros, $b_5^{\mathrm{II}} = 30$ euros, and $b_6^{\mathrm{II}} = 40$ euros. The winners are the third, fifth, and sixth bidder, who each acquires one item. Table 4.8 summarizes the two phases of the auction.

This model allows to implement different pricing rules. The simplest is the **first-price**, under which each bidder pays the bid placed. In this example, the payments would be $P_3^* = 50$ euros, $P_5^* = 30$ euros, and $P_6^* = 40$ euros, and the seller's revenue $R^* = P_3^* + P_5^* + P_6^* = 120$ euros.

Table 4.8 Rounds in a multi-unit Anglo-Dutch auction of homogeneous items

Phase I: ascending								
Rounds	$p^{\mathrm{I},t}$	$q_1^{\mathrm{I},t}$	$q_2^{\mathrm{I},t}$	$q_3^{\mathrm{I},t}$	$q_4^{\mathrm{I},t}$	$q_5^{\mathrm{I},t}$	$q_6^{\mathrm{I},t}$	$Q^{\mathrm{I},t}$
$t = 0$	$p^{\mathrm{I},0} = 5$	1	1	1	1	1	1	6
$t = 1$	$p^{\mathrm{I},1} = 10$	0	1	1	1	1	1	5
$t = 2$	$p^{\mathrm{I},2} = 15$	0	0	1	1	1	1	4
Phase II: sealed-bid		$b_{1,j}^{\mathrm{II}}$	$b_{2,j}^{\mathrm{II}}$	$b_{3,j}^{\mathrm{II}}$	$b_{4,j}^{\mathrm{II}}$	$b_{5,j}^{\mathrm{II}}$	$b_{6,j}^{\mathrm{II}}$	
$b_{i,j}^{\mathrm{II}}$ $(j = 1)$		0	0	50	15	30	40	

[16]According to the rules used for the 3G spectrum auction in the UK, each bidder could not earn more than one item.

4.6 Variables Used in This Chapter

In this chapter, we use the following variables:

- $I = (1, 2, \ldots, N)$: Bidders.
- $J = (1, 2, \ldots, M)$: Identical items.
- $V_i = (v_{i,1}, v_{i,2}, \ldots, v_{i,M})$: Vector of bidder i's values for each item.
- $v_{i,j}$: Value of bidder i for the item j.
- q_i^t: Number of items demanded by bidder i in round t of a dynamic auction (bidder i's individual demand in round t).
- $q_i^{\mathrm{I},t}$: Number of items demanded by bidder i in round t in Phase I of a hybrid auction whose Phase I corresponds to a dynamic auction.
- q_i^*: Number of items won by bidder i.
- q_i^{t*}: Number of items won by bidder i in round t of a dynamic auction.
- Q^t: Number of items demanded by all bidders in round t of a dynamic auction (aggregate demand in round t).
- $Q^{\mathrm{I},t}$: Number of items demanded by all bidders in round t in Phase I of a hybrid auction whose Phase I corresponds to a dynamic auction (aggregate demand in round t in Phase I).
- Q^{t*}: Number of items assigned in round t of a dynamic auction.
- $B_i = (b_{i,1}, b_{i,2}, \ldots, b_{i,M})$: Vector of bidder i's bids for each item in a sealed-bid auction (single-round).
- $b_{i,j}$: Bidder i's bid for item j in a sealed-bid auction (single-round).
- $b_{i,j}^{\mathrm{II}}$: Bidder i's bid for item j in Phase II of a hybrid auction whose Phase II corresponds to a sealed-bid auction (single-round).
- $b_{i,j}^*$: Winning bid submitted by bidder i for item j in a sealed-bid auction (single-round).
- $d_i(p)$: Bidder i's demand function.
- p^{s}: Starting bid (minimum bid) per item.
- $p^{\mathrm{I,s}}$: Starting bid (minimum bid) per item in Phase I of a hybrid auction.
- $p^{\mathrm{II,s}}$: Starting bid (minimum bid) per item in Phase II of a hybrid auction.
- p^*: Selling price in a uniform-price auction (clearing price).
- P_i^*: Bidder i's payment.
- M_i^t: Bidder i's residual supply in round t of an Ausubel auction.
- C_i^t: Number of units *clinched* by bidder i up to round t of an Ausubel auction.
- c_i^t: Number of units *clinched* by bidder i in round t of an Ausubel auction.
- C_{-i}: Vector of competing bids facing bidder i in a Vickrey auction.
- $c_{-i,j}$: Highest bid of the other bidders except bidder i for item j in a Vickrey auction.
- Γ_i^*: Income of bidder i from the acquired items.
- Π_i^*: Surplus of bidder i from the acquired items.
- R^*: Seller's revenue.

4.7 Exercises

1. A seller offers four identical items to two bidders. Values of the first bidder are $V_1 = (10, 8, 6, 4)$, and of the second bidder are $V_2 = (15, 10, 5, 0)$. The seller implements an ascending auction and sets a starting price per item of $p^s = 3$ euros. We assume that bidders bid for each item as long as the price is below his value. Calculate:
   (a) The quantity of items won by each bidder and at what price, assuming a price increment of one euro.
   (b) The final payment done by the winning bidders.
   (c) The seller's revenue.
   (d) The surplus of each bidder.
2. Using the same data as the previous exercise, if the seller had chosen a descending auction, set a starting price of $p^s = 16$ euros and the bidders decided to place bids when the price equals the marginal value of each unit. Calculate:
   (a) The quantity of items won by each bidder and at what price, assuming a price decrement of one euro.
   (b) The final payments of the winning bidders.
   (c) The seller's revenue.
   (d) The surplus of each bidder.
3. Using the same data as the previous exercise, if the seller had done an Ausubel auction with a starting price of $p^s = 3$ euros and the bidders bid for an item while the price is less than their marginal value, calculate:
   (a) The quantity of items won by each bidder and at what price, assuming a price increment of one euro.
   (b) The final payment done by the winning bidders.
   (c) The seller's revenue.
   (d) The surplus of each bidder.
4. A seller offers three identical items in a sealed-bid auction. The auction has three bidders who submit the following bid vectors in a single round: $B_1 = (200, 150, 50)$, $B_2 = (250, 100, 50)$, and $B_3 = (200, 50, 0)$. Solve the allocation process as follows:
   (a) Graphically represent the aggregate demand of all bidders, and distinguish winning bids from losing bids.
   (b) Indicate the winning bidders and how many items were won by each.
5. Using the data from the previous exercise, calculate the total payments of each winning bidder and the seller's revenue under the following assumptions.
   (a) The seller chooses a discriminatory auction.
   (b) The seller chooses an uniform-price auction.
   (c) The seller chooses a Vickrey auction.
   (d) Describe the effect of choosing one auction model over another on the seller's revenue and the bidder's strategies.

4.8 Solutions to Exercises

1. The ascending auction outcome is shown in Table 4.9. The first bidder has a marginal value for all items greater than the starting price, so he demands four units. The second bidder is not interested in the fourth item but his marginal value for the first, second, and third item is greater than three euros, so he demands three units in round $t = 0$. As the aggregate demand in this round is greater than supply ($Q^0 = 7 > 4 = M$), the seller increases the price. In the next round, $t = 1$, the first bidder reduces the demand by one unit because the marginal value of the fourth item is equal to the price set in this round $v_{1,4} = p^1$. The seller continues to increase the price in successive rounds, and the bidders decrease the quantity demanded until demand equal supply in round $t = 3$.
   (a) Each bidder wins two items ($q_1^* = q_2^* = 2$), and each pays $p^* = 6$ euros per item.
   (b) Both bidders have to do the same payments for the acquired items: $P_1^* = P_2^* = 2 \times 6 = 12$ euros.
   (c) The seller's revenue is equal to the sum of the payments made by all bidders, $R^* = P_1^* + P_2^* = 24$ euros.
   (d) The surplus of each bidder is equal to the difference between the income and the payment. Thus, $\Pi_1^* = \Gamma_1^* - P_1^* = (10 + 8) - 12 = 6$ euros and $\Pi_2^* = \Gamma_2^* - P_2^* = (15 + 10) - 12 = 13$ euros.
2. The descending auction in which all bidders bid when the price of each round equals the marginal value of the item ($p^t = v_{i,j}$) is presented in Table 4.10.
   (a) The first bidder wins two items ($q_1^* = 2$), one in round $t = 6$ at a price of $p^6 = 10$ euros and another in round $t = 8$ at a price of $p^8 = 8$ euros. The second bidder wins one item in round $t = 1$ at a price of $p^1 = 15$ euros and another item in round $t = 6$ at a price of $p^6 = 10$ euros.
   (b) The first bidder's payment is $P_1^* = \Sigma_{t=0}^{T} p^t q_1^{t*} = (1 \times 10) + (1 \times 8) = 18$ euros. The second bidder's payment is $P_2^* = \Sigma_{t=0}^{T} p^t q_2^{t*} = (1 \times 15) + (1 \times 10) = 25$ euros.
   (c) The seller's revenue is equal to the sum of the payments made by the winning bidders, $R^* = P_1^* + P_2^* = 18 + 25 = 43$ euros.

Table 4.9 Multi-unit ascending auction of homogeneous items (exercise 1)

Rounds (t)	Price (p^t)	q_1^t	q_2^t	Q^t
$t = 0$	$p^0 = 3$	4	3	7
$t = 1$	$p^1 = 4$	3	3	6
$t = 2$	$p^2 = 5$	3	2	5
$t = 3$	$p^3 = 6$	2	2	4
Values	$v_{i,j}$	$v_{1,j}$	$v_{2,j}$	
	$j = 1$	10	15	
	$j = 2$	8	10	
	$j = 3$	6	5	
	$j = 4$	4	0	

Table 4.10 Multi-unit descending auction of homogeneous items (exercise 2)

Round (t)	Round price (p^t)	q_1^{t*}	q_2^{t*}	$\Sigma_{t=0}^{l} Q^{t*}$
$t = 0$	$p^0 = 16$	0	0	0
$t = 1$	$p^1 = 15$	0	1	1
$t = 2$	$p^2 = 14$	0	0	1
$t = 3$	$p^3 = 13$	0	0	1
$t = 4$	$p^4 = 12$	0	0	1
$t = 5$	$p^5 = 11$	0	0	1
$t = 6$	$p^6 = 10$	1	1	3
$t = 7$	$p^7 = 9$	0	0	3
$t = 8$	$p^8 = 8$	1	0	4
Values	$v_{i,j}$	$v_{1,j}$	$v_{2,j}$	
	$j = 1$	10	15	
	$j = 2$	8	10	
	$j = 3$	6	5	
	$j = 4$	4	0	

Table 4.11 Ausubel auction (exercise 3)

Round (t)	Price (p^t)	q_1^t	q_2^t	Q^t	C_1^t	C_2^t
$t = 0$	$p^0 = 3$	4	3	7	1	0
$t = 1$	$p^1 = 4$	3	3	6	1	1
$t = 2$	$p^2 = 5$	3	2	5	2	1
$t = 3$	$p^3 = 6$	2	2	4	2	2
Values	$v_{i,j}$	$v_{1,j}$	$v_{2,j}$			
	$j = 1$	10	15			
	$j = 2$	8	10			
	$j = 3$	6	5			
	$j = 4$	4	0			

(d) The surpluses obtained by the winning bidders are: $\Pi_1^* = \Gamma_1^* - P_1^* = (10 + 8) - 18 = 0$ euros and $\Pi_2^* = \Gamma_2^* - P_2^* = (15 + 10) - 25 = 0$ euros.

In this auction model, the price to be paid by the winning bidders is equal to their bid (first-price rule), so they will never place bids equal to their valuations because they would obtain zero surplus, as in this example. The bidders thus tend to underbid.

3. In an Ausubel auction, items are awarded using the same method as a multi-unit ascending auction. However, final prices depend on the corresponding rounds in which each of the items was clinched. Table 4.11 presents the auction per round.
   (a) At the starting price (round $t = 0$) the first bidder clinches one item ($M_1^0 = C_1^0 = 1$) for three euros and in round $t = 2$ he clinches the second one ($M_1^2 = C_1^2 = 2$) for five euros. The second bidder clinches his first item in round $t = 1$ ($M_2^1 = C_2^1 = 1$) and the second one in round $t = 3$ ($M_2^3 = C_2^3 = 2$). He pays four and six euros, respectively, for each of them.

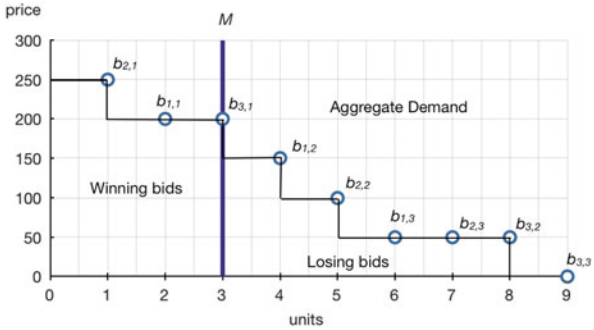


Fig. 4.8 Multi-unit sealed-bid auction of homogeneous items

(b) The first bidder's payment is $P_1^* = (1\times3)+(1\times5) = 8$ euros. For each item, the first bidder pays the price of the round in which the item was clinched. The second bidder's payment is $P_2^* = (1 \times 4) + (1 \times 6) = 10$ euros.
(c) The seller's revenue is equal to the sum of the payments made by the winning bidders $R^* = P_1^* + P_2^* = 8 + 10 = 18$ euros.
(d) The surplus of the first bidder is $\Pi_1^* = \Gamma_1^* - P_1^* = (10 + 8) - 8 = 10$ euros and the surplus of the second one is equal to $\Pi_2^* = \Gamma_2^* - P_2^* = (15 + 10) - 10 = 15$ euros.

4. The allocation process for this multi-unit sealed-bid auction follows below.
   (a) After all bids are received, the seller orders them from highest to lowest and calculates the aggregate demand, which is shown in Fig. 4.8. The vertical line represents the supply, and the bids on the left side correspond to the M highest bids, which are the winning bids. In this example, the three highest bids are $(b_{2,1}^*, b_{1,1}^*, b_{3,1}^*) = (250, 200, 200)$. The bids on the right side are the losing bids.
   (b) At the end of the auction, each bidder wins one item.
5. In the previous exercise, the allocation process for the sealed-bid auction was solved. However, the payments for each bidder depend on the pricing rule selected.
   (a) *Discriminatory auction:*
   If a discriminatory auction is chosen, each bidder pays his winning bids (*pay-your-bid rule*). In this example, the first bidder pays $P_1^* = 200$ euros, the second bidder pays $P_2^* = 250$ euros, and the third bidder pays $P_3^* = 200$ euros. The seller's revenue is $R^* = P_1^* + P_2^* + P_3^* = 650$ euros.
   (b) *Uniform-price auction:*
   If the seller chooses an uniform-price auction, all winning bidders pay the HRB for each item. In this example, the HRB is $b_{1,2} = p^* = 150$ euros, so $P_1^* = P_2^* = P_3^* = 150$ euros. The seller's revenue is thus $R^* = P_1^* + P_2^* + P_3^* = 450$ euros.
   (c) *Vickrey auction:*
   To calculate the price paid by each bidder in a Vickrey auction, the vector of competing bids facing each bidder must be obtained. For the first bidder $C_{-1} = (250, 200, 100)$, for the second bidder $C_{-2} = (200, 200, 150)$, and for the third bidder $C_{-3} = (250, 200, 150)$. When we compare the highest

bid of each bidder to the lowest competing bid, we find that $b_{i,1} > c_{-i,3}$ for all bidders: $b_{1,1} = 200 > 100 = c_{-1,3}$ for the first bidder, $b_{2,1} = 250 > 150 = c_{-2,3}$ for the second, and $b_{3,1} = 200 > 150 = c_{-3,3}$ for the third. Thus, each bidder wins one item and pays: $P_1^* = 100$ euros, $P_2^* = 150$ euros and $P_3^* = 150$ euros, respectively. The seller's revenue is $R^* = P_1^* + P_2^* + P_3^* = 400$ euros.

(d) The seller's revenues vary depending on the chosen auction model. Assuming that the bidders keep the same bids in the three models, the revenues obtained by the seller are $R^* = 650$ euros in a discriminatory auction, $R^* = 450$ euros in an uniform-price auction and $R^* = 400$ euros in a Vickrey auction. However, it is important to note that these results do not demonstrate that the discriminatory auction necessarily generates higher revenues than the other models. The selection of an auction model directly affects the bidding strategy. Bidders in discriminatory auctions will always tend to underbid to have a positive surplus.

Appendix

In dynamic multi-unit auctions, a price change may involve a decrease in the demanded quantity such that the demand does not cover the supply. In these circumstances, the seller may set a **rationing rule** to allocate all items. There are many different rationing rules but we are going to explain the proportional rationing rule with an example.[17]

A seller offers ten items in an ascending auction in which there are three bidders. Table 4.12 shows the submitted bids round by round. In rounds $t = 0$ and $t = 1$, the aggregate demand is greater than the supply $Q^t > 10 = M$, so the seller raises the price. However, as a result of this price increase, the aggregate demand in the final round is lower than the supply $Q^T = 8 < 10 = M$ for $T = 2$. Bidders are only willing to buy eight items, even though the seller is offering ten.

In this situation, the seller may return to round $T - 1$ and award all the items by applying a rationing rule. If the seller implements the proportional rationing rule, we compute the items obtained by bidder i according to the following equation[18]:

$$q_i^* = q_i^T + \frac{q_i^{T-1} - q_i^T}{Q^{T-1} - Q^T}(M - Q^T). \tag{4.32}$$

Table 4.12 Multi-unit ascending auction of homogeneous items

Round (t)	Price (p^t)	q_1^t	q_2^t	q_3^t	Q^t
$t = 0$	$p^0 = 5$	8	8	4	20
$t = 1$	$p^1 = 10$	6	6	0	12
$t = 2$	$p^2 = 15$	4	4	0	8

[17]To explore other possible rationing rules, see [34] or [69], among others.

[18]If an integer number is not obtained, the seller may round up.

In this example, the final quantities allocated to the first and second bidders are:

$$q_1^* = q_2^* = 4 + \frac{6-4}{12-8}(10-8) = 5.$$

Going back to round $T-1=1$ and applying the proportional rationing rule, the seller allocates the two unallocated items, which ensures that all supplied items are sold. Finally, both bidders win five items, and the price per unit is equal to $p^* = 10$ euros.

5 Sequential and Simultaneous Auctions

5.1 Introduction

In Chap. 4, we studied the main multi-unit auction models in which identical items are offered in a single auction. However, the seller also has the option of offering multiple items, both homogeneous and heterogeneous, in different auctions. These auctions may be performed at the same time (simultaneous auctions) or one after another (sequential auctions). In this chapter, we will present these two alternatives.

5.2 Sequential Auctions

When a seller wants to allocate M related objects, he may do so through **sequential auctions**, that is, by offering them in different consecutive auctions.[1] The seller may choose the auction model he prefers, either dynamic or single-round, but the auctions must be performed one after another.[2]

A seller offers M different but related items in M sequential sealed-bid (single-round) auctions. In each auction, an item j is offered, and each bidder i submits his bid for this item $b_{i,j}$. Bidder i obtains the item ($q^*_{i,j} = 1$) only if he submits the highest bid: $b_{i,j} = b^*_j$, for which he will have to pay a price of p^*_j, which will depend on the pricing rule selected. Only if the seller opts for the first-price rule, it will be true that $p^*_j = b^*_j$. However, if bidder i does not submit the highest bid $b_{i,j} \neq b^*_j$, he does not win the item ($q^*_{i,j} = 0$).

[1]For the auctions to be sequential, the time considered between one auction and the next must be short enough that the bidders do not discount the profits obtained in the previous periods.

[2]Supporters of sequential auctions claim that they are easy to implement and reduce the risk of collusion.

A. Mochón, Y. Sáez, *Understanding Auctions*, Springer Texts in Business and Economics, DOI 10.1007/978-3-319-08813-6_5

Bidder i's payment is equal to the sum of the prices for each acquired item:

$$P_i^* = \Sigma_{j=1}^{M} p_j^* q_{i,j}^*. \tag{5.1}$$

The income for bidder i is equal to the sum of the values of the items won:

$$\Gamma_i^* = \Sigma_{j=1}^{M} v_{i,j} q_{i,j}^*, \tag{5.2}$$

and the surplus is the difference between the income and the total payment:

$$\Pi_i^* = \Gamma_i^* - P_i^*. \tag{5.3}$$

Finally, the revenue of the seller is equal to the sum of the payments made by all of the winning bidders:

$$R^* = \Sigma_{i \in W} P_i^*, \tag{5.4}$$

where W is the set of winners.

To better understand sequential auctions, we analyze the following example. A seller wants to award three different but related items: for example, a theater ticket (item $j = 1$), a movie ticket (item $j = 2$), and a ticket to a musical (item $j = 3$). To allocate them, three sequential second-price sealed-bid auctions are performed. There are three bidders bidding sincerely whose values of the items are represented by the following vectors: $V_1 = (40, 3, 50)$; $V_2 = (35, 5, 40)$; and $V_3 = (15, 7, 55)$. The seller performs the three sealed-bid auctions consecutively and receives the bids reflected in Table 5.1.

The first bidder obtains the first item (the theater ticket), for which he pays $P_1^* =$ 35 euros (the second highest bid). The third bidder wins item $j = 2$ for a price of $p_2^* = 5$ euros and wins item $j = 3$ for $p_3^* = 50$ euros. In other words, this bidder pays $P_3^* = 55$ euros for the movie ticket plus the musical ticket. The income that these bidders obtain is equal to: $\Gamma_1^* = 40$ euros for the first and $\Gamma_3^* = 62$ for the third. The surplus is equal to $\Pi_1^* = 40 - 35 = 5$ euros and $\Pi_3^* = 62 - 55 = 7$ euros, respectively. After performing the auctions, the seller obtains a revenue of $R^* = 35 + 55 = 90$ euros.

Table 5.1 Sealed-bid sequential auctions

	$b_{1,j}$	$b_{2,j}$	$b_{3,j}$
First auction ($j = 1$)	$\boxed{\mathbf{40}}$	35	15
Second auction ($j = 2$)	3	5	$\boxed{\mathbf{7}}$
Third auction ($j = 3$)	50	40	$\boxed{\mathbf{55}}$
Values $v_{i,j}$	$v_{1,j}$	$v_{2,j}$	$v_{3,j}$
$j = 1$	40	35	15
$j = 2$	3	5	7
$j = 3$	50	40	55

In this example, we have assumed that the seller performs sequential second-price sealed-bid auctions, but the seller could have auctioned the three items in any type of sequential auction: first-price sealed-bid, ascending, and descending.[3]

5.3 Simultaneous Auctions

The main problem with sequential auctions is that the bids in the first auctions depend on the bidders' estimations of the prices in the future auctions. Therefore, bidders may regret either buying at high prices in the first auctions or not having acquired their desired items.[4] Given this uncertainty, predictions are frequently erroneous, which translates into inefficient allocations. To avoid these situations, the sellers may opt to perform **simultaneous auctions**. That is, they may auction all of the items in different auctions that are performed at the same time. These simultaneous auctions may be single-round (sealed-bid) or dynamic.

5.3.1 Simultaneous Sealed-Bid Auction

The easiest way to implement simultaneous auctions consists of performing sealed-bid auctions (single-round). The seller can offer the M items ($J = (1, 2, \dots, M)$) to N bidders ($I = (1, 2, \dots, N)$), in M sealed-bid auctions at the same time,[5] one auction per item. With this mechanism, all of the bidders bid the amount that they are willing to pay for each item, which is represented by the vector $B_i = (b_{i,1}, b_{i,2}, \dots, b_{i,M})$, in which $b_{i,j}$ indicates how much bidder i is willing to pay for item j. Bidder i will win item j ($q^*_{i,j} = 1$) if his bid is the highest bid made in the auction in which item j is offered, that is, if $b_{i,j} = b^*_j$. However, if another bidder $k \neq i$ makes the highest bid for item j, $b_{k\neq i,j} = b^*_j$, bidder i does not win this item ($q^*_{i,j} = 0$).

Bidder i's payment is equal to the sum of the prices for each item j that has been acquired, p^*_j:

$$P^*_i = \Sigma^M_{j=1} p^*_j q^*_{i,j}. \tag{5.5}$$

The price will depend on the pricing rule set by the seller. Only if the first-price rule is established it will be true that $p^*_j = b^*_j$. The income that bidder i obtains is equal to the sum of the values of the items won:

[3] In 1981, the US *Federal Communications Commission (FCC)* awarded seven satellite communication licenses through ascending sequential auctions.

[4] Ashenfelter [3] empirically analyzed sequential auctions of wines and art and observed that the final items were auctioned off at lower prices than the first ones.

[5] The seller could also offer M categories or types of items in M auctions, with a different quantity of items per category. This situation is described later in this chapter.

Table 5.2 Simultaneous sealed-bid auctions

Auction for item j	$b_{1,j}$	$b_{2,j}$	$b_{3,j}$
$j = 1$	10	15	$\boxed{\mathbf{20}}$
$j = 2$	3	2	$\boxed{\mathbf{7}}$
$j = 3$	$\boxed{\mathbf{19}}$	10	18
Values $v_{i,j}$	$v_{1,j}$	$v_{2,j}$	$v_{3,j}$
$j = 1$	15	20	25
$j = 2$	5	5	10
$j = 3$	25	15	20

$$\Gamma_i^* = \Sigma_{j=1}^M v_{i,j} q_{i,j}^*, \tag{5.6}$$

and the surplus is the difference between the income and the total payment:

$$\Pi_i^* = \Gamma_i^* - P_i^*. \tag{5.7}$$

The seller's revenue is equal to the sum of the payments of the winning bidders, $i \in W$:

$$R^* = \Sigma_{i \in W} P_i^*. \tag{5.8}$$

Let us analyze the next example to better understand this type of auctions. A seller offers three items in three simultaneous sealed-bid auctions. Three bidders simultaneously submit the bids that appear in Table 5.2, in which we can also observe the values of each bidder for the three items. In the auction for item $j = 1$, the third bidder is the winner because he makes the highest bid, $b_{3,1} = b_1^* = 20$ euros. This bidder is also the winner of item $j = 2$, $b_{3,2} = b_2^* = 7$ euros. However, the first bidder wins item $j = 3$ because $b_{1,3} = b_3^* = 19$ euros.

If the seller sets a first-price rule, the payments are equal to $P_1^* = 19$ euros (first bidder) and $P_3^* = 20 + 7 = 27$ euros (third bidder). The seller obtains a revenue of $R^* = 27 + 19 = 46$ euros. The profit that each winning bidder gets is equal to the difference between the values of the items won and the price paid, that is, $\Pi_1^* = 25 - 19 = 6$ euros for the first bidder and $\Pi_3^* = (25 + 10) - (20 + 7) = 8$ euros for the third one.

However, if the seller establishes a second-price rule, the amount that the winners pay is equal to $P_1^* = 18$ euros and $P_3^* = 15 + 3 = 18$ euros. Under this rule, they obtain the following profits: $\Pi_1^* = 25 - 18 = 7$ euros for the first bidder and $\Pi_3^* = (25 + 10) - (15 + 3) = 17$ euros for the third. The seller's revenue would be equal to $R^* = 18 + 18 = 36$ euros.

5.3.2 Simultaneous Ascending Auction

Within the framework of simultaneous dynamic auctions, the model that has been most frequently used to award multiple related items is the **simultaneous ascending auction** (**SAA**), also known as the **simultaneous multiple round auction** (**SMR auction**).[6] In this model, the items are simultaneously auctioned in a sequence of rounds and each item is offered with its corresponding price. The price of each item increases as bidders submit bids. The auction ends when there are no new bids for any item. Once the auction is finished, each item is awarded to the bidder with the highest bid for that item.[7]

Next, we describe a SAA. A seller offers M related items in M ascending auctions that are performed simultaneously,[8] and he establishes a starting price for each item and auction: $P^{\mathrm{s}} = (p_1^{\mathrm{s}}, p_2^{\mathrm{s}}, \ldots, p_M^{\mathrm{s}})$, in which p_j^{s} is the starting price of item j, that is, the minimum bid in round $t = 0$ for item j. In each round, bidders submit their bids, indicating how much they are willing to pay for each item. Bids made by bidder i in round t are represented by the vector $B_i^t = (b_{i,1}^t, b_{i,2}^t, \ldots, b_{i,M}^t)$, by which $b_{i,j}^t$ indicates how much bidder i is willing to pay for item j in round t. With all of the bids received in round t, the seller establishes a vector with the standing high bid for each item: $B_{\max}^t = (b_{\max 1}^t, b_{\max 2}^t, \ldots, b_{\max M}^t)$, in which $b_{\max j}^t$ is the highest bid made among all of the bidders for item j in round t. The highest standing bid in each round for an item is the maximum value among:

- the highest standing bid from the previous rounds,
- the highest new bid.

After each round, the seller also calculates the vector of the minimum bids that the bidders may submit for each item in the next round $(t + 1)$: $B_{\min}^{t+1} = (b_{\min 1}^{t+1}, b_{\min 2}^{t+1}, \ldots, b_{\min M}^{t+1})$. In this formula, $b_{\min j}^{t+1}$ is the minimum bid in round $t + 1$ for item j, which is an amount that is equal to the highest standing bid for item j in round t plus a bid increment λ:

$$b_{\min j}^{t+1} = b_{\max j}^{t} + \lambda, \tag{5.9}$$

where λ represents the bid increment that the seller establishes between rounds. The bid increment is frequently the highest value of some fixed amount or a particular

[6] This auction model was introduced for the first time in 1994 to award radio spectrum licenses in the USA and Professor Paul Milgrom's contributions were of major relevance (see [18, 48], or [51]).

[7] Simultaneous dynamic auctions can have either continuous bidding or discrete bidding (in rounds). In this book we will only focus on those examples in which bidders submit their bids by discrete rounds.

[8] The seller could also offer M categories or types of items in M auctions, with a different quantity of items per category. This situation is described later in the next section.

percentage over the highest standing bid.[9] The seller also identifies each highest bid with the bidder who made it because this bidder would be the winner of that item if the auction were to end in that round. This process continues until the auction ends. There are different ways of establishing the end of an SAA, among which the following are worth highlighting:

- Each auction closes independently: each individual auction closes when no new bids have been received during a set number of rounds for that specific auction.
- All of the auctions close at the same time: all auctions will remain open until there are no new bids for any item.

A potential strategy that a bidder could adopt is to avoid bidding in the first rounds and wait to see the bids of the others (without revealing his preferences). With this information about the opponents, the bidder could then decide which items to bid for in the last rounds. This strategy is known as the *snake in the grass strategy*. To prevent bidders from following this strategy and to make the auctions last longer, the seller may establish an **activity rule**. An activity rule is a restriction that the seller imposes on the allowable bids per round, which are calculated as a function of the level of bidding activity in the previous rounds. In this way, the seller encourages the rhythm of the auction to be maintained and promotes **price discovery**. This concept is explained in more detail in the next section.

SAA for substitute items

The next example shows a SAA in which there are three bidders and two items are offered. Table 5.3 shows the bidders' values for each item independently and altogether. As we can observe, the items are **substitutes** for all bidders as the value of the items combined is lower than the sum of the individual values: $v_{i,1+2} < v_{i,1} + v_{i,2}$.[10]

Given these values, a bid increment of $\lambda = 1$ euro and the assumption that bidders continue bidding for each item as long as the minimum bid of the next round is below their value, Table 5.4 shows the submitted bids.

Table 5.3 Values for substitutes

Item j	$v_{1,j}$	$v_{2,j}$	$v_{3,j}$
1	6	5	11
2	4	9	7
$1+2$	9	12	15

[9] Setting the bid increment is not a trivial decision because it affects the duration of the auction and the bidding strategy. In radio spectrum auctions, the bid increment is often 5 % of the highest standing bid.

[10] The items would be complements if $v_{i,1+2} > v_{i,1} + v_{i,2}$, see Chap. 7.

Table 5.4 Simultaneous ascending auction with substitutes

	$b_{1,j}^t$	$b_{2,j}^t$	$b_{3,j}^t$	$b_{\max j}^t$	$b_{\min j}^{t+1}$
$t=0$					
$j=1$	2	2	$\boxed{3}$	3	4
$j=2$	1	$\boxed{4}$	3	4	5
$t=1$					
$j=1$	4	4	$\boxed{6}$	6	7
$j=2$	0	$\boxed{6}$	5	6	7
$t=2$					
$j=1$	0	0	$\boxed{10}$	10	11
$j=2$	0	$\boxed{8}$	0	8	9

The auction begins with an opening bid of zero euros per item ($p_j^s = 0$), and bidders submit bids for all items. In round $t = 0$, the third bidder is the one who does the highest bid for the first item, and the second bidder for the second item, $b_{3,1}^0 = b_{\max 1}^0 = 3$ euros and $b_{2,2}^0 = b_{\max 2}^0 = 4$ euros, respectively. These bids become the highest standing bids in this round and are the bids that are taken into account to calculate the minimum bids in the following round.
In round $t = 1$, the first bidder stops bidding for the second item because his value is below the minimum bid ($v_{1,2} = 4 < 5 = b_{\min 2}^1$) and the items are substitutes.[11] This bidder only bids for the first item, and his bid matches the minimum bid. The second and third bidders bid for both items and the minimum prices for the following round are $b_{\min 1}^2 = b_{\min 2}^2 = 7$ euros for both items. In this new round ($t = 2$), the second bidder only bids for the second item ($b_{2,2}^2 = 8$ euros), which turns out to be the highest bid. The third bidder only bids for the first item, and his bid is also the highest bid of the round ($b_{3,1}^2 = 10$ euros).

After this round, no more bids are submitted and the auction ends. The final outcome can be summarized as follows. The second bidder wins item $j = 2$ and pays $P_2^* = b_{2,2}^2 = 8$ euros. This bidder obtains a surplus of $\Pi_2^* = v_{2,2} - P_2^* = 9 - 8 = 1$ euro. The third bidder wins the first item, for which he pays $P_3^* = b_{3,1}^2 = 10$ euros. His surplus is equal to $\Pi_3^* = v_{3,1} - P_3^* = 11 - 10 = 1$ euro. Finally, the seller's revenue is equal to the sum of the payments made by each winner: $R^* = P_2^* + P_3^* = 8 + 10 = 18$ euros.

In a SAA, the seller may establish multiple rules that will significantly affect the auction outcome.[12] Among these rules, the option of **bid withdrawal** between rounds is worth mentioning. According to this rule, after each round, the bidders can withdraw some (or all) of the items they have won in exchange for paying

[11] The seller could have set an activity rule whereby this participant could no longer continue bidding for this item in later rounds.

[12] Some of these rules are as follows: the payment mechanism, the bid increments, the number of rounds to be performed per day, the information that is made public after each round, etc.

a penalty.[13] This option becomes particularly relevant when the items that are auctioned are **complements**, so if the bidders do not win the combinations of items they are interested in, they may avoid losses by withdrawing certain bids.

SAA for complement items

Table 5.5 shows the values of three bidders involved in a SAA of two items. As can be observed, the items are complements for the first bidder because his value of the items combined is greater than the sum of the individual values ($v_{1,1+2} = 11 > 3 + 3 = v_{1,1} + v_{1,2}$). However, the other bidders are only interested in buying one item.

Given these values, the bidders could submit the bids included in Table 5.6 (assuming a bid increment of $\lambda = 1$ euro and an opening bid per item of $p_j^s = 0$ euros). In round $t = 0$, the first bidder bids one euro for each item, whereas the others only bid for the items for which they have a positive value. The highest bid for each item is 2 euros. Therefore, the minimum bids in round $t = 1$ are $b^1_{\min 1} = b^1_{\min 2} = 3$ euros. In this round, the first bidder once again bids for the two items $b^1_{1,1} = b^1_{1,2} = 3$ euros. Although the price per item is equal to his individual values, this bidder is trying to acquire both items together to obtain the superadditive values (complement items). However, the bids made by his rivals are higher, and the minimum bids for the following round are $b^2_{\min 1} = b^2_{\min 2} = 5$ euros for each item. Even though the minimum bid is higher than the first bidder's value for each item

Table 5.5 Values for complements

Item j	$v_{1,j}$	$v_{2,j}$	$v_{3,j}$
1	3	15	0
2	3	0	5
1 + 2	11	15	5

Table 5.6 Simultaneous ascending auction with complements

	$b^t_{1,j}$	$b^t_{2,j}$	$b^t_{3,j}$	$b^t_{\max j}$	$b^{t+1}_{\min j}$
$t = 0$					
$j = 1$	1	$\boxed{2}$	0	2	3
$j = 2$	1	0	$\boxed{2}$	2	3
$t = 1$					
$j = 1$	3	$\boxed{4}$	0	4	5
$j = 2$	3	0	$\boxed{4}$	4	5
$t = 2$					
$j = 1$	5	$\boxed{6}$	0	6	7
$j = 2$	$\boxed{5}$	0	0	5	6

[13]If the seller includes the withdrawal rule but without a penalty, there might be an incentive for bidders to distort their bids.

individually, he opts to continue bidding because if he were to win both items, he would still obtain a positive surplus of ($b_{1,1}^2 + b_{1,2}^2 = 5 + 5 = 10 < 11 = v_{1,1+2}$).

In round $t = 2$, the second bidder again bids for the first item, but the third bidder stops bidding. After this round, the minimum bids for each item in round $t = 3$ would be $b_{\min 1}^3 = 7$ euros for the first item and $b_{\min 2}^3 = 6$ euros for the second item. At these prices, the first bidder decides not to bid in $t = 3$ because the price of the combination of items with the minimum bids is greater than his value ($b_{\min 1}^3 + b_{\min 2}^3 = 7 + 6 = 13 > 11 = v_{1,1+2}$).

The auction ends and the winners are the bidders with highest standing bids in round $t = 2$. The first bidder wins the second item, for which he would have to pay $P_1^* = b_{1,2}^2 = 5$ euros, and the second bidder wins the first item for $P_2^* = b_{2,1}^2 = 6$ euros. With this outcome, the first bidder has a negative surplus: $\Pi_1^* = v_{1,1} - P_1^* = 3 - 5 = -2$ euros because he has not managed to win the two complements items for which he was bidding.

Given this situation, the first bidder would prefer to withdraw his winning bid and pay a penalty rather than keep the item. If the first bidder withdrawal is highest bid, then the third bidder will acquire the second item for an amount equal to his last bid $P_3^* = b_{3,2}^1 = 4$ euros. The surplus for the second and third bidders would be $\Pi_2^* = v_{2,1} - P_2^* = 15 - 6 = 9$ euros and $\Pi_3^* = v_{3,2} - P_3^* = 5 - 4 = 1$ euro, respectively. The seller's revenue would be equal to $R^* = P_2^* + P_3^* = 6 + 4 = 10$ euros.[14]

Silent auctions are another example of simultaneous auctions that are often used in charity. In these auctions, the items are simultaneously offered in a room, and the bidders make their bids. The price increments are not announced, so bidders must pay attention to the highest standing bid. The items are awarded simultaneously at the closing time established by the seller. As we mentioned in Chap. 3, the problem of using **deadline auctions** is that bidders tend to follow the strategy of **last minute bidding**.

5.3.3 Simultaneous Ascending Clock Auction

The **simultaneous ascending clock auction** is a simultaneous ascending auction (SAA or SMR auction) in which multiple related items (usually heterogeneous) are offered. The seller announces the price per item in each round and the bidders submit their bids indicating the quantity they are willing to buy at the standing prices. For all items with excess demand the price goes up and new bids are submitted. This iterated process goes until demand equals supply for all items. Winner bidders are those that are active in the last rounds and the selling price per item is equal to the price of the last round.

[14]In this example, the seller would obtain an additional revenue equal to the penalty paid by the first bidder for his bid withdrawal.

Before the auction starts, the seller describes the items being offered: number of categories or types of items $J = (1, 2, \ldots, M)$ and number of items included in each category j, $\overline{Q}_j$. Items within a category are homogeneous and heterogeneous among categories. For example, if a seller offers the items: A, A, B, B, B, C, C; this means that there are three categories and the supply per category is equal to $\overline{Q}_1 = 2$ (items type A) first category, $\overline{Q}_2 = 3$ (items type B) second category, and $\overline{Q}_3 = 2$ (items type C) third category. The seller will also set the starting price per item per category (p_j^s) and the price increment.

To avoid bidders waiting to see their rivals' behavior and bidding in the last rounds (*snake in the grass strategy*) the seller can establish an **activity rule**. A simple way to do so is to assign each item with a certain number of **eligibility points** (EP), so a bidder's **activity** within one round is calculated by adding the EP of the items for which the bidder has placed a bid. Bidder i's activity in round t will determine his **eligibility** for round $t + 1$, this is, the maximum number of items the bidder is eligible to bid in the next round. This rule is known as the **eligibility-based activity rule** and requires bidders to bid on packages of the same size or smaller (same or less EP) as prices increases.

Table 5.7 shows the information of the items to be auctioned in the example we are considering. There are five bidders and they all start the auction with the maximum eligibility points ($\text{Eligibility}_i^0 = (2 \times 10) + (3 \times 4) + (2 \times 2) = 36$), this means that, in the first round $t = 0$, they are allowed to bid on all items.[15]

If a bidder's activity in round t is below his eligibility, his eligibility for round $t + 1$ is reduced. In this example, we assume that the eligibility in round $t + 1$ is equal to the activity in round t.[16]

At the beginning of each round $t + 1$ the bidders obtain information about the price per item in each category (p_j^{t+1}) and the aggregate demand by category of the previous round (Q_j^t). The aggregate demand for category j in round t is equal to the sum of the bids submitted by each bidder for that category:

$$Q_j^t = \Sigma_{i=1}^n q_{i,j}^t, \tag{5.10}$$

Table 5.7 Items in a simultaneous ascending clock auction

Category j	Type	$\overline{Q}_j$	EP	p_j^s	Increment p_j^t
$j = 1$	A	2	10	100	10
$j = 2$	B	3	4	20	5
$j = 3$	C	2	2	5	5

[15] The eligibility points per bidder at the beginning of the auction can be calculated based on their financial situation, the market share, or the deposit done.

[16] In this example we are assuming that bidders must always bid up to 100 % of their EP in order to keep the same eligibility in the next round. Nevertheless this percentage could be lower, at least in the first rounds, thus giving more flexibility to the bidders.

Table 5.8 Simultaneous clock auction: Bidder 1

	Type A			Type B			Type C			Eligibility_1^t	Activity_1^t
Round t	p_1^t	$q_{1,1}^t$	Q_1^t	p_2^t	$q_{1,2}^t$	Q_2^t	p_3^t	$q_{1,3}^t$	Q_3^t		
$t = 0$	100	2	5	20	3	10	5	2	8	36	36
$t = 1$	110	1	3	25	3	5	10	2	6	36	26
$t = 2$	120	1	2	30	2	4	15	1	4	26	20
$t = 3$	120	1	2	35	2	4	20	0	3	20	18
$t = 4$	120	1	2	40	1	3	25	2	2	18	18

where $q_{i,j}^t$ is the number of items of category j that bidder i is willing to acquire in round t. The seller will increase in round $t + 1$ the price of those categories for which demand exceeds supply:

$$Q_j^t > \overline{Q_j}, \tag{5.11}$$

This process continues until there is no longer excess demand in all categories.

Table 5.8 depicts the auction per round from the perspective of the first bidder. This bidder bids on all items in round $t = 0$, so he has an activity of $\text{Activity}_1^0 = 36$, which will be his eligibility in the next round. In round $t = 1$ the bidder's activity is equal to $\text{Activity}_1^1 = 26$, as he stops bidding in one item of category A. In the next round $t = 2$ his activity is again below his eligibility $\text{Activity}_1^2 = 20$. In this round and in the following, demand equals supply for category A, so the price remains unchanged for this type of items. In round $t = 3$ the bidder only bids on one item A and two items B, so his activity is equal to $\text{Activity}_1^3 = 18$. After this round, the bidder will no longer be able to bid on two A items as his eligibility is below 20. In round $t = 4$ the bidder still has many options to bid on, as long as his activity does not excess his eligibility of 18. For example, he can keep bidding on one A item and two B items or he can bid on one A, one B, and two C. Both options involve the same activity level that equals the bidder's eligibility in this round. Finally, the bidder chooses the second option.

As rounds go by and prices increase, bidders' demand decreases. The auction ends in round $t = 4$ as demand equals supply in all categories. At last, the first bidder has obtained an A item for $p_1^4 = 120$ euros, a B item for $p_2^4 = 40$ euros, and two C items for $p_3^4 = 25$ euros each.

5.4 Variables Used in This Chapter

In this chapter, we use the following variables:

- $I = (1, 2, \ldots, N)$: Bidders.
- $J = (1, 2, \ldots, M)$: Items (homogeneous or heterogeneous).
- $V_i = (v_{i,1}, v_{i,2}, \ldots, v_{i,M})$: Vector of bidder i's values for each item.

- $v_{i,j}$: Value of bidder i for the item j.
- $B_i = (b_{i,1}, b_{i,2}, \ldots, b_{i,M})$: Vector of bidder i's bids for each item in a sealed-bid auction (single-round).
- $b_{i,j}$: Bidder i's bid for item j in a sealed-bid auction (single-round).
- b_j^*: Winning bid submitted by any bidder for item j in a sealed-bid auction (single-round).
- $q_{i,j}^*$: Function to determine if bidder i wins item j. $q_{i,j}^* = 1$ if bidder i wins the item j and $q_{i,j}^* = 0$ if he does not.
- $B_i^t = (b_{i,1}^t, b_{i,2}^t, \ldots, b_{i,M}^t)$: Vector of bidder i's bids for each item in round t of a SAA.
- $b_{i,j}^t$: Bidder i's bid for item j in round t of a SAA.
- $B_{\max}^t = (b_{\max 1}^t, b_{\max 2}^t, \ldots, b_{\max M}^t)$: Vector of highest bids per item in round t of a SAA.
- $b_{\max j}^t$: Highest bid submitted by any bidder for item j in round t of a SAA.
- $B_{\min}^{t+1} = (b_{\min 1}^{t+1}, b_{\min 2}^{t+1}, \ldots, b_{minM}^{t+1})$: Vector of minimum bids per item in round $t+1$ of a SAA.
- $b_{\min j}^{t+1}$: Minimum bid for item j in round $t+1$ of a SAA.
- λ: Bid increment that the seller establishes between rounds in a SAA.
- EP_i^t: Bidder i's eligibility points in round t.
- $\overline{Q}_j$: Number of items offered in category j in a simultaneous auction.
- $q_{i,j}^t$: Bidder i's demand of items in category j in round t in a simultaneous ascending clock auction.
- Q_j^t: Aggregate demand of items in category j in round t in a simultaneous ascending clock auction.
- $P^s = (p_1^s, p_2^s, \ldots, p_M^s)$: Vector of starting bids (minimum bids) per item in a SAA.
- p_j^s: Starting bid (minimum bid) for item j (or category j).
- p_j^*: Selling price for item j (or category j).
- P_i^*: Bidder i's payment for items won.
- Γ_i^*: Income of bidder i from the acquired items.
- Π_i^*: Surplus of bidder i from the acquired items.
- R^*: Seller's revenue.

5.5 Exercises

1. A seller does a SAA of two items, with an opening price per item of $p_j^s = 10$ euros, in which there are three bidders whose values are depicted in Table 5.9. Answer the following questions.

Table 5.9 Values

Item j	$v_{1,j}$	$v_{2,j}$	$v_{3,j}$
1	15	40	50
2	30	60	20
1 + 2	40	55	60

Table 5.10 Items in a simultaneous ascending clock auction

Category j	Type	$\overline{Q_j}$	EP	p_j^s	Increment p_j^t
$j = 1$	A	1	10	100	50
$j = 2$	B	2	5	50	10
$j = 3$	C	3	2	10	5

(a) Describe the auction from round to round, assuming that bidders always submit the minimum bid and stop bidding when the minimum bid is greater than or equal to their values. The seller sets a bid increment of $\lambda = 10$ euros.
(b) Determine the winning bidders and the prices they pay for the items they acquire.
(c) Calculate the surplus of each participant.
(d) Calculate the seller's revenue.

2. Table 5.10 shows the items being offered in a simultaneous ascending clock auction. Indicate:
   (a) Eligibility points needed to bid on all items.
   (b) A valid bid for a bidder with $\mathrm{EP}_i = 20$, for which his activity matches his eligibility.
   (c) A valid bid for a bidder with $\mathrm{EP}_i = 20$, for which his activity is below his eligibility.
   (d) A nonvalid bid for a bidder with $EP_i = 20$.

5.6 Solutions to Exercises

1. The results obtained after performing a simultaneous auction in which multiple substitutive items are offered are shown in the following items.
   (a) Table 5.11 shows the round-to-round auction, assuming that the bidders always submit the lowest possible bid and that they stop bidding when the minimum bid is greater than or equal to their personal value of the item.
   In the first round ($t = 0$), all of the bidders bid for both items. In round $t = 1$, the first player only bids for the second item, the second bidder continues bidding on both items, and the third bidder only bids for the first item. In round $t = 2$, the first bidder stops bidding, the second bidder still bids on both items, and the third bidder only bids for the first item. As the second bidder is the only one bidding on item $j = 2$, the price of this item stops increasing in the successive rounds. Finally, in round $t = 3$, the third bidder is the only one interested in the first item, so no more bids are submitted and the auction ends.
   (b) The second bidder wins the second item, for which he pays an amount equal to $P_2^* = 30$ euros. The first item is awarded to the third bidder for the amount of $P_3^* = 40$ euros.

Table 5.11 Simultaneous ascending auction

	$b^t_{1,j}$	$b^t_{2,j}$	$b^t_{3,j}$	$b^t_{\max j}$	$b^{t+1}_{\min j}$
$t = 0$					
$j = 1$	10	10	10	10	20
$j = 2$	10	10	10	10	20
$t = 1$					
$j = 1$	0	20	20	20	30
$j = 2$	20	20	0	20	30
$t = 2$					
$j = 1$	0	30	30	30	40
$j = 2$	0	$\boxed{30}$	0	30	40
$t = 3$					
$j = 1$	0	0	$\boxed{40}$	40	50
$j = 2$	–	–	–	–	–

(c) The surplus of the second bidder is equal to $\Pi_2^* = v_{2,2} - P_2^* = 60 - 30 = 30$ euros. The surplus of the third bidder is equal to $\Pi_3^* = v_{3,1} - P_3^* = 50 - 40 = 10$ euros.

(d) The seller's income is equal to $R^* = P_2^* + P_3^* = 30 + 40 = 70$ euros.

2. With the items being auctioned:
   (a) To be eligible to bid on all items, a bidder needs $\text{EP}_i = (1 \times 10) + (2 \times 5) + (3 \times 2) = 26$.
   (b) If a bidder bids on one A item and two B items, his activity is equal to 20, just as his eligibility.
   (c) If a bidder bids on one A item and two C items, his activity is equal to 14, below his eligibility.
   (d) A bidder with $PE_i = 20$ will not be eligible to bid on one A item, one B item, and three C items, as this bid implies an activity of $(1 \times 10) + (1 \times 5) + (3 \times 2) = 21$, which is above his eligibility.

Double Auctions

6

6.1 Introduction

In the auctions analyzed to this point, we have assumed that there is only one seller and many bidders, in which the seller acts as a monopolist (**forward auction**). However, the main characteristic of a **double auction** is that multiple buyers and sellers interact; it is a **two-sided auction**.

In a double auction the buyers submit *bids* for the amounts that they are willing to pay, whereas the sellers make offers, *asks*, to indicate the prices at which they are willing to sell. This auction model has been used for more than 100 years to exchange items such as stocks, bonds, and agricultural products. There are a significant number of different double-auction models, with both one round and multiple rounds. In the following sections, we will comment on several of these models, but we recommend reviewing the work of Friedman and Rust [31] to complement this study.

6.2 Sealed-Bid Double Auction

In a **sealed-bid double auction**, both the buyers and the sellers indicate, in a single round, the price at which they are willing to buy or sell an item.[1] The bid made by buyer i is represented by b_i, which is the maximum price for which the buyer is willing to buy the item. The ask made by seller k is represented by o_k, which is

[1]The **periodic double auction** is a sealed-bid auction in which potential buyers and sellers have one period of time to submit their bids or asks. Then the auctioneer determines the items to be exchanged and a new auction starts. The auctioneer establishes a new period in which buyers and sellers can submit bids for the next auction.

A. Mochón, Y. Sáez, *Understanding Auctions*, Springer Texts in Business and Economics, DOI 10.1007/978-3-319-08813-6_6

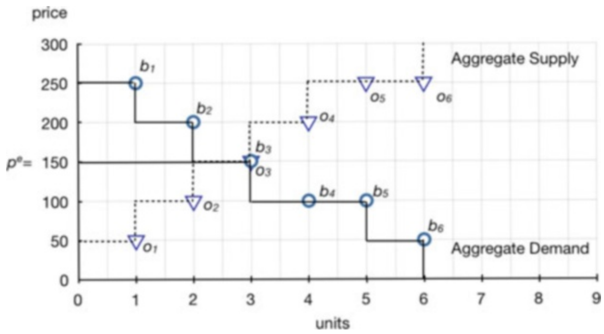


Fig. 6.1 Sealed-bid double auction

the minimum amount for which the seller is willing to sell the item.[2] The aggregate demand is obtained by rearranging the buyers' bids from the highest to the lowest. The aggregate supply is obtained by rearranging the sellers' asks from the lowest to the highest. The intersection between the aggregate supply and demand determines the amount that is bought and sold, as well as the **equilibrium price**, p^e.

To better understand how this type of auction works, let us examine the following example. There are six buyers who are willing to pay the following prices per item: $b_1 = 250$ euros, $b_2 = 200$ euros, $b_3 = 150$ euros, $b_4 = 100$ euros, $b_5 = 100$ euros, and $b_6 = 50$ euros. In addition, there are six sellers who are not willing to sell an item for less than the following amounts: $o_1 = 50$ euros, $o_2 = 100$ euros, $o_3 = 150$ euros, $o_4 = 200$ euros, $o_5 = 250$ euros, and $o_6 = 250$ euros. Figure 6.1 shows the aggregate supply and demand, in which we can observe that the demand equals supply when the price is equal to $p^e = 150$ euros. The winning bids are: $b_1^* = 250$ euros, $b_2^* = 200$ euros, and $b_3^* = 150$ euros; the winning offers are: $o_1^* = 50$ euros, $o_2^* = 100$ euros and $o_3^* = 150$ euros. Given the bids and asks that have been made, the first three sellers will effectively sell their items, and the first three buyers obtain one item each.

After determining the participants who will finally buy or sell an item, the next step is to calculate the prices that the buyers will pay and the revenue that the sellers will obtain. These amounts depend on the rules established by the auctioneer, that is, the **clearing house**. The rules that are most frequently used are uniform-price and discriminatory pricing rule.

[2]In a double auction, the bidders may also bid on or the sellers may offer multiple items, but to simplify the analysis, we will assume that each participant bids on and offers, at the most, one item.

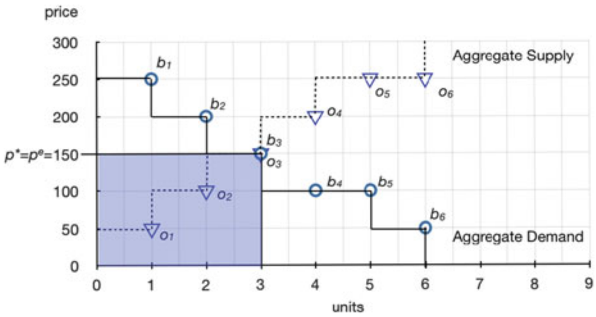


Fig. 6.2 Sealed-bid double auction with uniform-price rule

6.2.1 Sealed-Bid Double Auction with Uniform-Price Rule

If the **uniform-price rule** is established, all buyers will have to pay the same price for the acquired items, a price that is equal to the equilibrium price, that is, the price for which the aggregate demand equals the aggregate supply $p^e = p^*$. Continuing with the previous example and under this pricing rule, the winning buyers will pay: $P_1^* = P_2^* = P_3^* = 150$ euros. Each seller will also obtain a revenue of 150 euros. These payments are represented by the shaded area in Fig. 6.2.

Aggregate supply and demand may be equal not for a single equilibrium price but rather for an interval, in which case a rule must be applied to set the exchange price. The k**-double auction**, which is discussed later in this chapter, solves this problem.

6.2.2 Sealed-Bid Double Auction with Discriminatory Pricing Rule

If the seller sets the **discriminatory pricing rule** each bidder pays a different price for the items that he obtains, which will depend on the established rule. One option is to set the exchange price based on the bids made by the buyers (*pay-buyers' price*), so it is a *pay-your-bid* auction. With this rule the aggregate payment of all bidders is equal to the sum of the winning bids: $P^* = \Sigma_{i \in W} b_i^*$, where W is the set of winning bidders.

In the example of the previous section, the total amount that the buyers would pay is equal to $P^* = 250 + 200 + 150 = 600$ euros that corresponds to the shaded area in Fig. 6.3. The first buyer pays $P_1^* = 250$ euros, the second one $P_2^* = 200$ euros, and the third one $P_3^* = 150$ euros.

However, the sale price may also depend on the offers made by the sellers (*pay-sellers' price*). The total payment of all winning bidders with this rule is equal to the sum of the winning offers: $P^* = \Sigma_{k \in W} o_k^*$, where W is the set of winning sellers.

The total amount that the buyers would pay in this example is equal to $P^* = 50 + 100 + 150 = 300$ euros, represented by the shaded area in Fig. 6.4. So the

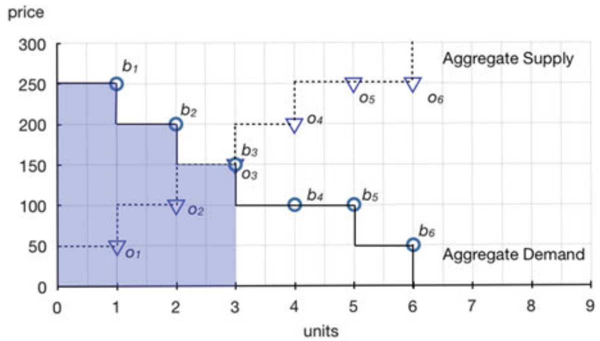


Fig. 6.3 Sealed-bid double auction with discriminatory pricing rule: pay-buyers' price

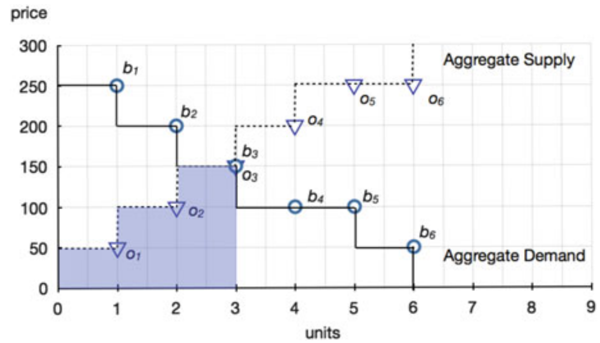


Fig. 6.4 Sealed-bid double auction with discriminatory pricing rule: pay-sellers' price

sellers' revenue is equal to $R_1^* = 50$ euros for the first one, $R_2^* = 100$ euros for the second one, and $R_3^* = 150$ euros for the third one.

The total payments done by all winning bidders using these two rules (pay-buyers' price and pay-sellers' price) are two extreme options but we could also consider an intermediate price-rule using the following equation:

$$P^* = \Sigma_{i,k \in W}(\alpha b_i^* + (1-\alpha)o_k^*), \tag{6.1}$$

where W is the set of winning buyers/sellers and α is a number between zero and one that shows the influence or bias of the buyer/seller at the time the exchange price is determined. For any value of α, no buyer pays more than what he would have bid, and no seller sells for less than what he would have offered.

Continuing with the previous example, the bids of the winning bidders are: $b_1^* = 250$ euros, $b_2^* = 200$ euros, and $b_3^* = 150$ euros. The winning asks are the following: $o_1^* = 50$ euros, $o_2^* = 100$ euros, and $o_3^* = 150$ euros. If, for example, $\alpha = 0.8$, then the aggregate payment done by the winning bidders is equal to $P^* = (0.8\times 250)+(0.8\times 200)+(0.8\times 150)+(0.2\times 50)+(0.2\times 100)+(0.2\times 150) = 540$ euros. Instead, if $\alpha = 0.2$, then $P^* = 360$ euros.

We can observe that, as the value of α increases, the total payment approaches the value with the pay-buyers' price rule. Conversely, as α approaches zero, the value is closer to the one obtained with the pay-sellers' price rule.

6.2.3 k-Double Auction

In a sealed-bid double auction (one round) in which the uniform-price rule is set, the supply and the demand may be equal for a range of prices instead of for a fixed price: $b^*_{\text{min}} \geq p^{\text{e}} \geq o^*_{\text{max}}$, where b^*_{min} is the lowest bid made by a buyer that exceeds an offer and o^*_{max} is the highest offer made by a seller that is below a bid. Under these circumstances, it is necessary to establish a rule to select the exchange price (p^*), which should be included in this interval. A solution that is frequently used is to draw on the k**-double auction**, as described by Satterthwaite and Williams [73]. In this model, the exchange price is calculated using the following equation:

$$p^* = kb^*_{\text{min}} + (1-k)o^*_{\text{max}}, \tag{6.2}$$

where k is a number between zero and one. Upon applying this rule, no buyer pays more than what he has bid, and no seller sells for less than his ask.

To understand this auction, let us analyze the following example. In a sealed-bid double auction, the potential buyers submit the following bids: $b_1 = 250$ euros, $b_2 = 200$ euros, $b_3 = 150$ euros, $b_4 = 100$ euros, $b_5 = 100$ euros, and $b_6 = 50$ euros. The potential sellers make the following offers: $o_1 = 50$ euros, $o_2 = 100$ euros, $o_3 = 100$ euros, $o_4 = 150$ euros, $o_5 = 250$ euros, and $o_6 = 250$ euros. The aggregate supply and demand are represented in Fig. 6.5. As shown, the supply and the demand are equal for the interval of $b^*_{\text{min}} = 150 \geq p^{\text{e}} \geq 100 = o^*_{\text{max}}$.

If we use a k-double auction to set the price paid, the price will depend on the value of the parameter k. The sale price (which will have to be included in the interval) is calculated by the following equation:

$$p^* = k150 + (1-k)100. \tag{6.3}$$

If $k = 0.5$, the exchange price is equal to $p^* = 125$ euros. If $k = 0.8$, the exchange price is equal to $p^* = 140$ euros. If $k = 0.2$, the exchange price is equal to $p^* = 110$ euros.

In this example, the buyers who effectively buy an item are the first, second, and third bidders. If we set a value of $k = 0.5$, the final payment that each of them

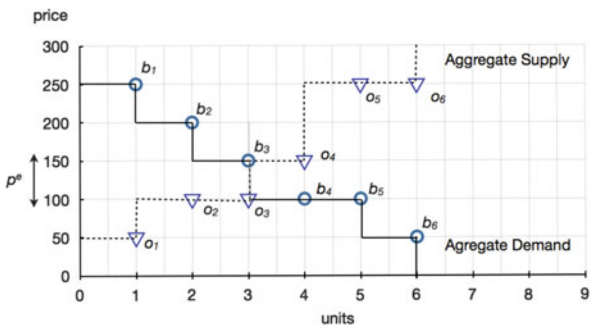
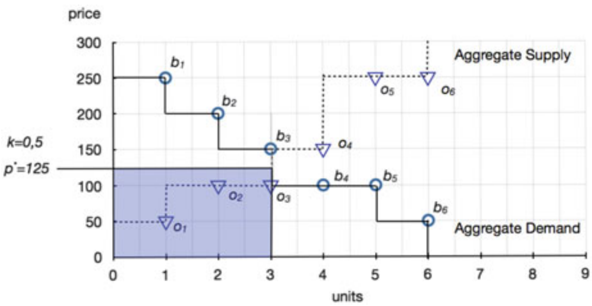


Fig. 6.5 k-double auction

Fig. 6.6 k-double auction for $k = 0.5$



makes is equal to $P_1^* = P_2^* = P_3^* = 125$ euros. The sellers who finally sell the items that they were offering are the first three sellers, who obtain revenues equal to $R_1^* = R_2^* = R_3^* = 125$ euros for $k = 0.5$. The shaded area in Fig. 6.6 represents the payments made by the buyers under this assumption.

6.3 Continuous Double Auction

The main characteristic of a sealed-bid double auction is that all of the participants submit their bids and offers in a single round, and the clearing house establishes all of the items to be exchanged and the sale prices. However, a **continuous double auction** (**CDA**) allows items to be bought and sold over the course of multiple rounds. According to rules that are determined, different types of CDAs can be established [31]. Below, we comment on some of these formats.

6.3.1 Synchronized CDAs

A **synchronized continuous double auction**[3] is composed of r rounds, which are divided into s periods. In each period, there is a fixed number of t steps with alternate steps in which the participants make bids (buyers) and asks (sellers) and steps in which buy and sell orders are placed. Each step t begins with a bid/ask step (BA step), in which, simultaneously, the buyers indicate the prices that they are willing to pay per unit, and the sellers indicate the prices at which they are willing to sell. $b_i^{s,t}$ represents the amount that bidder i is willing to pay for an item in step t in the period s, and $o_k^{s,t}$ represents the amount for which seller k is willing to sell an item in step t of period s. Among all of the bids and asks made in step t of period s of round r, the bidder who has the *highest outstanding bid* ($b_{\max}^{s,t}$) and the seller who has the *lowest outstanding ask* ($o_{\min}^{s,t}$) are determined. These participants pass on to the buy/sell step.

[3]Developed by the Santa Fe Institute [67].

Table 6.1 Synchronized continuous double auction

Round $r = 0$								
Period $s = 0$		Bids			Asks			Price
t	Step	$b_1^{0,t}$	$b_2^{0,t}$	$b_3^{0,t}$	$o_1^{0,t}$	$o_2^{0,t}$	$o_3^{0,t}$	$p^{*0,t}$
0	BA	5	$\boxed{10}$	7	$\boxed{20}$	25	28	
	BS		No		No			
1	BA	7	$\boxed{17}$	10	$\boxed{15}$	23	20	
	BS		Ok		Ok			$\boxed{17}$
Period $s = 1$		Bids			Asks			Price
t	Step	$b_1^{1,t}$	$b_2^{1,t}$	$b_3^{1,t}$	$o_1^{1,t}$	$o_2^{1,t}$	$o_3^{1,t}$	$p^{*1,t}$
0	BA	14	15	$\boxed{16}$	$\boxed{18}$	19	20	
	BS			Ok	Ok			$\boxed{18}$
1	BA	$\boxed{10}$	8	7	16	15	$\boxed{13}$	
	BS	No				No		

In the buy/sell step (BS step), both the buyers and the sellers may accept the bids or asks and the transaction is carried out. However, the participants may choose not to accept the bids or asks, and this round may end with no exchange. In the next round ($t + 1$), the BA step begins again, which later leads to another BS step.[4]

After multiple steps and periods, the participants learn about their rivals' values and their bidding strategies. In this auction model, participants must answer three key questions: (1) how much to bid or ask, (2) when to submit a bid or make an offer, (3) and when to accept a bid or an ask. Each buyer pays a different price for the units he gets, which is equal to the exchange price corresponding to period s of BS step t in which the buyer accepts the offer and the seller accepts the bid, $p^{*s,t}$.[5]

To better understand this auction, we analyze the following example, in which there is a single round ($r = 0$), composed of two periods, which include two BA and BS steps. Table 6.1 shows both the bids and the asks made in each step by the three buyers and the three sellers involved.

The auction begins with the first BA step ($t = 0$) in the first period ($s = 0$) of the first round ($r = 0$) in which buyers and sellers submit their bids and asks. The highest bid in this BA step is $b_{\max}^{0,0} = 10$ euros, submitted by the second buyer, and the minimum ask is $o_{\min}^{0,0} = 20$ euros, demanded by the first seller. These two participants go on to the BS step ($t = 0$); however, neither of them accepts the bid/ask of the other, and therefore, no transaction takes place.

The BA step $t = 1$ of period $s = 0$ then begins, in which the highest outstanding bid and the lowest outstanding ask are $b_{\max}^{0,1} = 17$ euros and $o_{\min}^{0,1} = 15$ euros, made

[4]The main characteristic of this CDA is the time discretization, in which the BA steps alternate with the BS steps. In auctions in which time is considered continuously, the fastest agents may have certain advantages. The speed of an agent may be affected by delays in the communication system or in the processors. However, with time discretization and with a sufficient time interval for each phase, it can be ensured that all of the participants have the same opportunities.

[5]McCabe et al. [50] proposed a CDA with a uniform-price rule, the uniform-price CDA.

by the second buyer and the first seller, respectively. These participants go on to the $t = 1$ BS step, and in this case, there is a transaction because the bid is greater than the ask, and both accept the exchange. Therefore, the second buyer buys one unit from the first seller, with a sale price equal to $p^{*0,1} = 17$ euros.

After this phase, the first period ends, and the second period ($s = 1$) begins with the BA step $t = 0$. In this step, the third buyer makes the highest bid, $b^{1,0}_{\max} = 16$ euros, and the first seller makes the lowest ask, $o^{1,0}_{\min} = 18$ euros. These participants go on to the BS step, in which, despite the bid being lower than the ask, both participants accept the transaction. Therefore, the third buyer pays $p^{*1,0} = 18$ euros to the first seller for one item.

Then, in BA step $t = 1$ takes place, in which $b^{1,1}_{\max} = 10$ euros and $o^{1,1}_{\min} = 13$ euros. The participants who have made this bid and ask go on to the BS step, but no transaction takes place because neither of them accepts the bid/ask of the other.

After this phase, the auction ends. The second buyer has won an item for which he has paid $P^*_2 = 17$ euros and the third buyer has won an item for $P^*_3 = 18$ euros. The first seller has sold two items, obtaining a total revenue of $R^*_1 = 17 + 18 = 35$ euros.

6.3.2 Double Dutch Auction

The **double Dutch auction**, developed by the University of Arizona [41, 49], operates as follows. An opening high price is established for the buyers (p^s_b) that progressively decreases in multiple rounds (t_b) until a buyer stops the process and is awarded an item. Similarly, for the sellers, an opening price is established (p^s_o), which is considered low and which increases from round to round (t_o) until a seller stops increasing the price and agrees to sell an item. The process continues on both sides until the prices intersect. At this point, all of the items that have been awarded during the process are exchanged. The final price that the buyers must pay is the overlapping price from both processes, that is, the price of the final round for both the buyers and the sellers: $p^{T_b}_b = p^{T_o}_o = p^*$, in which T_b is the last buyers' round and T_o is the last sellers' round.[6]

The next example illustrates a double Dutch auction in which $\hat{b}_i$ represents the intention of buyer i to bid, and $\hat{o}_k$ shows the intention of seller k to ask. There are five buyers willing to bid for an item at the following prices: $\hat{b}_1 = 40$ euros, $\hat{b}_2 = 35$ euros, $\hat{b}_3 = 25$ euros, $\hat{b}_4 = 10$ euros, and $\hat{b}_5 = 5$ euros. In addition, there are five sellers who are not willing to sell an item for less than the following prices: $\hat{o}_1 = 10$ euros, $\hat{o}_2 = 20$ euros, $\hat{o}_3 = 25$ euros, $\hat{o}_4 = 35$ euros, and $\hat{o}_5 = 40$ euros. Knowing these intentions to bid and ask and setting bid a increment/decrement at five euros, the evolution of the auction is shown in Table 6.2.

[6]It may happen that the bid of the last round on the buyers' side not to be equal to the offer of the last round on the sellers' side. Hence, the clearing house will have to set a selling price which will not be higher than the last bid nor lower than the last offer.

Table 6.2 Double Dutch auction

Buyers' rounds	$p_b^{t_b}$	$b_1^{t_b}$	$b_2^{t_b}$	$b_3^{t_b}$	$b_4^{t_b}$	$b_5^{t_b}$
$t_b = 0$	$p_b^0 = 55$	0	0	0	0	0
$t_b = 1$	$p_b^1 = 50$	0	0	0	0	0
$t_b = 2$	$p_b^2 = 45$	0	0	0	0	0
$t_b = 3$	$p_b^3 = 40$	1	0	0	0	0
Sellers' rounds	$p_o^{t_o}$	$o_1^{t_o}$	$o_2^{t_o}$	$o_3^{t_o}$	$o_4^{t_o}$	$o_5^{t_o}$
$t_o = 0$	$p_o^0 = 0$	0	0	0	0	0
$t_o = 1$	$p_o^1 = 5$	0	0	0	0	0
$t_o = 2$	$p_o^2 = 10$	1	0	0	0	0
Buyers' rounds	$p_b^{t_b}$	$b_1^{t_b}$	$b_2^{t_b}$	$b_3^{t_b}$	$b_4^{t_b}$	$b_5^{t_b}$
$t_b = 4$	$p_b^4 = 35$	0	1	0	0	0
Sellers' rounds	$p_o^{t_o}$	$o_1^{t_o}$	$o_2^{t_o}$	$o_3^{t_o}$	$o_4^{t_o}$	$o_5^{t_o}$
$t_o = 3$	$p_o^3 = 15$	0	0	0	0	0
$t_o = 4$	$p_o^4 = 20$	0	1	0	0	0
Buyers' rounds	$p_b^{t_b}$	$b_1^{t_b}$	$b_2^{t_b}$	$b_3^{t_b}$	$b_4^{t_b}$	$b_5^{t_b}$
$t_b = 5$	$p_b^5 = 30$	0	0	0	0	0
$t_b = 6$	$p_b^6 = 25$	0	0	1	0	0
Sellers' rounds	$p_o^{t_o}$	$o_1^{t_o}$	$o_2^{t_o}$	$o_3^{t_o}$	$o_4^{t_o}$	$o_5^{t_o}$
$t_o = 5$	$p_o^5 = 25$	0	0	1	0	0
Intention to bid/ask	i/k	1	2	3	4	5
$\hat{b}_i$		40	35	25	10	5
$\hat{o}_k$		10	20	25	35	40

The auction begins with a high opening price for the buyers, $p_b^s = 55$ euros. At this amount, no buyer makes a bid, which means that the price decreases until round $t_b = 3$, when the price is equal to $p_b^3 = 40$ euros, and the first bidder bids. At this point, the price stops decreasing, and the process turns to the sellers. The opening price is $p_o^s = 0$ euros, and no seller is willing to sell an item for this amount. The price successively increases until, in round $t_o = 2$, the price is equal to $p_o^2 = 10$ euros and there is a seller willing to exchange an item. The price stops increasing, and we return to buyers' rounds, in which the price again decreases. In the next buyers' round ($t_b = 4$), the second bidder bids at $p_b^4 = 35$ euros. Then, we return to the sellers' round to determine the next seller who is willing to exchange an item. In round $t_o = 3$, the price is equal to $p_o^3 = 15$ euros, and there is no seller who is interested. However, in the next round, the price is equal to $p_o^4 = 20$ euros, and another seller decides to offer his item at this amount. Now there are two items awarded, but because the last price on the buyers' side is greater than the last price on the sellers' side ($p_b^4 = 35 > 20 = p_o^4$), the auction continues. The buyers' price decreases in the next round, but because no bidder bids, the price continues to decrease. In round $t_b = 6$, the price is equal to $p_b^6 = 25$ euros and another bidder wants to buy an item, so the buyers' rounds stop again. The sellers' price increases in the next round to $p_o^5 = 25$ euros, and at this amount, another seller is willing to sell. In this round, the last buyers' price is equal to the last sellers' price, which marks the end of the auction, at $p_b^6 = p_o^5 = 25$ euros.

The final outcome of the auction is that the three first buyers each win an item in different rounds, but all of them pay the same price, which is the intersection price between the buyers and the sellers: $p^* = 25$ euros. The last two bidders, who are only willing to buy an item for $b_4 = 10$ euros and $b_5 = 5$ euros, respectively, do not manage to acquire any items. On the sellers' side, the three first sellers finally sell their items at a price of $p^* = 25$ euros. The two final sellers do not manage to award their items.

6.4 Variables Used in This Chapter

In this chapter, we use the following variables:

- $I = (1, 2, \ldots, N)$: Bidders.
- $J = (1, 2, \ldots, M)$: Identical items.
- $V_i = (v_{i,1}, v_{i,2}, \ldots, v_{i,M})$: Vector of bidder i's values for each item.
- $v_{i,j}$: Value of bidder i for the item j.
- b_i: Bidder i's bid for one item in a double auction.
- o_k: Seller k's ask for one item in a double auction.
- $b_i^{s,t}$: Bidder i's bid for one item in round t of period s of a synchronized CDA.
- $o_k^{s,t}$: Seller k's ask for one item in round t of period s of a synchronized CDA.
- $b_{\max}^{s,t}$: Highest outstanding bid in round t of period s of a synchronized CDA.
- $o_{\min}^{s,t}$: Lowest outstanding ask in round t of period s of a synchronized CDA.
- $\hat{b}_i$: Bidder i's intention to bid for one item in a double Dutch auction.
- $\hat{o}_k$: Seller k's intention to ask for one item in a double Dutch auction.
- $b_{\min}^*$: Lowest bid that matches an offer in a double Dutch auction.
- $o_{\max}^*$: Highest offer that matches a bid in a double Dutch auction.
- α: Parameter between 0 and 1, which determines the influence or bias of the buyer to set the final price in a double auction.
- $b_i^{t_b}$: Bidder i's bid in round t_b (buyers' rounds) in a double Dutch auction.
- $o_k^{t_o}$: Seller k's ask in round t_o (sellers' rounds) in a double Dutch auction.
- P^*: Payment done by all bidders for the items won.
- P_i^*: Bidder i's payment for the items won.
- p^{e}: Equilibrium price in a double auction.
- p^*: Selling price in a double auction.
- $p^{*s,t}$: Selling price of period s in round t of a synchronized CDA.
- p_b^s: Starting bid in a double Dutch auction.
- p_o^s: Starting ask in a double Dutch auction.
- $p_b^{t_b}$: Bidding price in round t_b in a double Dutch auction.
- $p_o^{t_o}$: Offering price in round t_o in a double Dutch auction.

- t_b: Round t on the buyers' side in a double Dutch auction.
- t_o: Round t on the sellers' side in a double Dutch auction.
- Γ_i^*: Income of bidder i from the acquired items.
- Π_i^*: Surplus of bidder i from the acquired items.
- R^*: Seller's revenue.

6.5 Exercises

1. In a sealed-bid double auction, the buyers submit the following bids in a single round: $b_1 = 25$ euros, $b_2 = 25$ euros, $b_3 = 25$ euros, $b_4 = 20$ euros, $b_5 = 20$ euros, and $b_6 = 10$ euros. In addition, the sellers make the following offers: $o_1 = 5$ euros, $o_2 = 5$ euros, $o_3 = 10$ euros, $o_4 = 15$ euros, $o_5 = 20$ euros, and $o_6 = 25$ euros. Given this information, solve the problems below.
   (a) Graphically represent the aggregate supply and demand.
   (b) Indicate which participants buy and sell an item.
   (c) Determine the final payments that the buyers will make if an uniform-price rule is established. Graphically illustrate the total payments of all the buyers.
   (d) Determine the final payments that the buyers will make if a discriminatory pricing rule with the pay-buyer's price is set. Show this result graphically.
   (e) Determine the final payments that the buyers will make if a discriminatory pricing rule with the pay-sellers' price is set. Show this result graphically.

6.6 Solutions to Exercises

1. The results obtained after performing a sealed-bid double auction are described in the following sections.
   (a) Figure 6.7 shows the aggregate supply and demand based on the bids and the offers made by the buyers and the sellers.

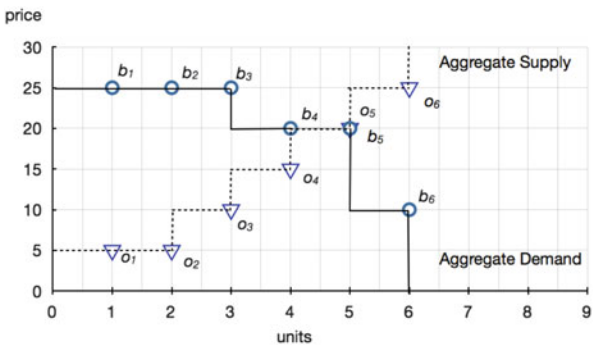


Fig. 6.7 Sealed-bid double auction

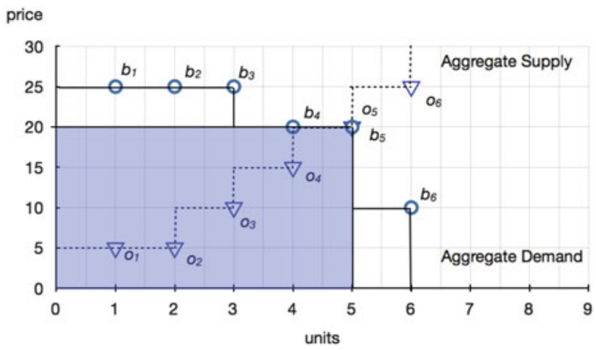


Fig. 6.8 Sealed-bid double auction with uniform-price rule

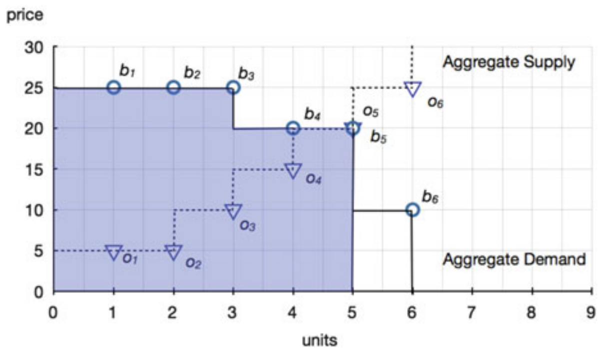
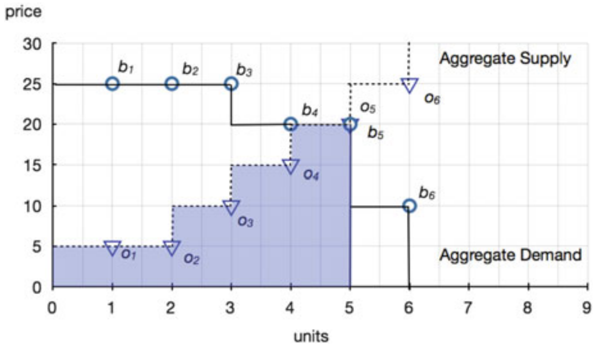


Fig. 6.9 Sealed-bid double auction with discriminatory pricing rule: pay-buyers' price

(b) Out of the six bidders, all except the last one (b_6) end up buying an item. Similarly, out of the six sellers, all except one (o_6) exchange the item that they were offering.
(c) If the selling price is established according to the uniform-pricing rule, all of the buyers pay the same amount for the acquired items. According to this rule, the exchange price is that for which the aggregate demand is equal to the aggregate supply, in this example, $p^* = p^e = 20$ euros. The total payment of all bidders is equal to $20 \times 5 = 100$ euros, represented by the shaded area in Fig. 6.8.
(d) If, instead of a uniform-price, a discriminatory price with the pay-buyers' price is set, the total amount that the buyers will pay will be equal to $(25 \times 3) + (20 \times 2) = 115$ euros. This amount is represented by the shaded area in Fig. 6.9.

Fig. 6.10 Sealed-bid double auction with discriminatory pricing rule: pay-sellers' price



(e) If a discriminatory price with the pay-sellers' price is set, the total amount that the buyers will pay is equal to $(5 \times 2) + 10 + 15 + 20 = 55$ euros, which is represented by the shaded area in Fig. 6.10.

7 Introduction to Combinatorial Auctions

7.1 Introduction

In this chapter, we introduce the concept of **combinatorial auction** (CA), or **package auction**. In this model, the seller offers multiple items (usually heterogeneous but related) in a single auction, in which the bidders are allowed to bid for the items or combinations of items that they want. These auctions are particularly suitable when substitutes and complements items are auctioned, because the risk of aggregation or exposure is reduced.

In working with CAs, the winner determination problem (WDP), that is, finding the winning combination of bids, is of particular relevance. Once this allocation problem is solved, the payments to be done by the winners for the acquired items must be established, which will depend on the pricing rule selected by the seller. We will study all of these concepts throughout this chapter.

7.2 Substitutes and Complements Items

When multiple related items are offered, it is important to recognize how winning the first item affects the marginal value of the other items. In Chap. 4, we mentioned that two items may be **substitutes** (when the value of a combination of items is lower than the sum of the individual values) or **complements** (when the value of a combination of items is greater than the sum of the individual values).

To better understand these concepts, let us analyze the following example. In a CA a seller offers an umbrella, a rain cap, and rain boots. A bidder may bid for all of the items or combinations. Table 7.1 shows the values of bidder i for each item and combination. For bidder i, the umbrella and the rain cap are substitutes because the sum of the individual values is greater than the value of both items together: $v_{i,1} + v_{i,2} = 18 > 15 = v_{i,1+2}$. In other words, when the bidder obtains one of the two items, the marginal value of the second item is lower because one

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may be substituted for the other. However, for this bidder, the umbrella and the rain boots are complements; that is, the value of obtaining both items together is greater than the sum of the individual values: $v_{i,1} + v_{i,3} = 16 < 18 = v_{i,1+3}$; there are **synergies** between the items. The rain cap and boots are also complements: $v_{i,2} + v_{i,3} = 14 < 16 = v_{i,2+3}$.

There are multiple markets in which the items auctioned may either be substitutes or complements, such as radio spectrum licenses [7], cable television licenses [33], transportation services [15], construction services [76], among others. For example, a mobile telephone company may be willing to pay 10 million euros for a radio spectrum license for zone A and 15 million euros for the same license in zone B. These licenses would be substitutes if the bidder were not willing to pay an amount greater than or equal to 25 million euros for both. However, both licenses would be complements if, by operating in both zones, synergies were obtained. In this case, the operator would be willing to pay an amount greater than 25 million euros to obtain both licenses.

7.3 The Exposure Problem

If bidders have complex preference structures (substitutes or complements items) the seller must choose an auction design that allows bidders to fully express their preferences through their bidding strategies. With complements, the bidders have to deal with the **exposure problem** (or *aggregation risk*), which means that if they decide to bid aggressively for a package of items but only win some items, they may incur in losses because they do not get the super-additivity value of the complete package.

For example, the items in Table 7.1 are auctioned in a sealed-bid (single-round) simultaneous auction and the first-price rule is established (see Chap. 5). With this mechanism, each item is offered in an independent auction, but all of the auctions are performed at the same time. Therefore, the bidders may bid for the item(s) that they want. A possible strategy for bidder i, based on his values and assuming sincere bidding, is to attempt to win the combination umbrella and boots, for which his package value is equal to $v_{i,1+3} = 18$ euros. With this aim, he could submit a bid for the umbrella of $b_{i,1} = 10$ euros and $b_{i,3} = 8$ euros for the boots. However, in

Table 7.1 Values for substitutes and complements items

Items and packages	j	$v_{i,j}$	
Umbrella	$j = 1$	10	
Cap	$j = 2$	8	
Boots	$j = 3$	6	
Umbrella and cap	$j = 1 + 2$	15	Substitutes
Umbrella and boots	$j = 1 + 3$	18	Complements
Cap and boots	$j = 2 + 3$	16	Complements
Umbrella, cap, and boots	$j = 1 + 2 + 3$	20	

Table 7.2 Bids for substitutes and complements items

Items and packages	j	$b_{i,j}$
Umbrella	$j = 1$	10
Cap	$j = 2$	8
Boots	$j = 3$	6
Umbrella and cap	$j = 1 + 2$	15
Umbrella and boots	$j = 1 + 3$	18
Cap and boots	$j = 2 + 3$	16
Umbrella, cap, and boots	$j = 1 + 2 + 3$	20

this auction model, winning one item does not mean winning the other. If the bidder loses the umbrella and only wins the boots he would have to pay eight euros (two euros more than his value). In this example, he would have incurred losses because he did not get the complete package.

In the presence of complements items, if a CA that allows bidding for complete packages is not used, the only possible strategy to avoid incurring losses would be not to bid above the individual value of each item. In other words, bidder i should at most bid $b_{i,1} = 10$ euros for the umbrella and $b_{i,3} = 6$ euros for the boots. Therefore, although he may not obtain both items, he will never be exposed to a loss.[1]

CAs are the best option to avoid the exposure problem. Continuing with the previous example in which the values of bidder i are summarized in Table 7.1, if the seller had chosen a sealed-bid combinatorial auction, bidder i could have bid individually for each item and combinations of items. Assuming sincere bidding, he would have made the seven bids shown in Table 7.2. With this mechanism, he could have bid up to 18 euros for the umbrella and rain boots combination and only six euros for the rain boots. With CAs, the bidders may fully express their preferences for the different combinations of items without the risk of incurring losses.[2]

7.4 The WDP in Combinatorial Auctions

In a CA, multiple items $J = (1, 2, \ldots, M)$ are offered among various players $I = (1, 2, \ldots, N)$. Each bidder i may submit as many bids as he likes for the items or combinations of these items. The combinations or packages are represented by $S \subseteq J$. Bidder i's value for the combination of items S is represented as $v_i(S)$,

[1]The **chopstick auction** is an example in which complements are auctioned and bidders have to face the exposure problem. The seller simultaneously offers three chopsticks, and the bidder with the highest bid wins two chopsticks. Therefore, the player with the second highest bid will be affected by the exposure problem because he will win one useless chopstick for which he must pay, see [29].

[2]In this chapter, we assume that bidders can only win with one of the bids; that is, the bids are mutually exclusive, XOR bidding language. With this rule, the bidders are assured that they will not face the exposure problem.

which is the maximum value that bidder i would be willing to pay for package S. Bidder i's bid for that package is represented as $b_i(S)$.

Among all bids submitted by all bidders, the seller will determine the winning bids that maximize his revenue, this is the feasible combination of bids that maximizes the sum of accepted bids under the constraint that each item is allocated, at most, to one bidder.[3] This allocation problem is known as the **winner determination problem** (WDP), which has the following mathematical formulation[4]:

$$\max \Sigma_{i\in I}\,\Sigma_{S\subseteq J} b_i(S)x_i(S), \tag{7.1}$$

subject to:

$$\begin{array}{ll} (1)\ \Sigma_{S\supseteq\{j\}}\Sigma_{i\in I} x_i(S) \leq 1 & \forall j \in J, \\ (2)\ \Sigma_{S\subseteq J} x_i(S) \leq 1 & \forall i \in I, \\ (3)\ x_i(S) \in \{0,1\} & \forall S \subseteq J, \forall i \in I. \end{array}$$

Solving this problem implies determining, among all of the bids ($b_i(S)$), the combination that maximizes the seller's revenue. According to this formulation, $x_i(S)$ is a binary variable, which is equal to one when a bidder wins an item or combination of items and equal to zero when he does not win any items, restriction (3). Restriction (1) ensures that each item is awarded to, at most, one bidder; that is, that a feasible allocation of items is made. Finally, restriction (2) limits the solution of the problem such that each bidder obtains, at most, one winning bid, meaning that the bids are mutually exclusive, **XOR bidding language**.[5]

The following example shows how to solve the WDP in a combinatorial auction in which the following items are offered: A, B, and C. Each bidder may submit up to seven bids, one for each item and combinations of items: A, B, C, AB, AC, BC, and ABC.[6] Assuming that there are three bidders in this auction, Table 7.3 presents the bids made by each of them $b_i(S)$ in a sealed-bid (single-round) CA in which the first-price rule is established.

With the submitted bids, there are many possible ways to allocate the items. However, solving the WDP requires identifying, among all of the possible solutions, the one that maximizes the accepted bids. One possible combination would be to award the AC items to the third bidder and the B item to the second one. With this

[3] In the final allocation the seller may not sell all items.

[4] There are different ways of mathematically expressing this problem; in this manual, we use the formulation presented by Day and Raghavan [24].

[5] The bidding languages that are most common in CAs are the **OR bidding language** and XOR bidding. With OR bidding, each bidder may win multiple bids. However, with XOR bidding, each bidder may win, at most, one bid. The problem with the OR bidding language is that when there are complements items, the bidders are affected by exposure problem, which does not occur when XOR bidding is used. In this book, all of the CAs will be explained using the XOR bidding language.

[6] The bidders are not obliged to bid for all the items or combinations.

Table 7.3 The winner determination problem

S	$b_1(S)$	$b_2(S)$	$b_3(S)$
A	$\boxed{\mathbf{200}}$	150	200
B	100	$\boxed{\mathbf{200}}$	100
C	100	100	$\boxed{\mathbf{200}}$
AB	200	200	200
AC	250	300	275
ABC	300	400	300
BC	150	150	150

allocation and the first-price rule, the seller's revenue is equal to $R = b_3(AC) + b_2(B) = 275 + 200 = 475$ euros. Although this combination is feasible (the same item is not awarded to different bidders) and meets the condition of XOR bids (each bidder wins at most one bid), is it really the combination that maximizes the seller's revenue?

This combination is not an efficient allocation with respect to the bids that have been received because it does not maximize the sum of the accepted bids. In this example, the combination of winning bids that solves the WDP is $b_1^*(A)$, $b_2^*(B)$, $b_3^*(C)$. The revenue that the seller obtains with this combination is the maximum and is equal to $R^* = 600$ euros.

Another combination for which the seller obtains the same revenue is $R^* = b_3(A) + b_2(B) + b_3(C) = 600$ euros. However, this combination does not satisfy restriction (2). In other words, the bids are not XOR because the third bidder has won two different bids. Therefore, this combination would not be a possibility with the established bidding language.[7]

In a CA, as the number of bidders and items increases, the possible allocations grow exponentially, which means that solving the WDP may be complicated and may require significant computation time. According to Sandholm [70], it is an *NP*-complete problem, which often requires the use of advanced optimization techniques to be solved.[8]

7.5 Payments to Be Done in a Combinatorial Auction

In a CA the bidders submit their bids and then, the seller solves the WDP by obtaining the winning bids, $b_i^*(S)$, thus working out the allocation problem. The next step consists of determining the payments that the winning bidders will have to make, which will depend on the **pricing rule**. Next, we present the two basic rules: first-price and VCG mechanism.

[7] If the seller were to opt for an OR bidding language, then this allocation could be another solution to the WDP.

[8] Several studies related to solving the WDP have been presented by Sandholm and Suri [71], Sandholm et al. [72], Saez et al. [68], among others.

Table 7.4 First-price combinatorial auction

S	$b_1(S)$	$b_2(S)$	$b_3(S)$	$b_4(S)$
A	$\boxed{\mathbf{20}}$	10		10
B		$\boxed{\mathbf{20}}$	10	10
C	10		$\boxed{\mathbf{20}}$	10
AB				28

7.5.1 First-Price Combinatorial Auction

If the seller sets the **first-price**, each bidder i will pay an amount equal to his bid for the item or combination that he has acquired:

$$P_i^{*1\text{st}} = b_i^*(S), \tag{7.2}$$

and the seller's revenue is equal to the sum of all of the payments made by the winning bidders:

$$R^{*1\text{st}} = \Sigma_{i \in W} P_i^{*1\text{st}}, \tag{7.3}$$

in which W is the set of winning bidders.

Table 7.4 shows the bids made by four bidders in a first-price sealed-bid CA.[9] The final allocation after solving the WDP is $b_1^*(A), b_2^*(B), b_3^*(C)$. The first bidder wins item A, the second item B, and the third item C. With the first-price rule, each bidder's payment is equal to his winning bid: $P_1^{*1\text{st}} = 20$ euros for the first bidder, $P_2^{*1\text{st}} = 20$ euros for the second bidder, and $P_3^{*1\text{st}} = 20$ euros for the third bidder. The seller's revenue is therefore equal to $R^{*1\text{st}} = 60$ euros.

The main drawback with this pricing rule is that the bidders tend to **underbid**, that is, bid below their values ($b_i(S) < v_i(S)$), in order to obtain a positive surplus. Hence this strategy yields inefficient allocation of the items because the bidder with the highest value is not always the winning bidder with the highest bid. To alleviate this problem, the seller may apply other pricing rules.

7.5.2 The Vickrey–Clarke–Groves Mechanism

An alternative way to solve the problem derived from the use of the first-price rule is to opt for the generalized Vickrey auction: the **Vickrey–Clarke–Groves mechanism** (**VCG mechanism**).[10] By implementing this mechanism, each winning bidder $i \in W$ pays an amount equal to the opportunity cost of the obtained item.

[9] With three items, there is a total of seven possible combinations. However, to simplify, in this example we have considered only four combinations and that all bidders do not bid for all of them.

[10] See Vickrey [79], Clarke [17], and Groves [35].

This amount depends only on the bids placed by the bidder's rivals and is calculated using the following formula:

$$P_i^{*\mathrm{VCG}} = \alpha_i - \Sigma_{k \neq i} b_k^*(S), \tag{7.4}$$

in which $\alpha_i = \max\{\Sigma_{k \neq i} b_i(S) | \Sigma_{k \neq i} S_k \leq J\}$ is the result of solving the WDP among all of the bids, ignoring those made by bidder i. The second term on the right side of Eq. (7.4) is equal to the sum of the initial winning bids ($b_k^*(S)$) made by all of the bidders except i. Once the payments to be made by each winning bidder have been calculated, the seller's revenue is equal to the sum of these payments:

$$R^{*\mathrm{VCG}} = \Sigma_{i \in W} P_i^{*\mathrm{VCG}}. \tag{7.5}$$

Using the data from the example included in Table 7.4, we will calculate the amount that each winning bidder would have to pay based on the VCG mechanism. Let us remember that the allocation of items is always the same, regardless of the pricing rule that has been chosen. In this case, the combination of winning bids is as follows: $b_1^*(A)$, $b_2^*(B)$, $b_3^*(C)$.

To compute the price that the first bidder has to pay, we must first calculate the value of α_1. In other words, we must omit the bids made by the first bidder and again calculate the WDP. Table 7.5 shows the winning combination if we do not take into account the bids of the first bidder: $b_4(A)$, $b_2(B)$, $b_3(C)$. Therefore, $\alpha_1 = 10 + 20 + 20 = 50$ euros. The second term on the right side of Eq. (7.4), calculated for the first bidder, is equal to $\Sigma_{k \neq i} b_i^*(S) = b_2^*(B) + b_3^*(C) = 20 + 20 = 40$ euros. Hence, the amount that the first player has to pay with this mechanism is equal to $P_1^{*\mathrm{VCG}} = 50 - 40 = 10$ euros.

We follow the same steps to calculate the final payments that the other winning bidders have to make. The second bidder's payment is equal to $P_2^{*\mathrm{VCG}} = (20 + 10 + 20) - (20 + 20) = 10$ euros, and the third bidder's payment is equal to $P_3^{*\mathrm{VCG}} = (20 + 20 + 10) - (20 + 20) = 10$ euros. The revenue that the seller obtains upon applying this pricing rule is equal to $R^{*\mathrm{VCG}} = 10 + 10 + 10 = 30$ euros.

There is also another way to compute VCG prices. The VCG price for bidder i is equal to the sum of the difference between the losing and winning bids per bidder for all bidders except him, $k \neq i$. The losing bid is the bid that would have become winning if bidder i would have not participated in the auction. Table 7.6 shows the

Table 7.5 VCG price for the first bidder

S	$\cancel{b_1(S)}$	$b_2(S)$	$b_3(S)$	$b_4(S)$
A	~~20~~	10		$\boxed{\mathbf{10}}$
B		$\boxed{\mathbf{20}}$	10	10
C	~~10~~		$\boxed{\mathbf{20}}$	10
AB				28

Table 7.6 VCG price for the first bidder

Bidder	Losing bid	Winning bid	Losing–winning
b_2	20	20	0
b_3	20	20	0
b_4	10	0	10
			$\boxed{\mathbf{10}}$

winning and losing bids for the second, third, and fourth bidder and computes the VCG price for the first bidder.

As we mentioned in Chap. 2, this is an **incentive compatible** mechanism; that is, the dominant strategy for all of the bidders is to bid according to their true values ($b_i(S) = v_i(S)$), which yields an efficient allocation of the items (see **efficient auction**, Chap. 3).

Despite the advantages of the VCG mechanism, this pricing rule may also have significant drawbacks, such as: low revenues for the seller, non-monotonicity of the seller's revenues in the set of bidders and bids,[11] vulnerability to **collusion** among the bidders, and vulnerability to **shill bids**.[12] Because of these drawbacks, the VCG mechanism is not frequently used in real auctions, see the work done by Ausubel and Milgrom [10].[13]

Implications of the VCG mechanism

The main feature of the VCG mechanism is that the winners' payments depend on the bids submitted by their rivals. To understand the implications of this pricing rule, consider the following example. In a CA items A, B, B, C, C are offered and the first bidder submits a single bid of 100 euros for the combination (A, B, C). The following scenarios can happen depending on his rival's bids, see Table 7.7.

1. **Without rivals:** In this scenario, bidder one is the only bidder, so he wins the combination (A, B, C) for zero euros.
2. **A rival with a matching bid:** The bid of the second bidder is compatible with the first bidder's bid, i.e., both bidders can win the items they have requested. In this scenario the combination that solves the WDP is $b_1^*(ABC) + b_2^*(BC) = 180$. Both bidders get their packages and pay nothing: $P_1^{*\mathrm{VCG}} = 80 - 80 = 0$ euros and $P_2^{*\mathrm{VCG}} = 100 - 100 = 0$ euros.

[11] An auction has **bidder monotonicity** if, upon including another bidder, the bidders' surplus always decreases (weakly) and the seller's revenue increases (weakly).

[12] Multiple bidding identities by a single bidder.

[13] These weaknesses do not surface in environments in which all of the items are **substitutes** for all of the bidders. However, when this condition is violated, even for a single bidder, these problems can occur. The fact that the bidders have budget restrictions may also affect the auction result when applying the VCG mechanism.

Table 7.7 Different scenarios

Scenario	Bidders	A	BB	CC	Bid	P_i^{*VCG}
1	b_1^*	A	B	C	100	0
2	b_1^*	A	B	C	100	0
	b_2^*		B	C	80	0
3	b_1^*	A	B	C	100	80
	b_2	A	B		80	
4.a	b_1	A	B	C	100	
	b_2^*	A	B		90	30
	b_3^*		B	C,C	70	10
4.b	b_1^*	A	B	C	100	90
	b_2	A	B		50	
	b_3		B	C,C	40	

3. **A rival with a non matching bid:** In the third scenario bids are not compatible, as there is only one A item and both bidders have bid for it. $b_2(AB)$ turns to be the *losing bid* and $b_1^*(ABC) = 100$ the winning bid. The first bidder gets his package for $P_1^{*VCG} = 80 - 0 = 80$ euros (the value of the losing bid).
4. **Two rivals with matching bids among them but non matching bids respect to the first bidder:** The winning combination is that which maximizes the value of the bids.
   - **Scenario 4.a:** $b_2^*(AB) + b_3^*(BCC) = 160$ is the winning combination of bids. The second bidder gets the package AB and pays $P_2^{*VCG} = 100 - 70 = 30$ euros. The third bidder wins the combination BCC for $P_3^{*VCG} = 100 - 90 = 10$ euros. The first bidder does not win his package.
   - **Scenario 4.b:** $b_1^*(ABC) = 100$ is the winning bid, and the first bidder pays $P_1^{*VCG} = 90 - 0 = 90$ euros.

7.6 Core-Selecting Package Auctions

One of the problems that may emerge when using the VCG mechanism is that the seller's revenue may be very low (or even zero). Let us analyze the example presented by Ausubel and Milgrom [10], as shown in Table 7.8. With these bids, the winning combination is $b_2^*(A)$, $b_3^*(B)$.[14] With this allocation, the second bidder's payment is equal to $P_2^{*VCG} = 2000-2000 = 0$ euros and the third bidder's payment is equal to $P_3^{*VCG} = 2000-2000 = 0$ euros. This example illustrates that the use of this mechanism may generate an unacceptable outcome because the seller's revenue is zero although the bidders had positive values for the items offered.

The question that emerges at this point is, how low do the seller's revenues have to be for the outcome to be considered unacceptable? In auction literature, a solution

[14]In this example, the winning combination could be either $b_2^*(A)$, $b_3^*(B)$ or $b_3^*(A)$, $b_2^*(B)$, but the result would be the same.

Table 7.8 Combinatorial auction (example 1)

S	$b_1(S)$	$b_2(S)$	$b_3(S)$
A	0	$\boxed{\mathbf{2000}}$	2000
B	0	2000	$\boxed{\mathbf{2000}}$
AB	2000	0	0

Table 7.9 Combinatorial auction (example 2)

S	$b_1(S)$	$b_2(S)$	$b_3(S)$	$b_4(S)$
A	$\boxed{\mathbf{20}}$			10
B		$\boxed{\mathbf{25}}$		10
C			$\boxed{\mathbf{30}}$	10
AB	20			
AC				
ABC	25	25	25	50
BC				

is considered to be acceptable if the payments are the result of a **core allocation** with respect to the bids that have been received. Given several bids, an auction generates a **core outcome** if and only if there is no group of bidders that would strictly prefer an alternative outcome that would also be strictly preferred by the seller. This group of bidders would form a **blocking coalition** against the original outcome.

To understand this concept, let us examine the following example. Table 7.9 shows the bids that have been received and the winning bidders after solving the WDP.

The efficient allocation that maximizes the value with respect to the submitted bids, that is, the combination of bids after solving the WDP, is as follows: $b_1^*(A)$, $b_2^*(B)$, and $b_3^*(C)$. Under the VCG mechanism, the bidders' payment for the acquired items is equal to $P_1^{*\mathrm{VCG}} = (25 + 30 + 10) - (25 + 30) = 10$ euros for the first bidder, $P_2^{*\mathrm{VCG}} = (20 + 30 + 10) - (20 + 30) = 10$ euros for the second bidder, and $P_3^{*\mathrm{VCG}} = (20 + 25 + 10) - (20 + 25) = 10$ euros for the third bidder. In other words, the seller awards all of the items and obtains a revenue equal to $R^{*\mathrm{VCG}} = 30$ euros.

However, the result obtained cannot be considered a core outcome because there is a bidder who strictly prefers another outcome, which would also benefit the seller. The fourth bidder is willing to pay 50 euros for the combination ABC, such that he would form a coalition that blocks the outcome obtained with the VCG mechanism. Therefore, the initial result is considered socially unacceptable because there is another solution that would be better for one or more bidders in addition to the seller.

As an alternative to the VCG mechanism in situations in which complementary items are offered, Day and Milgrom [23] proposed the **core-selecting package auction**. In core-selecting package auctions, first the feasible allocation of items that maximizes the revenue of the seller is calculated. In other words, the WDP is solved. Then a pricing rule that ensures that a result is achieved in the core with respect to the bids that have been received is implemented. The final payments are obtained by increasing the VCG prices in a way that a core outcome is obtained.[15] Several authors highlight core-selecting package auctions, such as Day and Raghavan [24], Day and Cramton [22], Erdil and Klemperer [30], and Ausubel and Baranov [5], among others.

We now continue with the example in Table 7.9, but instead of using the VCG mechanism, the seller opts for the core-selecting auction proposed by Day and Raghavan [24]. The combination of winning bids will still be $b_1^*(A)$, $b_2^*(B)$ and $b_3^*(C)$, but now each winning bidder i would have to increase his payments with respect to $P_i^{*\text{VCG}}$ to obtain a core outcome. With this core-selecting auction, the final payments of the bidders are as follows: $P_1^{*\text{CORE}} = P_2^{*\text{CORE}} = P_3^{*\text{CORE}} = 16.67$ euros. As can be observed, with these amounts, the seller obtains a revenue equal to $R^{*\text{CORE}} = 50$ euros, meaning that the fourth bidder stops blocking the outcome because there is no other combination in which both the bidders and the seller are better off.

Given a set of bids, the seller's revenue with a core-selecting CA is at an intermediate point between the revenue generated by a first-price and a VCG mechanism. In other words, the following holds:

$$R^{*\text{VCG}} \leq R^{*\text{CORE}} \leq R^{*1\text{st}}.$$

With the bids included in Table 7.4, the seller's revenue with the three aforementioned price mechanisms is as follows:

$$R^{*\text{VCG}} = 30 \leq R^{*\text{CORE}} = 50 \leq R^{*1\text{st}} = 60.$$

It is mentioning that this relationship between revenues only holds when the comparison is made for bids already submitted. However, when the seller sets a price mechanism, this decision significantly affects the bidding strategy of the bidders, and the bids made with each mechanism cease to be equal, meaning that, before the auction, it cannot be assured that a mechanism will generate more or less revenue to the seller.

7.7 Variables Used in This Chapter

In this chapter, we use the following variables:

- $I = (1, 2, \ldots, N)$: Bidders.
- $J = (1, 2, \ldots, M)$: Items (homogeneous or heterogeneous).
- S: Package or combination of item, $S \subseteq J$.
- $v_i(S)$: Value of bidder i for the combination S in a CA.

[15] Given the complexity of the core-selecting package auction, the calculation of the final payments made by the winning bidders is beyond the scope of this book.

- $b_i(S)$: Bidder i's bid for the combination S in a CA.
- x_i: A binary variable that is equal to one when a bidder wins an item or combination of items and zero when he does not win any items.
- $b_i^*(S)$: Bidder i's winning bid for the combination S in a CA.
- $P_i^{*1\text{st}}$: Bidder i's payment for the items won under the first-price rule.
- $P_i^{*\text{VCG}}$: Bidder i's payment for the items won under the VCG mechanism.
- $P_i^{*\text{CORE}}$: Bidder i's payment for the items won under a core-selecting package auction.
- R^*: Seller's revenue.
- $R^{*1\text{st}}$: Seller's revenue with the first-price rule.
- $R^{*\text{VCG}}$: Seller's revenue with the VCG mechanism.
- $R^{*\text{CORE}}$: Seller's revenue with a core-selecting package auction.
- α_i: The result of solving the WDP among all of the bids, ignoring those submitted by bidder i.

7.8 Exercises

1. In a sealed-bid, first-price simultaneous auction, a seller offers a plane ticket to Paris, a train ticket to Paris, and lodging in a hotel in Paris. Bidder i's values for each item and combination are shown in Table 7.10.
   (a) Which items are substitutes?
   (b) Which items are complements?
   (c) Provide an example of a bidding strategy in which the bidder could be affected by the exposure problem upon bidding for complements if they are offered in simultaneous auctions.
   (d) Point out the maximum bid that the bidder may submit for complements items so as not to be affected by the exposure problem in a simultaneous auction.
   (e) If the seller opts for a CA, point out the maximum bid that the bidder may submit for the combinations of complements items and still not incur losses.
2. In a CA, a seller receives the offers that appear in Table 7.11. Indicate the following:

Table 7.10 Values for substitutes and complements items

Items and packages	j	$v_{i,j}$
Plane ticket	$j = 1$	100
Train ticket	$j = 2$	80
Lodging	$j = 3$	150
Plane and train ticket	$j = 1 + 2$	120
Plane ticket and lodging	$j = 1 + 3$	300
Train ticket and lodging	$j = 2 + 3$	250
Plane ticket, train ticket, and lodging	$j = 1 + 2 + 3$	320

Table 7.11 Combinatorial auction (exercise 2)

S	$b_1(S)$	$b_2(S)$	$b_3(S)$
A	10	4	1
B	3	9	3
C	2	3	9
AB	18	15	11
AC	18	12	16
ABC	20	20	25
BC	10	19	17

Table 7.12 Combinatorial auction (exercise 3)

S	$b_1(S)$	$b_2(S)$
A	30	10
B	20	15
AB	50	20

Table 7.13 Combinatorial auction (exercise 4)

S	$b_1(S)$	$b_2(S)$
A	30	10
B	20	15
AB	40	20

   (a) A feasible allocation of items that does not maximize the total value of the accepted bids.
   (b) A feasible allocation of items that does not satisfy the XOR bidding language.
   (c) The efficient allocation of items that satisfies the WDP with XOR bidding.
3. Table 7.12 shows the bids received in a CA of two items with two bidders. Calculate the following:
   (a) All of the possible allocations with the XOR bidding language.
   (b) The combination of bids that solves the WDP.
   (c) The payments that the winning bidders must make and the revenue that the seller obtains if the first-price rule is established.
   (d) The payments that the winning bidders must make and the revenue that the seller obtains if the VCG mechanism is employed.
4. Table 7.13 shows the bids received in a CA of two items with two bidders. Calculate the following:
   (a) All of the possible allocations with the XOR bidding language.
   (b) The combination of bids that solves the WDP.
   (c) The payments that the winning bidders must make and the revenue that the seller obtains if the first-price rule is established.
   (d) The payments that the winning bidders must make and the revenue that the seller obtains if the VCG mechanism is employed.
5. Given the bids shown in Table 7.14, solve the following:
   (a) The combination of feasible bids that maximizes the seller's revenue.
   (b) Are the payments of the winning bidders with VCG mechanism a core outcome? If not, identify a coalition that will block this result.

Table 7.14 Combinatorial auction (exercise 5)

S	$b_1(S)$	$b_2(S)$	$b_3(S)$
A	100	50	50
B	100	50	300
C	100	50	50
AB	600	100	100
AC	600	500	100
ABC	1000	300	300
BC	600	100	100

Table 7.15 Combinatorial auction (exercise 6)

S	$b_1(S)$	$b_2(S)$	$b_3(S)$	$b_4(S)$
A	200	150	200	100
B	100	250	100	100
C	100	100	300	100
AB	200	200	200	200
AC	250	300	275	200
ABC	500	400	400	500
BC	150	150	150	200

6. Given the bids shown in Table 7.15 compute the following:
   (a) The combination of feasible bids that maximizes the seller's revenue.
   (b) Are the payments of the winning bidders with VCG mechanism a core outcome? If not, identify a coalition that will block this result.

7.9 Solutions to Exercises

1. With the data from the exercise, we obtain the following results.
   (a) Plane and train tickets are substitutes: $v_{i,1} + v_{i,2} = 180 > 120 = v_{i,1+2}$.
   (b) Plane tickets and the lodging are complements: $v_{i,1} + v_{i,3} = 250 < 300 = v_{i,1+3}$, as are the train tickets and the lodging: $v_{i,2} + v_{i,3} = 230 < 250 = v_{i,2+3}$.
   (c) If the bidder decides to bid for the plane ticket and lodging combination, he could be affected by the exposure problem in a simultaneous auction if $b_{i,1} > 100$ euros or $b_{i,3} > 150$ euros, because he may not win the two items for which he is bidding and may only obtain one of them. Similarly, if he opts for the train tickets and lodging combination, he may incur losses if $b_{i,2} > 80$ euros or $b_{i,3} > 150$ euros.
   (d) The bidder will not incur losses if his bids in a simultaneous auction are as follows: $b_{i,1} \leq 100$ euros, $b_{i,2} \leq 80$ euros, and $b_{i,3} \leq 150$ euros. However, this strategy implies not including the synergy value in his bids.
   (e) If, instead of a simultaneous auction, the seller were to opt for a CA, bidder i could include the value of the synergies without the fear of incurring losses.

Table 7.16 The winner determination problem (exercise 3)

S	$b_1(S)$	$b_2(S)$
A	30	10
B	20	15
AB	$\boxed{\mathbf{50}}$	20

Table 7.17 VCG mechanism for the first bidder (exercise 3)

S	$\cancel{b_1(S)}$	$b_2(S)$
A	~~30~~	10
B	~~20~~	15
AB	~~50~~	$\boxed{\mathbf{20}}$

In this case, he may bid as much as $b_{i,1+3} \leq 300$ euros and $b_{i,2+3} \leq 250$ euros for each package.

2. With the bids submitted by the bidders:
   (a) There are several feasible combinations, that is, combinations in which an item is not awarded to more than one bidder. Some of these combinations include the following: $b_1(AB) + b_3(C)$ or $b_3(ABC)$, in which the total values of the accepted bids are 27 euros and 25 euros, respectively. However, neither of these combinations solves the WDP because there is another feasible combination in which the value of the accepted bids is maximized.
   (b) The combination $b_1(A) + b_1(BC)$ would also be feasible but would not be valid as a solution as defined in this chapter because the bids are not XOR. With this allocation, the first bidder would win two bids.
   (c) The feasible combination of bids that maximizes the seller's revenue with XOR bids and that therefore implies an efficient allocation of items with respect to the received bids is $b_1^*(A) + b_2^*(BC)$, for which the total value of the accepted bids is maximized and is equal to 29 euros.
3. With the bids received in this auction, we obtain the following results.
   (a) The feasible allocations using XOR bidding language are:
   $b_1(A) + b_2(B) = 45$ euros.
   $b_2(A) + b_1(B) = 30$ euros.
   $b_1(AB) = 50$ euros.
   $b_2(AB) = 20$ euros.
   (b) The efficient allocation after solving the WDP, that is, the combination in which the value of the accepted bids is maximized, is $b_1^*(AB) = 50$ euros, see Table 7.16. Therefore, the first bidder wins both items.
   (c) If the seller establishes the first-price rule, the winning bidder will pay $P_1^{*1st} = 50$ euros, which will coincide with the seller's revenue: $R^{*1st} = 50$ euros (there is only one winning bidder).
   (d) The payment of the winning bidder under the VCG mechanism is calculated with Eq. (7.4). The value of α_1 is equal to the winning combination after eliminating the bids of the first bidder. As can be observed in Table 7.17, the winning allocation would be $b_2(AB) = 20$ euros, so $\alpha_1 = 20$ euros. The second term of the right side of Eq. (7.4) is equal to 0 because there are no

Table 7.18 The winner determination problem (exercise 4)

S	$b_1(S)$	$b_2(S)$
A	$\boxed{\mathbf{30}}$	10
B	20	$\boxed{\mathbf{15}}$
AB	40	20

Table 7.19 VCG mechanism for the first bidder (exercise 4)

S	$\cancel{b_1(S)}$	$b_2(S)$
A	~~30~~	10
B	~~20~~	15
AB	~~40~~	$\boxed{\mathbf{20}}$

Table 7.20 VCG mechanism for the second bidder (exercise 4)

S	$b_1(S)$	$\cancel{b_2(S)}$
A	30	~~10~~
B	20	~~15~~
AB	$\boxed{\mathbf{40}}$	~~20~~

other winning bidders. Therefore, the payment of the winning bidder with the VCG mechanism is equal to $P_1^{*\mathrm{VCG}} = R^{*\mathrm{VCG}} = 20$ euros. This is also the seller's revenue.

4. With the bids submitted in this auction, we obtain the following results.
   (a) The feasible allocations using XOR bidding language are:
   $b_1(A) + b_2(B) = 45$ euros.
   $b_2(A) + b_1(B) = 30$ euros.
   $b_1(AB) = 40$ euros.
   $b_2(AB) = 20$ euros.
   (b) The efficient allocation of items that solves the WDP, in which the value of the accepted bids is maximized, is $b_1^*(A) + b_2^*(B) = 45$ euros, as shown in Table 7.18. The first bidder wins item A, and the second bidder wins item B.
   (c) If the seller establishes the first-price rule, the payments of the winning bidders are as follows: $P_1^{*1\mathrm{st}} = 30$ euros the first bidder and $P_2^{*1\mathrm{st}} = 15$ euros the second bidder. The seller's revenue is the sum of both amounts: $R^{*1\mathrm{st}} = 45$ euros.
   (d) Under the VCG mechanism, the first bidder's payment according to Eq. (7.4) is equal to $P_1^{*\mathrm{VCG}} = \alpha_1 - b_2^*(B)$. As shown in Table 7.19, the winning combination after eliminating the bids of the first bidder is $\alpha_1 = b_2(AB) = 20$ euros. Therefore, the payment of the first bidder for the item that he acquires is equal to $P_1^{*\mathrm{VCG}} = 20 - 15 = 5$ euros.
   In the same way, we calculate the second bidder's payment for the item that he acquires: $P_2^{*\mathrm{VCG}} = \alpha_2 - b_1^*(A)$. Table 7.20 shows the winning combination after omitting the bids of the second bidder: $\alpha_2 = b_1(AB) = 40$ euros. Therefore, $P_2^{*\mathrm{VCG}} = 40 - 30 = 10$ euros.
   The seller's income with this mechanism is equal to $R^{*\mathrm{VCG}} = P_1^{*\mathrm{VCG}} + P_2^{*\mathrm{VCG}} = 15$ euros.

Table 7.21 The winner determination problem (exercise 5)

S	$b_1(S)$	$b_2(S)$	$b_3(S)$
A	100	50	50
B	100	50	300
C	100	50	50
AB	600	100	100
AC	600	500	100
ABC	**1000**	300	300
BC	600	100	100

Table 7.22 The winner determination problem (exercise 6)

S	$b_1(S)$	$b_2(S)$	$b_3(S)$	$b_4(S)$
A	**200**	150	200	100
B	100	**250**	100	100
C	100	100	**300**	100
AB	200	200	200	200
AC	250	300	275	200
ABC	500	400	400	500
BC	150	150	150	200

5. With the bids received in this auction, we obtain the following outcome.
   (a) The efficient allocation of items that solves the WDP implies that the first bidder wins all three items $b_1^*(ABC)$, see Table 7.21.
   (b) If the VCG pricing rule is established, the payment that the winning bidder will have do is equal to $P_1^{*\text{VCG}} = (500 + 300) - 0 = 800$ euros, so $R^{*\text{VCG}} = 800$ euros. In this example, this is a core outcome, there is no blocking coalition.
6. With the bids submitted in this auction, we obtain the following outcome.
   (a) The efficient allocation of items after solving the WDP is $b_1^*(A)$, $b_2^*(B)$, and $b_3^*(C)$; in other words, the first bidder wins item A, the second wins item B, and the third wins item C, see Table 7.22.
   (b) If the VCG mechanism is established, the payment of each winning bidder is equal to $P_1^{*\text{VCG}} = (250 + 300 + 100) - (250 + 300) = 100$ euros for the first, $P_2^{*\text{VCG}} = (200+300+100)-(200+300) = 100$ euros for the second, and $P_3^{*\text{VCG}} = (200 + 250 + 100) - (200 + 250) = 100$ euros for the third. The seller obtains a revenue equal to $R^{*\text{VCG}} = 300$ euros. However, this outcome is not a core outcome because there is a blocking coalition: there is a bidder who would strictly prefer an alternative result that would also be strictly preferred by the seller. The fourth bidder is willing to pay 500 euros for the three items ($b_4(ABC) = 500$ euros), an outcome that the seller would also prefer.

Combinatorial Auction Models

8

8.1 Introduction

Combinatorial auctions (CAs) are characterized by offering multiple related items and allowing the bidders to bid for the items or combinations of items in which they are interested. When a combinatorial auction is being designed, the seller must set up all the details, such as: dynamic or sealed-bid (single-round), pricing rule, activity rule, and starting price. Given the wide spectrum of possibilities, in this chapter, we will only focus on analyzing a few of these models in which the first-price rule will be applied.

8.2 Sealed-Bid Combinatorial Auctions

Among all of the possible designs for combinatorial or package auctions, the simplest one is the **combinatorial sealed-bid auction**, in which each bidder submits all of his bids for the items and combinations of items that are being auctioned in a single round. Once all of the bids have been received, the seller calculates the efficient allocation of the items according to the received bids, for which he will need to solve the **winner determination problem** (WDP).

According to the following example, a seller receives the bids shown in Table 8.1 in a single round and solves the WDP. In this example, the feasible allocation of items that maximizes the seller's income is $b_1^*(A)$, $b_2^*(B)$ and $b_3^*(C)$, that is, the first bidder wins item A, the second wins item B, and the third wins item C. If the seller sets the first-price rule, the price that each winning bidder pays is equal to his bid, that is, $P_1^{*1\text{st}} = 50$ euros, $P_2^{*1\text{st}} = 50$ euros, and $P_3^{*1\text{st}} = 50$ euros. The seller's revenue is equal to $R^{*1\text{st}} = 150$ euros.

This is the basic mechanism of a sealed-bid auction. Nevertheless, the seller can implement many rules in order to achieve his goals. For example:

A. Mochón, Y. Sáez, *Understanding Auctions*, Springer Texts in Business and Economics, DOI 10.1007/978-3-319-08813-6_8

Table 8.1 Sealed-bid combinatorial auction (example 1)

S	$b_1(S)$	$b_2(S)$	$b_3(S)$
A	$\boxed{\mathbf{50}}$	10	10
B	10	$\boxed{\mathbf{50}}$	10
C	10	10	$\boxed{\mathbf{50}}$
AB	60	70	20
AC	60	30	60
ABC	100	100	100
BC	40	60	50

Table 8.2 Sealed-bid combinatorial auction (example 2)

S	$b_1(S)$	$b_2(S)$	$b_3(S)$
A	0	20	0
B	0	0	30
AB	$\boxed{\mathbf{100}}$	0	0

Table 8.3 Sealed-bid combinatorial auction with multipliers

	α_j	$b_1(S)\alpha_j$	$b_2(S)\alpha_j$	$b_3(S)\alpha_j$
A	2	0	$\boxed{\mathbf{40}}$	0
B	3	0	0	$\boxed{\mathbf{90}}$
AB	1	100	0	0

- One possible rule is that if a bidder bids on a package, he must also bid on the items individually. For example if a bidder submits a bid for package AB, he will also have to bid on items A and B. With this rule, the seller increases the number of possible allocations.
- The seller might be interested in allocating the items among the largest possible number of bidders. With this aim, he can set a different multiplier per type of item and solve the WDP with the bids multiplied by these multipliers. For example, Table 8.2 shows the submitted bids in a sealed-bid auction. With these bids, the efficient allocation that solves the WDP is $b_1^*(AB)$, and the seller's revenue is equal to $R^* = 100$ euros (with the first-price rule).

  Nevertheless, the seller can use a multiplier α_j per item or package. According to the values of α_j included in Table 8.3 the new efficient allocation is $b_2^*(A)$, $b_3^*(B)$, although the seller's revenue is equal to $R^* = 20 + 30 = 50$ euros as payments are calculated according to real bids. In this example, the seller has set higher multipliers for individual items rather than for package, promoting the distribution of items among bidders.[1]

A problem commonly faced by the bidders in a CA is finding their values per item and package (given the large number of combinations and the interrelationships among the items). This problem is exacerbated in sealed-bid auctions because, in only a single round, the bidders do not obtain information regarding the possible

[1] A similar rule was applied in the French auction of radio spectrum in 2011.

bids of their rivals (*no price discovery*), which affects their bidding strategy and makes the efficient allocation of the items more difficult.

8.3 Dynamic Combinatorial Auctions

The best option to reduce the problem resulting from the use of single-round auctions is to resort to dynamic or **iterative combinatorial auctions** (ICAs).[2] In these models, the bidders can submit bids for the items or combinations of items in which they are interested during a multiple-round process. At the end of each round, the seller provides information regarding the provisional allocation of the items and the prices reached in that round. With this mechanism, the bidders obtain information about their rivals' bids and can modify their own bids as the process advances (**price discovery**). There are various ICA models, several of which are discussed in the following sections.[3]

8.3.1 Ascending Proxy Auction

In an **ascending proxy auction**,[4] each bidder reports to a **proxy bidder**[5] his preferences for the items or packages that are being offered, that is, the maximum bid that bidder i is willing to offer for each item or package, $v_i^{\text{proxy}}(S)$.[6] Using this information, the proxy bidder bids throughout successive rounds on the bidder's behalf: $b_i^t(S)$. At the end of each round, the seller calculates the feasible allocation that maximizes his revenue, taking into account all of the bids throughout the rounds. That is, he solves the WDP considering all bids from the current and past rounds and announces provisional winners and payments. If a bidder is not within the list of provisional winners, the proxy bidder will then bid again for the item or package that will provide him the greatest potential profits. The process goes on until no new bids are submitted and the auction ends. In this auction model, the bids are mutually exclusive, that is, the **XOR bidding language** is implemented.[7]

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[2]Parkers [63] studies the characteristics of the main models of ICAs. In this chapter, we will only focus on a subset of these models.

[3]The **simultaneous ascending auction** (SAA) that we analyze in Chap. 5 could be considered to be an ICA if package bidding throughout the different rounds were allowed.

[4]For more information regarding this auction model, see [9].

[5]The **proxy bidder** can be either the seller or an electronic proxy bidder.

[6]The preferences communicated by bidder i to his proxy bidder for each item, $v_i^{\text{proxy}}(S)$, do not necessarily have to coincide with their true values, $v_i(S)$.

[7]For the specific case of a single item, this model is similar to auctions conducted on the Internet, in which the bidder indicates the maximum bid that he is willing to submit, and the system automatically updates the bids as long as the maximum bid is higher than the actual bid, and the bidder is not the provisional winner.

Table 8.4 Bids made by proxy bidders in an ascending proxy auction

	A	B	AB
$v_i^{\text{proxy}}(S)$	5	4	15
Round t			
Price in t	2	3	10
Potential surplus in t	3	1	$\boxed{\mathbf{5}}$
$b_i^t(S)$			10
Round $t+1$			
Price in t	2	3	11
Potential surplus in t	3	1	$\boxed{\mathbf{4}}$
$b_i^{t+1}(S)$			11

To understand the bidding process which the proxy bidders follow in each round, we will analyze the following example. A seller offers items A and B, and bidder i indicates his preferences to the proxy bidder $v_i^{\text{proxy}}(S)$, see Table 8.4. In each round t, the proxy bidder must identify the possible bids that bidder i can make at the standing prices if he is not among the provisional winners. These prices are calculated based on the bids of the previous round. With this information, the proxy bidder calculates the potential profit that the bidder would obtain with each item or combination (the difference between the preferences $v_i^{\text{proxy}}(S)$ and the price). Finally, the proxy bidder bids for the item or package that, if won, would provide the greatest profit to bidder i. In this example, the possible surplus that could be obtained with each bid in round t are as follows: three euros for item A, one euro for item B, and five euros for the combination AB. Thus, in round t, the proxy bidder would submit a bid equal to $b_i^t(AB) = 10$ euros, because, if he wins, this bid provides the greatest surplus to the bidder.

If, after this bid, bidder i is still not among the provisional winners, the proxy bidder must again determine the bid to be made in the following round, $t+1$. Table 8.4 presents the new data of this round, in which we will assume that the prices for items A and B did not change but increased by one euro for the package AB. As we can see, the proxy bidder will bid again for the AB combination ($b_i^{t+1}(AB) = 11$ euros), given that this bid yields the greatest potential surplus.

This auction model is an alternative to the **Vickrey auction** for multiple heterogeneous items, and, unlike the **VCG mechanism**, the ascending proxy auction avoids the low revenues, reduces the possibility of having **shill bids** and **collusion** among the bidders. Furthermore, this model fosters the attainment of competitive revenues and efficient allocation with respect to the preferences reported to the proxy bidder, see [9].

8.3.2 Clock-Proxy Auction

The **clock-proxy auction**, developed by Ausubel, Cramton and Milgrom [8], is a hybrid auction with two phases. The first phase corresponds to a **simultaneous**

ascending clock auction (see Chap. 5), except that package bidding is allowed. After completing this phase, the process ends with an **ascending proxy auction**. To better understand this design, we will take a closer look at each phase.

This auction involves multiple heterogeneous items, so the seller must describe the items being offered: the number of categories or types of items being offered, $J = (1, 2, \dots, M)$ and the number of items included in each category j, $\overline{Q}_j$.[8] For example, if a seller offers the items A, B, and B, we would have two categories, and the number of items offered by type is equal to $\overline{Q}_1 = 1$ (item A) for the first type and $\overline{Q}_2 = 2$ (items B) for the second type.

The clock phase begins, and the seller indicates in round $t = 0$ the opening bid per item for each type (p_j^s). Given these unitary prices per category and round (p_j^t), each bidder i indicates the number of items of the type j he is willing to buy in round t ($q_{i,j}^t$). The aggregate demand for type j in round t is equal to the sum of the bids submitted for that type by all of the bidders:

$$Q_j^t = \Sigma_{i=1}^n q_{i,j}^t. \tag{8.1}$$

At the end of each round, the seller determines the categories for which demand exceeds supply:

$$Q_j^t > \overline{Q_j}, \tag{8.2}$$

and increases the prices for round $t + 1$, while other prices remain unchanged. Bidders submit their new bids in the next round. This iterative process continues until there is no more excess demand for any category.[9]

Continuing with the example in which the seller offers one A item ($\overline{Q_1} = 1$) and two B items ($\overline{Q_2} = 2$), Table 8.5 presents the evolution of the clock phase. The auction begins with round $t = 0$, in which the starting price per item for both types is equal to $p_1^0 = p_2^0 = 1$ euro. In this round, the first bidder bids for the A item ($q_{1,1}^0 = 1$) and for both of the B items ($q_{1,2}^0 = 2$). The second bidder submits the same bids ($q_{2,1}^0 = 1$ and $q_{2,2}^0 = 2$), and therefore in both types, the aggregate demand is greater than the supply: $Q_1^0 = 2 > 1 = \overline{Q_1}$ and $Q_2^0 = 4 > 2 = \overline{Q_2}$.

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[8] Items within the same category are homogeneous items and heterogeneous among categories.

[9] Throughout the entire clock phase and in the transition to the proxy phase, the seller may establish an **activity rule** to encourage the bidders to bid according to their true preferences from the beginning of the auction. One option is to select an activity rule that presents monotonicity in quantity, which implies that as the price increases, the demanded quantity cannot increase. A weaker activity rule is monotonicity of aggregate quantity across a group of items, such that the aggregate quantity demanded cannot increase as the price increases, which allows shifting quantities among products (this is the **eligibility-based activity rule** described in Chap. 5). However, the activity rule proposed by the inventors of this model is the **revealed-preference activity rule** because it gives more flexibility to the bidders when there are substitutes and complements items. For more details regarding this rule, see the appendix at the end of this chapter or [8].

Table 8.5 Clock phase in a clock-proxy auction

	Type A					Type B				
Round t	p_1^t	$q_{1,1}^t$	$q_{2,1}^t$	Q_1^t	$\overline{Q_1}$	p_2^t	$q_{1,2}^t$	$q_{2,2}^t$	Q_2^t	$\overline{Q_2}$
$t = 0$	1	1	1	2	1	1	2	2	4	2
$t = 1$	2	1	1	2	1	2	1	2	3	2
$t = 2$	3	1		1	1	3	1	2	3	2
$t = 3$	3	1		1		4		2	2	2

Table 8.6 Package bids during the clock phase

Bidder 1			Bidder 2		
Round (t)	Package	Bid amount	Round (t)	Package	Bid amount
$t = 0$	ABB	3	$t = 1$	ABB	6
$t = 2$	AB	6	$t = 3$	BB	8
$t = 3$	A	3			

In round $t = 1$, the seller increases the price per item in both categories, but at the end of the round, the demand still exceeds supply: $Q_1^1 = 2 > 1 = \overline{Q_1}$ and $Q_2^1 = 3 > 2 = \overline{Q_2}$ (although the first bidder reduced his demand for the type $j = 2$ by one unit). In the following round ($t = 2$), the second bidder stops bidding for item A, which eliminates the excess demand for that type $Q_1^2 = 1 = \overline{Q_1}$, and therefore the price stops increasing. In round $t = 3$, the seller increases only the price of type $j = 2$ (item B). In this round, the first bidder stops being interested in the B items, so there is no excess demand: $Q_2^3 = 2 = \overline{Q_2}$. At this point, the clock phase finishes.

It is important to note that the bids submitted by the bidders in each round of the clock phase are considered to be bids for complete packages, that is, the bidder obtains either all or none of the items for which he was bidding in that round. Thus, the total price that bidder i offers for the package he is bidding on in round t is calculated by the following equation:

$$P_i^t = \Sigma_{j=1}^{M} p_j^t q_{i,j}^t, \tag{8.3}$$

that is, the sum of the price per item multiplied by the number of items demanded for each type. In the previous example, the price that the first bidder offered in round $t = 0$ for the combination of items he requested (ABB) equals $P_1^1 = (1 \times (1)) + (1 \times (2)) = 3$ euros. Considering the bids in each round as packages implies that the bidder could have won the three items simultaneously at the price of that round. In no case he could have obtained any of them individually. Hence, the **exposure problem** is eliminated, and the bidders can include the synergy value of the complements items in their bids without the fear of incurring any losses. The package bids submitted by each bidder in the clock phase are shown in Table 8.6. If a bidder has bid for the same package in more than one round, the bid with the highest bid amount (the latest round) would be the one included to compute the WDP.

After the clock phase, the proxy phase begins, and an ascending proxy auction is conducted, as discussed in the previous section. The bidders report their preferences to the proxy bidders and they bid on the bidders' behalf in each round, while the price is below the reported preference. The minimum bid of the proxy phase must always be higher than or equal to the last price reached in the clock phase for each type of item. This second phase will continue until the proxy bidders stop submitting new bids.[10]

With all the submitted bids, both in the clock and proxy phase, the seller calculates the combination of feasible bids that maximizes his revenue, that is, he solves the WDP. In this auction model, bids of both phases are considered to be mutually exclusive: XOR bidding language.

The clock-proxy auction is a hybrid auction, so it incorporates the advantages of the mechanisms used in each phase. By including the clock phase, the bidders obtain information regarding the preferences of their rivals (**price discovery**), so it is easier for them to make their bidding decisions. Furthermore, collusive strategies are eliminated as no individual bid information is reported after each round, only aggregate demand. The proxy phase promotes obtaining competitive revenues for the seller and an efficient allocation with respect to the values reported to the proxy bidders, that is, it fosters yielding a core outcome.[11]

8.3.3 Combinatorial Clock Auction

The **combinatorial clock auction** (CCA) is a hybrid auction with two stages: allocation stage and assignment stage, used to allocate multiple heterogeneous items.[12]

1. **Allocation stage**:
   In the allocation stage winning bidders and base prices are settled. This phase consists of two rounds:
   (a) **Clock round**: This round is composed of multiple rounds in which the price per item for all the categories in which there is excess demand increases. In each round, bidders submit a single bid for a package of items at the current prices. The clock round ends when there is no longer excess demand for any category.

[10] The proxy phase with mandatory proxy bidding (bidders not allowed to change their reported preferences) is observationally equivalent to a sealed-bid auction, see [8].

[11] Mochon et al. [55] analyzed the possible outcomes obtained when using this auction model to allocate radio spectrum licenses by means of genetic algorithms.

[12] Different governments have used this auction mechanism to award spectrum licenses: Switzerland (2012), United Kingdom (2012), or Canada (2013).

Table 8.7 Items offered in a CCA

Category j	Type	$\overline{Q_j}$	EP	p_j^s	Increment p_j^t
$j = 1$	A	2	20	150	50
$j = 2$	B	3	15	100	10

Table 8.8 Clock rounds for bidder 1

	Type A			Type B			Eligibility$_1^t$	Activity$_1^t$	Bid$_1^t$
Round t	p_1^t	$q_{1,1}^t$	Q_1^t	p_2^t	$q_{1,2}^t$	Q_2^t			
$t = 0$	150	2	5	100	2	3	70	70	AABB
$t = 1$	200	1	4	100	3	4	70	65	ABBB
$t = 2$	250	1	2	110	3	4	65	65	ABBB
$t = 3$	250	1	2	120	2	3	65	50	ABB

Throughout the clock rounds the seller may set an **eligibility-based activity rule**, see Chap. 5. The seller can also incorporate the **revealed-preference activity rule**, see appendix at the end of this chapter.

(b) **Supplementary round**: When the clock round is over, bidders submit, in a single round, as many bids as they want for the packages they are interested in. This round is a **combinatorial sealed-bid auction**.[13]

With the bids received in both rounds (clock and supplementary), the seller solves the **winner determination problem** (WDP), and winning bidders are established. The base prices according to the pricing rule selected are also settled.

2. **Assignment stage**:

In this stage specific items are assigned to winning bidders. If a bidder has different values for the specific items within the same category, the bidder can bid in this stage in order to get those that yields him the highest value.

The following example describes a CCA from a bidder's perspective. A seller offers two A items and three B items, see Table 8.7. The seller sets the eligibility-based activity rule in the clock round, so bidders are required to bid on packages with the same EP or smaller as prices rise.

The allocation stage begins with the first clock round. We assume that the first bidder starts the auction with Eligibility$_1^0 = 70$ and in round $t = 0$ he bids for the package AABB (two items of category A: $q_{1,1}^0 = 2$ and two items of category B: $q_{1,2}^0 = 2$). Hence, his activity in this round equals his eligibility: $(2\times20)+(2\times15) = 70$, so his eligibility for round $t = 1$ remains constant Eligibility$_1^1 = 70$.

Table 8.8 shows that there is excess demand for items of category A ($Q_1^0 = 5 > 2 = \overline{Q_1}$), so that the price increases in the next round. However, the supply equals the demand for items of category B ($Q_2^0 = 3 = \overline{Q_2}$) and the price remains constant.

[13]The seller can set a limit on the maximum amount of the supplementary bids of each bidder based on his clock rounds bidding. Thus bidders have incentives to bid according to their true values throughout the auction.

Table 8.9 Package bids during the clock rounds for bidder 1

Round (t)	Package	Bid amount
$t = 0$	AABB	500
$t = 2$	ABBB	580
$t = 3$	ABB	490

In round $t = 1$ the first bidder drops one A item and increases one B item, so he bids for the package ABBB. While bidding for the same number of items, his activity has gone down, and so has his eligibility for the next round $\text{Eligibility}_1^2 = (1 \times 20) + (3 \times 15) = 65$.[14]

Price increases in round $t = 2$ for both categories and the bidder bids for the same package. In this round there is no excess demand for category A, so the price stops increasing. Finally, in round $t = 3$ the bidder bids on package ABB and the clock round ends as there is no excess demand in any category.

The individual bids placed by a bidder in a certain round are considered to be single package bids, and the price of each package is equal to the sum of the bids for the individual items at the clock prices of that round (all-or-nothing bid). For example, in round $t = 0$ the first bidder has submitted a bid for the package AABB. The bid amount for this package is $(2 \times 150) + (2 \times 100) = 500$ euros.

All these package bids will be added to the sealed-bids of the supplementary round to determine the winners of the auction, solve the WDP. In this example, the package bids submitted by the first bidder during the clock rounds are shown in Table 8.9. Given that in rounds $t = 1$ and $t = 2$ he submitted a bid for the same package (ABBB), we just include the last one as it implies a higher price for the same package.

The supplementary phase is a single round process in which bidders can submit additional bids:

- Bids to improve bids on packages previously submitted in the clock rounds.
- Bids for new packages. In this case, the seller can set a minimum price, such as the sum of the starting price of the items included in the package.

The seller may constrain the supplementary bids per bidder based on his bidding history during the clock rounds.

In this example the first bidder submits the supplementary bids shown in Table 8.10. He improves the clock bid for the package ABB and submits bids for three new packages: AB, A, and B; the first two at the opening price and B at a higher price.

All valid bids received from bidders in the clock rounds (Table 8.9 for bidder 1) and in the supplementary round (Table 8.10 for bidder 1) are considered to solve the

[14]We assume that according to the activity rule, the eligibility of round t equals the activity of round $t - 1$.

Table 8.10 Supplementary bids for bidder 1

Package	Bid amount
ABB	550
AB	250
A	150
B	300

Table 8.11 Assignment round for B items: bidder 1's bids

Item	Bid
B^1	10
B^2	25
B^3	0

WDP and get the base prices. We assume that, in this example, the seller sets the first-price rule and the fist bidder wins one B item for 300 euros.

After the allocation stage, the items within a category are no longer generic items and the winners of the allocation stage can bid for specific items in the assignment stage. Only bidders who have won one or more generic items in the allocation stage can bid in the assignment stage. In this example the assignment stage is composed by two sequential rounds. In the first one the A items are auctioned but the first bidder cannot bid as he has not been awarded any item of this category. In the second assignment round there are three B items, that we now call: B^1, B^2, and B^3. The first bidder submits in a single round the bids included in Table 8.11:

With these bids the first bidder indicates that he is willing to pay 25 euros extra (in addition to 300 euros) to get the item B^2, 10 euros for the item B^1 and he is not willing to pay extra for the item B^3. With the bids submitted by all bidders who acquire at least one B item in the previous stage, the seller determines the winners of the specific items. In this example we assume that the first bidder wins B^2, so his final price is the allocation price plus the assignment price: $300 + 25 = 325$ euros.

This bidder could have also not submitted any bid in the assignment stage. He would have risked not winning the B^2 item but, in any case, he would have been awarded one B item for 300 euros.

8.4 Variables Used in This Chapter

In this chapter, we use the following variables:

- $I = (1, 2, \ldots, N)$: Bidders.
- $J = (1, 2, \ldots, M)$: Categories or type of items.
- S: Package or combination of items, $S \subseteq J$.
- $v_i(S)$: Value of bidder i for the combination S.
- $v_i^{\text{proxy}}(S)$: Value of bidder i reported to his proxy bidder for the combination S.
- $b_i(S)$: Bidder i's bid for the combination S in a sealed-bid combinatorial auction.

- $b_i^t(S)$: Bidder i's bid for the combination S in round t of a dynamic combinatorial auction.
- $b_i^*(S)$: Bidder i's winning bid for the combination S in a combinatorial auction.
- p_j^s: Starting bid (minimum bid) per item of type j in a clock phase of a dynamic auction.
- p_j^t: Price per item of type j in round t in a clock phase of a dynamic auction.
- P_i^t: Total price for bidder i's demanded package in round t in a clock phase of a dynamic auction.
- P_i^*: Bidder i's payment for items won.
- $q_{i,j}^t$: Bidder i's demanded quantity of type j in round t in a clock phase of a dynamic auction.
- Q_j^t: Aggregate demanded quantity of type j in round t in a clock phase of a dynamic auction.
- $\overline{Q}_j$: Quantity supplied of type j in a clock phase of a dynamic auction.
- R^*: Seller's revenue.

8.5 Exercises

1. In a sealed-bid first-price combinatorial auction with XOR bids, the seller receives the bids shown in Table 8.12. Compute:
   (a) The allocation after solving the WDP.
   (b) The winning bidders' payments.
   (c) The seller's revenue.
2. Table 8.13 presents the bidders' values for the items offered in a clock-proxy auction. Illustrate:
   (a) The clock phase assuming that: the bidders continue bidding until the price is equal to 50 % of their value, the starting price per item and price increment for all types is one euro.
   (b) The proxy phase assuming that: the bidders report the values shown in Table 8.13 to the proxy bidders, the first bid submitted by each proxy bidder is equal to the last price per item reached in the clock phase, and the price increment is one euro.

Table 8.12 Sealed-bid combinatorial auction

	A	B	AB
$b_1(S)$	5	5	10
$b_2(S)$	5	3	7
$b_3(S)$	6	9	12

Table 8.13 Values in a clock-proxy auction

	A	B	AB
$v_1(S)$	4		
$v_2(S)$		6	
$v_3(S)$			14

(c) The auction outcome: the winning bidders, their payments, and the seller's revenue.

8.6 Solutions to Exercises

1. Given the bids submitted by the bidders in this first-price sealed-bid auction, the auction outcome is as follows.
   (a) The winning bids after solving the WDP are $b_1^*(A)$ and $b_3^*(B)$, that is, the first bidder wins item A and the third wins item B (see Table 8.14).
   (b) It is a first-price auction, so the first bidder's payment is equal to $P_1^* = 5$ euros and the third bidder's payment is equal to $P_3^* = 9$ euros.
   (c) The seller's revenue is equal to $R^* = P_1^* + P_3^* = 14$ euros.
2. We will now discuss the clock-proxy auction in detail.
   (a) The bidders will bid in the clock phase until the price reaches 50 % of their value, so the first bidder will bid for item A until the price is equal to two euros, the second bidder will bid for item B until the price is equal to three euros, and the third bidder will bid for the package AB until the price is equal to 7 euros. With these bidding strategies, the bidding rounds of the clock phase are shown in Table 8.15.

      In the clock phase, the price for item A increases until it reaches three euros in round $t = 2$, whereas the price for item B increases until it reaches four euros in round $t = 3$.
   (b) When there is no excess demand for any type, the second phase begins. The proxy bidders submit an offer equal to the price of the last round of the clock phase for each type of item. Table 8.16 shows the submitted bids for all of the proxy bidders during the first round of this phase ($t^{\text{proxy}} = 1$).

      With all the bids submitted in the clock phase and in this first round of the proxy phase, the seller solves the WDP. In this example there is a tie, as there are two possible allocations that yield the same revenue to the seller:

Table 8.14 Sealed-bid combinatorial auction (solved)

	A	B	AB
$b_1(S)$	$\boxed{\mathbf{5}}$	5	10
$b_2(S)$	5	3	7
$b_3(S)$	6	$\boxed{\mathbf{9}}$	12

Table 8.15 Clock phase in a clock-proxy auction

	Type A						Type B					
Round t	p_1^t	$q_{1,1}^t$	$q_{2,1}^t$	$q_{3,1}^t$	Q_1^t	$\overline{Q_1}$	p_2^t	$q_{1,2}^t$	$q_{2,2}^t$	$q_{3,2}^t$	Q_2^t	$\overline{Q_2}$
$t = 0$	1	1		1	2	1	1		1	1	2	1
$t = 1$	2	1		1	2	1	2		1	1	2	1
$t = 2$	3			1	1	1	3		1	1	2	1
$t = 3$	3			1	1	1	4			1	1	1

Table 8.16 Bids of proxy bidder in round $t^{\text{proxy}} = 1$

$t^{\text{proxy}} = 1$	A	B	AB
$b_1^1(S)$	$\boxed{3}$		
$b_2^1(S)$		$\boxed{4}$	
$b_3^1(S)$			7

Table 8.17 Bids of proxy bidder in round $t^{\text{proxy}} = 2$

$t^{\text{proxy}} = 2$	A	B	AB
$b_1^2(S)$	3		
$b_2^2(S)$		4	
$b_3^2(S)$			$\boxed{8}$

Table 8.18 Bids of proxy bidder in round $t^{\text{proxy}} = 3$

$t^{\text{proxy}} = 3$	A	B	AB
$b_1^3(S)$	$\boxed{4}$		
$b_2^3(S)$		$\boxed{5}$	
$b_3^3(S)$			8

Table 8.19 Bids of proxy bidder in round $t^{\text{proxy}} = 4$

$t^{\text{proxy}} = 4$	A	B	AB
$b_1^4(S)$	4		
$b_2^4(S)$		5	
$b_3^4(S)$			$\boxed{10}$

$b_1^1(A)+b_2^1(B) = 7$ and $b_3^1(AB) = 7$. If the seller chooses the first allocation as the provisional winning bids ($b_1^{1*}(A)$ and $b_2^{1*}(B)$) the proxy agent of the third bidder will bid on his behalf in the next round (because he is not among the winning bidders and his reported value is higher than the actual price). Table 8.17 shows the bids in round $t^{\text{proxy}} = 2$.

In this round, the new combination that solves the WDP is $b_3^{2*}(AB) = 8$. Nevertheless, the proxy bidders of the first and second bidders would submit new bids on their behalf's in round $t^{\text{proxy}} = 3$, see Table 8.18.

Once again, the third bidder is not a provisional winner ($b_1^{3*}(A) + b_2^{3*}(B) = 9$), so his proxy bidder will bid on his behalf in round $t^{\text{proxy}} = 4$, see Table 8.19. With this new bid, the third bidder is the provisional winner: $b_3^{4*}(AB) = 10$.

In round $t^{\text{proxy}} = 5$ the first bidder's proxy bidder stops bidding, as the price equals his reported value. Nevertheless, the proxy bidder of the second bidder will submit a new bid, see Table 8.20.

In this round, there is a tie again, as there are two possible allocations that solve the WDP (taking into account all bidders in the clock and proxy phases): $b_3^{5*}(AB) = 10$ and $b_1^{4*}(A) + b_2^{5*}(B) = 4 + 6 = 10$ (first bidder's bid in round $t^{\text{proxy}} = 4$ plus second bidder's bid in round $t^{\text{proxy}} = 5$). If the seller selects the second combination, the proxy bidder of the third bidder will bid again in round $t^{\text{proxy}} = 6$ (Table 8.21). After this bid, there are no more bids, the auction ends and the final allocation is equal to $b_3^{6*}(AB) = 11$.

Table 8.20 Bids of proxy bidder in round $t^{\text{proxy}} = 5$

$t^{\text{proxy}} = 5$	A	B	AB
$b_1^5(S)$			
$b_2^5(S)$		6	
$b_3^5(S)$			$\boxed{\mathbf{10}}$

Table 8.21 Bids of proxy bidder in round $t^{\text{proxy}} = 6$

Round $t^{\text{proxy}} = 6$	A	B	AB
$b_1^6(S)$			
$b_2^6(S)$			
$b_3^6(S)$			$\boxed{\mathbf{11}}$

(c) With all the bids submitted during the clock phase and the proxy phase, the seller determines the efficient allocation of the items that will maximize his revenue, that is, he solves the WDP. In this example, the winning bidder is the third one, who obtains the AB package for a price equal to $P_1^* = 11$ euros. The seller's revenue is equal to $R^* = 11$ euros.

As can be seen, an efficient allocation of the items has been achieved with respect to the values reported to the proxy bidders.

Appendix

In multi-unit auctions, the seller usually sets an **activity rule** to encourage sincere bidding from early rounds.

The **revealed-preference activity rule** developed by Ausubel et al. [8] allows bidder i to bid on a certain package in round t if it satisfies the revealed preference constraint with respect to the package bid in round l, for $l < t$:

$$\sum_{j=1}^{m}(q_{i,j}^t \times (p_j^t - p_j^l)) \leq \sum_{j=1}^{m}(q_{i,j}^l \times (p_j^t - p_j^l)), \tag{8.4}$$

where:

- j refers to the category of the item;
- m is the total number of categories or types of items;
- $q_{i,j}^t$ is bidder i's quantity of the jth type bid in round t;
- $q_{i,j}^l$ is bidder i's quantity of the jth type bid in round l;
- p_j^t is the clock price of the jth type in round t;
- p_j^l is the clock price of the jth type in round l.

For bidder i, the package bid in round t ($\sum_{j=1}^{m} q_{i,j}^t$) satisfies the revealed preference constraint with respect to an earlier clock round l if it has become relatively less expensive than the package on which the bidder bid in clock round l

Table 8.22 Items offered in a CCA

Category j	Type	$\overline{Q}_j$	EP	p_j^s	Increment p_j^t
$j = 1$	A	5	20	100	50
$j = 2$	B	3	15	100	50

Table 8.23 Clock rounds for bidder 1

	Type A			Type B			Eligibility_1^t	Activity_1^t	Bid_1^t
Round t	p_1^t	$q_{1,1}^t$	Q_1^t	p_2^t	$q_{1,2}^t$	Q_2^t			
$t = 0$	100	2	5	100	3	6	85	85	AABBB
$t = 1$	100	2	5	150	1	5	85	55	AAB
$t = 2$	100	2	6	200	1	5	55	55	AAB
$t = 3$	150	3	6	250	0	4	55	$\boxed{\mathbf{60}}$	AAA

($\sum_{j=1}^{m} q_{i,j}^{l}$, for $l < t$) due to the clock price increment between both rounds. With this activity rule, a bidder can bid on round t on a bigger package (more **eligibility points**, EP) than in round l, as long as it is less expensive.

An example of this activity rule is described below. Table 8.22 shows the items offered in a **combinatorial clock auction** (CCA).

The first bidder submits the clock bids depict in Table 8.23. In round $t = 1$ his activity is below his eligibility ($\text{Activity}_1^1 = 55 < 85 = \text{Eligibility}_1^1$), so his eligibility decreases for the next round. Between rounds $t = 0$ and $t = 2$ the clock price for category A does not increase as there is no excess demand. Nevertheless, there is excess demand in all rounds for items in category B, so the clock price in $t = 2$ is equal to 200 euros.

Because of the differential in the price increment between the two categories, the bidder would like to switch in round $t = 3$ from the package AAB to the package AAA. Nevertheless, the package AAA requires a higher activity than his actual eligibility: $\text{Activity}_1^3 = 60 > 55 = \text{Eligibility}_1^3$. Would the bidder be able to submit this package bid in round $t = 3$?

If the seller sets the **eligibility-based activity rule**, bidders are required to bid on packages of the same size or smaller (same or less EP) as prices go up, so the bidder will not be able to bid on package AAA in round $t = 3$. Nevertheless, if the seller sets the revealed-preference activity rule the bidder could submit this bid if it has become relatively less expensive than the package bid submitted in round $t = 1$ (AAB) in which he reduced his eligibility. The bidder could submit the bid if it satisfies the following constraint:

$$\sum_{j=1}^{m}(q_{i,j}^{3} \times (p_j^3 - p_j^1)) \leq \sum_{j=1}^{m}(q_{i,j}^{1} \times (p_j^3 - p_j^1)).$$

The first part of the inequality refers to the price increment between $t = 3$ and $t = 1$ for the package AAA (package bid in round $t = 3$):

- Price of the package AAA in round $t = 3$: $\sum_{j=1}^{m}(q_{i,j}^{3} \times p_{j}^{3}) = 150 \times 3 = 450$ euros.
- Price of the package AAA in round $t = 1$: $\sum_{j=1}^{m}(q_{i,j}^{3} \times p_{j}^{1}) = 100 \times 3 = 300$ euros.
- Price increment between $t = 3$ and $t = 1$ for package AAA: $450 - 300 = 150$ euros.

The second part of the inequality refers to the price increment between $t = 3$ and $t = 1$ for the package AAB (package bid in round $t = 1$):

- Price of the package AAB in round $t = 3$: $\sum_{j=1}^{m}(q_{i,j}^{1} \times p_{j}^{3}) = (150 \times 2) + 250 =$ 550 euros.
- Price of the package AAB in round $t = 1$: $\sum_{j=1}^{m}(q_{i,j}^{1} \times p_{j}^{1}) = (100 \times 2) + 150 =$ 350 euros.
- Price increment between $t = 3$ and $t = 1$ for package AAB $t = 3$ y $t = 1$: $550 - 350 = 200$ euros.

This example satisfies the constraint: $150 \leq 200$, so package AAA has become relatively less expensive than package AAB because of the clock price evolution. Hence, if the revealed-preference rule is settle, the bidder could bid for package AAA in round $t = 3$, even though this is a bigger package (more EP).

It is important to mention that, although a bidder may bid above his eligibility in one round, his eligibility for the next round will not increase. Therefore, it may happen that a bid is no longer valid in a later round if prices increase and the revealed preference constraint is no longer fulfilled.

The advantage of the revealed-preference rule is that bids are not constrained by the aggregate demand and it allows bidders to modify their bids according to the price increments, giving greater flexibility to bid on packages that have become relatively cheaper.

9 Online Auctions

9.1 Introduction

In recent years, Internet auctions have experienced strong growth, increasing the volume of online trades. Driven by this fact as well as the possibility offered by certain platforms to obtain field data, research of **online auctions** has become a significant field.[1] In this chapter, we will focus on the main auction models that are found on the Internet, as well as possible strategies used by the bidders. In this context, the work done by Ockenfels et al. [57] is worth mentioning because it comprises an excellent compilation of the main theoretical, experimental, and empirical work describing online auctions.

9.2 Sellers in Online Auctions

Platforms such as *eBay* offer the possibility of selling any item to other consumers using an Internet auction (C2C; *consumer-to-consumer commerce*). A potential seller needs to be registered as a client and a secure online payment system is recommended, for which an account must be opened in a system such as *PayPal*.

The most widely used model in these platforms is the **ascending-bid auction**, in which the seller establishes an opening bid that is relatively low and increases as bidders submit their bids.[2] The winning bidder is the one who submits the highest bid, which is equal to the second highest bid plus the bid increment (dynamic auction in which the **second-price** rule is implemented). It is worth mentioning that

[1] The Internet also offers the possibility of re-transmitting auctions that are taking place in real time through the web, so that bidders can bid directly in person as well as through the Internet. These auctions are known as **live auctions**.

[2] The bid increment depends on the price of the item.

A. Mochón, Y. Sáez, *Understanding Auctions*, Springer Texts in Business and Economics, DOI 10.1007/978-3-319-08813-6_9

these auctions are **deadline auctions**, given that they do not end when the bidders stop bidding but rather at a time previously established by the seller.[3]

When a seller chooses to offer his item through an online auction, he must make certain decisions regarding his selling conditions that may affect the final outcome[4]:

- **Starting bid**, p^s: The seller must set the price at which he will offer the item. Before taking this decision, it can be helpful to analyze the price at which the item has been sold in the past and the prices at which similar items are being auctioned at that time, including the shipping costs.
- **Reserve price**, p^r: If the seller has a minimum selling price below which he does not wish to sell the item, he has the option to establish a private reserve price.[5] With this option, the bidders know that there is a reserve price but do not know what this price is. The seller also has the option of setting a starting bid that matches the minimum price at which he wishes to sell the item, which is equivalent to a public reserve price. Another alternative to prevent selling for less than a certain price is for the seller to arrange **shill bids** or **bid padding**, that is, to bid for his items using another account. However, this practice is usually prohibited in online auction platforms.[6]
- **Buyout option**! or **buy-it-now**!: The seller can fix a price at which the buyer can acquire the item immediately, without the need to bid or wait until the end of the auction. This option is more relevant when there are impatient or risk-averse bidders. The price fixed with the buy-it-now option is always higher than the opening bid (minimum 10 % higher).[7]
- **Starting date and auction duration**: The seller decides on the exact day and time when the auction begins, as well as how long it would last for. The duration may be 1, 3, 5, 7, or up to 10 days.
- **Other factors**: The final result of the auction may also be influenced by other factors, such as the item description, the photos taken, the shipping cost, and the seller's reputation.

[3]Given that *eBay* is the world leader in online auctions among consumers, in this chapter we will use the options offered on this platform as a reference.

[4]eBay charges the seller a fee for including an item in an auction (*insertion fee*) and an additional fee if the item is sold (*final value fee*), therefore the cost of selling is the sum of both.

[5]Establishing a private reserve price carries an additional fee (*reserve price fee*).

[6]To review these concepts in more detail, we recommend Chap. 1.

[7]On *eBay*, the buy-it-now option stops being active when a bidder submits a bid or when the reserve price is exceeded, if there is one.

9.3 Buyers in Online Auctions

In platforms such as *eBay*, the consumers have the option of buying any item by bidding in online auctions.[8] To be able to participate in these auctions, the buyers need to be registered as clients in the system, and before placing a bid, it is recommended that the buyers analyze various offers and the reputations of the respective sellers. When bidding in an online auction, the buyer must take into account the following factors:

- **Proxy bid** (*maximum bid*): These platforms ask the buyers to indicate the maximum price they are willing to pay for an item. This information is never made public neither to the other bidders nor to the seller, but it allows the system to automatically increase the bid of a buyer without that buyer having to constantly monitor the progress of the auction. Instead, a buyer is guaranteed to be the winner if his maximum bid is higher than those of his rivals. The winner pays a price equal to the highest bid made by his rivals plus the increment of the bid.

  For example, if there are two bidders who have maximum bids of $b_1^{\text{proxy}} = 100$ euros and $b_2^{\text{proxy}} = 50$ euros, and the bid increment is 10 euros, the first bidder wins the item, and the sale price is $p^* = 60$ euros.
- **Last minute bidding**: Usually, online auctions end at a specific time, that is, they are deadline auctions. This ending rule encourages bidders to bid at the last moment of the auction, even in the last minutes or seconds (**sniping**). Through this strategy, the bidders avoid entering into a price war with their rivals or transmitting information regarding their valuations. However, the use of this strategy carries a certain risk for the buyer because the time that it takes him to submit a bid may vary depending on Internet traffic or connection time, which can lead to the bid coming in too late and the auction being closed.[9]
- **Cross-bidding**: On the Internet, a buyer can frequently find, in a given time period, different online auctions that offer exactly the same item, known as **competing auctions**. In these competing auctions, a group of sellers competes with each other to offer a homogeneous item to a group of heterogeneous bidders, thus losing their monopoly power.[10] Thus, the buyer can choose from among all of the active homogeneous auctions the one in which he is most interested in

[8] On the Internet, there are also other platforms in which the sellers are companies; some examples include *Bidz* or *uBid* (B2C; *business-to-consumer commerce*).

[9] To avoid this risk, *eBay* always recommends to submit a true proxy bid with the real value from the beginning, although several authors have shown that bidding at the last minute can be the best strategy in certain scenarios. See the work of Ockenfels and Roth [60], among others.

[10] In the auctions analyzed to this point, with the exception of double auctions, we have always assumed that there is only one seller and many bidders, that is, situations that are similar to a monopoly. However, in competing auctions, the sellers lose the classic market power observed in a monopoly.

participating. The strategy of cross-bidding implies that, before submitting a bid, the bidder takes into consideration all of the active competing auctions.

Researches, such as Roth and Ockenfelds [66], Anwar et al. [1], Peters and Severinov [64], or Pages and Mochon [62], have observed that the bidders in competing auctions tend to engage in last minute bidding and to bid in multiple auctions. In this context, bidders also frequently use the *minimal increment bidding* strategy.

- **Auction fever** (*competitive arousal, bidding frenzy, or bidding war*): Some researches claim that bidders in an auction involves a certain degree of emotion because of the competition with other bidders, which influences the auction outcome, see [2]. As a consequence of this emotion and the possible satisfaction of being the winner, some bidders tend to bid in a frenzy, even bidding above the prices that they would be willing to pay in fixed-price sales. This phenomenon could explain why many sellers prefer to establish a low opening bid, to increase the number of bidders and thus the competition and the final sale price.[11]

9.4 Penny Auctions

Another auction model that is growing on the Internet is what is called a **penny auction**. In this model, certain companies offer items for an opening bid that is very low, and the bidders increase the price by bidding a penny in each round. What is peculiar about this auction is that each bidder pays for each penny that he bids, but the one who makes the last bid before the end of the auction is the only one who wins the item.[12] This is an **all-pay auction**, studied in Chap. 3.

To better understand how these auctions work, let us analyze the following example. A seller auctions an item with an opening bid of zero euros. The three bidders involved in the auction submitted the bids shown in Table 9.1. The auction finished in round $t = 6$, and the winner is the second bidder because he bids for the last penny, b_2^5. The price that he pays is equal to the sum of the bids that he submitted, which in this example is $P_2^* = b_2^1 + b_2^3 + b_2^5 = 0.03$ euros. In this example, the first and third bidders pay $P_1^* = b_1^0 + b_1^2 = 0.02$ euros and $P_3^* = b_3^4 = 0.01$ euros, respectively, even though they did not win the item. The seller's revenue is equal to $R^* = 0.06$ euros.

Usually these auctions have fixed deadlines and finish at the end of a countdown that is activated at the beginning of the auction (although frequently the timer is reset or a specific amount of time is added when a bid is made). Additionally, certain platforms offer the option to the bidder of establishing a price range based on which the system submits bids in his name. By doing this, the bidder ensures that he will

[11]Delgado et al. [25] conducted an analysis of blood oxygen levels during auctions and observed that the phenomenon of bidding above personal valuations (*overbidding*) is a result of the fear of losing in a social competition in front of rivals.

[12]Platforms such as *Bidrivals* and *Beezid* conduct this type of auction.

Table 9.1 Penny auction

Round (t)	Price (p^t)	b_i^t
$t = 0$	$p^0 = 0.00$	$b_1^0 = 0.01$
$t = 1$	$p^1 = 0.01$	$b_2^1 = 0.01$
$t = 2$	$p^2 = 0.02$	$b_1^2 = 0.01$
$t = 3$	$p^3 = 0.03$	$b_2^3 = 0.01$
$t = 4$	$p^4 = 0.04$	$b_3^4 = 0.01$
$t = 5$	$p^5 = 0.05$	$b_2^5 = 0.01$
$t = 6$	$p^6 = 0.06$	

win the item if he has set an upper limit that is higher than those of his rivals, even if he is not monitoring the auction at that time.[13] Likewise, certain systems allow the bidders to bid until the last moment, as long as the established range of the auction has not been reached, and thus the bidder had not had the option to bid.[14]

9.5 Multi-unit Online Auctions

Multi-unit online auctions are becoming more frequent for transactions between companies (B2B; *business-to-business commerce*). With this mechanism, companies can buy and sell any item from anywhere in the world, thus decreasing the number of intermediaries (reducing costs). In auctions between companies, the participant conducting the auction can be the seller as well as the buyer. In the first case, **forward auction**,[15] the winner will be the bidder who makes the highest bid. However, the buyer is also frequently the one conducting the auction, **reverse auction**, in which case the winner of the auction will be the seller who offers the item for the lowest price.

Governments are also using multi-unit online auctions to award public items, such as electricity, spectrum licenses, and transportation licenses. In these transactions, governments try to identify and implement an auction model that is best adapted to the specific circumstances. For this, it is fundamental to consider variables such as item characteristics (homogeneous or heterogeneous), possible bidders, goals to reach, and so on. Furthermore, governments are implementing multi-unit online auctions to buy items at the lowest price **online procurement auctions**, so-called *e-procurement auctions*.

[13]In the *Beezid* platform, this option is called *AutoBeezid*.

[14]In the *Beezid* platform, this option is called *Beezid Sniper*.

[15]These auctions are called reverse auctions because the roles of the buyers and sellers have been reversed.

9.5.1 Position Auctions

Search engines frequently use multi-unit online auctions to sell the positions on the website on which the ads will be published, called **position auctions**.[16] In these auctions, one seller auctions M positions ($J = (1, 2, \ldots, M)$) between N advertisers ($I = (1, 2, \ldots, N)$). The process works in the following manner.

Each advertiser chooses a set of keywords related to the item he is offering and submits a bid for each of the selected words (b_i). When a user conducts a search using a keyword, a set of ads appear that are ordered based on the bids submitted by the advertisers for that specific keyword. For example, $b^*_{i,j}$ represents the bid submitted by advertiser i with which he obtained position j. The advertisers with the highest bids are placed in the first positions. If a user finally clicks on an ad, the advertiser will have to pay the highest bid after his, that is, the bid placed by the advertiser who is in the position following him. This price, named **cost per click** (CPC) is calculated using the following equation:

$$P^*_{i,j} = b^*_{k,j+1}. \tag{9.1}$$

In other words, if a user clicks on the ad in position j, the advertiser i who obtained position j will have to pay the bid placed by advertiser k, who obtained the $j + 1$ position. If the users do not click on a specific ad, the corresponding advertiser will not have to make any payment.

In this example there are four advertisers that offer the same item and that select the same keyword to be included in the search engine. The bids placed by these advertisers for the keywords are $b_1 = 1$ euro, $b_2 = 0.5$ euros, $b_3 = 2$ euros, and $b_4 = 0.1$ euros, respectively. If a user introduces the selected keyword, the search engine displays the ads in the order shown in Table 9.2.

If the user clicks on the first ad, the price that advertiser $i = 3$ will have to pay is equal to the bid placed by the advertiser who is in second place, in this case, the bid of advertiser $i = 1$, which is $P^*_{3,1} = b^*_{1,2} = 1$ euro. In the same way, if the user clicks on the second ad ($j = 2$), which corresponds to advertiser $i = 1$, he will have to pay the bid made by the advertiser who is in third place, $P^*_{1,2} = b^*_{2,3} = 0.5$ euros, and so on.

Table 9.2 Position auction

Position (j)	Advertiser (i)	Bid (b_i)
First $j = 1$	$i = 3$	$b_3 = 2$
Second $j = 2$	$i = 1$	$b_1 = 1$
Third $j = 3$	$i = 2$	$b_2 = 0.5$
Fourth $j = 4$	$i = 4$	$b_4 = 0.1$

[16] Milgrom and Levin [53] analyzed position auctions in several markets.

Table 9.3 Google Ad auction

Position (j)	Advertiser (i)	Bid (b_i)	Quality (c_i)	$b_i c_i$
First $j = 1$	$i = 1$	$b_1 = 1$	$c_1 = 8$	8
Second $j = 2$	$i = 3$	$b_3 = 2$	$c_3 = 3$	6
Third $j = 3$	$i = 2$	$b_2 = 0.5$	$c_2 = 4$	2
Fourth $j = 4$	$i = 4$	$b_4 = 0.1$	$c_4 = 10$	1

9.5.1.1 Google Ad Auctions

When analyzing position auctions, it is worth taking time to understand how the auctions conducted by the search engine *Google* work each time users perform a search.[17] In **Google Ad auctions**, the position of each ad is calculated by multiplying the bid submitted by the advertiser (b_i) times the quality of the ad (c_i).[18]

Table 9.3 shows the final position of the ads from the previous example if we include the quality factor in the calculation. If we assume that the quality of each ad is as shown, we can see that the order has been modified. Advertiser $i = 3$, who previously appeared in the first position, is now in the second position. Thus, *Google* encourages its advertisers to produce quality ads to increase user satisfaction and thereby promote the use of its services.

In these auctions, the price to be paid by advertiser i with position j when the user clicks on his ad (CPC) is calculated using the following equation:

$$P^*_{i,j} = (b^*_{k,j+1} c^*_{k,j+1})/c^*_{i,j}, \tag{9.2}$$

where k is the advertiser with position $j + 1$ and $c^*_{i,j}$ is the quality of the ad of advertiser i who obtained position j. In other words, the CPC is equal to the product of the bid and the ad quality of the advertiser in the following position ($j + 1$), all divided by the quality of the ad in position j.

In this example, the CPC for advertiser $i = 1$ is equal to $P^*_{1,1} = (b^*_{3,2} c^*_{3,2})/c^*_{1,1} = (3 \times 2)/8 = 0.75$ euros. The CPC for the other advertisers are calculated in the same way. In these auctions, the advertisers who are in the last place pay the minimum bid established by the seller.

9.6 Variables Used in This Chapter

In this chapter, we use the following variables:

- $I = (1, 2, \ldots, N)$: Bidders.
- $J = (1, 2, \ldots, M)$: Items (which correspond to positions in a position auction).

[17] Varian [78] analyzed the equilibrium in a model based on Google Ad auctions.

[18] The quality of each ad is measured based on three variables: the ratio of clicks that it obtained, the relevance of the ad, and the quality of the website reached with the ad.

- b_i^{proxy}: Bidder i's proxy bid in an online auction.
- b_i^t: Bidder i's bid in round t of a dynamic auction.
- b_i: Advertiser i's bid for this ad position in a position auction.
- $b_{i,j}^*$: Advertiser i's bid to have his ad in position j in a position auction.
- c_i: Advertiser i's ad quality in a position auction.
- $c_{i,j}^*$: Advertiser i's ad quality who obtained position j in a position auction.
- p^{s}: Starting bid (minimum bid).
- p^{r}: Reserve price. If the reserve price is public, it holds that $p^{\text{r}} = p^{\text{s}}$.
- p^t: Price of the item in round t of a dynamic auction.
- p^*: Selling price. Only if the auctioneer sets the first-price rule the selling price matches the highest bid $p^* = b^*$.
- P_i^*: Bidder i's payment for items won.
- $P_{i,j}^*$: Advertiser i's CPC who obtained position j in a position auction.

9.7 Exercises

Select the correct answer.

1. The auction model used in online auction platforms such as *eBay* corresponds to:
   (a) A sealed-bid second-price auction.
   (b) An ascending second-price auction.
   (c) A dynamic third-price auction.
   (d) A hybrid second-price auction.
2. Three bidders submit the following proxy bids in an auction on *eBay*: $b_1^{\text{proxy}} = 100$ euros, $b_2^{\text{proxy}} = 120$ euros, and $b_3^{\text{proxy}} = 50$ euros. Determine the winning bidder and the price that he will have to pay assuming a bid increment of 10 euros.
   (a) The winner is the second bidder and will pay 120 euros.
   (b) The winner is the second bidder and will pay 100 euros.
   (c) The winner is the second bidder and will pay 110 euros.
   (d) The winner is the second bidder and will pay 130 euros.
3. A seller sets an opening bid of $p^{\text{s}} = 10$ euros and a reserve price of $p^{\text{r}} = 50$ euros. If the highest bid of the auction is 40 euros
   (a) The seller is not required to sell the item.
   (b) The winner buys the item for 50 euros.
   (c) The winner buys the item for 40 euros.
   (d) The winner buys the item for 10 euros.
4. Indicate which of the following strategies are frequently observed in competing auctions:
   (a) Last minute bidding or sniping.
   (b) Cross-bidding.
   (c) Making the lowest possible bid increments.
   (d) All of these strategies are frequently found in competing auctions.

5. In penny auctions:
   (a) Each bidder, winners as well as losers, pays for the bids made.
   (b) The winning bidder pays an amount equal to the last bid of the auction.
   (c) Only the winning bidder pays.
   (d) All of the bidders pay the final price of the auction.
6. In a reverse auction:
   (a) The seller who offers the lowest price makes the sale.
   (b) The seller who offers the item for the highest price makes the sale.
   (c) The buyer who submits the highest bid obtains the item.
   (d) The buyer who submits the lowest bid obtains the item.
7. In a position auction, three advertisers submit the following bids for the same keywords: $b_1 = 0.5$ euros, $b_2 = 1$ euro, and $b_3 = 2$ euros. Given these bids, select the order in which the ads will appear after conducting the search.
   (a) First position for $i = 3$, second position for $i = 1$, and third position for $i = 2$.
   (b) First position for $i = 3$, second position for $i = 2$, and third position for $i = 1$.
   (c) First position for $i = 2$, second position for $i = 3$, and third position for $i = 1$.
   (d) First position for $i = 1$, second position for $i = 2$, and third position for $i = 3$.
8. In a position auction, three advertisers submit the following bids for the same keyword: $b_1 = 0.5$ euros, $b_2 = 1$ euros, and $b_3 = 2$ euros. Given these bids, determine how much the second advertiser will have to pay if a user clicks on his ad.
   (a) $P^*_{2,2} = 1$ euro.
   (b) $P^*_{2,2} = 2$ euros.
   (c) $P^*_{2,2} = 0.5$ euros.
   (d) $P^*_{2,2} = 1.5$ euros.

9.8 Solutions to Exercises

The correct answers are provided below.

1. Answer: (b)
2. Answer: (c)
3. Answer: (a)
4. Answer: (d)
5. Answer: (a)
6. Answer: (a)
7. Answer: (b)
8. Answer: (c)

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Glossary

Activity A bidder's activity in one round of a dynamic multi-unit auction is calculated by adding the *eligibility points* of the items of which the bidder has bid, see *activity rule* and *eligibility*.

Activity rule Restriction set by the seller on eligible bids in each round to encourage price discovery in a simultaneous ascending auction (SAA or SMR auction). The ability to bid will depend on the activity level in the previous rounds, so it prevents bidders from adopting the "snake in the grass strategy".

Affiliation A situation in which the bidders have interdependent values whose signals regarding the item's value are positively correlated: *affiliated signals*.

Aggregation risk Risk bidders face when complements items are auctioned if they only win some (but not all) items included in a complementary package. The bidders are exposed to a loss if they bid aggressively for one package that they do not get, losing the super-additivity value of the complete package. Also known as *Exposure problem*.

All-pay auction An auction in which the bidder with the highest bid wins the item, but all of the other bidders also pay what they bid.

Anglo-Dutch auction Hybrid auction carried out in two phases. The first phase is an ascending auction, and the second phase is a first-price sealed-bid auction.

Ascending clock auction *Ascending-bid auction* in which the price increases continuously and the bidders submit their bids.

Ascending proxy auction In an ascending proxy auction, each bidder communicates his preferences to a proxy bidder, who will then bid on the bidders' behalf in each round for the item or package that has the greatest potential surplus for the bidder. After each round, the seller determines the feasible combination of bids that maximizes his revenue (solves the WDP) and announces provisional winners and payments. The process will continue as long as the proxy bidders submit new bids.

Ascending-bid auction Single-unit auction that begins with a starting price which is considered to be relatively low and increases until only one bidder remains. He is awarded the item and pays an amount equal to his final bid. It is also called *English auction*.

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Asymmetric bidders The bidders in an auction are asymmetric if their valuations are not distributed according to the same distribution function.

Auction Market mechanism with precise rules capable of determining to whom one or multiple items will be awarded and at what price.

Auction fever Theory according to which the bidders in an auction feel a certain emotion when competing against other rivals that makes them bid above what they would pay at a fixed-price sale. It is also known as *Competitive arousal*, *bidding frenzy*, or *bidding war*.

Ausubel auction The Ausubel auction is a particular case of a multi-unit ascending auction in which each bidder pays the price of the round in which they clinched each item. Thus, the auction implements the *Vickrey rule*.

Bayesian Nash equilibrium An extension of the Nash equilibrium to incomplete information games. In the equilibrium, each player chooses a best response to the strategies of the other players (evaluated after a player obtains his own private information but before he learns his rivals' private information). Each player maximizes his expected utility given his private information, the joint distribution of others' private information, and the strategies of the other players. Private information is drawn from a common joint distribution and beliefs about strategies are consistent.

Bid The offer submitted by a bidder in an auction indicating his intention to buy in a forward auction (or sell in *Reverse auctions*).

Bid padding A bid that can be made by the seller through multiple identities to shift the auction in his favor. This practice is usually prohibited. It is also known as *Shill bid*.

Bid withdrawal In a dynamic multi-unit auction (usually an SAA or an SMR auction), the bidders may be able to withdraw some (or all) of their winning bids in exchange for paying a penalty. This option is particularly relevant when complements items are auctioned.

Bidder monotonicity An auction has bidder monotonicity if, upon including another bidder, the bidders' surplus always decrease (weakly) and the seller's revenue increases (weakly).

Bidding rule Rule established by the auctioneer to determine how to bid in an auction.

Blocking coalition A coalition of bidders who block the result of a *combinatorial auction* because they would strictly prefer an alternative result that is also strictly preferred by the seller.

Buy-it-now! An option that can be used by online auction sellers to fix the price at which the bidder can buy the item directly without having to wait until the end of the auction.

Buy-side auction An auction conducted by the buyer with the goal of buying one or multiple items from the seller who makes the lowest offer. It is also known as *Reverse auction*.

Candle auction Traditionally, an auction that ends when one candle is consumed.

Chopstick auction Auction in which the seller simultaneously offers three chopsticks, and the bidder with the highest bid wins two chopsticks. The player with the second highest bid will be affected by *exposure problem* because he will win one useless chopstick for which he must pay.

Clearing house An intermediary between the buyers and sellers in *double auctions*.

Clock auction A type of dynamic auction in which the seller increases prices continuously and the bidders submit their bids at the standing price.

Clock-proxy auction Hybrid auction with two phases. The first phase corresponds to a simultaneous clock auction with package bidding, and the second phase corresponds to an ascending proxy auction.

Collusion Cooperative behavior between bidders.

Combinatorial auction (CA) In a combinatorial auction multiple items are offered, and the bidders are allowed to submit as many bids as they want for the items or combinations of items in which they are interested. This mechanism is also known as *Package auction*.

Combinatorial clock auction (CCA) Hybrid auction with two stages: allocation stage and assignment stage, used to allocate multiple heterogeneous items. In the allocation stage winning bidders and base prices are settled. This phase consists of two rounds: clock and supplementary. In the assignment stage specific items are assigned to winning bidders.

Combinatorial sealed-bid auction *Combinatorial auction* in which the bidders submit their bids in a single round for the items or combination (packages) in which they are interested. Once all the bids have been received, the seller solves the *Winner Determination Problem* (WDP), thus awarding the items.

Common value The extreme case of an *interdependent value* in which, before the auction, each bidder has his own value of an item based on his estimations, but after the auction, when uncertainty is eliminated, all bidders have the same value.

Competing auctions In competing auctions a group of sellers compete against each other through different auctions that take place during the same period of time in which a homogeneous item is being offered to a group of heterogeneous bidders.

Complements Two items are complements, or there are *synergies* between them, when the marginal value of winning the second item is higher than the marginal value of winning the first one, such that the bidder's value for the combination of items is greater than the sum of the individual values.

Continuous double auction Double auction with multiple rounds.

Core allocation A *combinatorial auction* yields a core allocation when there is no coalition of bidders who, after the auction ends, would strictly prefer an alternative outcome that would also be strictly preferred by the seller.

Core outcome A *combinatorial auction* generates a core outcome if and only if, given several bids, there is no group of bidders that would strictly prefer an alternative outcome that would also be strictly preferred by the seller.

Core-selecting package auction *Combinatorial auction* that establishes a pricing rule that ensures that a *core outcome* with respect to the bids that have been received.

Cost per click (CPC) Search engines frequently use position auctions to establish the order in which ads will appear. In these auctions, the advertiser of the ad that has been clicked on pays a price that is equal to the bid placed by the advertiser in the next lower position, that is, the next highest bid after his. This is his CPC.

Cross-bidding Strategy used in competing auctions, in which bidders, before making a bid, consider all active auctions offering the same item in the same period of time.

Deadline auction An auction that ends at a specific day and time fixed by the seller.

Descending-bid auction A single-unit auction in which the seller establishes a starting price that is considered relatively high but that decreases until a bidder makes a bid. The winner pays the current price at the time of the bid. Also called *Dutch auction*.

Discriminatory auction Sealed-bid auction of multiple homogeneous items in which each winner pays an amount equal to the sum of the bids placed for the items obtained, also called *Pay-your-bid auction*.

Discriminatory pricing rule Pricing rule in which each winning bidder of a multi-unit or double auction pays a different price for the acquired items.

Dollar auction A particular case of an *all-pay auction* in which a seller offers a dollar that is awarded to the bidder who submits the highest bid, but all bidders must pay their placed bids. This game is used as a reference in the analysis of escalating conflict.

Dominant strategy A strategy that does at least as well as any other strategy for one bidder, no matter how that bidder's opponents may play.

Dominant strategy equilibrium A refinement of a Nash equilibrium (or a Bayesian Nash equilibrium for games of incomplete information). In the equilibrium each player's strategy is a best response, regardless of the strategies of the other players. Behavior in dominant strategy equilibria is robust to uncertainty about rivals' strategies and private information. With private values, bidding one's value is a dominant strategy equilibrium in the Vickrey auction.

Double auction Market mechanism in which multiple buyers submit their bids indicating the items that they are willing to buy and at what price; and multiple sellers submit their asks indicate the items that they are willing to sell for no less than a specific amount. See *two-sided auction*.

Double Dutch auction Double auction in which a high price is established that progressively decreases until a buyer makes a bid. In the same way, a low price is established that progressively increases until a seller accepts the offer. The process continues until both prices intersect. At this time, all of the purchases and sales are made at the intersection price.

Dutch auction A single-unit auction in which the seller establishes a starting price that is considered relatively high but that decreases until a bidder makes a bid. The winner pays the current price at the time of the bid. Also called *Descending-bid auction*.

Dutch–English auction A hybrid auction in two phases. The first phase is a descending auction (or Dutch auction), and the second phase is an ascending auction (or English auction).

Dynamic auction An auction model that allows bidders to submit multiple bids and where some information about the bidding is shown during the auction.

Dynamic auctions may have discrete rounds (*iterative or multi-round auctions*) or continuous bidding (*clock auctions*).

Efficient auction Auction that awards the items to the bidders who most value them (*ex post*).

Eligibility Maximum quantity of items that a bidder may bid on a round of a dynamic multi-unit auction, that depends on the activity rule selected. The eligibility of one round depends on the activity of the previous round. The eligibility of the first round is based on the bidder's financial situation, its market position, or a bid deposit, see *activity rule*.

Eligibility points (EP) In a multi-unit dynamic auction, the seller can allocate a certain number of EP per item. Adding the EP of each bidder per round, the seller obtains his activity, which will constrain his eligibility for the next round, see *Activity rule* and *Eligibility*.

Eligibility-based activity rule *Activity rule* proposed for multi-unit auctions that requires bidders to bid on packages of the same size or smaller (same or less *eligibility points*) as prices go up.

English auction Single-unit auction that begins with a starting price which is considered to be relatively low and increases until only one bidder remains. He is awarded the item and pays an amount equal to his final bid. It is also called *Ascending-bid auction*.

Equilibrium price Price for which the aggregate supply equals the aggregate demand.

Expected payment The expected payment *before* the auction that a bidder will have to make when placing a bid.

Expected surplus The expected surplus *before* the auction that a bidder will obtain when placing a bid.

Exposure problem Risk bidders face when complements items are auctioned if they only win some (but not all) items included in a complementary package. The bidders are exposed to a loss if they bid aggressively for one package that they do not get, losing the super-additivity value of the complete package. Also known as *Aggregation risk*.

First-price Pricing rule under which the winner pays a price equal to his bid for the awarded item. Also known as *Pay-what-you bid*.

First-price sealed-bid auction An auction in which bidders submit bids simultaneously in one round. The winning bidder is the one with the highest bid and pays his bid.

Forward auction An auction conducted by the seller with the goal of selling one or more items to the bidder who submits the highest bid. Also known as *Sell-side auction*.

Google Ad auction In Google Ad auctions, the position of each ad is calculated by multiplying the bid submitted by the advertiser times the quality of the ad.

Incentive compatible An allocation mechanism is incentive compatible if the dominant strategy of all bidders is to make sincere bids.

Income of the bidder The income obtained by a bidder when winning an item, that is, his value of the item.

Independent-private-values A bidder has an independent-private-value when his private value is independent of his rival's values, values are distributed independently.

Interdependent value The estimation made by a bidder of an item's value before the auction. This value can change if the bidder receives signals regarding his rivals' values.

Iterative auction A dynamic auction with multiple discrete rounds, also known as *Multi-round auction*.

Iterative combinatorial auction (ICA) *Combinatorial auction* in which the bidders can submit, throughout a multiple-round process, bids for the items or combinations of items that are being offered. At the end of each round, the seller releases information regarding the provisional winners and the actual prices so that the bidders obtain information regarding the bids of their rivals and can then modify their own bids as the rounds advance.

Japanese auction *Ascending-bid auction* in which the price increases continuously and as the bidders stop being interested in the item, they leave the auction without the possibility of returning.

***k*-double auction** A *sealed-bid double auction* in which the *uniform-price rule* is established but in which supply equals demand over a price range. The k-double auction offers an exchange price within that range.

Last minute bidding Strategy used in deadline auctions in which bidders tend to bid at the last moment of the auction, even in the last minutes or seconds. This strategy is also known as *Sniping*.

Live auction An auction conducted in a given place and time that may be retransmitted on the Internet, enabling the bidders to bid either physically or through the Internet.

Maximum bid In online auctions, buyers may indicate the maximum price they are willing to pay for an item, information that is not made public to their rivals but that allows the system to bid in their name, thereby increasing their bid automatically without exceeding the established limit. It is also known as *Proxy bid*.

Minimum bid The price per item at which the auction begins. Also known as *Starting bid* or *Opening bid*.

Multi-round auction A dynamic auction with multiple discrete rounds, also known as *Iterative auction*.

Multi-unit ascending auction Multi-unit auction of homogeneous items in which the seller sets a minimum bid per item, and the bidders indicate the number of units that they are willing to purchase at that price. The seller increases the price until the aggregate demand does not exceed supply. This design is also called *Multi-unit English auction*.

Multi-unit descending auction Multi-unit auction of homogeneous items in which the seller sets a relatively high starting price per item, which decreases round by round. Bidders submit bids indicating the number of units they are willing to purchase at each price. The bidders are awarded the items they bid on. The auction ends when there is no excess supply. It is also called *Multi-unit Dutch auction*.

Multi-unit Dutch auction Multi-unit auction of homogeneous items in which the seller sets a relatively high starting price per item, which decreases round by round. Bidders submit bids indicating the number of units they are willing to purchase at each price. The bidders are awarded the items they bid on. The auction ends when there is no excess supply. It is also called *Multi-unit descending auction*.

Multi-unit English auction Multi-unit auction of homogeneous items in which the seller sets a minimum bid per item, and the bidders indicate the number of units that they are willing to purchase at that price. The seller increases the price until the aggregate demand does not exceed supply. This design is also called *Multi-unit ascending auction*.

Multi-unit sealed-bid auction A sealed-bid auction of M homogeneous items in which each bidder submits, in a single round, a bid vector indicating how much he is willing to pay for each item. The seller awards the items to the bidders with the M highest bids. The final price to be paid will depend on the pricing rule that was established: uniform-price, discriminatory, or Vickrey rule.

Nash equilibrium The set of strategies, one for each player of the game, such that each player's strategy is a best response to the other's strategies.

Online auction An auction conducted over the Internet.

Open bid Bid submitted by a bidder in a dynamic auction.

Opening bid The price per item at which the auction begins. Also known as *Starting bid* or *Minimum bid*.

Optimal auction Auction that maximizes the expected revenue of the seller.

OR bidding language In *combinatorial auction* with OR bidding language, bidders can win any number of their bids, the bids are not mutually exclusive.

Overbidding When a bidder submits a bid that is higher than his valuation.

Package auction In a package auction multiple items are offered, and the bidders are allowed to submit as many bids as they want for the items or combinations of items in which they are interested. This mechanism is also known as *Combinatorial auction*.

Pay-what-you bid (PWYB) Pricing rule under which the winner pays a price equal to his bid for the awarded item. Also known as *First-price*.

Pay-your-bid auction Sealed-bid auction of multiple homogeneous items in which each winner pays an amount equal to the sum of the bids placed for the items obtained, also called *Discriminatory auction*.

Penny auction Auctions in which all of the bidders pay the pennies that they have bid, but only one bidder (who submitted the last bid) wins the item.

Periodic double auction Sealed-bid auction in which potential buyers and sellers have one period of time to submit their bids or asks. Then the auctioneer sets the exchanged items and a new auction starts. The auctioneer establishes a new period in which buyers and sellers can submit bids for the next auction.

Position auction Multi-unit auction used frequently by search engines to sell the positions of their ads. The price paid by the advertiser each time a user clicks on his ad is equal to the bid placed by the advertiser in the next lower position, that is, the next highest bid after his.

Price discovery Characteristic of dynamic auctions in which the bidders receive information regarding the progression of prices, which they can use to adjust their bids.

Pricing rule A mechanism established by the seller to set the price to be paid by the winning bidder for the awarded item.

Private value The personal value determined by a bidder on an item before the auction and that is not affected by his rivals' values.

Procurement auction An auction conducted by the government to buy items or services from the private sector. The companies that will provide the items or services will be those that make the lowest offers (see *Reverse auction* or *Buy-side auction*).

Proxy bid In online auctions, buyers may indicate the maximum price they are willing to pay for an item, information that is not made public to their rivals but that allows the system to bid in their name, thereby increasing their bid automatically without exceeding the established limit. It is also known as *Maximum bid*.

Proxy bidder Bids on behalf a bidder in an *ascending proxy auction*. The proxy bidder can be either the seller or an electronic proxy bidder.

Rationing rule A mechanism to address the case in which, in multiple-unit auctions, when varying the price, the demand does not cover the supply. Therefore, the seller could establish a mechanism to proceed in the allocation of all of the items.

Reserve price An option that the seller has to fix a minimum price that the selling price must reach at the end of the auction for him to be required to actually make the sale.

Revealed-preference activity rule *Activity rule* proposed for the clock-proxy auction that gives more flexibility to bidders when there are complements or substitutes items as it allows bidders to exceed their *eligibility points* in order to bid on packages that have become comparatively less expensive.

Revenue Equivalence Theorem Theorem that states that, if bidders have private values and other assumptions hold, the four standard single-unit auctions yield the same expected revenue to the seller.

Reverse auction An auction conducted by the buyer with the goal of buying one or multiple items from the seller who makes the lowest offer. It is also known as *Buy-side auction*.

Risk averse A risk averse bidder prefers to increase his probabilities of winning an item by submitting higher bids even if doing so involves a decrease in his potential surplus.

Risk loving A risk loving or risk seeking bidder tends to submit a low bid because he prefers to have a lower probability of winning an item but a greater potential surplus.

Risk neutral A risk neutral bidder will follow the bidding strategy that maximizes his expected surplus.

Sealed-bid Bid submitted by a bidder in a *single-round* or *sealed-bid auction*.

Sealed-bid auction An auction with only one round in which bidders submit their bids, also known as a *Single-round auction*.

Sealed-bid double auction Two-sided auction in which buyers and the sellers indicate, in a single round, the price at which they are willing to buy or sell an item.

Second-price A pricing rule under which the winner pays a price equal to the second highest bid made for the awarded item.

Second-price sealed-bid auction An auction in which bidders submit bids simultaneously in one round. The winning bidder is the one with the highest bid and pays the second highest bid. It is also called *Vickrey auction*.

Sell-side auction An auction conducted by the seller with the goal of selling one or more items to the bidder who submits the highest bid. Also known as *Forward auction*.

Seller's revenue The income obtained by the seller after awarding an item in an auction; equal to the price to be paid by the winning bidder (or bidders in auctions of multiple items).

Selling price The final price that the winning will pay for the acquired item.

Sequential auctions Auctions of multiple related items that are performed consecutively in time.

Shill bid A bid that can be made by the seller through multiple identities to shift the auction in his favor. This practice is usually prohibited. It is also known as *Bid padding*.

Silent auction Simultaneous ascending multi-unit auction in which the items are simultaneously offered in a room, and the bidders place their bids. The price increments are not announced, so bidders must pay attention to the highest standing bid. The items are awarded simultaneously at the closing time established by the seller.

Simultaneous ascending auction (SSA) Auction format to award multiple related items simultaneously in several rounds. The seller sets a starting price per item, which increases round by round as bidders submit their bids. The auction ends when there are no new bids. This auction is also known as the *Simultaneous multiple round auction (SMR auction)*.

Simultaneous ascending clock auction *Simultaneous ascending auction* (SAA or SMR auction) in which the seller announces the price per item in each round and the bidders indicate the quantity they are interested to by at that price. The auction ends when there are no new bids.

Simultaneous auctions Auctions of multiple related items that are performed simultaneously.

Simultaneous multiple round auction (SMR auction) Auction format to award multiple related items simultaneously in several rounds. The seller sets a starting price per item, which increases round by round as bidders submit their bids. The auction ends when there are no new bids. This auction is also known as the *Simultaneous Ascending Auction (SAA)*.

Sincere bidding Strategy in which a bidder bids according to his true value of the item.

Single-round auction An auction with only one round in which bidders submit their bids, also known as a *Sealed-bid auction*.

Sniping Strategy used in deadline auctions in which bidders tend to bid at the last moment of the auction, even in the last minutes or seconds. This strategy is also known as *Last minute bidding*.

Starting bid The price per item at which the auction begins. Also known as *Opening bid* or *Minimum bid*.

Substitutes Two items are substitutes when the marginal value of winning the second item is lower than the marginal value of winning the first one, so the bidder's value for the combination of items is lower than the sum of the individual values.

Surplus of the bidder Equal to the difference between the income (value) and the cost (final price to pay) obtained by a bidder when winning an item.

Symmetric bidders The bidders in an auction are symmetric if all of their valuations are distributed according to the same distribution function F.

Synchronized continuous double auction Continuous double auction composed of r rounds, which are divided into s periods. In each period s there is a fix number of alternating bid/ask (BA) and buy/sell (BS) steps.

Synergies A bidder has synergies for a combination of items if those items are *complements*.

Third-price A pricing rule under which the winner pays a price equal to the third highest bid made for the awarded item.

Two-sided auction Auction mechanism in which multiple buyers and sellers interact, see *Double auction*.

Underbidding When a bidder submits a bid that is lower than his valuation.

Uniform-price rule Pricing rule in which all of the winning bidders of a multi-unit auction or a *double auction* pay the same price for the acquired items.

Uniform-price auction Sealed-bid auction of multiple homogeneous items in which all bidders pay the same price for the acquired items. The price is that for which the demand equals supply.

Value The maximum price that a bidder is willing to pay for an item.

Vickrey auction Sealed-bid auction of multiple homogeneous items in which the winning bidders pay the opportunity cost of the items that they won. For a single-unit auction it would be a *second-price sealed-bid auction*. The approach for multiple heterogeneous items is called the *generalized Vickrey auction* or the *Vickrey–Clarke–Groves mechanism*.

Vickrey rule The Vickrey rule establishes that each bidder pays an amount equal to the opportunity cost of the item won.

Vickrey–Clarke–Groves (VCG) mechanism A VCG mechanism for allocating items states that each bidder has to pay an amount equal to the opportunity cost of the item won. For a single-unit auction, this corresponds to the second-price sealed-bid auction because the opportunity cost of the awarded item is equal to the second highest bid. The VCG mechanism is *incentive compatible*.

War of attrition Auction in which all the bidders pay their bids, but the winner pays only the second highest bid.

Winner determination problem (WDP) Among all bids submitted by all bidders in a *combinatorial auction*, the seller will determine the winning bids that maximize his revenue, this is the combination of bids that maximizes the sum of accepted bids under the constraint that each item is allocated, at most, to one bidder.

Winner's curse The risk incurred by the winner of an auction (specially when bidders have *common values*) of having overestimated the value of the item and having placed a bid that is higher than the item's real worth.

XOR bidding language In *combinatorial auctions* with XOR bidding language, bidders can only win one of their bids; the bids are mutually exclusive.

Analytic index

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