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Wolfgang Marty

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An Introduction to Return and Risk Measurement

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Wolfgang Marty
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Preface

While getting acquainted with a series of topics relating to the banking industry I realized that scientific methods and terminologies describe many aspects of financial markets and that many notions that are used in finance have a scientific background. In the daily activities of the finance industry, we often fail to realize that mathematical concepts are being applied. The use of mathematics ranges from basic arithmetic manipulations such as adding positions on an account or calculating interest payments to sophisticated problems such as investing times series or portfolio optimization. In my view, the connection between science and financial mathematics is particularly apparent in portfolio analysis. The illustration of this fact is one of the goals of this book.

In the traditional asset management industry, I was confronted with statements like the following: "I am a practitioner; I don't need some abstract theory." Nothing could be further from the truth. Using portfolio analysis, I intend to explain why concepts that are based on mathematics can enhance the decision-making process of the participants in the financial community. Furthermore, the approach we describe here helps to standardize the terminology and provides a concise use of the notions.

Financial markets are inundated by data. In other words, they are literally produced continuously. In portfolio analysis, the amalgamation and reconciliation of data is an important aspect. We seek to answer the question "What information is relevant in portfolio analysis?" and to explain the figures that are used to assess a specific portfolio.

In this book, the basic notions in portfolio analysis are introduced using mathematical notation. For more advanced concepts and ideas, we limit our exposition to intuitive descriptions. The book presents not only conventional material but also some current research topics in portfolio analysis. We also refer to relevant literature for further reading.

As the book's approach is independent of market movement and market inferences, it allows us to describe the basic principles more clearly and more accurately. In addition, we only introduce the most essential tools for being able to describe the material in this book, providing ad hoc definitions of specific mathematical notions accompanied by intuitive illustrations.

Preface to the Second Edition

Portfolio Analytics has been published about a year ago. I have received many valuable feedbacks. The contents of the book led to many interesting discussions. I realized that my book has been studied thoroughly. Improving a text is a nice task and I hope that the book gets more readable. I am grateful that the Editor gives me the opportunity for publishing a second edition.

I have published a contribution in the “BOND YEARBOOK 2015” (www.fixed-income.org) entitled “Portfolio Analytics.” The text does not use any formula and gives the main ideas of the book.

A book is only a snapshot of my professional development and I decided to include two additions reflecting my more recent research activities.

Firstly I added a section on Bond Analytics. I discuss the approximation of the solution of the internal rate of return equation. I summarize the findings of the SBC (Swiss Bond Commission) working group “Portfolio Analytics.”

Secondly I added an example in Sect. 3.5. It illustrates the calculation introduced in this chapter and opens further research activities. For instance, the application of this example assesses the difference between normal distributed times series and distributions observed in the marketplace.

Stallikon, Switzerland
27.03.2015

Wolfgang Marty

Conventions

This book consists of five chapters. The chapters are divided into sections and where needed sections are divided into subsections. (1.2.3) denotes formula (3) in Section 2 of Chapter 1. If we refer to formula (2) in Section 2 of Chapter 1, we only write (2); otherwise, we use the full reference (1.2.3). Within the chapters, assumptions, definitions, figures, remarks, tables, theorems, and examples are numerated continually, e.g., Theorem 2.1 refers to Theorem 1 in Chapter 2. These points are numerated continually.

The end of a theorem, lemma, or example is marked with $\diamond$. Square brackets [ ] indicate references. The details of the references are provided at the end of the book.

Acknowledgments

This book is based on several courses and seminars held at different institutions. The transformation of a series of presentations into a book is a challenge that should not to be underestimated. The book has been written over several years and reflects my own professional development. However, it could not have been completed without the continuing support of Dr. Beat Niederhauser and Miriam Munari.

The basic ideas of this book go back to my long collaboration with Prof. Berc Rustem and Dr. Stefan Illmer.

I am grateful to Dominik Büsser and Petar Ilic for reviewing the first draft. Many discussions throughout my daily activities contributed substantially to the material presented here. The thoughts and ideas of the following colleagues (in alphabetical order) also substantially enriched this book: Rolf Bertschi, Florian Böhringer, Ralph Häfliger, Thomas Kellersberger, Wojciech Marcinczyk, Elias Mulky, Kurt Oberhänsli, Dominik Studer, and Thomas Widmer. Thanks also go to Matt Fentem (www.translation-hotline.com) for copyediting the work.

The ongoing discussions with Prof. Janos Mayer helped me concerning mathematical aspects of this book. In addition, I was able to establish an excellent platform with Yuri Shestopaloff for exchanging thoughts on different issues in performance measurement.

Contents

Chapter 1
Introduction

A portfolio is a set of investments. The ***return*** on a portfolio is probably the most important information for the investor. In most cases the investor has an intuitive perception of the return. If the value of the portfolio rises the return is positive, and if the value of the portfolio declines the return is negative. However, there is no unique method for calculating the return. Discussing different approaches to return calculation is one of the main aspects of this book.

Generally speaking, prices are provided by the market or by theoretical considerations. In the following ***we assume*** that the prices of the different investments are available and consequently that it is possible to calculate the return. Returns can be calculated on the basis of past prices on the marketplace (ex post), of actual prices or of projected prices (ex ante) (see Fig. 1.1).

The return depends of the input data, and the more frequent the available data is, the more precisely the return on the investments and the portfolio can be monitored. ***Chapter 2*** is concerned with the two main concepts for measuring the return. The ***first concept*** starts with the evaluation of the portfolio, parts of the portfolio or an investment at different points of time in the past. The return is then calculated between two specified times in advance, i.e., it always refers to a time span. A value for a return is always accompanied by a time span. For instance, a return over multiple years is usually annualized. This concept neutralizes external cash flows in the portfolios. Thus, it is appropriate for comparing portfolio managers, i.e., it provides a basis for peer analysis. The unit of the return is percentage. This concept is the subject of Sect. 2.3, entitled ***Time-weighted rate of return***. Examples are e.g. the total return on an investment and considerations of profit and loss.

The ***second concept*** is based on an ***arbitrage relationship***, i.e., a condition that avoids a situation involving profit without risk. We refer here to the saying: 'There is no free lunch in the financial market.' The corresponding return measurement is based on the value of portfolio, parts of the portfolio or an investment at one point in time. This concept includes cash flow and it is thus the return from the client's perspective. Profit and loss are also considered. This concept is the subject of Sect. 2.4, entitled ***Money-weighted rate of return***. The ***Internal rate of return (IRR)***

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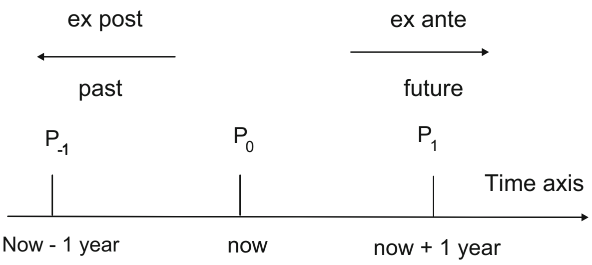

Fig. 1.1 The setting for portfolio analytics

equation is an example of the MWR approach. TWR and MWR are applied on an absolute and a relative basis. We discuss the relationship between TWR and MWR in Sect. 2.5.

Chapter 3 starts with descriptions of the different types of risk. With the development of portfolio analysis, ***risk*** considerations have become more influential. Here we assess the risk of a portfolio with the elementary measure ***standard deviation*** used throughout in statistics; in other words we quantify the risk using a value for the fluctuation of the return. Intuitively this means that if the return is constant there is no risk and the greater the risk value is, the greater the uncertainty of the return is.

As will be shown in Chaps. 2 and 3, the return and the risk are considered on an absolute and on a relative basis. The ***tracking error*** originally stems from control theory and is a relative risk measure. We will see that the concept of returns is easier to understand than the concept of risk. Returns are additive, i.e., when referring to the weights of the investments, the sum of the return on investment adds up to the returns on the portfolio. Adding risks of investments obeys a generalization of the Pythagorean theorem, which states that under the assumption that a and b are at right angles to one another, the sum of a^2 and b^2 is equal to c^2 (Fig. 1.2).

The following statement explains the important notion of ***performance***:

Performance encompasses return and risk

Remark 1.1 In physics, power is introduced through work by unit of time and in the transition from physics to finance, performance is analogous to power, just as return is to work and risk is to time.

Performance deals with the past, the present and the future, as seen in Fig. 1.1. The time axis and the Price P_j ($j = -1, 0, 1$) of an asset in the past are shown on the left side and the potential outcome in the future on the right side. In the past we deal with statistics, whereas in the present and the future we are confined to applying models that provide forecasts and projections. The calculations have to reflect cash flows such as dividends from equities or coupons from bonds and liabilities that are typically important for pension funds. Many assets do not have a price history. Estimations of their future behavior can solely depend on future scenarios and a

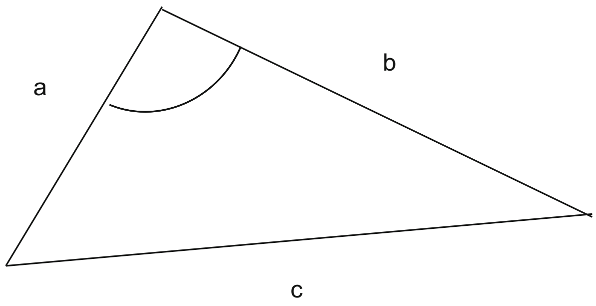

Fig. 1.2 A geometrical interpretation: adding risk

statistical price analysis is not possible. Examples include Initial Public Offerings (IPOs), recently built houses or bonds just launched on the market.

The ***ex post return*** analysis is a time series analysis and an ***ex ante return*** is the same as a return forecast. The ***ex post risk*** analysis of a portfolio is a time series analysis of the returns on the portfolio, i.e., all past changes in the portfolio are reflected in the risk figures. An ***ex ante risk analysis***, however, is based on the actual portfolio and the following mathematical question:

- What are the main drivers of risk in the portfolio?

A set of factors is used to explain the risk of a portfolio. Choosing the appropriate factors requires intensive research. They are modeled by time series and do not depend of the portfolios under investigation. They are estimated on the basis of the underlying market universe of the portfolio. The factors are objective estimations of the markets considered, while the portfolio is subjective. Thus a risk model can also be used for risk management. Updating the risk model is a slow process, i.e., updating once a month is sufficient. Data is mostly based on monthly frequency and in some cases on daily frequency.

We proceed by summarizing the main topics addressed here:

1. We introduce some basic notions for describing and assessing the properties of portfolios.
2. The decomposition of the portfolio into its constituents is discussed.
3. The portfolio is investigated on an absolute basis and relative to a reference portfolio.

The following definition unites the notions of return and risk.

Definition 1.1 ***Portfolio Analytics*** is concerned with quantifying the sources of the return and assessing the risk of a portfolio. It not only measures the evolution of wealth over a certain time period but also provides a comprehensive performance breakdown for specific portfolios.

We proceed with the following remarks:

Remark 1.2 In portfolio analytics we primarily consider variance and covariance as risk measures. They are introduced in Chap. 3.

Remark 1.3 We distinguish between performance measurement on a portfolio level and performance measurements on the ***segment level***, ***constituent level*** and ***multi-period level***. We intend to discuss the complete portfolio analysis problem.

Performance attribution is an important aspect of Portfolio Analytics and is essentially the decomposition of a real number, i.e., $5 = 1 + 4$ or $5 = 2 + 3$. In the two additions just considered the operation usually goes from right to left. In performance attribution, however, the operation goes from left to right and is therefore not unique. We elaborate on this fact throughout the material presented here. A portfolio's return and risk numbers are decomposed. However, what is paramount is that this decomposition reflects the requirements of the client, portfolio manager or adviser. In this book we will above all focus on the return attribution; on the risk side we will only provide an introduction to decomposing the risk.

We present returns attribution for the TWR and for the MWR. We also illustrate the IRR with multiple solutions by means of easy examples. Returns attribution for the TWR has become common practice, but the approach presented here for the MWR is new.

Chapter 4 introduces a holistic approach from a portfolio-based perspective, i.e., return and risk considerations are simultaneously taken into account. We start with the portfolio construction aspect of the investment process and describe the key factors for the portfolio construction. The material presented here is basic and largely conventional, but can also be considered a concise summary of content otherwise found in voluminous textbooks. The basic characteristics of Modern Portfolio Theory (MPT) are described. We present an introduction to absolute and relative optimization, using examples to show how constraints can change optimal portfolios. In ***Chap. 5*** we focus on the investment controlling aspect of the investment process and illustrate a performance review of a portfolio.

Chapter 2
Return Analysis

2.1 Interest Rate, Return and Compounding

Retail accounts give the clients of a bank the opportunity to deposit their money. The client ***lends*** money to the bank, whereas the bank ***borrows*** money from the client. Money on accounts is also called a retail deposit. Normally banks recompense their clients by paying an ***interest rate***, yet these accounts pay little or no interest. Low interest rates can be justified by the fact that the costs of the banks for the infrastructure, staff and paperwork are substantial. In the following we present the basic interest calculation. The ***beginning value BV*** and the ***end value EV*** of a money account are related to the interest rate r:

$$EV = BV(1 + r). \tag{2.1.1}$$

We assume that BV and EV are quoted in US dollars ($). However, our considerations in the following are applicable to any other reference currency. In relationship (1) the timespan between BV and EV is not specified and there is no reference point on the time axis. Sometimes interest rates are also called ***growth rates***. Assuming that one investor $100 in an account and gets $200 out after 10 years and another investor turns $100 into $200 in just a year, the growth rates are obviously different. Interest rates are usually quoted on an annual basis. However, other timespans like days, months or 6-month (semi-annual) periods can also be appropriate in specific situations. We can see that interest rates always refer to a specific timespan. In this section and in Sect. 2.2, the timespan is arbitrary but fixed. We refer to this as the ***base period***.

In most cases the relationship (1) is applied in three versions. *First* if we know BV and r, EV can be calculated. BV is the ***present value*** und EV is the ***future value***. Referring to Fig. 1.1 the future value is on the right-hand side of the present value on the time axis.

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Example 2.1 If $BV = \$100$ and $r = 5\ \%$ then we have

$$EV = \$100 \cdot (1 + 0.05) = \$105.$$

◇

Secondly if EV and BV are known and if we assume that the investor starts from an initial lump sum $BV > 0$, then the solution r of (1) is called ***the return*** expressed by

$$r = \frac{EV - BV}{BV} = \frac{EV}{BV} - 1. \tag{2.1.2}$$

We divide the profit or loss $(EV - BV)$ by the invested capital BV. EV cannot be negative, as a complete loss translates into $EV = 0$, that is, we have

$$EV \geq 0,$$

i.e., by (1)

$$r \geq -1.$$

Example 2.2 If $BV = \$100$ and $EV = \$200$ then we have by (2)

$$r = \frac{EV - BV}{BV} = \frac{\$200 - \$100}{\$100} = 1.$$

We could say the amount is doubling in this easy example, i.e., in percentage

$$r = 100\%.$$

◇

Thirdly if EV and r with $r > -1$ are given we have

$$BV = \frac{EV}{1 + r}.$$

By using the ***discount factor d*** defined by

$$d = \frac{1}{1 + r} \tag{2.1.3}$$

we find

$$BV = d\,EV.$$

Broadly speaking, interest rates with the accompanying discount factors translate into money amounts through time.

Definition 2.1 ***Compounding*** is the reinvestment of the income to earn more income in the subsequent periods. If the income and the gains are retained within the investment vehicle or reinvested, they will accumulate and contribute to the starting balance for each subsequent period's income calculation.

We proceed by assuming that the interest is distributed twice and is reinvested at the same interest rate in the middle of the period. We consider

$$\mathrm{EV}_2 = \left[\mathrm{BV} \cdot \left(1 + \frac{r}{2}\right)\right] \cdot \left(1 + \frac{r}{2}\right).$$

If interest is distributed n-times per period we find

$$\mathrm{EV}_n = \mathrm{BV}\left(1 + \frac{r}{n}\right)^n. \tag{2.1.4}$$

We define e by the limit

$$\lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n = e.$$

The number e appears in many contexts in science and is named after the mathematician who discovered it, Leonard Euler. The 14-digit numerical value is

$$e = 2.71828'\ 18284'\ 5905.$$

The faculty n! is defined by

$$n! = n(n-1)\,(n-2)\ldots..$$

In order to calculate e numerically it is favorable to use the series

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \ldots$$

and find

$$\lim_{n\to\infty}\left(1 + \frac{r}{n}\right)^n = e^r$$

which yields

$$\mathrm{EV}_\infty = \mathrm{BV}\lim_{n\to\infty}\left(1 + \frac{r}{n}\right)^n = \mathrm{BV}\ e^r. \tag{2.1.5}$$

We see that the end values $EV_1, EV_2, \ldots, EV_\infty$ increase with the compounding frequency. EV_∞ is the end value (see page 7) using ***continuously compounding*** as the money is reinvested momentarily. The above formulae assume that the beginning value BV, the interesting rate r and the compounding assumption are given and the ending values $EV_1, EV_2, \ldots, EV_n, \ldots, EV_\infty$ are computed.

Example 2.3 For $r = 0.05$ (=5 %) annually we get in decimals and $BV = \$100$

$$\begin{aligned} EV_1 &= \$105.000000 \text{ (Annual)},\\ EV_2 &= \$105.062500 \text{ (Semi-annual)},\\ EV_4 &= \$105.094534 \text{ (Quarterly)},\\ EV_\infty &= \$105.127110 \text{ (Continuous)}. \end{aligned}$$

◇

If the beginning value BV, the ending value EV and the number of compounding annually are given, the underlying interest is based on (4)

$$r(n) = \left(\sqrt[n]{\frac{EV}{BV}} - 1\right) n.$$

Consistency with (2) yields $r = r(1)$. If the beginning value BV and the ending value EV are given, the underlying return is found based on (5)

$$r_\infty = \ln \frac{EV}{EV},$$

where ln is the natural logarithm, i.e., the logarithm for the basis e.

In the following we look at N annual base periods with $n = 1$ in the above formulae and distinguish between different ways of compounding. ***Simple or arithmetic simple return*** means that the interests are added

$$EV_s = BV \cdot (1 + N \cdot r). \tag{2.1.6}$$

We use simple interest calculation if the earned income is withdrawn from the account. ***Geometric compounding*** is mostly used in practice

$$EV_c = BV \cdot (1 + r)^N. \tag{2.1.7}$$

For $N = 1$ (6) is the same as (7). In the following we denote by $\mathbf{R}^1$ the set of real numbers. Then the continuous versions, i.e., the formulae for arbitrary $t \in \mathbf{R}^1$ are

$$EV_s(t) = BV \cdot (1 + t \cdot r), \quad \text{or} \tag{2.1.8a}$$

$$EV_c(t) = BV \cdot (1 + r)^t. \tag{2.1.8b}$$

We illustrate (8) in Table 2.1 with $r = 0.01$ (=10 %) for $t \in [0, 1]$

Table 2.1 Arithmetic versus geometric

Time	(8a)	(8b)	(8a) – (8b)
0.0	1.000000	1.000000	0.000000
0.1	1.010000	1.009577	0.000423
0.2	1.020000	1.019245	0.000755
0.3	1.030000	1.029006	0.000994
0.4	1.040000	1.038860	0.001140
0.5	1.050000	1.048809	0.001191
0.6	1.060000	1.058853	0.001147
0.7	1.070000	1.068993	0.001007
0.8	1.080000	1.079230	0.000770
0.9	1.090000	1.089566	0.000434
1.0	1.100000	1.100000	0.000000

Remark 2.1 We consider (8a) and (8b) for $t \in [0, 1]$. For no compounding we have

$$EV_s(t) = BV(1 + t \cdot r), \quad \forall t \geq 0$$

and we assume that EV (see also Example 2.2) is achieved by means of continuous compounding

$$EV(t) = BVe^{r_1 t}.$$

For $t = 1$ there follows from (8)

$$r_1 = \ln(1 + r)$$

thus

$$r_1 t = \ln(1 + r)t, \quad \forall t \geq 0 \tag{2.1.9}$$

and we conclude

$$EV(t) = BVe^{\ln(1+r)t} = BVe^{\ln[(1+r)]^t} = BV \cdot (1 + r)^t, \quad \forall t \geq 0.$$

In (8a) there is no compounding and in (8b) we have continuous compounding and by consistency (8a) and (8b) are the same for $t = 1$. A similar formula to (9) can be derived for n-arbitrary compounding periods on the interval $t \in [0, 1]$. For $t > 1$ the ending values are different because of different compounding.

Remark 2.2 Considering semi-annual compounding by

$$\left(1 + \frac{r}{2}\right)^2 \tag{2.1.10}$$

is faulty, as the time is scaled by 2. In other words a semi-annular interest stream of \$1 at the end of the year is

$$\left(1+\frac{r}{2}\right)\ (1+r_1)^{0.5}-1+\frac{r}{2}$$

where r_1 reflects an assumed reinvestment of the first cash flow. This formula assumes that the proceeds are paid out after the first half year and the interest is r_1 for the second half of the year.

$$EV=\left(\left(1+\frac{r}{2}\right)\ (1+r_1)^{0.5}+\frac{r}{2}\right)BV.$$

Example 2.4 (semi-annual coupon) We consider a semi-annual bond with coupon $C=8\ \%$, yield to maturity $r=4\ \%$ and time to maturity $t=0.5$ years. Referring to discount factor (3) the coupon stream in the first year based on (8b) and (10) after half a year is

$$\frac{\frac{C}{2}}{(1+r)^{0.5}}=3.922323,\ \frac{\frac{C}{2}}{1+\frac{r}{2}}=3.921569$$

and after 1 year is

$$\frac{\frac{C}{2}}{1+r}=3.846154,\ \frac{\frac{C}{2}}{\left(1+\frac{r}{2}\right)^2}=3.844675.$$

We note that they are not the same. The size of the difference depends of the values for C and r.

$\diamond$

Furthermore continuous compounding

$$EV=BV\cdot e^{rN}$$

is mostly used in the academic literature because it allows us to formulate theoretical contexts in an elegant and efficient way. The continuous version for arbitrary time $t\in\mathbf{R}^1$ is

$$EV=BV\cdot e^{rt}.$$

2.2 Return Contribution and Attribution

In the following we start by examining a set of investments like stocks.

Definition 2.2 A ***portfolio*** is a set of investments.

The building blocks of portfolio analysis are investments. On the time axis we consider ***a beginning portfolio value PB*** and ***an end portfolio value PE***.

Definition 2.3 ***The simple or arithmetic rate of return r_p of a portfolio P*** is

$$r_P = \frac{PE - PB}{PB} = \frac{PE}{PB} - 1, \tag{2.2.1}$$

i.e., it is measured as the change of the portfolio value relative to its beginning value over a pre-specified timespan in the past.

Remark 2.3 By multiplying PE and PB by the scalar $\lambda \in \mathbf{R}^1$ we have

$$r_P = \frac{PE - PB}{PB} = \frac{\lambda \cdot PE - \lambda \cdot PB}{\lambda \cdot PB}.$$

Thus we conclude that r_P is invariant by $\lambda \in \mathbf{R}^1$. Using

$$\lambda = \frac{1}{PB}$$

we conclude that we can assume without loss of generality $PB = 1$. Thus r_P does not depend on the size of the portfolio.

Remark 2.4 (2.1.2) and (1) express the same idea. Assuming that an interest rate is given, however, (2.1.1) is applied much more often than (2.1.2) and (1) is prominent in portfolio analysis, as PE and PB are measured on the marketplace by calculating the return.

Example 2.5 A portfolio grows in value from 80 to 100. What is its return r_P in decimals and in %?

$$r_P = \frac{100 - 80}{80} = 0.25, \text{ which is in percentage equal to } 25\ \%.$$

In this formula we assume that there is no cash flow between PB and PE. Furthermore we did not specify how long it takes to achieve this return. Is this the return over a day, a month or a year?

◇

Definition 2.4 The ***investment universe*** is given by the securities the portfolio manager is allowed to invest in.

In the following we consider a given investment universe and a portfolio of n stocks with Price PB_j, $1 \leq j \leq n$, PE_j, $1 \leq j \leq n$ with units N_j from the investment universe. The values of the portfolio are

$$PB = \sum_{j=1}^{n} N_j \cdot PB_j, \quad PE = \sum_{j=1}^{n} N_j \cdot PE_j, \quad \text{or} \tag{2.2.2}$$

which yields by (1)

$$r_P = \frac{\sum_{j=1}^{n} N_j \cdot PE_j - \sum_{i=1}^{n} N_j \cdot PB_j}{\sum_{j=1}^{n} N_j \cdot PB_j} = \frac{\sum_{j=1}^{n} N_j \cdot \left(PE_j - PB_j\right)}{\sum_{j=1}^{n} N_j \cdot PB_j} = \frac{\sum_{j=1}^{n} N_j \cdot PB_j \cdot \frac{\left(PE_j - PB_j\right)}{PB_j}}{\sum_{j=1}^{n} N_j \cdot PB_j} \tag{2.2.3a}$$

We define for j = 1, 2,. . ..,n the weights

$$w_j = \frac{N_j \cdot PB_j}{\sum_{i=1}^{n} N_i \cdot PB_i}, \quad j = 1, 2, \ldots, n \tag{2.2.3b}$$

and the return r_j of the stock j

$$r_j = \frac{PE_j - PB_j}{PB_j}, \quad j = 1, 2, \ldots, n\,. \tag{2.2.3c}$$

By inserting (3b) and (3c) in (3a) we find that the ***(absolute) return r_P of a portfolio P*** is the weighted average of the returns:

$$r_P = \sum_{j=1}^{n} w_j r_j = w_1 r_1 + w_2 r_2 + \ldots + w_n r_n. \tag{2.2.4}$$

The terms $w_j\ r_j,\ 1 \leq j \leq n$ are called the ***return contribution***. They are the contribution of the return on the asset j to the return on the portfolio, whereas the returns r_j allows us to compare the returns between the assets j because they are ***non-weighted*** or ***unweighted*** returns.

Remark 2.5 With (3b) we introduce the weights at the beginning of the period. From (3b) there follows

$$\sum_{j=1}^{n} w_j = \sum_{j=1}^{n} \frac{N_j \cdot PB_j}{\sum_{i=1}^{n} N_i \cdot PB_i} = 1. \tag{2.2.5a}$$

If for instance $n = 2$ and $w_1 = 0.3$ there follows $w_2 = 0.7$. In portfolio theory this condition is thus called the ***budget constraint***. If there is no short selling we have the ***non-negativity constraint***

$$w_j \geq 0, j = 1, 2, \ldots, n. \tag{2.2.5b}$$

From (5) follows

$$0 \leq w_j \leq 1, \quad j = 1, 2, \ldots, n.$$

Example 2.6 (absolute contribution) We consider a portfolio with stocks A, B and C and the following price movements:

The returns on the stocks A, B, C in the last column of Table 2.2 follow from the beginning value and the ending value by using (2.1.2). Furthermore we assume the weights

$$w_1 = 15\%, w_2 = 25\%, w_3 = 60\%.$$

The absolute return contribution is shown in the last column of Table 2.3. This table summarizes the analysis of the return contribution

◇

We see that the elements or the basis of the return analysis are the returns on the individual securities in the portfolio. In an absolute return contribution analysis the return is decomposed according to a breakdown of the investment universe. Often the return is broken down into the return on the different countries or the different industries in the investment universe.

Table 2.2 Return calculation

Stock	Beginning	End	Return (%)
A	120	160	33.3
B	100	120	20.0
C	30	50	66.7

Table 2.3 Absolute return decomposition

Stock	Weights (%)	Return (%)	Absolute contribution (%)
A	15	33.3	5
B	25	20.0	5
C	60	66.7	40
Portfolio return			50

Definition 2.5 A ***segment*** is a set of investments in the investment universe.

We illustrate (4) by considering a breakdown of the universe into two segments. One part consists of the first m securities with returns $r_1,\ldots,r_m$ and in the second part we have the n − m securities with returns $r_{m+1},\ldots,r_n$. Using the abbreviations

$$W_1 = \sum_{j=1}^{m} w_j, W_2 = \sum_{j=m+1}^{n} w_j,$$

$$R_1 = \frac{w_1 r_1}{\sum_{j=1}^{m} w_j} + \frac{w_2 r_2}{\sum_{j=1}^{m} w_j} + \ldots + \frac{w_m r_m}{\sum_{j=1}^{m} w_j} \quad \text{and}$$

$$R_2 = \frac{w_{m+1} r_{m+1}}{\sum_{j=m+1}^{n} w_j} + \frac{w_{m+2} r_{m+2}}{\sum_{j=m+1}^{n} w_j} + \ldots + \frac{w_n r_n}{\sum_{j=m+1}^{n} w_j}$$

(4) is the same as

$$r_P = W_1 \cdot R_1 + W_2 \cdot R_2. \tag{2.2.6}$$

Here we see that (6) is the special case $n = 2$ in (3). R_1 and R_2 are the return on the two segments. If, for instance, $R_1 > R_2$ the return on segment 1 is higher than that on segment 2. The overall return on the portfolio is then the weighted sum of the returns on the two segments. We summarize as follows

The return is linear, i.e., the return of a portfolio is equal to the sum of the weighted returns of its investments.

In return contribution it is assumed that we only measure the return of a single portfolio manager or in other words it is a one-dimensional breakdown of the return.

This is determined for instance by the orientation of a fund or by the contract between the client and an asset management company. If the portfolio manager's goal is to maximize the absolute return, the return analysis has to account for this. Here we have an important principle in return analysis or more general in performance measurement. The analysis of the performance has to account for the investment strategy of the portfolio manager. Given appropriate risk consideration, in an absolute return strategy the portfolio manager tries to maximize the absolute return on his or her portfolio. We see that the elements of the return analysis are the returns on the individual securities in the portfolio. In an absolute return contribution analysis the return is decomposed according to a breakdown of the investment universe into segments. Often the return is broken down into the returns of the different countries or the different industries in the investment universe. In order to evaluate the return on an absolute contribution we need:

Definition 2.6 A ***risk-less asset*** is defined as an investment that has no capital loss over a predetermined period and the ***risk-free return r_f*** is the return that we can earn on such an investment.

Typical examples of risk-less assets are Treasury bills or certificates of deposit. They are considered to be risk-less, as the issuers of these money market instruments are guaranteed to repay the money the investor has invested and to pay them the interest rate due.

Definition 2.7 The ***excess return*** is the difference between the return on a portfolio r_P and the risk-free return r_f

$$r_P - r_f.$$

If we consider a relative return analysis the return is measured against a reference portfolio.

Definition 2.8 A reference portfolio is called a ***benchmark portfolio*** or simply a ***benchmark***.

In the finance industry there are two kinds of benchmarks. For equity or fixed income portfolios most investors consider ***industry-standard benchmarks***. In the equity world the most important index providers are MSCI and FTSE and in the fixed income area names like J.P. Morgan, Citigroup, Barclays and Merrill Lynch are market leaders in producing indices. For balanced portfolios, i.e., portfolios that consist of different asset classes, investors mostly use ***tailor-made benchmarks***. In most cases the benchmark is a mix of equity, bond and money market indices. The proportion of the different asset classes is subject to discussions between the client and the portfolio manager. It is important to realize that the performance of the portfolio manager is measured against the benchmark. Thus a ***relative return contribution*** has to be considered. Relative return measures the contributions of the securities in the portfolio relative to the contributions of securities in the benchmark.

We proceed by adopting the following notation for a benchmark portfolio. We consider n investments with beginning price BB_i, $1 \leq i \leq n$, and end prices BE_i, $1 \leq i \leq n$ with M_i units. The values of the benchmark are

$$BB = \sum_{i=1}^{n} M_i \cdot BB_i, \quad BE = \sum_{i=1}^{n} M_i \cdot BE_i \quad \text{resp}\ldots \tag{2.2.7a}$$

Similar to (3b), with

$$b_j = \frac{M_j \cdot BB_j}{\sum_{i=1}^{n} M_i \cdot BB_i}, \quad 1 \leq j \leq n \tag{2.2.7b}$$

we have the weights of the benchmark securities in the benchmark. The return r_B on the benchmark is

$$r_B = \sum_{j=1}^{n} b_j \; r_j. \tag{2.2.7c}$$

Definition 2.9 With (4) and (7) the ***arithmetical relative return ARR*** is the difference

$$ARR = r_P - r_B.$$

With (7) we have for the relative return

$$ARR = \sum_{j=1}^{n} \left(w_j - b_j\right) \; r_j. \tag{2.2.8}$$

From

$$0 = \sum_{j=1}^{n} \left(w_j - b_j\right)$$

we have

$$0 = \sum_{j=1}^{n} \left(w_j - b_j\right) \; \left(-r_B\right)$$

and by adding (8) there follows

$$ARR = \sum_{j=1}^{n} \left(w_j - b_j\right) \; \left(r_j - r_B\right). \tag{2.2.9}$$

The decomposition (8) is called the ***Brinson-Hood-Beebower (BHB)*** method and the decomposition (9) is called the ***Brinson-Fachler (BF)*** method.

From (8) and (9) we can see that return analysis is not a unique process and in fact in many practical situations (9) is used instead of (8). In the decomposition (9) if the term of the sum in (9) is positive, the portfolio manager contributed positively to the relative return; otherwise, they did so negatively. Thus (9) makes a meaningful discussion of the relative return possible. Summarizing, in return contribution we analyze the portfolio return in view of different investment universes. It is a one-dimensional process.

If the portfolio manager is allowed to invest in assets outside the benchmark, i.e., the benchmark weights b_j in (7b) are zero, then (4), (8) and (9) can also be used for the return analysis. However, stocks outside the benchmark contribute with the absolute return to the relative return on the portfolio. From the portfolio manager's view, investments outside the benchmark are thus considered to be risky.

Example 2.7 (relative contribution) We consider the same three stocks as in Example 2.6 and the universe of the portfolio and the benchmarks are the same with

$$w_1 = 15\%,\ w_2 = 25\%,\ w_3 = 60\%$$

in (3b) and

$$b_1 = 25\%,\ b_2 = 25\%,\ b_3 = 50\%$$

in (7b) (see Table 2.4). Inputs are in bold. Table 2.4 shows two decompositions of the relative return 1.00 % of the portfolio versus the benchmark. The last line shows the absolute return on the portfolio to be −1.50 % and on the benchmark to be −2.50 %, i.e., we find that the portfolio outperforms the benchmark by 1.00 %. The value added indicates that stocks A and C are underperforming the benchmark, while B is outperforming it. However, Stock B has a neutral position and does not contribute to the relative return. As A is an underweighting position versus the benchmark, the investment decision is favorable, that is, the contribution is positive, namely 2.00 % on an absolute basis (BHB) and 1.75 % on a relative basis (BF). As C is an overweighting position versus the benchmark, the investing decision is unfavorable, that is, the contribution is negative, namely −1.00 % on an absolute basis (BHB) and −0.75 % on a relative basis (BF). Table 2.4 shows the BHB and BF decompositions.

If we consider a capitalization-weighted benchmark, the benchmark weights are the capitalization weights of the securities in the benchmark portfolio. However, from (7) we can see that they can be chosen differently from capitalization weights, that the benchmark portfolio is of the same nature as the investment portfolio, and that the elements of the return analysis are the securities in the investment universe.

◇

We proceed by explaining the analysis of the portfolio manager's performance. We attribute the return to the appropriate decision makers and to the different steps

Table 2.4 Relative return decomposition

		Portfolio	Benchmark	Value added $r_j - r_B$	Under/over weight	Contribution BHB	Contribution BF
	Return	Weight	Weight				
A	**−20 %**	**15 %**	**25 %**	−17.5 %	−10 %	2.00 %	1.75 %
B	**30 %**	**25 %**	**25 %**	32.5 %	0	0.00 %	0.00 %
C	**−10 %**	**60 %**	**50 %**	−7.5 %	10 %	−1.00 %	−0.75 %
Return		−1.50 %	−2.50 %			1.00 %	1.00 %

in the investment process. We analyze different factors in the investment process. As in the previous section we distinguish between an absolute and relative attribution:

Definition 2.10 ***Return attribution*** is the decision-oriented decomposition of the return.

Here we limit the discussion to an explanatory illustration for absolute attribution, as the absolute attribution is not widely used. However, as the following example shows, the formulae nonetheless become intricate. The basis for the attribution is:

- The segmentation of the investment universe and (or) the benchmark universe
- The multi-level investment process

Example 2.8 (absolute attribution) We consider the Example 2.6 shown in Table 2.4 and we assume the universes of the portfolio and the benchmark are the same and consist of stock A with weight $w_1 = 15$ %, stock B with weight $w_2 = 25$ % and stock C with weight $w_3 = 60$ %. We introduce a two-step investment process and three portfolio managers (PMs).

- PM1 decides on the country allocation (W_1, W_2),
- PM2 decides on the portfolio in country X (w_1, w_2),
- PM3 decides on the portfolio in country Y (w_3).

We ask what the return of the individual managers is and we have a two-level decision. First the PM1 (country manager) decides on $W_1 = w_1 + w_2$ and $W_2 = w_3$. Referring to the notation in (7) the return r_X and r_Y, respectively of the country benchmark in countries X and Y, respectively is

$$r_X = \frac{b_1}{b_1 + b_2} r_1 + \frac{b_2}{b_1 + b_2} r_2, \quad r_Y = r_3.$$

Using the data from Example 2.6 we find

$$\frac{b_1}{b_1 + b_2} = 0.5, \frac{b_2}{b_1 + b_2} = 0.5, r_X = \ 5\% \text{ and } r_Y = -10\%.$$

PM1's return is measured by

$$r_{PM1} = (w_1 + w_2)\left(\frac{b_1}{b_1 + b_2} r_1 + \frac{b_2}{b_1 + b_2} r_2\right) + w_3 r_3 = 2\% - 6\% = -4\%.$$

r_{PM1} is the return arising from investing in countries X and Y. PM1 has opted for country X, which has an underweight position: a poor choice. It is called ***the asset allocation effect***. The PM2 (country X) decides on w_1 and w_2 and his or her return is

$$r_{PM2} = w_1 r_1 + w_2 r_2 - (w_1 + w_2)\left(\frac{b_1}{b_1 + b_2} r_1 + \frac{b_2}{b_1 + b_2} r_2\right) =$$
$$4.5\% - 0.4 \cdot 5\% \ = \ 2.5\%.$$

r_{PM1} is the return from deciding on the securities in X. PM2 has made the right choice, as he has decided for an underweight in Stock A and an overweight in Stock B. This is called ***the stock picking effect***.

If PM3 (country Y) is not allowed to buy stocks outside the benchmark, there is no decision for PM3 to make and we have

$$r_{PM3} = r_Y = -10\%.$$

We see that the definition of the universe is crucial in return attribution. If PM3 had more choices it could influence the return. Moreover, it is not clear whether the allocation is an asset allocation effect or a sector effect. The decisions of PM1 and PM3 coincide. In the following section we will see that decision makers can have an influence on part of the return.

◇

In the ***return attribution*** it is assumed that the investment process consists of different steps and different decision makers. It is a multi-step process. Again ***absolute*** and ***relative return attribution*** can be considered. In the following we focus on relative return attribution that consists of two layers. As in the return contribution a breakdown of the investment universe like for instance by country or industry is considered. In most cases investors are interested in breakdowns for a country and (or) industry. However, classifications concerning the security capitalization on the equity market or duration buckets in the fixed income area are also possible.

In the following we distinguish between arithmetic attribution (addition) and geometric attribution (product).

2.2.1 The Arithmetic Attribution

In the following $W_j,\ 1 \le j \le n$ n represents the weights of the segments of the portfolio with returns $R_j,\ 1 \le j \le n$ and $V_j,\ 1 \le j \le n$ are the weights of the segments of the benchmark with returns $B_j,\ 1 \le j \le n$. In most cases the market capitalizations are the weights of the segments considered in the benchmark. Following Definition 2.9 the arithmetic relative return is

$$ARR = \sum_{j=1}^{n} W_j \cdot R_j - \sum_{j=1}^{n} V_j \cdot B_j = \sum_{i=1}^{n} \left(W_j \cdot R_j - V_j \cdot B_j\right) \qquad (2.2.10)$$

and consider the identity

$$W_j \cdot R_j - V_j \cdot B_j = \\ (W_j - V_j) \cdot B_j + (R_j - B_j) \cdot V_j + (W_j - V_j) \cdot (R_j - B_j) \tag{2.2.11}$$

(see Fig. 2.1). This identity allows us to discuss the relative return.

We proceed with the first part of the sum and in (10) we refer to Brinson-Hood-Beebower (see (9)) and introduce by A_i^{BHB} in the following the ***asset allocation effect A_j^{BHB} in segment j*** based on

$$A_j^{BHB} = (W_j - V_j) \cdot B_j.$$

By referring to Brinson-Fachler (see (8)) we denote by A_i^{BF} in the following the ***asset allocation effect A_j^{BF} in segment j*** based on

$$A_j^{BF} = (W_j - V_j) \cdot (B_j - r_B).$$

We note that we can consider A_j^{BHB} as the special case when the benchmark return vanishes in A_j^{BF}. The asset allocation effect is the overweight or underweight of the segment of the breakdown of the benchmark. However, the weight of the constituents within the segments stays the same. $A_j^{BHB} > 0$ and $A_j^{BHB} < 0$, respectively show whether the asset manager's decision to overweight or underweight the segment adds positively (negatively) to the return on the portfolio. $A_j^{BF} > 0$ and $A_j^{BF} < 0$, respectively show whether the asset manager's decision to overweight or underweight the segment adds positively (negatively) to the relative return on the portfolio versus the benchmark. As in (8) and (9) at the portfolio level A_j^{BHB} and A_j^{BF} add up to the same:

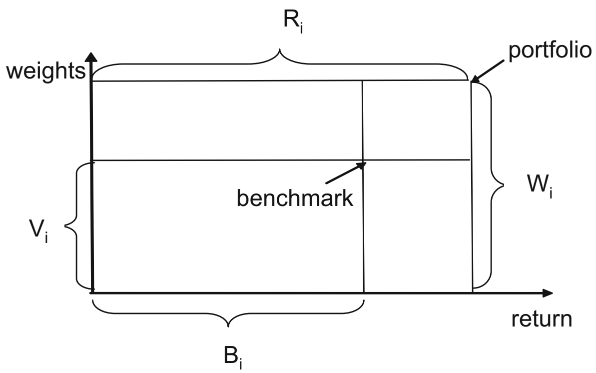


Fig. 2.1 Brinson-Hood-Beebower

$$A_{tot} = \sum_{j=1}^{n} A_j^{BF} = \sum_{j=1}^{n} \left(W_j - V_j\right)\left(B_j - r_B\right) = \sum_{j=1}^{n} A_j^{BHB}. \tag{2.2.12a}$$

This measures the return of the portfolio manager who takes the asset allocation decisions relative to the considered breakdown of the investment universe into account. There are two possibilities for a positive asset allocation effect, namely an overweight of that segment followed by an outperformance or an underweight followed by an underperformance of the segment relative to the benchmark. An analogous statement holds true for a negative asset allocation effect.

Stock selection S_j effect in segment j defined by

$$S_j = \left(R_j - B_j\right) \cdot V_j.$$

measures the effect of the security selection within the universe. We can measure the overall stock selection in the portfolio or the stock selection within a specific segment.

In the relative performance attribution there is also a third component of the return, which is called ***interaction I_j effect in segment j*** and defined by

$$I_j = \left(W_j - V_j\right) \cdot \left(R_j - B_j\right).$$

This interaction effect is a part of the performance that cannot be uniquely attributed to a particular decision maker. For instance if the asset allocation effect and the stock selection effect are positive, then the overall return is not merely the sum of the two components, but returns accumulate and as a result the return is greater. It is important to calculate and to account for the interaction effect, though determining which decision maker is responsible for the interaction effect can be highly subjective.

Taken together, the asset allocation effect, the stock selection effect and the interaction effect are called ***the management effect***.

With

$$W_j \cdot R_j - V_j \cdot B_j = A_j^{BHB} + S_j + I_j$$

and

$$S_{tot} = \sum_{j=1}^{n} S_j, \quad I_{tot} = \sum_{i=1}^{n} I_j \tag{2.2.12b}$$

we find by (10), (11)

$$\text{ARR} = \sum_{j=1}^{n} W_j \cdot R_j - \sum_{j=1}^{n} V_j \cdot B_j = \sum_{j=1}^{n} (A_j + S_j + I_j) =$$
$$\sum_{i=1}^{n} A_j^{\text{BHB}} + \sum_{j=1}^{n} S_j + \sum_{j=1}^{n} I_j = \sum_{i=1}^{n} A_j^{\text{BF}} + \sum_{j=1}^{n} S_j + \sum_{j=1}^{n} I_j =$$
$$A_{\text{tot}} + S_{\text{tot}} + I_{\text{tot}}.$$

We note that in general:

$$W_j \cdot R_j - V_j \cdot B_j \neq A_j^{\text{BF}} + S_j + I_j$$

and summarize as follows:

The BF decomposition does not hold at the segment level, but at the portfolio level the relative return is equal to the three management effects. The BHB decomposition holds at both the segment level and portfolio level.

This concludes our discussion of the BF and the BHB decomposition of the arithmetic relative return. The return attribution is the reverse problem, i.e., the return is known and observable and the decomposition of return is not unique. In the following examples we illustrate different decompositions based on the different user requests in a return attribution system.

Example 2.9 We expand the portfolio in Example 2.7 by adding stock D. In Table 2.5, we find the analysis from Table 2.4 for four stocks. The inputs in Table 2.5 are in bold.

We now consider two segments. One consists of stocks A and B and another consists of stocks C and D. Table 2.6 shows the absolute return on the benchmark and the portfolio.

Table 2.7 illustrates (10)–(12).

This example shows that all investment decisions except for the investment in stock B are favorable and the portfolio outperforms the benchmark by 14.00 %.

◇

Table 2.5 Relative return decomposition on a stock level

		Portfolio	Benchmark	Value added $r_j - r_B$	Under/over weight	Contribution BHB	Contribution BF
	Return	Weight	Weight				
A	**−20 %**	**15 %**	**25 %**	−10.5 %	−10 %	2.00 %	1.95 %
B	**10 %**	**25 %**	**25 %**	30.5 %	0 %	0.00 %	0.00 %
C	**−60 %**	**10 %**	**20 %**	−50.5 %	−10 %	6.00 %	5.95 %
D	**30 %**	**50 %**	**30 %**	30.5 %	20 %	6.00 %	6.10 %
Return		13.5 %	−0.5 %			14 %	14 %

Table 2.6 Segment return

Segment j	W_j (%)	R_j (%)	V_j (%)	B_j (%)
1	40	11.25	50.00	5.00
2	60	15.00	50.00	−6.00
Total		13.50		−0.50

Table 2.7 Relative return decomposition on a segment return

	Relative return (BHB) (%)	BHB (%)	BF (%)	S (%)	IA (%)
	2.000	−0.500	−0.550	3.125	−0.625
	12.000	−0.600	−0.550	10.500	2.100
Total	14.000	−1.100	−1.100	13.625	1.475

Example 2.10 (two investment processes) We assume

$$r_P - r_B = 2.0\%$$

over a month. We decompose the relative return on a balanced or multi-asset-class portfolio in two different ways.

The first decomposition assumes a three-step investment process that starts with the benchmark definition, asset allocation and stock picking. Performing this type of analysis also assumes that one decision maker does both the asset allocation and the stock picking. Following (11) the report of an attribution system over a month might find the asset allocation effect with a return of 3.0 %, the stock picking effect with a return of −0.7 % and the interaction effect with a return of −0.3 %. Then the decomposition is

$$2.0\% = 3.0\% - 0.7\% - 0.3\%.$$

In the second decomposition of the relative return we consider an investment process that is a more complex decision making process. We assume that the investment process consists of six steps and that we have the following returns: benchmark definition, the internal benchmark selection with a return of 0.3 %, an asset allocation effect with a return of 1.0 %, fixed income asset allocation with a return of −0.5 %, an equity asset allocation with a return of 0.8 % and portfolio implementation or stock picking with a return of 0.7 %. The interaction has a return of −0.3 %. Then we have

$$2.0\% = 0.3\% + 1.0\% - 0.5\% + 0.8\% + 0.7\% - 0.3\%.$$

Looking at the figures, in the first decomposition one would conclude that the balanced account manager is a poor stock picker, but if the second investment process is correct, on the contrary the balanced account manager would be a good stock picker. Neglecting the real investment process and not reflecting it in the performance attribution can potentially skew the interpretation and lead to the wrong conclusions.

◇

Example 2.11 (two different benchmarks) We consider a European equity mutual fund for which MSCI Europe is a formal external benchmark. We assume that over a certain period the relative return of 2.0 % on the portfolio versus MSCI Europe is decomposed in the asset allocation effect with an assumed return of 3.0 %, the stock picking effect with a return of −0.7 % and the interaction effect with a return of −0.3 %, i.e.,

$$2.0\% = 3.0\% - 0.7\% - 0.3\%.$$

Comment: If the portfolio manager is measured against his or her competitor then a benchmark European growth index is an objective measure of the market. However, if an investor wants to know how the market European growth index return was in the past, the return of the portfolio manager is an indication of the European growth index and his/her benchmark ought to be the MSCI Europe.

The product management of the mutual fund company positioned this mutual fund internally as a growth product with the internal benchmark of a European growth index. In order to arrive at an accurate image, the setup of the performance attribution has to be changed so that the excess return versus the external benchmark is split into four effects: (a) the benchmark selection with a return of 1.8 %, (b) the asset allocation effect with a return of −0.5 %, (c) a stock picking effect with a return of 1.0 % and (d) an interaction effect with a return of −0.3 %. This yields the following decomposition

$$2.0\% = 1.8\% - 0.5\% + 1.0\% - 0.3\%.$$

Such a return decomposition ensures that the contribution of the positioning of the product and the contribution of the asset manager can be isolated and assessed. In our case the sign of the asset allocation changed from positive to negative and the stock picking effects changed from negative to positive simply because we changed the relevant benchmark to the European growth index.

◇

Example 2.12 (two different segmentations) Performance attribution software is normally quite flexible in setting up the analysis, so that the performance analysts can choose between decomposing the return by using different segments such as countries or sectors. Depending on the segment chosen to decompose the return, the performance attribution may come up with different management effects. We assume an asset manager of a European equity account and consider the following decomposition of the relative return of 2.0 % into the asset allocation effect, stock selection and interaction effect using different segmentations of the MSCI Europe benchmark.

With a sector approach

$$2.0\% = 3.0\% - 0.7\% - 0.3\%$$

and with a country approach

$$2.0\% = -0.5\% + 2.7\% - 0.2\%$$

we come up with different management effects. Setting up the performance attribution incorrectly may adversely effect the interpretation and lead to the wrong conclusions. This again shows that setting up the performance attribution according to the investment process is essential in order to arrive at a meaningful analysis and valid feedback on the investment process.

◇

In (2) it is assumed that

- The investments in the portfolio relate only to a single currency. An illustration is a fixed income portfolio with bonds income that is denominated in a single currency.
  or
- The return arising from different currencies is not of interest. For instance, the currencies effect of equities of internationally active companies, or of the quotes at different stock exchanges, is of little real interest and a currency attribution is accordingly not requested.

2.2.2 The Geometrical Attribution

The reminder of this section is a preparation for the following section as we consider multiplying the return. Here in b. and c. we consider still one time period and afterwards we investigate multi-time-periods.

Definition 2.11 The ***geometrical relative return GRR*** of the portfolio P and the benchmark B is defined by

$$\mathrm{GRR} = \frac{1 + \mathrm{r_P}}{1 + \mathrm{r_B}} - 1. \tag{2.2.13}$$

Remark 2.6 (absolute return of the portfolio) If $\mathrm{r_B} = 0$ then

$$\mathrm{GRR} = \mathrm{ARR} = \mathrm{r_P}.$$

Remark 2.7 (interpretation of GRR) Based (12) we find

$$\mathrm{GRR} = \frac{1+\mathrm{r_P}}{1+\mathrm{r_B}} - 1 = \frac{\mathrm{r_P}-\mathrm{r_B}}{1+\mathrm{r_B}} = \frac{\dfrac{\mathrm{PE}-\mathrm{PB}}{\mathrm{PB}} - \dfrac{\mathrm{BE}-\mathrm{BB}}{\mathrm{BB}}}{1+\dfrac{\mathrm{BE}-\mathrm{BB}}{\mathrm{BB}}}.$$

According to Remark 2.2 we scale PB or BB such that we can assume PB = BB and it follows:

$$\mathrm{GRR} = \frac{\frac{\mathrm{PE}-\mathrm{BE}}{\mathrm{PB}}}{\frac{\mathrm{BE}}{\mathrm{PB}}} = \frac{\mathrm{PE}-\mathrm{BE}}{\mathrm{BE}} = \frac{\mathrm{PE}}{\mathrm{BE}} - 1.$$

We conclude that

- GRR can be expressed by the end value of the portfolio and the benchmark; more specifically, GRR is the rate of return of the portfolio ending value and the benchmark ending value.

Example 2.13 (geometrical versus arithmetic relative return) Let us assume that the difference between the portfolio and the benchmark is

$$\mathrm{ARR} = 4.00\%.$$

Firstly this can be found with $\mathrm{r_P} = 8.00\ \%$ and $\mathrm{r_B} = 4.00\ \%$. This yields

$$\mathrm{GRR} = 3.85\%.$$

Secondly assuming $\mathrm{r_P} = 12.00\ \%$ and $\mathrm{r_B} = 8.00\ \%$ we find

$$\mathrm{GRR} = 3.70\%.$$

The arithmetic relative return is the same for both cases. The user of the geometrical relative return can argue that with a higher benchmark return a higher portfolio return is easier to achieve than a lower portfolio return with a lower benchmark return. The geometrical relative return GRR is relative to the benchmark return.

◇

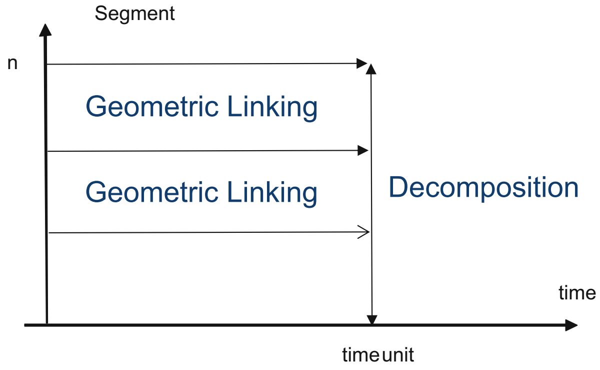


Fig. 2.2 Decomposing on segment level

Referring to e.g. [24] and Fig. 2.2, we proceed with the decomposition of GRR at the segment level over a time unit. For the notation in the following definitions we refer to the notation in the previous paragraph **a** 'arithmetic attribution'.

The asset allocation effect for segment j, j = 1,....,n is defined by

$$A_j = (W_j - V_j) \cdot \left(\frac{1 + B_j}{1 + r_B} - 1 \right) \tag{2.2.14a}$$

and ***the stock selection effect*** for segment j, j = 1,....,n is defined by

$$S_j = W_j \cdot \left(\frac{1 + R_j}{1 + B_j} - 1 \right) \cdot \frac{1 + B_j}{1 + r_S}. \tag{2.2.14b}$$

We continue by defining the return r_S.

Definition 2.12 A ***notional (or reference) portfolio*** is a synthetic portfolio that allows us to decompose the relative return.

We have already considered notional portfolios in the BHB and BF decomposition and here such a notional portfolio with return r_S is

$$r_S = \sum_{j=1}^{n} W_j B_j \tag{2.2.14c}$$

and the ***geometrical relative return GRR$_j$*** for segment j is defined by

$$1 + GRR_j = (1 + A_j) \cdot (1 + S_j).$$

We proceed with

$$A_{tot} = \sum_{j=1}^{n} A_j = \sum_{j=1}^{n} (W_j - V_j)\ \left(\frac{1+B_j}{1+r_B} - 1\right) = \\ \sum_{j=1}^{n} (W_j - V_j)\ \left(\frac{r_B - B_j}{1+r_B}\right) = \\ \frac{1}{1+r_B}\sum_{j=1}^{n} (W_j - V_j)(B_j - r_B) = \\ \frac{r_S - r_B}{1+r_B} = \frac{1+r_S}{1+r_B} - 1$$

and

$$S_{tot} = \sum_{j=1}^{n} S_i = \sum_{j=1}^{n} W_j \cdot \left(\left(\frac{1+R_j}{1+B_j} - 1\right) \cdot \frac{1+B_j}{1+r_S}\right) = \\ \sum_{j=1}^{n} W_j \cdot \left(\left(\frac{R_j - B_j}{1+B_j}\right) \cdot \frac{1+B_j}{1+r_S}\right) = \\ \frac{r_P - r_S}{1+r_S} = \frac{1+r_P}{1+r_S} - 1. \tag{2.2.15}$$

From (13) we have

$$1 + GRR = \frac{1+r_P}{1+r_B} = \frac{1+r_P}{1+r_S}\ \frac{1+r_S}{1+r_B}.$$

We define the asset allocation effect of the portfolio as

$$A_{tot} = \frac{1+r_S}{1+r_B} - 1$$

and the selection effect of the portfolio as

$$S_{tot} = \frac{1+r_P}{1+r_S} - 1.$$

Then we have by (15)

$$1 + GRR = (1 + A_{tot})\ (1 + S_{tot}).$$

Remark 2.8 We do not propose any relationship between GRR and GRR_j, $j = 1,....,n$ and the decomposition of A_{tot} and S_{tot} is arithmetic and not geometric. In addition our consideration in (14) and (15) are not multi periodic.

2.2.3 Currency Attribution Basics

The return on the portfolio depends of the currency and can be expressed in different currencies. In the following we discuss the relation between the different returns.

Definition 2.13 The ***base currency*** relates to the investor's accounting currency.

Definition 2.14 The ***local currency*** relates to the currency in which the investment is made.

We start with an investment value PB in the base currency. We assume an investment in a market, the return on which r_L is not measured in the base currency, but in the local currency.

Definition 2.15 A ***currency exchange rate*** translates prices from one currency in another currency. More specially we denote by $e_{b,t}$ the translation of the price to the base currency of the portfolio from another currency.

Remark 2.9 Mathematically the currency exchange rate is a scalar referring to a point in time.

The ***return*** r_C from the ***currency exchange rate*** is

$$r_C = \frac{e_{b,E} - e_{b,B}}{e_{b,B}} = \frac{e_{b,E}}{e_{b,B}} - 1. \tag{2.2.16}$$

Starting with a beginning value PB (see Fig. 2.3), the end value of the investment PE is

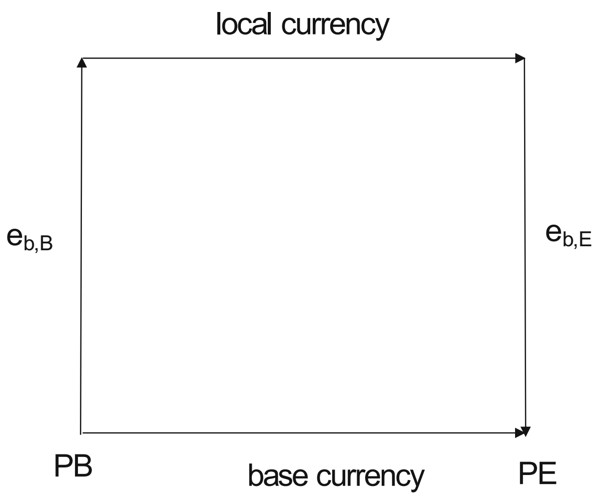


Fig. 2.3 Evaluation in different currencies

$$\mathrm{PE} = \mathrm{PB}\frac{\frac{1}{e_{b,B}}(1+r_L)}{\frac{1}{e_{b,E}}}.$$

By (16) the total return r_T in the base currency is

$$r_T = \frac{\mathrm{PE}-\mathrm{PB}}{\mathrm{PB}} = \frac{\mathrm{PB}\,\dfrac{e_{b,E}(1+r_L)}{e_{b,B}} - \mathrm{PB}}{\mathrm{PB}} = \frac{e_{b,B}(1+r_L)}{e_{b,E}} - 1 = (1+r_L)\ (1+r_C) - 1,$$

i.e.,

$$r_T = r_L + r_C + r_C r_L.$$

The expression (16) is independent of PB. We see that there is a geometric link between the market and the currency return.

Example 2.14 (two bonds in foreign currencies) We consider a CHF investor who buys a bond denominated in Danish kroner and Polish zloty. We want to calculate the return in CHF.

The bonds have the following prices in the beginning

$$P_{B,DKK} = 111.110, \quad P_{B,POL} = 111.160 \tag{2.2.17a}$$

and the end

$$P_{E,DKK} = 111.946, \quad P_{E,POL} = 111.049. \tag{2.2.17b}$$

In Tables 2.8 and 2.9, the bold figures indicate the currency exchange rate against US$ and are market data. In Table 2.9, the exchange rates for Polish zloty and Danish kroner are shown against the US dollar. With the value of Table 2.8 against the Swiss franc we find for example

Table 2.8 Base currency

	t_B	t_E
CHF/USD	**1.15535**	**1.07835**

Table 2.9 Local currency

	US	US	CHF	CHF
	t_B	t_E	t_B	t_E
DKK	**6.06270**	**6.08235**	5.24750	5.64042
POL	**3.32075**	**3.37715**	2.87424	3.13177

$$5.24750\,\mathrm{DKK/CHF} = \frac{6.06270\ \mathrm{DKK/USD}}{1.15535\ \mathrm{CHF/USD}}.$$

The other three values in Table 2.9 are computed analogously.

1. For the Danish bond we have by (16)

$$r_{DKK} = \frac{5.24750}{5.64149} - 1 = -0.06966, \text{i.e.,} \ -6.966\%.$$

   The CHF has a negative return because the Danish krone is inflating. (16) and (17) yields in CHF the following total return

$$r_T = \frac{111.946}{111.110}\frac{5.24750}{5.64149} - 1 = -0.06266, \ \text{i.e.,} \ -6.266\%.$$

2. For the Polish bond we have a negative return

$$r_{POL} = \frac{2.87424}{3.13177} - 1 = -0.08223, \text{i.e.,} \ -8.223\%.$$

   Thus we have in CHF the following total return

$$r_T = \frac{111.049}{111.160}\frac{2.87424}{3.13177} - 1 = -0.08315, \ \text{i.e.,} \ -8.315\%.$$

◇

We proceed by considering the currency attribution of a segment j with a unique currency

$$r_{T,\,j} = r_{C,\,j} + r_{M,\,j} + r_{C,\,j}r_{M,\,j},\, 1 \le j \le n,$$

$$r_{T,P} = \sum_{j=1}^{n} w_j r_{T,\,j} = \sum_{j=1}^{n}\left(w_j\left(r_{M,\,j} + r_{C,\,j} + r_{C,\,j}r_{M,\,j}\right)\right) =$$
$$\sum_{j=1}^{n} w_j r_{M,\,j} + \sum_{j=1}^{n} w_j r_{C,\,j} + \sum_{j=1}^{n} w_j r_{M,\,j} r_{C,\,j}.$$

If the investments in the base currency are in the segment 1 we have $r_{C,1} = 0$, i.e.,

$$\sum_{j=1}^{n} w_j r_{M,\,j} + \sum_{j=2}^{n} w_j r_{C,\,j} + \sum_{j=2}^{n} w_j r_{M,\,j} r_{C,\,j}.$$

The total return on the portfolio consists of three parts. The first and second part are the market return and the currency return of the segment. The third part cannot be assigned to the currency return or the market return.

2.3 Time-Weighted Rate of Return

2.3.1 *Measuring Absolute Return*

In the last two sections we have considered a time interval with a beginning knot and an ending knot. In the following we consider an arbitrary, but finite number of additional time knots in-between. In investment reporting a knot usually refers to a valuation of the portfolio. Typical examples are the end-of-day valuation or closing and opening of an account. In return measurement, however, a knot is mandatory if there is a cash flow in or out of the portfolio.

The unit of the time axis can be a day, a month or a year, but in the following notation non-equidistant knots can also be considered. We start with the notation

$$t_0 = 0, \ldots, t_N = T \tag{2.3.1}$$

as time points and corresponding cash flows $C_1,\ldots,C_{N-1}$ on the time axis, where the indices k and j refer to the numeration of the time points. Furthermore

$${}_k r_{P,k+j}, 0 < j \leq N - k \tag{2.3.2a}$$

denotes the return of the portfolio P between t_k and t_{k+j}, $k = 1,\ldots,N$ (see Fig. 2.4).

Based on the beginning portfolio values PB_k and the ending portfolio values PE_{k+1} we calculate the return ${}_k r_{P,k+1}$ by

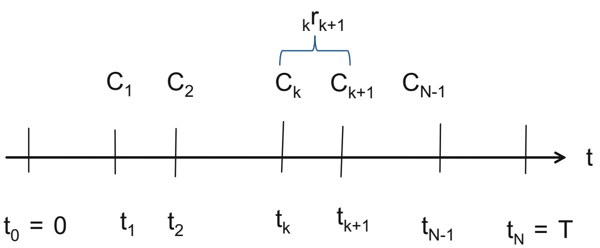
$${}_k r_{P,k+1} = \frac{PE_{k+1} - PB_k}{PB_k}, \quad k = 0, \ldots, N - 1. \tag{2.3.2b}$$

In order to measure the return between time 0 and time k we link the returns multiplicatively by

$${}_0 r_{P,k+1} = (1 + {}_0 r_{P,1}) \cdot (1 + {}_1 r_{P,2}) \ldots (1 + {}_k r_{P,k+1}) - 1, \quad k = 1, \ldots, N - 1. \tag{2.3.2c}$$

Furthermore it is assumed in (2c) that the returns are compounded at the end of each sub-period. As we multiply we allude to an area or the volume of a geometrical

Fig. 2.4 The time axis and cash flows



object because the area or the volume is usually calculated by using multiplications. For instance the area of a rectangle is the multiplication of its lattices. We refer to this as ***geometrically linked returns***.

Definition 2.16 For equidistant knots $t_k = k$, $k = 0,\ldots, N = T$, we consider the ***geometric mean return*** $\hat{r}_P$

$$\hat{r}_P = \sqrt[N]{(1 + {}_0r_{P,T})} - 1. \tag{2.3.3a}$$

It is seen when compounding with $\hat{r}_P$ over the time period that we get $\hat{r}_P$ by (3a), i.e.,

$$\hat{r}_P = \sqrt[N]{(1 + \hat{r}_P)} - 1. \tag{2.3.3b}$$

Definition 2.17 For annual data over N years, ***the annualized return*** $a\hat{r}_P$ is defined by the right-hand side of (3a). For monthly data over N months ***the annualized return*** $a\hat{r}_P$ is

$$a\hat{r}_P = \sqrt[12N]{(1 + \hat{r}_P)} - 1.$$

Remark 2.10 We see that we have here according to (3) two different attributions along the time axis. The first attribution is according to (2c) and the second is described by (3), where the returns over ${}_0r_{P,K}$ are uniformly distributed or attributed with $\hat{r}_P$ defined by (3b).

We note that the returns in (2) are commutative, i.e., the returns stay unchanged by considering for example r_1 in period 2 and r_2 in period 1. The commutative law of the returns and the independence of the size of the portfolio by geometrical linking are illustrated by the following example.

Example 2.15 (basic properties of geometrical linking) (a) We assume annual data with $T = 2$, $N = 2$, $t_0 = 0$, $t_1 = 1$ and $t_2 = 2$ in (1) and $PB_0 = \$1$, $PB_1 = \$100$, ${}_0r_{P,1} = 7\ \%$, ${}_1r_{P,2} = 5\ \%$ in (2). Then we have with (2b)

$$PE_1 = PB_0(1 + {}_0r_{P,1}) = \$1 \cdot (1 + 0.07) = \$1.07$$

and

$$PE_2 = PB_1(1 + {}_1r_{P,2}) = \$100 \cdot (1 + 0.05) = \$105.$$

(2c) yields

$${}_0r_{P,2} = (1 + {}_0r_{P,1}) \cdot (1 + {}_1r_{P,2}) - 1 = 0.123500.$$

There is an **inflow** $C_1 = \$100 - \$1.07 = \$98.93$ at $t_1 = 1$. It is seen that C_1 does not infer the calculation of ${}_0r_{P,2}$.

(b) We assume annual data with $T = 2$, $N = 2$, $t_0 = 0$, $t_1 = 1$, $t_2 = 2$ in (1) and $PB_0 = \$100$, $PB_1 = \$1$, ${}_0r_{P,1} = 5\ \%$, ${}_1r_{P,2} = 7\ \%$. Then we have with (2b)

$$PE_1 = PB_0(1+{}_0r_{P,1}) = \$100(1 + 0.05) = \$105$$

and

$$PE_2 = PB_1(1+{}_1r_{P,2}) = \$1(1 + 0.07) = \$1.07.$$

(2c) yields

$${}_0r_{P,2} = (1+{}_0r_{P,1}) \cdot (1+{}_1r_{P,2}) - 1 = 0.123500.$$

There is an **outflow** $C_1 = \$1 - \$105 = -\$104$ at $t_1 = 1$. It is seen that C_1 does not infer the calculation of ${}_0r_{P,2}$.

We see that there is a cash inflow in **(a)** and a cash outflow in **(b)** at time 1, although the result is the same in both cases. We see that the cash flows are reinvested by ${}_1r_{P,2}$. In **(a)** and **(b)** we find by (3a) for the annualized return

$$\hat{r}_P = \sqrt[2]{(1 + {}_0r_{P,2})} - 1 = \sqrt[2]{(1 + 0.123500)} - 1 = 0.059953.$$

(c) We assume annual data with $T = 1$, $N = 2$, $t_0 = 0$, $t_1 = 0.5$, $t_2 = 1$ in (1) and $PB_0 = \$1$, $PB_1 = \$100$, ${}_0r_{P,1} = 7\ \%$, ${}_1r_{P,2} = 5\ \%$ in (2). Following case **(a)** above we find with (2b)

$${}_0r_{P,2} = (1+{}_0r_{P,1}) \cdot (1+{}_1r_{P,2}) - 1 = 0.123500.$$

(d) We assume annual data with $T = 1$, $N = 2$, $t_0 = 0$, $t_1 = 0.5$, $t_2 = 1$ in (1) and $PB_0 = \$100$, $PB_1 = \$1$, ${}_0r_{P,1} = 5\ \%$, ${}_1r_{P,2} = 7\ \%$ in (2). Following case **(b)** above we find with (2b)

$${}_0r_{P,2} = (1+{}_0r_{P,1}) \cdot (1+{}_1r_{P,2}) - 1 = 0.123500.$$

We see that in both cases **(c)** and **(d)** we have with (3a)

$$\hat{r}_P = \sqrt[1]{(1 + {}_0r_{P,1})} - 1 = 0.123500.$$

We conclude from the considerations **(a)**, **(b)** resp. and **(c)**, **(d)** resp. that the return does not depend on the cash flow but is instead dependent on the underlying timespan.

◇

The mean defined in the following definition does not apply to compounding. It is used extensively in Chap. 4 on risk.

Definition 2.18 For equidistant knots $t_k = k,\ k = 0,\ldots,\ N = T$, we consider the ***arithmetic mean return*** $\bar{r}_P$ of a portfolio P. It is calculated by the sum of the returns in (2a) divided by the number N of returns in (2a):

$$\bar{r}_P = \frac{1}{N}\sum_{k=0}^{N-1} {}_k r_{P,k+1}. \tag{2.3.4}$$

Remark 2.11 As can be seen from (4), $\bar{r}_P$ is a function of N. However, with the definition of $\bar{r}_P$ in (4) we presume that for N tending to infinity, the limit exists and is unique.

By expanding (2.2.2) we denote by $N_{j,k}$ the units and by $P_{j,k}$ the price of stock j at the time t_k, $k = 0,\ldots,N-1$. With

$$PB_k = \sum_{j=1}^{n} N_{j,k}P_{j,k},\ PE_k = \sum_{j=1}^{n} N_{j,k}P_{j,k+1}$$

it follows that (2b) is equivalent to

$$1 + {}_k r_{P,k+1} = \frac{\sum_{j=1}^{n} N_{j,k}P_{j,k+1}}{\sum_{j=1}^{n} N_{j,k}P_{j,k}}.$$

With (2b) we have by ***chain linking*** for $k = p,\ldots,q$, $0 \le p < q \le T-1$

$$1 + {}_p r_{P,q+1} = \prod_{p=k}^{q} \frac{\sum_{j=1}^{n} N_{j,k}P_{j,k+1}}{\sum_{j=1}^{n} N_{j,k}P_{j,k}}$$

and for $p = 0$ and $q = N - 1$ we find

$$
{}_0r_{P,T} = \prod_{k=0}^{N-1} \frac{\sum_{j=1}^{n} N_{j,k} P_{j,k+1}}{\sum_{j=1}^{n} N_{j,k} P_{j,k}} - 1. \tag{2.3.5}
$$

At the end of each period, we divide by the portfolio's value and observe the price movement of the securities in the portfolio over the next period. In other words we neutralize the portfolio value, and the number n of stock with outstanding units $N_{j,k}$ remains unchanged over the time period.

Definition 2.19 (5) is called the ***time-weighted rate of return calculation (TWR)***.

TWR is based on the geometrical linking presented in (2) and has the properties described in Example 2.15. As potential changes in the money invested in the portfolio do not affect the return, we simply look at relative changes over multiple time periods. Time-weighted rates of return are used by the index providers. Most market indices are calculated on a daily basis, i.e., the time unit is days. The value of the market portfolio is calculated at the end of each business day and formula (5) is evaluated. The returns are then published by data providers. It is well known that indices can be rebased at any point in time. This again reflects the fact that (5) is independent of the portfolio size, i.e., if we consider for instance an index with base currency USD and rebase the portfolio to 100 we can observe the development of an initial portfolio of \$100.

We assume that there is a $\lambda_k \in \mathbf{R}^1$ independent of j for each knot $k = 1, \ldots, N - 1$ such that

$$
N_{j,k} = \lambda_k \cdot N_{j,k-1} \tag{2.3.6}
$$

and consider (5) over two periods $[\lambda_{k-1}, \lambda_k]$ and $[\lambda_k, \lambda_{k+1}]$, $0 \leq p < k < q \leq T$. We apply (6) twice

$$
(1 + {}_{k-1}r_{P,k})(1 + {}_k r_{P,k+1}) = \frac{\sum_{j=1}^{n} N_{j,k-1} P_{j,k} \cdot \sum_{j=1}^{n} N_{j,k} P_{j,k+1}}{\sum_{j=1}^{n} N_{j,k-1} P_{j,k-1} \cdot \sum_{j=1}^{n} N_{j,k} P_{j,k}} = \tag{2.3.7}
$$

$$\frac{\sum_{j=1}^{n} N_{j,k} P_{j,k+1}}{\sum_{j=1}^{n} N_{j,k-1} P_{j,k-1} \cdot \lambda_k} = \frac{\sum_{j=1}^{n} N_{j,k-1} P_{j,k+1}}{\sum_{i=1}^{n} N_{j,k-1} P_{j,k-1}} = (1 +_{k-1} r_{P,k+1}).$$

By applying (6) to the denominator and using (7) repeatedly we arrive at

$${}_0 r_T = \frac{\sum_{i=1}^{n} N_{i,T} P_{i,T}}{\sum_{i=1}^{n} N_{i,0} P_{i,0}} - 1$$

and consequently

$${}_0 r_T = \frac{PV_T - PV_0}{PV_0},$$

i.e., the return reduces to the return based on the change of the portfolio value at time 0 and T. The portfolio values at the time t_1 to t_{N-1} do not affect the return, as the intermediate portfolio values are dropping.

We can summarize as follows:

Theorem 2.1 If (6) holds, then the return of a portfolio over multiple periods is equal to the geometrically linked returns of the portfolio over the sub-periods.

Proof The assertion follows by applying (7) to the interval $[p, q], 0 \leq p < q + 1 \leq T$ consecutively.

◇

We distinguish between two cases:

Case 1

$$\sum_{j=1}^{n} N_{j,k-1} P_{j,k} = \sum_{j=1}^{n} N_{j,k} P_{j,k}, k = 1, \ldots, N - 1 \tag{2.3.8}$$

This case includes the case $\lambda_k = 1$, $k = 1, \ldots, N - 1$ in (6), in other words, as the number of securities is held constant and the 'buy and hold strategy' is reflected. However, case 1 leaves open the possibility of a rebalancing within the portfolio and the portfolio is restructured (see Fig. 2.5). By evaluating (8) for the 'buy and hold strategy' and by assessing the value added from rebalancing, the return of the portfolio is measured. In both cases the return is only determined by the portfolio's beginning value and the ending value.

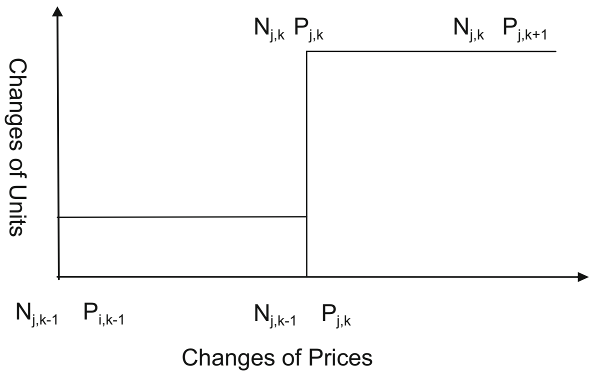


Fig. 2.5 Chain linking at a seam

Theorem 2.2 If (8) holds, then the return of a portfolio over multiple periods is equal to its geometrically linked returns over the sub-periods.

Proof The assertion follows by considering the interval [p, q], $0 \leq p < q + 1 \leq T$ and hence we have by considering (7) over two periods $[t_{k-1}, t_k]$ and $[t_k, t_{k+1}]$

$$(1+{}_{k-1}r_{P,k})(1+{}_{k}r_{P,k+1}) = \frac{\sum_{j=1}^{n} N_{j,k-1}P_{j,k} \cdot \sum_{j=1}^{n} N_{j,k}P_{j,k+1}}{\sum_{j=1}^{n} N_{j,k-1}P_{j,k-1} \cdot \sum_{j=1}^{n} N_{j,k}P_{j,k}} = \frac{\sum_{j=1}^{n} N_{j,k}P_{j,k+1}}{\sum_{j=1}^{n} N_{j,k-1}P_{j,k-1}}$$

$$= \frac{PE_{k+1}}{PB_{k-1}} = (1+{}_{k-1}r_{P,k+1}).$$

◇

Case 2 There is a k, $k = 1, \ldots, N-1$ with

$$\sum_{j=1}^{n} N_{j,k-1}P_{j,k} \neq \sum_{j=1}^{n} N_{j,k}P_{j,k}. \qquad (2.3.9)$$

This case reflects a change in the portfolio value due to an external cash flow as we divide in (5) by the portfolio value. However, the cash flow does not affect the return calculation, as the return depends only on the movement between the knots t_k, $k = 0, \ldots, N$. Taking the argument further, chain linking return measurement does not take into account the timing of external cash flows. This property underpins the importance of this methodology in the index industry. The finance industry is only interested in objective market movements. For the difference in the return from a client's or a portfolio manager's perspective, please consult Sect. 2.5.

Furthermore this case is important to understanding the computation of Total Return Indices, i.e., indices that not only measure the price movement but also the return from cash flow out of its constituents. If an index reinvests overnight or instantaneously, it is assumed that a cash flow such as dividends or coupons stays in the index, but is distributed between the securities of the index, i.e., the portfolio value is unchanged, but the cash flow is distributed such that the weights for e.g. the market capitalization are unchanged.

Theorem 2.3 If (9) holds, then we can conclude the following: If the external cash flow $C \neq 0$ in t_k

$$C = \sum_{j=1}^{n} N_{j,k-1} P_{j,k} - \sum_{j=1}^{n} N_{j,k} P_{j,k}$$

is distributed with (6) the return over $[t_{k-1}, t_{k+1}]$ is independent of the portfolio value in t_k; otherwise the return over $[t_{k-1}, t_{k+1}]$ is dependent firstly on the value

$$\sum_{j=1}^{n} N_{j,k-1} P_{j,k}$$

and secondly on the allocation of the external cash flow C in $N_{j,k}$, $1 \leq j \leq n$ in t_k, i.e., consideration (7) does not hold.

Proof The assertion follows from Theorem 2.1 or from

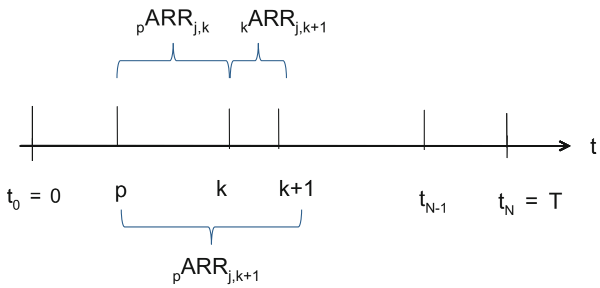
$$(1 + {}_{k-1}r_{P,k})(1 + {}_{k}r_{P,k+1}) = \frac{\sum_{j=1}^{n} N_{j,k-1} P_{j,k} \cdot \sum_{j=1}^{n} N_{j,k} P_{j,k+1}}{\sum_{j=1}^{n} N_{j,k-1} P_{j,k-1} \cdot \sum_{j=1}^{n} N_{j,k} P_{j,k}}.$$

◇

2.3.2 Measuring Relative Return

In this section we expand on the Brinson-Hood-Beebower method (Sect. 2.2) from one base period to multiple time periods. We decompose the arithmetic relative return (see Definition 2.9) in the segments of the portfolio and the management effects (see also Fig. 2.6). Algebraic formulation and reformulation are presented here and there are no ‘rest terms’. The decompositions are consistent with the overall return of the portfolio and the benchmark. We derive recursive formulae for generating the arithmetic relative return over multiple time periods. It is important to note that the recursive formulae derived depend only on two data points in the past and not on the whole data history. The methodology here is often implemented in commercially available software. We refer to [1, Bonafede] and [9, Feibel, p. 279].

Fig. 2.6 Recursive formula



Let t_p, t_k and t_q, $p < k < q$ denote times on the time axis (Fig. 2.4). The following formulae are valid at the portfolio level. Based on (2c) we consider the geometrically linked return between the time from t_p to t_k and from t_k to t_q of the portfolio

$$_pr_{P,q} = \left(1+{}_pr_{P,k}\right)\left(1+{}_kr_{P,q}\right) - 1 = {}_pr_{P,k} + {}_kr_{P,q} + {}_pr_{P,k} \cdot {}_kr_{P,q} \tag{2.3.10a}$$

and the benchmark

$$_pr_{B,q} = \left(1+{}_pr_{B,k}\right)\left(1+{}_kr_{B,q}\right) - 1 = {}_pr_{B,k} + {}_kr_{B,q} + {}_pr_{B,k} \cdot {}_kr_{B,q}. \tag{2.3.10b}$$

Definition 2.20 The term $r_{\bullet,k}\,p \cdot r_{\bullet,q}k$ with • either a portfolio or a benchmark is called a ***cross term***.

Definition 2.21 The ***arithmetical relative returns*** $_pARR_q$, $_pARR_k$, $_kARR_q$, $p < k < q$, $p = 0,\ldots.,N$, $k = 0,\ldots.,N$, $q = 0,\ldots.,N$ are defined by (10)

$$_pARR_q = {}_pr_{P,q} - {}_pr_{B,q},$$

$$_pARR_k = {}_pr_{P,k} - {}_pr_{B,k},$$

$$_kARR_q = {}_kr_{P,q} - {}_kr_{B,q}.$$

For $_pARR_q$ and $p < k < q$ we find

$$\begin{aligned} {}_pARR_q &= {}_pr_{P,k} + {}_kr_{P,q} + {}_pr_{P,k} \cdot {}_kr_{P,q} - {}_pr_{B,k} - {}_kr_{B,q} - {}_pr_{B,k} \cdot {}_kr_{B,q} = \\ &\quad {}_pr_{P,k} - {}_pr_{B,k} + {}_kr_{P,q} - {}_kr_{B,q} + {}_pr_{P,k} \cdot {}_kr_{P,q} - {}_pr_{B,k} \cdot {}_kr_{B,q} = \\ &\quad {}_pARR_k + {}_kARR_q + {}_pr_{P,k} \cdot {}_kr_{P,q} - {}_pr_{B,k} \cdot {}_kr_{B,q}. \end{aligned} \tag{2.3.11}$$

We proceed by introducing two different decompositions of the relative arithmetical return for *two periods*.

Based on (11) ***the first decomposition*** of the cross terms is

$$ {}_{p}r_{P,k} \cdot {}_{k}r_{P,q} - {}_{p}r_{B,k} \cdot {}_{k}r_{B,q} = $$

$$ {}_{p}r_{P,k} \cdot {}_{k}r_{P,q} - {}_{p}r_{P,k} \cdot {}_{k}r_{B,q} + {}_{p}r_{P,k} \cdot {}_{k}r_{B,q} - {}_{p}r_{B,k} \cdot {}_{k}r_{B,q} = $$

$$ {}_{p}r_{P,k} \left({}_{k}r_{P,q} - {}_{k}r_{B,q} \right) + {}_{k}r_{B,q} \left({}_{p}r_{P,k} - {}_{p}r_{B,k} \right), $$

thus by (11) again we have

$$ {}_{p}ARR_{q} = \left(1 + {}_{k}r_{B,q}\right) \cdot \left({}_{p}r_{P,k} - {}_{p}r_{B,k} \right) + \left(1 + {}_{p}r_{P,k}\right) \cdot \left({}_{k}r_{P,q} - {}_{k}r_{B,q} \right), $$

i.e.,

$$ {}_{p}ARR_{q} = \left(1 + {}_{k}r_{B,q}\right) \cdot {}_{p}ARR_{k} + \left(1 + {}_{p}r_{P,k}\right) \cdot {}_{k}ARR_{q}. \tag{2.3.12a} $$

We see in (12a) that in addition to the sum of the arithmetical relative return the relative return in the first period is compounded with the total benchmark return in the second period and the relative return of the second period is compounded with the total portfolio return in the first period.

Based on (11) ***the second decomposition*** of the cross terms is

$$ {}_{p}r_{P,k} \cdot {}_{k}r_{P,q} - {}_{p}r_{B,k} \cdot {}_{k}r_{B,q} = $$

$$ {}_{p}r_{P,k} \cdot {}_{k}r_{P,q} - {}_{p}r_{B,k} \cdot {}_{k}r_{P,q} + {}_{p}r_{B,k} \cdot {}_{k}r_{P,q} - {}_{p}r_{B,k} \cdot {}_{k}r_{B,q} = $$

$$ {}_{k}r_{P,q} \left({}_{p}r_{P,k} - {}_{p}r_{B,k} \right) + {}_{p}r_{B,k} \left({}_{k}r_{P,q} - {}_{k}r_{B,q} \right), $$

thus by (11)

$$ {}_{p}ARR_{q} = \left(1 + {}_{k}r_{P,q}\right) \cdot \left({}_{p}r_{P,k} - {}_{p}r_{B,k} \right) + \left(1 + {}_{p}r_{B,k}\right) \cdot \left({}_{k}r_{P,q} - {}_{k}r_{B,q} \right), $$

i.e.,

$$ {}_{p}ARR_{q} = \left(1 + {}_{k}r_{P,q}\right) \cdot {}_{p}ARR_{k} + \left(1 + {}_{p}r_{B,k}\right) \cdot {}_{k}ARR_{q}. \tag{2.3.12b} $$

We see in (12b) that in addition to the sum of the arithmetical relative return the relative return in the first period is compounded with the total portfolio return in the second period and the relative return of the second period is compounded with the total benchmark return in the first period.

We proceed by presenting an example that illustrates (12):

Example 2.16 We consider two base periods with $t_0 = 0$, $t_1 = 1$ and $t_2 = 2$ in (1) with returns ${}_{0}r_{P,1} = 10\ \%$, ${}_{0}r_{B,1} = 2\ \%$, ${}_{1}r_{P,2} = 2\ \%$ and ${}_{1}r_{B,2} = 10\ \%$. As the portfolio has an outperformance of 8 % in the first period and an underperformance of 8 % in the second period, we find

$$_0\mathrm{ARR}_2 = 0.$$

(12a) reduces to

$$_0\mathrm{ARR}_2 = (1+{_1}r_{B,2})\cdot{_0}\mathrm{ARR}_1 + (1+{_0}r_{P,1})\cdot{_1}\mathrm{ARR}_2.$$

and the numerical values of the decomposition are

$$(1+0.1)\cdot 8\% - (1+0.1)\cdot 8\% = 0$$

Similarly (12b) reduces to

$$_0\mathrm{ARR}_2 = (1+{_1}r_{P,2})\cdot{_0}\mathrm{ARR}_1 + (1+{_0}r_{B,1})\cdot{_1}\mathrm{ARR}_2.$$

and the numerical values of the decomposition are

$$(1+0.02)\cdot 8\% - (1+0.02)\cdot 8\% = 0.$$

This example shows two different decompositions of the arithmetical relative returns over time.

◇

> The attribution of the return over time is not unique and depends on reinvestment assumptions.

$_p\mathrm{ARR}_q$ is in general critically dependent on the compounding knots between p and q (see Theorem 2.3). This property is encountered with regard to the absolute return (see (2.1.6) and (2.1.7)) and the relative return. As previously shown, the value for $_p\mathrm{ARR}_q$ is not necessarily unique.

In the following we assume that the compounding pattern is given and we respect all compounding times t_k, $k = 0,\ldots,N$. We use the iterative character of (12) and with $q = k + 1$ we find:

$$_p\mathrm{ARR}_{k+1} = (1+{_k}r_{B,k+1})\cdot{_p}\mathrm{ARR}_k + \left(1+{_p}r_{P,k}\right)\cdot{_k}\mathrm{ARR}_{k+1}, \tag{2.3.13a}$$

resp.

$$_p\mathrm{ARR}_{k+1} = (1+{_k}r_{P,k+1})\cdot{_p}\mathrm{ARR}_k + \left(1+{_p}r_{B,k}\right)\cdot{_k}\mathrm{ARR}_{k+1}. \tag{2.3.13b}$$

We continue by considering ***the segmentation of the portfolio and the benchmark*** over time (Fig. 2.6). The weights change in time but are fixed at each time knot. By adapting the notation from (2.2.10) for the segment level we consider $k = 0,....,N - 1$

$$ {}_{k}r_{P,\,k+1} = \sum_{j=0}^{n} W_{j,k} R_{j,k},\ {}_{k}r_{B,\,k+1} = \sum_{j=0}^{n} V_{j,k} B_{j,k}. \tag{2.3.14} $$

Definition 2.22 The ***arithmetical relative return*** $_{k}\boldsymbol{ARR_{j,k+1}}$, $\boldsymbol{k = 0,....,N - 1}$ in segment j is defined by

$$ {}_{k}ARR_{j,k+1} = W_{j,k} R_{j,k} - V_{j,k} B_{j,k} \tag{2.3.15a} $$

with

$$ \sum_{j=0}^{n} {}_{k}ARR_{j,k+1} = {}_{k}ARR_{k+1}. \tag{2.3.15b} $$

Based on the ***first*** decomposition (13a) we consider the iteration for $_{p}ARR^{1}_{j,k}$ with respect to k

$$ {}_{p}ARR^{1}_{j,k+1} = (1 + {}_{k}r_{B,k+1}) \cdot {}_{p}ARR^{1}_{j,k} + \left(1 + {}_{p}r_{P,k}\right) \cdot {}_{k}ARR_{j,k+1} \tag{2.3.16a} $$

with $p = 0, 1,....,T - 2$, $k = p + 1, p + 2,....., T - 1$. p is fixed and k propagates the portfolio through time (see Fig. 2.6). $_{k}ARR_{j,k+1}$ is given by (15). The initial values are

$$ \begin{aligned} {}_{p}ARR^{1}_{j,\,p+1} = {}_{p}ARR_{j,\,p+1} &= W_{j,\,p} \cdot R_{j,\,p} - V_{j,\,p} \cdot B_{j,\,p} \\ {}_{p+1}ARR_{j,\,p+2} &= W_{j,\,p+1} \cdot R_{j,\,p+1} - V_{j,\,p+1} \cdot B_{j,\,p+1}. \end{aligned} \tag{2.3.16b} $$

Reinvestment assumption The relative segment return on p to k is compounded with the benchmark return in the period from k to k + 1 and the relative segment return from k to k + 1 is compounded with the portfolio return on p to k.

Based on the ***second*** decomposition (13b) we consider the iteration for $_{p}ARR^{2}_{j,k}$ with respect to k

$$ {}_{p}ARR^{2}_{j,k+1} = (1 + {}_{k}r_{P,k+1}) \cdot {}_{p}ARR^{2}_{j,k} + \left(1 + {}_{p}r_{B,k}\right) \cdot {}_{k}ARR_{j,k+1} \tag{2.3.17a} $$

with $p = 0, 1,....,T - 2$, $k = p + 1, p + 2, ..., T - 1$. p is fixed and k propagates the portfolio through time. $_{k}ARR_{j,k+1}$ is given by (15). The initial values are

$$ \begin{aligned} {}_{p}ARR^{2}_{j,\,p+1} = {}_{p}ARR_{j,\,p+1} &= W_{j,\,p} \cdot R_{j,\,p} - V_{j,\,p} \cdot B_{j,\,p} \\ {}_{p+1}ARR_{j,\,p+2} &= W_{j,\,p+1} \cdot R_{j,\,p+1} - V_{j,\,p+1} \cdot B_{j,\,p+1}. \end{aligned} \tag{2.3.17b} $$

Reinvestment assumption The relative segment return on p to k is compounded with the portfolio return in the period from k to k + 1 and the relative segment return from k to k + 1 is compounded with the benchmark return on p to k.

The iterations (16) and (17) introduce ${}_p\mathrm{ARR}^{\mathrm{i}}_{\mathrm{j,k+1}}$, $k > p$, $i = 1, 2$. They calculate the arithmetic relative return if a new data point ${}_k\mathrm{ARR}_{\mathrm{j,k+1}}$ is available. The following lemma ensures that they are a decomposition of ${}_p\mathrm{ARR}_{\mathrm{k+1}}$.

Lemma 2.1 We claim that

$$\sum_{j=1}^{n} \mathrm{pARR}^{\mathrm{i}}_{\mathrm{j,k+1}} = {}_p\mathrm{ARR}_{\mathrm{k+1}}, i = 1, 2 \tag{2.3.18}$$

where ${}_p\mathrm{ARR}^{\mathrm{i}}_{\mathrm{j,k+1}}$, $i = 1, 2$ is calculated by the recursive formulae (16), (17), respectively.

Proof We start with the right-hand side from (18) and by combining it with (16a), (17a) we have

$$\sum_{j=1}^{n} {}_p\mathrm{ARR}^1_{\mathrm{j,k+1}} =$$
$$(1 + {}_k r_{B,k+1}) \cdot \sum_{j=1}^{n} {}_p\mathrm{ARR}^1_{\mathrm{j,k}} + \left(1 + {}_p r_{P,k}\right) \cdot \sum_{j=1}^{n} {}_k\mathrm{ARR}_{\mathrm{j,k+1}},$$
$$\sum_{j=1}^{n} {}_p\mathrm{ARR}^2_{\mathrm{j,k+1}} =$$
$$(1 + {}_k r_{B,k+1}) \cdot \sum_{j=1}^{n} {}_p\mathrm{ARR}^2_{\mathrm{j,k}} + \left(1 + {}_p r_{P,k}\right) \cdot \sum_{j=1}^{n} {}_k\mathrm{ARR}_{\mathrm{j,k+1}}, \text{resp..}$$

We now need to show that the right-hand side of the equation is equal the left-hand side of the assertion. We adopt a proof by induction with respect to k and start with $k = p + 1$. By adding (16b), (17b), resp. for the first sum and (15b) for the second sum the assertion follows from (13). For general k the assertion follows from the induction assumption, (15b) and (13) again.

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Example 2.16 (continued) We consider again two periods with $N = 2$, $T = 2$, $t_0 = 0$, $t_1 = 1$, $t_2 = 2$ with ${}_0r_{P,1} = 10\ \%$, ${}_0r_{B,1} = 2\ \%$, ${}_1r_{P,2} = 2\ \%$ and ${}_1r_{B,2} = 10\ \%$ and assume in addition the portfolio and the benchmark consist of two segments with the following data:

$$W_{1,0} = W_{1,1} = W_{2,1} = W_{2,2} = V_{1,0} = V_{1,1} = V_{2,1} = V_{2,2} = 0.5;$$
$$R_{1,0} = R_{1,1} = R_{2,1} = R_{2,2} = 2\%;$$
$$B_{1,0} = B_{1,1} = B_{2,1} = B_{2,2} = 10\%.$$

(16a) reduces to

$${}_0ARR^1_{j,2} = (1+{}_1r_{B,2}) \cdot {}_0ARR^1_{j,1} + \left(1+{}_pr_{P,k}\right) \cdot {}_1ARR_{j,2}.$$

and the numerical values of the decomposition are

$$(1 + 0.1) \cdot 0.5 \cdot (10\% - 2\%) - (1 + 0.1) \cdot 0.5 \cdot (10\% - 2\%) = 0$$

i.e.,

$${}_0ARR^1_{j,2} = 0, \; j = 1, 2$$

Similarly (17a) yields the decomposition

$${}_0ARR^2_{j,2} = (1+{}_1r_{P,2}) \cdot {}_0ARR^2_{j,1} + (1+{}_0r_{B,1}) \cdot {}_1ARR_{j,2}$$

and the numerical values of the decomposition are

$$(1 + 0.02) \cdot 0.5 \cdot 8\% - (1 + 0.02) \cdot 0.5 \cdot 8\% = 0,$$

i.e.,

$${}_0ARR^2_{j,2} = 0, \; j = 1, 2$$

This example shows two different decompositions of the return over time.

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We proceed by investigating ***the management effects*** over multiple periods by adopting the Brinson-Hood-Beebower approach. We have a set of management effects in each segment and at each time point (Fig. 2.7).

Definition 2.23 The ***asset allocation effect*** $_kA_{j,k+1}$, $k = 0, \ldots, N-1$ in segment j is defined by

$${}_kA_{j,k+1} = \left(W_{j,k} - V_{j,k}\right) \cdot B_{j,k}. \tag{2.3.19a}$$

The ***stock selection effect*** $_kS_{j,k+1}$, $k = 0, \ldots, N-1$ in segment j is defined by

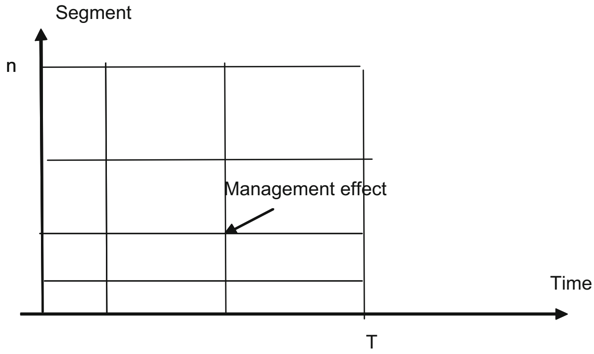

Fig. 2.7 The full portfolio return problem (management effect)

$$_{k}S_{j,k+1} = \left(R_{j,k} - B_{j,k}\right) \cdot V_{j,k}. \tag{2.3.19b}$$

The ***interaction effect*** $_{k}I_{j,k+1}$, $k = 0,\ldots,N-1$ in segment j is defined by

$$_{k}I_{j,k+1} = \left(W_{j,k} - V_{j,k}\right) \cdot \left(R_{j,k} - B_{j,k}\right). \tag{2.3.19c}$$

Based on the decomposition in (12), (16) and (17) we consider the iteration for the management effect

$$_{p}A^{1}_{j,k+1} = (1 + {}_{k}r_{B,k+1}) \cdot {}_{p}A^{1}_{j,k} + \left(1 + {}_{p}r_{P,k}\right) \cdot {}_{k}A_{j,k+1} \tag{2.3.20a}$$

$$_{p}S^{1}_{j,k+1} = (1 + {}_{k}r_{B,k+1}) \cdot {}_{p}S^{1}_{j,k} + \left(1 + {}_{p}r_{P,k}\right) \cdot {}_{k}S_{j,k+1} \tag{2.3.20b}$$

$$_{p}I^{1}_{j,k+1} = (1 + {}_{k}r_{B,k+1}) \cdot {}_{p}I^{1}_{j,k} + \left(1 + {}_{p}r_{P,k}\right) \cdot {}_{k}I_{j,k+1} \tag{2.3.20c}$$

$$_{p}A^{2}_{j,k+1} = (1 + {}_{k}r_{P,k+1}) \cdot {}_{p}A^{2}_{j,k} + \left(1 + {}_{p}r_{B,k}\right) \cdot {}_{k}A_{j,k+1} \tag{2.3.20d}$$

$$_{p}S^{2}_{j,k+1} = (1 + {}_{k}r_{P,k+1}) \cdot {}_{p}S^{2}_{j,k} + \left(1 + {}_{p}r_{B,k}\right) \cdot {}_{k}S_{j,k+1} \tag{2.3.20e}$$

$$_{p}I^{2}_{j,k+1} = (1 + {}_{k}r_{P,k+1}) \cdot {}_{p}I^{2}_{j,k} + \left(1 + {}_{p}r_{B,k}\right) \cdot {}_{k}I_{j,k+1} \tag{2.3.20f}$$

with $p = 0, 1,\ldots,T-2$, $k = p+1, p+2,\ldots,T-1$ and the initial values are given by (19).

Lemma 2.2 Based on (20) we claim that

$$_{p}ARR_{k+1} = \sum_{j=1}^{n} {}_{p}A^{i}_{j,k+1} + \sum_{j=1}^{n} {}_{p}S^{i}_{j,k+1} + \sum_{j=1}^{n} {}_{p}I^{i}_{j,k+1}, i = 1,2.$$

Proof We adopt a proof by induction and start with $k = p+1$. We consider for $i = 1, 2$

$$
\begin{aligned}
{}_{p}\mathrm{ARR}_{p+2} &= \sum_{j=1}^{n} {}_{p}A^{i}_{j,\,p+2} + \sum_{j=1}^{n} {}_{p}S^{i}_{j,\,p+2} + \sum_{j=1}^{n} {}_{p}I^{i}_{j,\,p+2} \\
&= \sum_{j=1}^{n} \left({}_{p}A^{i}_{j,\,p+2} + {}_{p}S^{i}_{j,\,p+2} + {}_{p}I^{i}_{j,\,p+2} \right)
\end{aligned}
$$

The assertion follows from (12), (19) and (20). For general k the assertion follows from induction assumptions (12), (19) and (20).

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We now conclude our exposition on the arithmetic relative return and proceed with the geometrical approach. We will limit, however, our discussion to the portfolio and benchmark levels, respectively.

Definition 2.24 For the ***geometrical relative return*** $_{p}GRR_{q}$ $_{p}GRR_{k}$, $_{k}GRR_{q}$, $p < k < q$, $p = 0,....,N$, $k = 0,....,N$, $q = 0,....,N$ we define

$$
\begin{aligned}
{}_{p}\mathrm{GRR}_{q} &= \frac{1 + {}_{p}r_{P,q}}{1 + {}_{p}r_{B,q}} - 1, \\
{}_{p}\mathrm{GRR}_{k} &= \frac{1 + {}_{p}r_{P,k}}{1 + {}_{p}r_{P,k}} - 1, \\
{}_{k}\mathrm{GRR}_{q} &= \frac{1 + {}_{k}r_{P,q}}{1 + {}_{k}r_{B,q}} - 1.
\end{aligned}
$$

As a result of the definition we have

$$
1 + {}_{p}\mathrm{GRR}_{q} = \frac{1 + {}_{p}r_{P,q}}{1 + {}_{p}r_{B,q}} = \frac{\left(1 + {}_{p}r_{P,k}\right)\left(1 + {}_{k}r_{P,q}\right)}{\left(1 + {}_{p}r_{B,k}\right)\left(1 + {}_{k}r_{B,q}\right)} = \left(1 + {}_{p}\mathrm{GRR}_{k}\right) \cdot \left(1 + {}_{k}\mathrm{GRR}_{q}\right).
$$

We see that the geometrical relative return has no cross term.

Example 2.17 We consider two periods with returns ${}_{0}r_{P,1} = 2\%$, ${}_{1}r_{P,2} = 10\%$, ${}_{0}r_{B,1} = 10\%$ and ${}_{1}r_{B,2} = 4\%$. For the geometrical linked return we have for the portfolio

$$1.02{*}1.10 - 1 = 1.1220 - 1 = 0.1220 \ \text{or} \ 12.20\%$$

and for the benchmark

$$1.10{*}1.04 - 1 = 1.144 - 1 = 0.1440 \ \text{or} \ 14.40\%.$$

The geometrical relative return is

$$\frac{1.1220}{1.1440} - 1 = 0.0192 \ \text{or} - 1.92\%.$$

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2.4 Money-Weighted Rate of Return

2.4.1 First Properties of the Internal Rate of Return Equation

In this section we concentrate on methods for calculating rates of return that take cash flows into account. We consider

$$\mathrm{PB_0} = \sum_{k=1}^{N-1} \frac{C_k}{(1+r)^{t_k}} + \frac{\mathrm{PE_T}}{(1+r)^T} \tag{2.4.1a}$$

where PB_0, PE_T, respectively are as in the previous section the beginning and ending values of the portfolio, respectively and C_k are the cash flows at time t_k, $k = 0, 1, 2, \ldots, N$ with $t_0 = 0$, $t_N = T$. The solutions of (1a) are called ***internal rates of return (IRR)***. We denote the solutions of (1a) with ***IR***.

The fundamental properties are:

- (1a) is based on the condition that the value today is equal to the discounted cash flow in the future (No arbitrage condition).
- The investment assumption is that the cash flows are reinvested by the internal return.

(1a) does not in general have an explicit solution, however, there are several numerical methods, such as the Secant, Newton-Raphson or modified Newton-Raphson methods for computing a solution of the equation for r. Starting with an initial approximation for the value of IR, more accurate approximations for IR are then computed. In [24–26] different methods for computing IR are compared. In numerical analysis it is well known for instance that the Secant method has the advantage that we do not need any derivatives, although the convergence speed is lower than that of the Newton-Raphson method.

Remark 2.12 (invariance of the solution IR) The multiplication by $\frac{1}{\mathrm{PE_T}}$ or $\frac{1}{\mathrm{PB_0}}$ of (1a) leaves the solution of (1a) invariant. Thus without loss of generalization we can assume

$$\mathrm{PB_0} = \sum_{k=1}^{N-1} \frac{C_k}{(1+r)^{t_k}} + \frac{1}{(1+r)^T}, 0 < t_1 < \ldots < t_N = T \tag{2.4.1b}$$

or

$$1 \quad = \quad \sum_{k=1}^{N-1} \frac{C_k}{(1+r)^{t_k}} + \frac{PE_T}{(1+r)^T}, 0 < t_1 < \ldots . < t_N = T. \tag{2.4.1c}$$

More generally the solutions of (1) are invariant under multiplication by $\lambda \in \mathbf{R}^1$ (see also Remark 2.3).

Remark 2.13 (normalization of the solution IR to [0,1]) By considering the transformation $\widetilde{t}_k = \frac{t_k}{T}$, k = 0, 1,...., N in (1a) we have

$$PB_0 = \sum_{k=1}^{N-1} \frac{C_k}{(1+r)^{\widetilde{t}_k}} + \frac{PE_T}{1+r}, 0 < \widetilde{t}_1 < \ldots . < \widetilde{t}_N = 1. \tag{2.4.1d}$$

With $d = \frac{1}{1+IR}$ in (1a) and with the solution IR_1 of (1d) we consider $d_1 = \frac{1}{1+IR_1}$. Then we find with

$$d_1 = d^T$$

that d_1 satisfies (1d)

$$PB_0 - \sum_{k=1}^{N-1} \frac{C_k}{(1+r)^{\widetilde{t}_k}} = PB_0 - \sum_{k=1}^{N-1} C_k d_1^{-\widetilde{t}_k} =$$

$$PB_0 - \sum_{k=1}^{N-1} C_k d_1^{-\frac{t_k}{T}} = PB_0 - \sum_{k=1}^{N-1} C_k d^{-t_k} = PE_T \cdot d^T = PE_T \cdot d_1.$$

Furthermore we have

$$IR_1 = (1 + IR)^{-T} - 1. \tag{2.4.1e}$$

The basic idea of (1a) is to discount all cash flows at a single point in time. In (1a) we discount to t = 0. This time is chosen somewhat arbitrarily; we could discount to another time point and the solutions of (1a) would not change. In other words, we see that the solution of (1) is unaffected by the multiplication by a discount factor. It is important to realize that IR is thus independent of time and only depends on the cash flows.

Definition 2.25 The concept of ***money-weighted rate of return (MWR)*** refers to a method for assessing the return reflecting the timing and size of cash flows.

Thus the IRR method is a type of MWR. Furthermore if there are no cash flows, the solution of (1a) is equal to ${}_0r_1$ in (2.3.5), i.e., the time-weighted rate of return is equal to the money-weighted rate of return (see Example 2.20). The difference between the TWR and MWR is called the ***timing effect***, as the timing of the cash flows explains the difference between the two concepts (see Sect. 2.5).

As solving (1a) is mathematically challenging, we concentrate on the analytics of (1a). With the discount factors (2.1.3) we consider the function P defined by

$$P(d) = PE_T d^T + \sum_{k=1}^{N-1} C_k \cdot d^k - \quad PB_0 d^0 \tag{2.4.2}$$

where

$$d^0 = 1.$$

We proceed by illustrating (2):

Example 2.18 (discussing different signs of the cash flow) We consider one cash flow and equidistant knots, i.e., $T = 2$, $N = 2$, $t_0 = 0$, $t_1 = 1$, $t_2 = 2$. From (1a) we have

$$PB_0 = C_1 d + PE_2 d^2.$$

Hence

$$PE_2 d^2 + C_1 d - PB_0 = 0. \tag{2.4.3a}$$

The solutions $d_{1/2}$ are

$$d_{1/2} = \frac{-C_1 \pm \sqrt{(C_1)^2 + 4PB_0 PE_2}}{2PE_2}, \tag{2.4.3b}$$

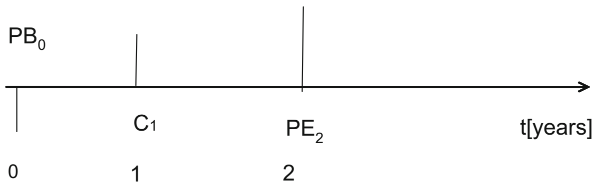
i.e.,

$$IR_{1/2} = \frac{1}{d_{1/2}} - 1. \tag{2.4.3c}$$

We note that in this special case of (1), (3a) is a polynomial of order two that generally has two solutions. We consider the following choices in (3a)

(a) $PB_0 = \$2$, $C_1 = \$4/3$ in (1b) or $C_1 = \$2/3$, $PE_2 = \$0.5$ in (1c).
As can be seen from Fig. 2.8, this case represents the cash flow of a bond with time to maturity 2 base period, i.e., there is an investment at the beginning, one intermediate cash outflow and one cash outflow at the end. Then (3b) yields

Fig. 2.8 Cash flow positive



$$d_1 = 0.896805 \text{ and } d_2 = -2.230139$$

and by (3c) the solutions for the internal rate of return are

$$IR_1 = 0.115069, \quad IR_2 = -1.448403.$$

Figure 2.9 reflects the function (2) with the choices of the parameter considered here.

(b) $PB_0 = \$2$, $C_1 = \$1.0$ in (1b) or $C_1 = \$0.5$, $\quad PE_2 = \$0.5$ in $(1c)$.
Then (3b) yields

$$d_1 = 1 \text{ and } d_2 = -2$$

and by (3c) the solutions for the internal rate of return are

$$IR_1 = 0, IR_2 = -1.5.$$

(c) $PB_0 = \$2$, $C_1 = -\$4/3$ in (1b) or $C_1 = -\$2/3$, $\quad PE_2 = \$0.5$ in $(1c)$.
Then (3b) yields

$$d_1 = 2.230139 \text{ and } d_2 = -0.896805$$

and by (3c) the solutions for the internal rate of return are

$$IR_1 = -0.551597, IR_2 = -2.115069.$$

In **(a)**, **(c)**, resp. the minimum is positive, negative, resp. for the discount factor d coordinate (x coordinate). In all cases **(a)**–**(c)**, it has a negative P (d) coordinate (y coordinate) and there is a unique positive solution. We see that the minimum in Figs. 2.8, 2.9, 2.10, 2.11 and 2.12 changes from negative to positive. Although we have two solutions, the business-relevant solution in

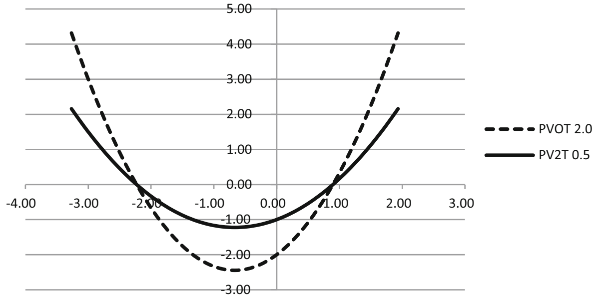


Fig. 2.9 Minimum positive—cash positive

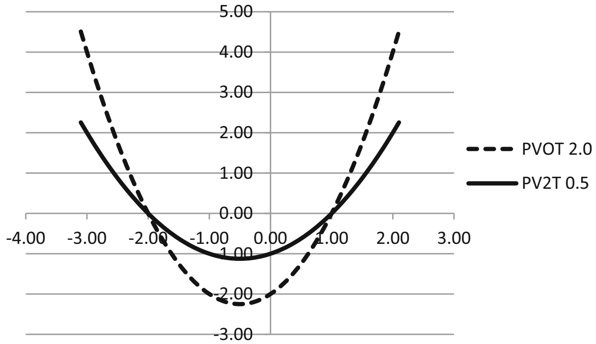


Fig. 2.10 IR is zero

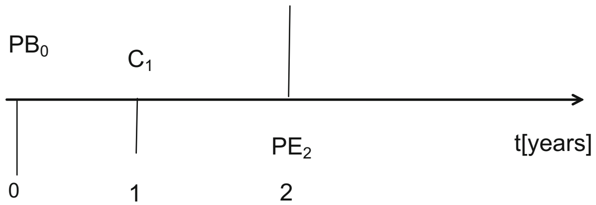


Fig. 2.11 Cash flow negative

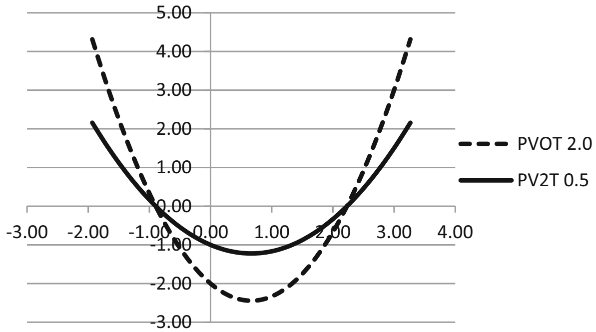


Fig. 2.12 Minimum negative—cash negative

all three cases is IR_1. This example can be solved explicitly because the knots are assumed to be equidistant and we have only one cash flow.

◇

From the example above it can be seen that from $PB_0 < C_1 + PE_2$ resp. $PB_0 > C_1 + PE_2$ it follows $IR_1 > 0$ resp. $IR_1 < 0$.

Example 2.19 (transition from two real solutions to no real solution by continuously changing the cash flow) We assume equidistant knots (2.3.1) with $N = 2$, $T = 2$, $t_0 = 0$, $t_1 = 1$, $t_2 = 2$ and

$$PB_0 = -\$1, \;\; C_1 = -\$\mathbf{8/3}, \;\; PE_2 = \$1$$

in (1b). Then the quadratic equation for the internal rate of return yields the solutions

$$IR_1 = -0.548584, \quad IR_2 = 1.215250.$$

We assume

$$PB_0 = -\$1, \quad C_1 = -\$\mathbf{2}, \quad PE_2 = \$1.$$

Then we have (Fig. 2.13)

$$IR_1 = 0.00, \quad IR_2 = 0.00.$$

We assume

$$PB_0 = -\$1, \quad C_1 = -\$\mathbf{4/3}, \quad PE_2 = \$1.$$

Then there is no real solution. We see that the solution behaves differently in all three cases.

◇

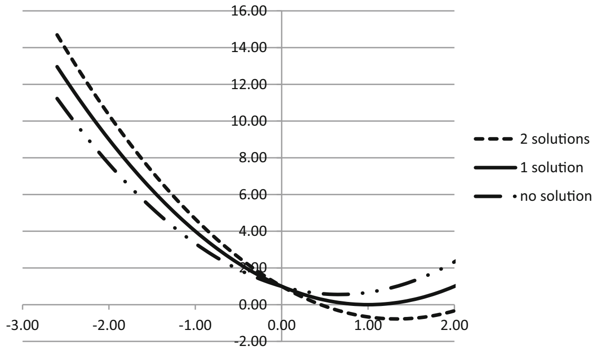


Fig. 2.13 Different number of solutions

2.4.2 Approximating the Solution of the IRR Equation Near Zero

In the following we assume $T = 1, 2, \ldots.$ *and* $t_1 = 1,\ t_2 = 2, \ldots.., t_{N-1} = N - 1,\ t_N = T.$ Then (2) in (1) has integer exponents in the discount factors and is called a polynomial. In keeping with the fundamental theorem of algebra the equation P (d) = 0 has in general T zeros or solutions that can be complex. We denote by **C** the set of complex numbers and i is the unit of the imaginary numbers.

We consider the special case where there are no cash flows during the timespan considered from 0 to T. For $C_k = 0$, k = 1,...., N − 1, the condition P(d) = 0 in (2) is then equal to

$$d^T = \frac{PE_T}{PB_0}.$$

Hence

$$d^T \frac{PB_0}{PE_T} = 1$$

and consequently

$$d_{1,\ldots,T} = \sqrt[T]{\frac{PE_T}{PB_0}}\ e^{\frac{2\pi i}{k}} \in \mathbf{C}, k = 1, \ldots, T,$$

where $\sqrt[T]{\frac{PE_T}{PB_0}}$ is the positive root. By (2.1.3) we find for the internal rate of return

$$IR_{1,\ldots,T} = \frac{1}{d_{1,\ldots,T}} 1 - \in \mathbf{C}.$$

If T is odd we have a real positive zero d and T − 1 complex zeros d, whereas if T is even we have a positive and a negative real zero d. In addition we have T − 2 complex zeros d (see also [24]). If there is no cash flow, MWR is equal to TWR. TWR is not a special case of MWR when the cash flow is zero. As illustrated in Sect. 2.5.2 TWR measures the return from the perspective of the portfolio manager and MWR measures the return from the client's perspective.

Example 2.20 (TWR coincides with MWR) We assume T = 2, N = 2, $t_0 = 0$, $t_1 = 1$, $t_2 = 2$ and no cash flow at $t_1 = 1$ with

$$PE_0 = \$40, \quad PB_2 = \$90$$

in the IRR equation (1). Then we have

$$\mathrm{PE}_0 = \frac{\mathrm{PB}_2}{(1+\mathrm{r})^2}.$$

By using the discount factor we have

$$\mathrm{d}^2 = \frac{40}{90} = 0.444444$$

With the solutions

$$\mathrm{d}_1 = \sqrt{\frac{40}{90}} = 0.666667, \mathrm{d}_2 = -\sqrt{\frac{40}{90}} = -0.666667.$$

Thus

$$\mathrm{IR}_1 = 0.5, \mathrm{IR}_2 = -2.5.$$

From a business perspective, the solution $\mathrm{IR}_1 = 0.5$, i.e., 50 %, has to be considered as IRR. (2.3.5) yields

$$1 +_0 \mathrm{r}_2 = \frac{\mathrm{PE}_1 \cdot \mathrm{PE}_2}{\mathrm{PB}_0 \cdot \mathrm{PB}_1}$$

and as there is no cash flow we have $\mathrm{PE}_1 = \mathrm{PB}_1$, thus

$$\frac{\mathrm{PE}_2}{\mathrm{PB}_0} = \frac{90}{40}$$

and (2.3.2) and (2.3.3) demonstrate that IRR and TWR yield the same results.

◇

A straight bond, i.e., a fixed income instrument with an initial value followed by periodic cash flows and an ending value, is a typical example of the IRR equation (1). The following example illustrates this type of bond.

Example 2.21 (cash flows of a bond) Again considering (1b) and referring to Fig. 2.14 we evaluate (1b) between $\mathrm{r} = 0$ and $\mathrm{r} = 1.6$ for a bond with four cash flows or coupons of 0.2 and a last coupon including the face value 1.2. Then for $\mathrm{r} = 0$ the sum of the face value and the coupon amounts to 2.0. We see that if a price PB_0 lies between 0.132364 and 2, a numerical procedure can compute the corresponding internal rate of return. Also for $\mathrm{r} = 0.2$ the price of the bond PB_0 is equal to 1. Furthermore it is seen that the discount factors are monotonic decreasing functions, just as the sum of those factors is, and as a result there is a unique solution r for a given price between 0 and 2. In bond analytics, the price yield behavior depicted in Fig. 2.14 is referred to as being positive convex. Cases in

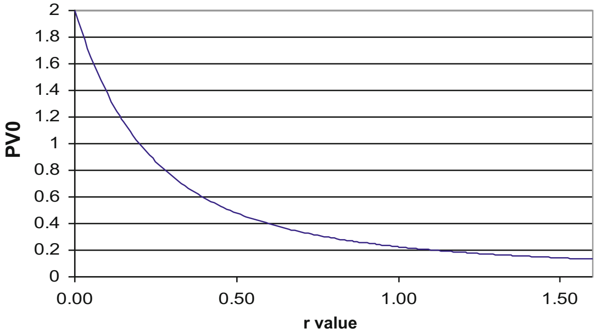


Fig. 2.14 Convexity of a bond ($C = 2.0\ \%$)

which the cash flows are not positive throughout are the subject of further research [25].

◇

Negative real zeros can also be found in cases with two periods and one cash flow by solving the quadratic equation explicitly. In summary, we see that the solution of (1) is not unique.

As we are assuming that the discount factors are positive, we see in Fig. 2.15 that we are interested in varying r from -1 to infinity and as a result $d, d^2, d^3, \ldots$ should be in the interval $[-1, \infty]$. In Fig. 2.15 we illustrate the discount factors d, d^2 and d^3 defined for all r in $\mathbf{R}^1$ except $r = -1$.

In the following we assume T in $\boldsymbol{R}^1$. As interest rates are small real numbers, we develop the discount factors in a neighborhood of $r = 0$. They have a singularity in $r = -1$. Furthermore for $|r| < 1$ the Taylor expansion is 0

$$\frac{1}{(1+r)^t} = 1 + \frac{a_1}{1!}r + \frac{a_2}{2!}r^2 + \ldots + \frac{a_k}{k!}r^k + \ldots, t \in \mathbf{R}^1. \tag{2.4.4}$$

By considering the continuing derivatives of the left-hand side we find that

$$a_k = (-1)^k \prod_{j=1}^{k} (t + j - 1)$$

holds. For $t = 1$, (4) reduces to

$$\frac{1}{1+r} = 1 - r + r^2 + \ldots + (-1)^k \cdot r^k + \ldots$$

and for $k = 1$ we have

$$a_1 = -\ t.$$

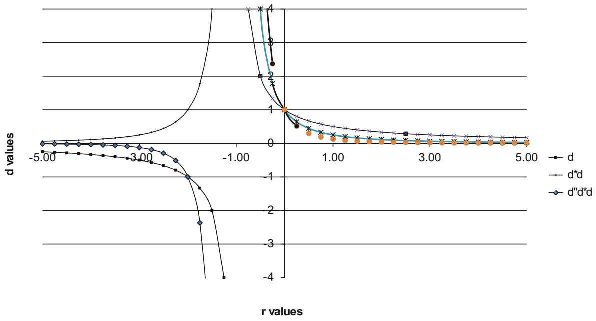


Fig. 2.15 Discount factors

With $T = 1$, $t_0 = 0$, $t_N = 1$ and $0 < t_0 < t_1 << t_{N-1} < 1$ in (2.3.1) we find by multiplying with $\frac{1}{1+r}$

$$(1 + r) \cdot PB_0 = \sum_{k=1}^{N-1} \frac{C_k}{(1 + r)^{t_k - 1}} + PE_1.$$

We consider the first order approximation and have by (4) with $t = t_k - 1$

$$(1 + r) \cdot PB_0 = \sum_{k=1}^{N-1} C_k(1 - r(t_k - 1)) + PE_1.$$

This type of approximation can either be good or bad in nature. We have here the full problem and the (linear) approximation. My colleagues have often marvelled at how accurate the approximations are, but this is hardly a mystery because the chosen parameters like the interest rates are often small and we are fairly close to the full problem; the approximation is then good. However, there's nothing to stop us from changing the parameters, as a result of which the approximation can suddenly turn quite bad. In mathematical finance we generally look at interest rates between 0 and 10 %, i.e., at comparatively small numbers. The accuracy of the approximation depends on the magnitude of the increments r and C_k. The smaller the interest rates, the time period and the cash flows are, the better the series converges, or in other words the better the first order approximation is. In addition, the more terms in the Taylor series (4) of (1) are reflected, the better the approximation is. We solve

$$r = \frac{PE_1 + \sum_{k=1}^{N-1} C_k - PB_0}{PB_0 - \sum_{k=1}^{N-1} C_k(1 - t_k)}. \tag{2.4.5a}$$

r is not IR because our derivation includes the approximation by the Taylor series (4). By replacing C_k by $-C_k$, $k = 1,\ldots,N-1$ we have:

Definition 2.26 The ***Modified Dietz return MD*** is defined by

$$MD = \frac{PE_1 - \sum_{k=1}^{N-1} C_k - PB_0}{PB_0 + \sum_{k=1}^{N-1} C_k (1 - t_k)}. \tag{2.4.5b}$$

In the denominator we have the sum of the portfolio's initial value and the time-weighted cash flows. In the following ***Original Dietz return OD*** the cash flows are centered in the middle of the interval.

Definition 2.27 The ***Original Dietz return OD*** is defined by:

$$OD = \frac{PE_1 - \sum_{k=1}^{N-1} C_k - PB_0}{PB_0 + \sum_{k=1}^{N-1} 0.5 \cdot C_k}. \tag{2.4.6}$$

Example 2.22 (compare Example 2.15) (a) We assume $T = 1$, $N = 2$ with $t_0 = 0$, $t_1 = 0.5$ and $t_2 = 1$ in (2.3.1) and $PB_0 = \$1$, $PE_1 = \$107$, $C_1 = \$98.93$. Then we have with (5) and (6)

$$MD = OD = \frac{PE_1 - C_1 - PB_0}{PB_0 + 0.5 \cdot C_1} = \frac{107 - 98.93 - 1}{1 + 0.5 \cdot 98.93} = 0.140097.$$

(b) We assume $T = 1$, $N = 2$ with $t_0 = 0$, $t_1 = 0.5$ and $t_2 = 1$ in (2.3.1) and $PE_0 = \$100$, $PE_1 = \$1.07$, $C_1 = -\$104$. Then we have with (5) and (6)

$$MD = OD = \frac{PE_1 - C_1 - PB_0}{PB_0 + 0.5 \cdot C_1} = \frac{1.07 + 104 - 100}{100 - 0.5 \cdot 104} = 0.105625.$$

We see that OD and MD do not respect the commutative law shown in Example 2.15 and the results for the returns of OD and MD here are different from the geometrical linking.

◇

Definition 2.28 The ***Profit and Loss PL*** is defined by

$$\mathrm{PL} = \mathrm{PE}_1 - \sum_{k=1}^{N-1} \mathrm{C}_k - \mathrm{PB}_0$$

and the ***average investment capital AIC*** of a portfolio is defined as follows

$$\mathrm{AIC}^{\mathrm{IR}}(\mathrm{IR}) = \begin{cases} \dfrac{\mathrm{PL}}{\mathrm{IR}}, & \mathrm{IR} \neq 0 \\ \mathrm{PB}_0 + \sum_{k=1}^{N-1} \mathrm{C}_k(1 - \mathrm{t}_k), & \mathrm{IR} = 0 \end{cases}, \tag{2.4.7a}$$

$$\mathrm{AIC}^{MD} = \mathrm{PB}_0 + \sum_{k=1}^{N-1} \mathrm{C}_k(1 - \mathrm{t}_k), \tag{2.4.7b}$$

$$\mathrm{AIC}^{OD}\mathrm{PB}_0 + \sum_{k=1}^{N-1} 0.5 \cdot \mathrm{C}_k. \tag{2.4.7c}$$

Remark 2.14 (L'Hôpital's rule) From various examples we have seen that the IR can vanish and that the IR appears for instance in the denominator of the definition (7a). We will use this rule only in the following lemma and we explain the rule in brief here. If the nominator is not zero, the limit becomes unbounded. However, if the nominator is also zero, the ratio can be finite. **L'Hôpital's** rule allows us to calculate such ratios. It states that if the limit of such a ratio is equal to the ratio of the first derivative of the denominator and the first derivative of the nominator, then the first derivative is essentially the slope of a function. In our illustration we consider the exponential function e. From elementary mathematics we know that the slope of the exponential function is the exponential function itself and the slope of the function x is equal to 1. We consider

$$\frac{\mathrm{e}^{\mathrm{x}} - 1}{\mathrm{x}}$$

and we want to calculate the limit to 0. We see that the nominator and denominator vanish. The limit in question can then be found by using the rule

$$\lim_{\mathrm{x}\to 0} \frac{\mathrm{e}^{\mathrm{x}} - 1}{\mathrm{x}} = \lim_{\mathrm{x}\to 0} \frac{\mathrm{e}^{\mathrm{x}}}{1} = 1.$$

Lemma 2.1 (AIC^{IR} is continuous in $IR = 0$) With (7a) we have

$$\lim_{IR\to 0} AIC^{IR}(IR) = AIC^{IR}(0) = AIC^{MD} = PB_0 + \sum_{k=1}^{N-1} C_k(1 - t_k).$$

Proof We consider a portfolio with beginning value PB_0, ending value PE_T and cash flows $C_k \neq 0$, $k = 1,...., N - 1$ and $N \geq 2$ over the time unit interval. The IR is the solution of

$$PB_0(1 + IR) + \sum_{k=1}^{N-1} C_k(1 + IR)^{1-t_k} = PE_T.$$

We subtract the total cash flow

$$PB_0(1 + IR) + \sum_{k=1}^{N-1} C_k(1 + IR)^{1-t_k} - \sum_{k=1}^{N-1} C_k = PE_T - \sum_{k=1}^{N-1} C_k.$$

By

$$PL = PE_T - PB_0 - \sum_{k=1}^{N-1} C_k$$

we have

$$PB_0 \cdot IR + \sum_{k=1}^{N-1} C_k(1 + IR)^{1-t_k} - \sum_{k=1}^{N-1} C_k = PL$$

and for the average invested capital we find

$$\lim_{IR\to 0} AIC^{IR}(IR) = PB_0 + \lim_{IR\to 0} \frac{\sum_{k=1}^{N-1} C_k\left((1 + IR)^{1-t_k} - 1\right)}{IR}.$$

The above relationship shows that the average invested capital is equal to the initial value corrected by the difference of the cash flows and the discounted cash flow to the invested starting point divided by the internal rate of return. In the absence of cash flows, the average invested capital is equal to the portfolio's beginning value. For $IR \to 0$ we use L'Hôpital's rule (Remark 2.2) and find

$$AIC^{IR}(0) = PB_0 + \sum_{k=1}^{N-1} C_k(1 - t_k).$$

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If we neglect second and higher terms in the Taylor series (4) we find for the difference of the discount factor and the first order approximation the expression

$$\frac{1}{(1+r)^t} - (1 - t \cdot r).$$

This difference is sometimes called the ***second order effect***. In Fig. 2.16 it is illustrated that the error term above grows with the power t reflecting the time and the interest rate r.

Example 2.23 (sign change of interest) We consider $T = 1$, $N = 2$ with $t_0 = 0$, $t_1 = 0.5$ and $t_2 = 1$ in (2.3.1) and one cash flow

$$PB_0 + C_1 d^{\alpha} = PE_1 d, \quad 0 < \alpha < 1, C_1 \neq 0$$

but in contrast to Example 2.22 the time intervals are non-equidistant, and do not have length 1.

Referring to Table 2.10 we calculate the figures in bold. The values for the IRR are computed by means of the Excel Routine XIRR. However, for the value 25.36 % ($\alpha = 0.5$) we find by using (1e), (3a), (7a) annualizing

$$0.2536 = (1 + 0.1196) \cdot (1 + 0.1196) - 1,$$
$$AIC^{IR} = \frac{60 - 20 - \ 30}{25.36} = 39.44.$$

By (5)

$$MD = \frac{PE_1 - C_1 - \ PB_0}{PB_0 + (1 - t_k)C_1} = \frac{10}{45} = 22.22\%,$$

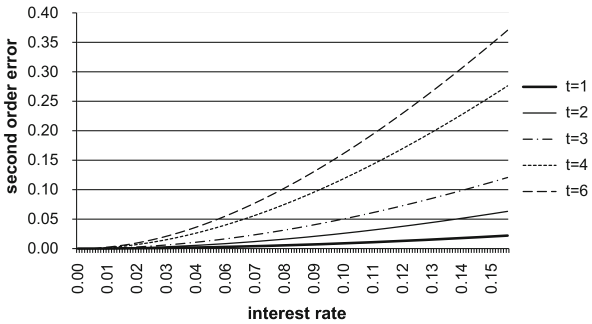


Fig. 2.16 Second order effect

Table 2.10 Different return calculation

α	PV0	C1	PV1	IRR (%)	AIC (IRR)	MD (%)	AIC (MD)	OD (%)	AIC (OD)
0.25	30	−20	60	21.51	46.50	**22.22**	**45.00**	25.00	40.00
	30	−30	60	0.00	52.50	0.00	52.50	0.00	45.00
	30	−40	60	−15.63	63.99	−16.67	60.00	−20.00	50.00
0.5	30	−20	60	**25.36**	**39.44**	25.00	40.00	25.00	40.00
	30	−30	60	0.00	45.00	0.00	45.00	0.00	45.00
	30	−40	60	−19.57	51.09	−20.00	50.00	−20.00	50.00
0.75	30	−20	60	27.66	36.15	28.57	35.00	25.00	40.00
	30	−30	60	0.00	37.50	0.00	37.50	0.00	45.00
	30	−40	60	−22.46	44.52	−25.00	40.00	**−20.00**	**50.00**

by (7b)

$$\mathrm{AIC}^{\mathrm{MD}} = 30 + (1 - 0.25) \cdot 20 = 45.$$

By (5)

$$\mathrm{OD} = \frac{\mathrm{PE}_1 - \mathrm{C}_1 - \ \mathrm{PB}_0}{\mathrm{PB}_0 + 0.5\ \ \mathrm{C}_1} = \frac{60 - 40 - 30}{50} = -\frac{10}{20} = -20\%$$

and by (7c)

$$\mathrm{AIC}^{\mathrm{OD}} = 30 + \frac{1}{2} \cdot 40 = 50.$$

We see from Table 2.10 that the Modified Dietz (MD) is a better approximation than Ordinary Dietz (OD) for the internal rate of return. As expected, if the cash flow is in the middle of the time period ($\alpha = 0.5$), MD is equal to OD. By reducing the cash for all α, the different measurements for the return show that the sign of the return changes from positive to negative. These properties are preserved for the approximations of MD and OD for IR.

◇

2.4.3 *The Solution of the IRR and Bond Portfolios*

In this section we expand the discussion of the solution of the equation of the internal rate of return (IRR) as introduced in (1). Example 2.18 and 2.19 are first illustrations for (1) and give only a first insight in the behaviors of its solutions. The cash flows can be in past (ex post) and in the future (ex ante). In Bond analytics futures cash flow in the future are important and often invested in detail.

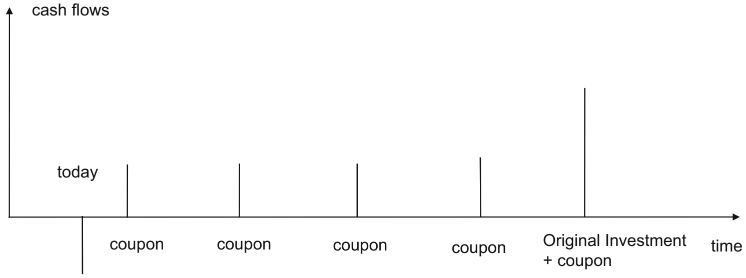


Fig. 2.17 Straight bond

In the following we focus on straight bonds as illustrated in Example 2.21. We introduce the following notations. A ***straight bond*** with ***price P*** will pay back ***the face or nominal value F*** on its ***maturity date T*** and will pay ***a Coupon C*** on specific dates (see Fig. 2.17).

We start with the knots $t_0 = 0,\ldots,t_N = T$ on the time axis that are equidistant, i.e., $t_k = k$, $k = 0,\ldots, N = T$ and with cash flows $C_1,\ldots,C_{N-1}$. In the following we consider exclusively future cash-flows.

As depicted in Fig. 2.17 for straight bond we have an investment today followed by positive cash flows and a face value paid back and the end of the investment time. (1a) reduces to

$$P(r) = \sum_{k=0}^{N} \frac{C}{(1+r)^k} + \frac{F}{(1+r)^T}, \quad \forall r \in \mathbf{R}^1 . \tag{2.4.8}$$

This formula is applied in two versions.

- In a scenario analysis the behaviour of the Price of a straight bond is investigated by considering different interest rate r.
- Assuming that we have a market price on the left-hand on (8) side and the right-side is given by the description of the Bond (reference data) the solution of the IRR equation is not called the ***internal rate of return (IRR)*** but the **yield to maturity (YTM)**. Solving (1) is based on the assumptions: that all future Coupons are paid, i.e., there are no defaults and the investor holds the bond till time to maturity.
- It is not clear how yield to maturity are adding for different bonds.

We proceed by extending (8) to a Bond portfolio with Price Po that have n straight bonds

$$Po(r) = \sum_{j=1}^{n} N_j P_j(r) = \sum_{j=1}^{n} \left(\sum_{k=1}^{T_j} \frac{N_j C_{j,k}}{(1+r)^k} + \frac{N_j F_j}{(1+r)^{T_j}} \right) \tag{2.4.9a}$$

With the abbreviations

$$P_j = P_j(r_j), \quad j = 1, \ldots, n$$

and

$$P_j(r) = \sum_{k=1}^{T_j} \frac{C_j}{(1+r)^k} + \frac{F_j}{(1+r)^{T_j}}$$

we have

$$\sum_{j=1}^{n} N_j P_j - \sum_{j=1}^{n} N_j P_j(r) = 0. \tag{2.4.9b}$$

As the computation of the solution IR of (9) needs in general a numerical procedure and as the collection of the coupon can be tedious, an approximation is often used instead in practical situation. We consider a weighted combination of the yield to maturities of the individual bonds for the approximation of the solution IR:

$$r_{nom} = \sum_{j=1}^{n} w_j r_j \tag{2.4.10a}$$

$$w_j = \frac{N_j}{\sum_{i=1}^{n} N_i} \tag{2.4.10b}$$

and

$$r_{lin} = \sum_{j=1}^{n} \hat{w}_j r_j \tag{2.4.11a}$$

$$\hat{w}_j = \frac{N_j P_j}{\sum_{i=1}^{n} N_i P_i}. \tag{2.4.11b}$$

(10) is nominal weighted and (11) is asset weighed approximation of IR. The software providers and the benchmark providers however, recommend a Macaulay Duration weighted approximation, i.e.,

$$r_{mac} = \sum_{j=1}^{n} v_j r_j \tag{2.4.12a}$$

with the abbreviation in (9b) we consider

$$v_j = \frac{N_j P_j D^j_{mac}(r_j)}{\sum_{i=1}^{n} N_i P_i D^i_{mac}(r_j)} \tag{2.4.12b}$$

where $D^j_{Mac}(r_j)$ is the Macaulay Duration defined by

$$D^j_{Mac}(r_j) = \frac{\sum_{k=1}^{N} k \frac{C}{(1+r_j)^k} + N \frac{F}{(1+r_j)^N}}{\sum_{k=1}^{N} \frac{C}{(1+r_j)^k} + \frac{F}{(1+r_j)^N}}. \tag{2.4.12c}$$

This is the same as

$$D^j_{Mac}(r_j) = \frac{\sum_{k=1}^{N} k \frac{C}{(1+r_j)^k} + n \frac{F}{(1+r_j)^N}}{P(r_j)}.$$

As the transition from a straight Bond (8) to a portfolio (9) we consider the Macaulay Duration of a Portfolio

$$D^{Po}_{Mac}(r) = \frac{\sum_{j=1}^{n} \left(\sum_{k=1}^{T_j} k \frac{N_j\ C_j}{(1+r)^k} + T_j \frac{N_j\ F_j}{(1+r)^{T_j}} \right)}{\sum_{j=1}^{n} \left(\sum_{k=1}^{T_j} \frac{N_j\ C_j}{(1+r)^k} + \frac{N_j\ F_j}{(1+r)^{T_j}} \right)}. \tag{2.4.13}$$

As shown in Fig. 2.18 there are different ways of evaluating (13). In most software the yield of maturities of the individual is substituted (YTM approach). This makes

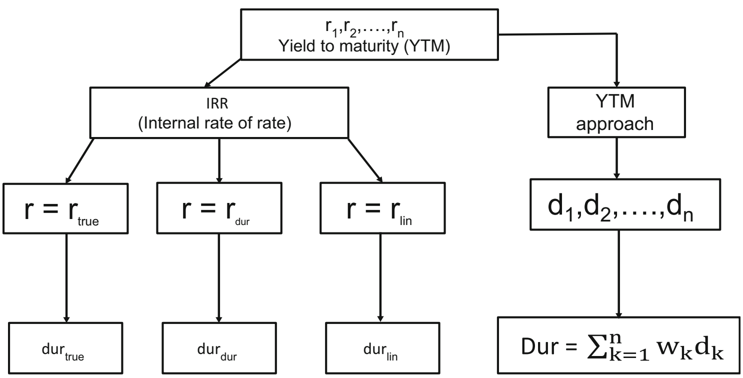


Fig. 2.18 The Macaulay duration

little sense as we cannot have different yield in a portfolio. We propose therefore the exact solution of (8) or an approximation thereof.

The analysis of the approximation of (10)–(12) of the solution of (9) in subject in current research and is in preparation for a publication. Here we confine our exposition to a typical example.

Definition 2.29 We consider a yield curve considering the following yields of maturity

$$r_1 > 0, \ldots, r_j > 0, \ldots r_n > 0, \quad 1 \leq j \leq n$$

and the time points

$$t_1 > 0, \ldots, t_j > 0, \ldots, t_n > 0, \quad 1 \leq j \leq T_n .$$

We discuss three cases:

(a) If

$$r_1 = r_2, \ldots r_j = r_{j+1} \ldots, r_{n-1} = r_n, \quad 1 \leq j \leq n$$

the yield curve is said to be ***flat***.

(b) If

$$r_1 > r_2, \ldots r_j > r_{j+1} \ldots, r_{n-1} > r_n, \quad 1 \leq j \leq n$$

the yield curve is said to be ***normal***.

(c) If

$$r_1 < r_2, \ldots r_j < r_{j+1} \ldots, r_{n-1} < r_n, \quad 1 \leq j \leq n$$

the yield curve is said to be ***inverted***.

Example 2.24 (different yield, different time to maturities) We consider three straight bonds with Coupon $C = 9\ \%$ with $N_j = 1$, $j = 1, 2, 3$ and the time to maturities are $T_1 = 4$, $T_2 = 9$, $T_3 = 14$. We assume $r_2 = 9\ \%$ and consider

1. $r_1 = r_2 - \alpha, r_3 = r_2 + \alpha$ (Normal yield curve).
2. $r_1 = r_2 - \alpha, r_3 = r_2 - \alpha$, (Inverted yield curve).

We chose $\alpha \in \mathbf{N}$ between $1\% \leq \alpha \leq 5\%$. The Figs. 2.19 and 2.20 shows the difference of the different approximation for IR. We see that the duration approximation (5) yields the best approximation. If the yield is flat the approximations are equal to the solution of (1). The Figs. 2.21 and 2.22 are the accompanying Macaulay Duration calculations.

◇

Fig. 2.19 Approximation of IR for increasing yields

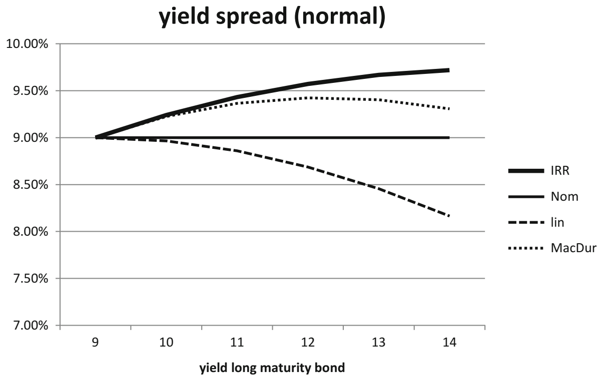


Fig. 2.20 Approximation of IR for decreasing yield spread

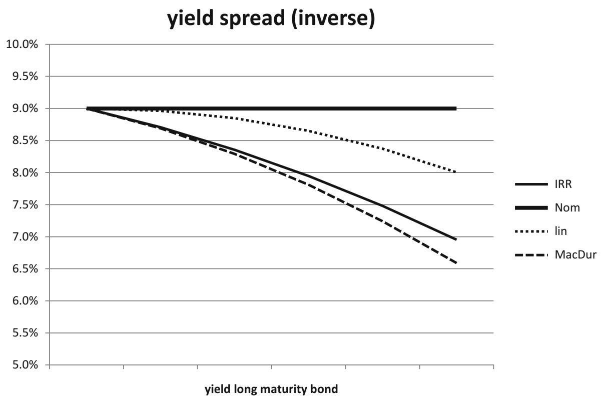
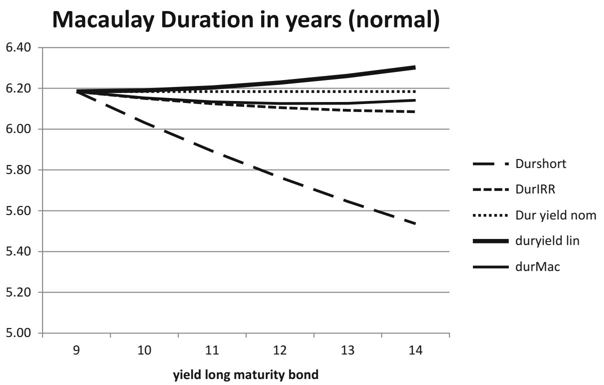


Fig. 2.21 Macaulay duration calculation for increasing yields



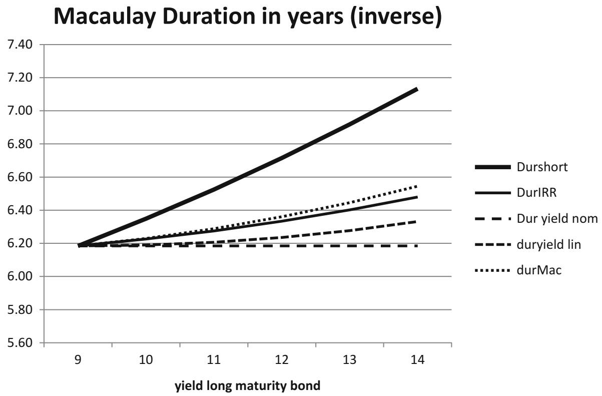


Fig. 2.22 Macaulay duration calculation for decreasing yields

2.5 TWR Attribution Versus MWR Attribution

2.5.1 Absolute Decomposition of the MWR

In this section we consider a portfolio with n constituents at discrete points in time

$$t_0 = 0, \quad 0 < t_1 < t_2 \ldots < t_N, \quad t_N = T. \tag{2.5.1a}$$

For the value PB_0 of the portfolio and the value $PB_{j,0}$, $j = 1,\ldots, n$ of its segments at the beginning $t_0 = 0$ of the time period considered we have

$$PB_0 = \sum_{j=1}^{n} PB_{j,0} \tag{2.5.1b}$$

and for the value PE_T of the portfolio and the value $PE_{j,T}$, $j = 1,\ldots, n$ of its segments at the end $t_N = T$ of the time period we have

$$PE_T = \sum_{j=1}^{n} PE_{j,T}. \tag{2.5.1c}$$

Remark 2.15 Contrary to (2.3.5) and (2.3.7), (1) specifies the time the portfolio is considered. In (1) the products in (2.3.5) and (2.3.7) are summarized to values of segments expressed in the base currency of the portfolio, i.e., this notation allows us to investigate not only investments like equity or bonds but also cash and cash flows.

As we consider in this section not only a portfolio but also a benchmark, we distinguish between cash flows affecting the portfolio and cash flows affecting the benchmark. The ***cash flows*** $\boldsymbol{PC_{j,k}}$, $j = 1,\ldots, n$, $k = 1,\ldots, N - 1$ with the ***segment j*** of

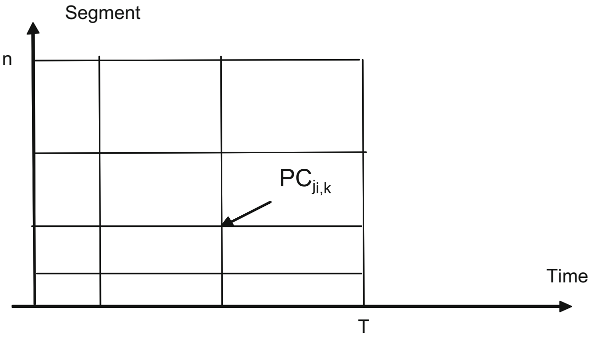


Fig. 2.23 The full portfolio return problem (cash management)

the portfolio P are composed of the ***internal cash flow*** $PIC_{j,k}$ and ***external cash flow*** $PEC_{j,k}$ with (Fig. 2.23)

$$PC_{j,k} = PIC_{j,k} + PEC_{j,k} \tag{2.5.1d}$$

where the overall cash flow PC_k in and out of the portfolio is

$$PC_k = \sum_{j=1}^{n} PC_{j,k} = \sum_{j=1}^{n} \left(PIC_{j,k} + PEC_{j,k}\right).$$

The solution IR_P of (2.4.1) for a portfolio is a function of an initial investment PB_0, a series of cash flows PC_k, $k = 1,\ldots, N-1$, followed by a terminal value PE_T:

$$IR_P = IR(PB_0, PC_1, \ldots., PC_{N-1}, PE_T). \tag{2.5.2}$$

We have shown in Sect. 2.4 that (2) can have more than one solution. We assume here that there is a unique business-relevant solution. If there are no cash flows, the MWR is equal to the TWR (see Example 2.20). In the case of cash flows the difference of the TWR and IR is reflected in the timing effect (see paragraph c '**A first discussion of the difference between the MWR and TWR'** of this section). The following example is essentially based on the article [15]:

Example 2.25 (difference between TWR and MWR, external cash flow) We assume $T = 2$, $N = 2$, $t_0 = 0$, $t_1 = 1$, $t_2 = 2$, $PB_0 = \$200$ and

$${}_0r_{P,1} = 0.25, \quad {}_1r_{P,2} = -0.2$$

in (2.3.2). Then

$$PE_1 = \$200 \cdot (1 + 0.25) = \$250.$$

We distinguish between three cases:

(a)

$$PIC_1 = 0, \quad PEC_1 = 0,$$

i.e., by (1c),

$$\mathrm{PC}_1 = 0.$$

By (2.3.5) we have for the TWR

$$_0\mathrm{r} = \frac{\mathrm{PE}_1 \cdot \mathrm{PE}_2}{\mathrm{PB}_0 \cdot \mathrm{PB}_1} - 1 = \frac{250 \cdot 200}{200 \cdot 250} - 1 = (1 + 0.25) \cdot (1 - 0.2) - 1 = 0.$$

For the profit and loss PL (see Definition 2.28) we have

$$\mathrm{PL} = 0.$$

We have $\mathrm{PB}_0 = \mathrm{PE}_2$, thus by (2.4.1) and (2.4.3) we find

$$\mathrm{d}_1 = 1, \quad \mathrm{d}_2 = -1,$$

i.e., the MWR realized here with the internal rate of return is

$$\mathrm{IR} = 0.$$

(b)

$$\mathrm{PIC}_1 = 0, \quad \mathrm{PEC}_1 = \$50,$$

i.e., by (1c)

$$\mathrm{PC}_1 = \$50.$$

By (2.3.5) we have for the TWR

$$_0\mathrm{r} = \frac{\mathrm{PE}_1 \cdot \mathrm{PE}_2}{\mathrm{PB}_0 \cdot \mathrm{PB}_1} - 1 = \frac{250 \cdot 240}{200 \cdot 300} - 1 = 0.$$

We have

$$\mathrm{PL} = \$90$$

and the solution of (2.4.1)

$$\mathrm{d}_1 = 1.0229615, \quad \mathrm{d}_2 = -0.814628$$

i.e., we find for MWR

$$\mathrm{IR} = -0.0224461.$$

(c)

$$\mathrm{PIC}_1 = 0, \quad \mathrm{PEC}_1 = -\$50,$$

i.e., by (1c)

$$PC_1 = -\$50.$$

By (2.3.5) we have for the TWR

$$_0r = \frac{PE_1 \cdot PE_2}{PB_0 \cdot PB_1} - 1 = \frac{250 \cdot 160}{200 \cdot 200} - 1 = 0$$

and we have

$$PL = -\$90$$

and the solution of (1) is

$$d_1 = -1.285149, \quad d_2 = 0.9726495,$$

i.e., for the business-relative solution we find

$$IR = 0.0281196.$$

In all three cases the TWR vanishes, as it is unaffected by cash flows after the first period; the MWR, however, reflects the PL, i.e., selling in a bear market is a good decision and buying in a bear market is a bad decision. In addition, the MWR equals the TWR if there is no cash flow. We continue the discussion concerning the MWR and TWR in the following section.

◇

We proceed by introducing the portfolio value and its segments in the investment period. With PB_k, $PB_{j,k}$, resp. we denote the portfolio value, the portfolio value of segment j, $j = 1,\ldots, n$, resp. at the beginning of period k, $k = 0,\ldots, N-1$ and with PE_k, $PE_{j,k}$, resp. the portfolio value, the portfolio value of segment j, $j = 1,\ldots, n$, resp. at the end of period $k = 1,\ldots, N$.

Remark 2.16 The notation introduced above is consistent with the notation in (1) with $k = 0$ and $k = N$.

Then the weights $W_{j,k}$ of portfolio segment j at the beginning of period k are

$$W_{j,k} = \frac{PB_{j,k}}{\sum_{i=1}^{n} PB_{i,k}}, \quad k = 0, \ldots, N-1, \quad j = 1, \ldots, n,$$

and for the portfolio return $R_{j,k}$ over the period $[k, k+1]$, $k = 0,\ldots, N-1$ of segment j , $j = 1,\ldots,n$ we have

$$R_{j,k} = \frac{PE_{j,k+1} - PB_{j,k}}{PB_{j,k}}, \quad k = 0, \ldots, N-1, \quad j = 1, \ldots, n.$$

As a result, for the market value of the portfolio $PE_{j,k+1}$ at time $k+1$ in segment j we have:

$$PE_{j,k+1} = \left(1 + R_{j,k}\right) PB_{j,k}. \tag{2.5.3}$$

Furthermore $PC_{j,k}$ denotes the portfolio cash flow in or out of segment j at time k given by

$$PC_{j,k} = PE_{j,k} - PB_{j,k}, \quad k = 1, \ldots, N-1. \tag{2.5.4}$$

As in (2) for a portfolio we consider the IR_j for segment j

$$\begin{aligned} IR_j = IR\left(PB_{j,0}, PC_{j,k}, k = 1, \ldots, N-1, PE_{j,T}\right) = \\ IR\left(PB_{j,0}, R_{j,k}, W_{j,k}, k = 0, \ldots, N-1\right). \end{aligned} \tag{2.5.5}$$

The Profit and Loss PL_P of the portfolio over the investment period [0, T] is equal to the sum of the PL_j, $j = 1, \ldots, n$ of the individual segments:

$$PL_P = \sum_{j=1}^{n} PL_j. \tag{2.5.6}$$

Definition 2.30 Assuming that there exists a solution $IR_j \neq 0$, $IR_P \neq 0$, resp. ***the average invested capital*** for segments AIC_j^{IR}, AIC_P^{IR}, resp. is

$$AIC_j^{IR} = \frac{PL_j}{IR_j}, \tag{2.5.7a}$$

$$AIC_P^{IR} = \frac{PL_P}{IR_P}, \quad \text{resp}. \tag{2.5.7b}$$

Using the average invested capital for the internal rate of return we have

$$AIC_P^{IR} IR_P = \sum_{j=1}^{n} AIC_j^{IR} IR_j,$$

thus we find a decomposition of IR

$$IR_P = \frac{1}{AIC_P^{IR}} \sum_{j=1}^{n} AIC_j^{IR} IR_j = \sum_{j=1}^{n} \frac{AIC_j^{IR}}{AIC_P^{IR}} IR_j. \tag{2.5.8}$$

Definition 2.31 By defining ***the return contribution*** $\boldsymbol{RC_j}$ by

$$\mathrm{RC}_j = \frac{\mathrm{AIC}_j^{\mathrm{IR}}}{\mathrm{AIC}_P^{\mathrm{IR}}} * \mathrm{IR}_j. \tag{2.5.9}$$

We have

Theorem 2.4 As described at the beginning of this section with equation (1) we consider a portfolio P with n segments on the time interval [0, T]. Then the decomposition of the internal rate of return IRp based on the return contribution defined in Definition 2.30 satisfies

$$\mathrm{IR}_P = \sum_{j=1}^{n} \mathrm{RC}_j \tag{2.5.10}$$

and is consistent with the PL of the portfolio and its segments.

Proof (10) follows by combining (6)–(9).

□

We proceed by discussing two special cases of (9) and show that

$$\sum_{j=1}^{n} \mathrm{AIC}_j^{\mathrm{IR}} = \mathrm{AIC}_P^{\mathrm{IR}}.$$

Case 1 (no cash flows) We assume $T = 1$ in (1a) and $C_{j,k} = 0$, $j = 1,\ldots., n$, $k = 1,\ldots.,$ $N - 1$, $N \geq 2$. Then we have

$$\mathrm{IR}_j = \frac{\mathrm{PE}_{j,T} - \mathrm{PB}_{j,0}}{\mathrm{PB}_{j,0}}$$

and

$$\mathrm{IR}_P = \frac{\mathrm{PE}_T - \mathrm{PB}_0}{\mathrm{PB}_0}.$$

Thus by the definition (2.4.7a)

$$\mathrm{AIC}_j^{\mathrm{IR}} = \frac{\mathrm{PL}_j}{\mathrm{IR}_j} = \frac{\mathrm{PE}_{j,T} - \mathrm{PB}_{j,0}}{\mathrm{IR}_j} = \mathrm{PB}_{j,0}$$

and

$$\mathrm{AIC}_{\mathrm{P}}^{\mathrm{IR}} = \frac{\mathrm{PL}_{\mathrm{P}}}{\mathrm{IR}_{\mathrm{P}}} = \frac{\mathrm{PE}_0 - \mathrm{PB}_{\mathrm{T}}}{\mathrm{IR}_{\mathrm{P}}} = \mathrm{PB}_0.$$

Thus by (1b)

$$\mathrm{AIC}_{\mathrm{P}}^{\mathrm{IR}} = \sum_{j=1}^{n} \mathrm{AIC}_{j}^{\mathrm{IR}}.$$

Case 2 (no compounding, linearity of Modified Dietz, IR = 0) Here we are concerned with the modified Dietz formula at the portfolio level

$$\mathrm{MD}_{\mathrm{P}} = \frac{\mathrm{PE}_1 - \sum_{k=1}^{N-1}\sum_{j=1}^{n} \mathrm{PC}_{j,k} - \mathrm{PB}_0}{\mathrm{PB}_0 + \sum_{k=1}^{N-1}(1-t_k)\sum_{j=1}^{n}\mathrm{PC}_{j,k}}.$$

By using (1b) and (1c) this is the same as

$$\mathrm{MD}_{\mathrm{P}} = \frac{\sum_{j=1}^{n}\left[\mathrm{PE}_{j,T} - \sum_{k=1}^{N-1}\mathrm{PC}_{j,k} - \mathrm{PB}_{j,0}\right]}{\mathrm{PB}_0 + \sum_{k=1}^{N-1}(1-t_k)\sum_{i=1}^{n}\mathrm{PC}_{i,k}}.$$

Hence

$$\mathrm{MD}_{\mathrm{P}} = \sum_{j=1}^{n}\left[\frac{\mathrm{PB}_{j,0} + \sum_{k=1}^{N-1}(1-t_k)\mathrm{PC}_{j,k}}{\mathrm{PB}_0 + \sum_{k=1}^{N-1}(1-t_k)\sum_{j=1}^{n}\mathrm{PC}_{j,k}} \cdot \frac{\mathrm{PE}_{j,T} - \sum_{k=1}^{N-1}\mathrm{PC}_{j,k} - \mathrm{PB}_{j,0}}{\mathrm{PB}_{j,0} + \sum_{k=1}^{N-1}(1-t_k)\mathrm{PC}_{j,k}}\right].$$

By identifying

$$w_j^{\mathrm{MD}} = \frac{\mathrm{PB}_{j,0} + \sum_{k=1}^{N-1}(1-t_k)\mathrm{PC}_{j,k}}{\mathrm{PB}_0 + \sum_{k=1}^{N-1}(1-t_k)\sum_{i=1}^{n}\mathrm{PC}_{i,k}}, \quad j = 1, 2, \ldots, n$$

and

$$\mathrm{MD}_{j} = \frac{\mathrm{PE}_{j,T} - \sum_{k=1}^{N-1} \mathrm{PC}_{j,k} - \mathrm{PB}_{j,0}}{\mathrm{PB}_{j,0} + \sum_{k=1}^{N-1} (1 - t_k)\, \mathrm{PC}_{j,k}}, \quad j = 1, 2, \ldots, n$$

we find

$$\mathrm{MD}_{P} = \sum_{j=1}^{n} w_j^{MD} \mathrm{MD}_{j}.$$

By using the definition of the average invested capital for the modified Dietz (IR = 0) in (2.4.7) and Lemma 2.1 we find

$$\sum_{j=1}^{n} w_j^{MD} = \sum_{j=1}^{n} \frac{\mathrm{PB}_{j,0} + \sum_{k=1}^{N-1} (1 - t_k)\, \mathrm{PC}_{j,k}}{\mathrm{PB}_0 + \sum_{k=1}^{N\text{-}1} (1 - t_k) \sum_{i=1}^{n} \mathrm{PC}_{i,k}} = 1.$$

$$\mathrm{AIC}_P^{MD} = \sum_{j=1}^{n} \mathrm{AIC}_j^{MD}.$$

As OD is a special case of MD we conclude

$$\mathrm{AIC}_P^{OD} = \sum_{j=1}^{n} \mathrm{AIC}_j^{0D}.$$

In the following we use only the AIC for the internal rate of return and suppress the superscript IR in AIC throughout.

As illustrated in Example 2.26 the decomposition (9) has a weighting scheme that does not add up to one, i.e., in general AIC_P^{IR} is different from $\sum_{j=1}^{n} \mathrm{AIC}_j$.

Example 2.26 (internal cash flow) We consider **a portfolio** with initial value $\mathrm{PB}_0 = \$100$ over 2 years. The portfolio consists of two segments, the first of which has an initial value of $\mathrm{PB}_{1,0} = \$60$ and the second of which has an initial value of $\mathrm{PB}_{2,\,0} = \$40$. In the first period the returns of the first, second segment, resp. are

$$R_{1,0} = 25\%, \quad R_{2,0} = 37.5\%, \tag{2.5.11a}$$

resp. and in the second period the returns are

$$R_{1,1} = 100\%, \quad R_{2,1} = 40\%, \quad \text{resp.} \tag{2.5.11b}$$

There is no external cash flow, i.e., $PEC_1 = 0$, $PEC_2 = 0$ (see (1c)) and there is an internal cash flow of

$$PC_{1,1} = PIC_{1,1} = -\$15, \quad PC_{2,1} = PIC_{2,1} = \$15 \tag{2.5.11c}$$

in the middle of the 2-year period.

We proceed with the following calculations. Based on (3) and (11) we have

$$\begin{aligned} PE_{1,1} &= \$60 \cdot (1 + 0.250) = \$75 \\ PE_{2,1} &= \$40 \cdot (1 + 0.375) = \$55. \end{aligned}$$

Furthermore by (4)

$$\begin{aligned} PB_{1,1} &= \$75 - \$15 = \$60, \\ PB_{2,1} &= \$55 + \$15 = \$70. \end{aligned}$$

Again with (3) we have

$$\begin{aligned} PE_{1,2} &= \$60 \cdot (1 + 1.0) = \$120, \\ PE_{2,2} &= \$70 \cdot (1 + 0.4) = \$98, \\ PE_2 &= PVE_{1,2} + PVE_{2,2} = \$218. \end{aligned}$$

With (6), (7) and (8) we have

$$\begin{aligned} PL_1 &= \$120 - \$15 - \$60 = \$45, \\ PL_2 &= \$98 + \$15 - \$40 = \$73, \\ PL_P &= \$218 - \$100 = \$118. \end{aligned}$$

By using an Excel routine we find

$$IR_1 = 29.47\%, \quad IR_2 = 76.39\%, \quad IR_P = 47.65\%,$$

$$\begin{aligned} AIC_1 &= \frac{\$45}{29.47} = \$1.5268, \\ AIC_2 &= \frac{\$73}{76.39} = \$0.9556, \\ AIC_P &= \frac{\$118}{47.65} = \$2.4765. \end{aligned}$$

The return contributions (9) are

$$\begin{aligned} RC_1 &= \frac{AIC_1}{AIC_P} IR_1 = \frac{45*47.65\%}{118} = 18.17\%, \\ RC_2 &= \frac{AIC_2}{AIC_P} IR_2 = \frac{73*47.65\%}{118} = 29.48\%, \end{aligned}$$

which illustrates and is consistent with (10)

$$\mathrm{RC}_1 + \mathrm{RC}_2 = 47.65\%.$$

◇

2.5.2 A First Discussion of the Difference Between the MWR and TWR

In this section we discuss the impact of the client's investment decision on the overall return of the portfolio and we are concerned with the following question: what part of the overall performance can be credited to the asset manager and what part is due to the client's decisions? In the following discussion we assume that only the client has control over the external cash flows.

In Example 2.15 we illustrated that the TWR does not depend on changes in the invested capital. As a result, the TWR measures the return that is due to the asset manager's investment decision. As we can see from e.g. the IRR equation, the MWR incorporate the external cash flows in the portfolio and thus measures the return from the client's perspective. Unlike the MWR:

- The TWR allows us to make comparisons with a benchmark as well as with peers and competitors
- The calculation, decomposition and reporting of the TWR are common practice in the asset management industry
- Determining the TWR is one of the main principles of the GIPS standard (see e.g. [3])

In the following we discuss the return attribution from a client's point of view and illustrate the decomposition of the MWR and its relation to the return attribution based on the TWR (see also [16]). We start by explaining the notions in the middle line of Fig. 2.24.

- The "***benchmark effect***" is the return contribution due to the decision to invest the initial money into a specific benchmark strategy and is equal to the benchmark return over the investment period.
- The "***management effect***" is the return contribution due to the decision to change the asset allocation and stock selection of the account relative to the benchmark over the investment period.
- The "***timing effect***" is the return contribution due to the decision to change the money invested in the benchmark strategy and in the active asset allocation of the account over the investment period.

What is indicated in bold in Fig. 2.24 is measured and computed on the basis of portfolio and market data**.** First we calculate the management effect from the difference between the TWR and benchmark effect, then calculate the timing effect by subtracting the benchmark effect and the management effect from the MWR.

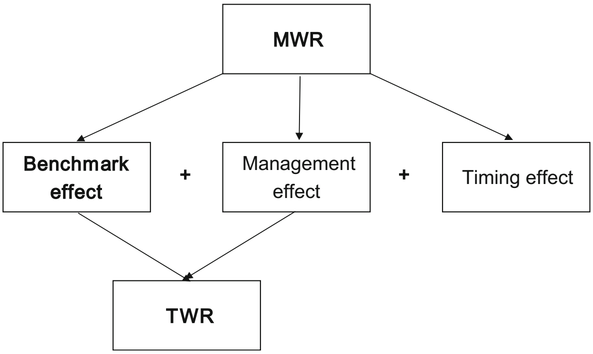


Fig. 2.24 Overview of decomposing the MWR

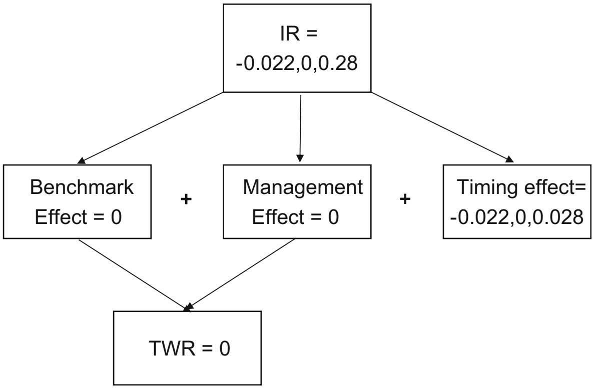


Fig. 2.25 A first example of decomposing the MWR

Example 2.25 (continuation) We discuss Example 2.25 in relation to Fig. 2.24. We have calculated the IRR (MWR) and the TWR is zero. We make the assumption that the benchmark return is zero. Furthermore the management effect is also zero. In Fig. 2.25 we see the impact of investing or withdrawing \$50. We see that withdrawing the cash was a good decision and an additional investment was a bad decision, as we simulate here a bear market in the second investment phase.

◇

2.5.3 Relative Decomposition of the IRR

In (2.2.8) and (2.2.9) we decomposed the relative return of a portfolio. Essentially the formula is valid for a single period, i.e., the weights are given at the beginning of the investment period and it is assumed that there is no cash flow. In the following section we introduce and discuss the case of the IR Attribution on an absolute and on a relative basis. In a portfolio the weights are given at the beginning of a period, however, by the end of the period they are different due to market changes. Similar to the previous section, for a portfolio we proceed by introducing ***a benchmark (portfolio)*** and its segments in the investment period. With BB_k, $BB_{j,k}$, resp. we denote the benchmark value, the benchmark value of segment j, $j = 1, \ldots, n$, resp. at the beginning of period k, $k = 0, \ldots, N-1$ and with BE_k, $BE_{j,k}$, resp. the benchmark value, the benchmark value of segment j, $j = 1, \ldots, n$, resp. at the end of period $k = 1, \ldots, N$. For the beginning value of the benchmark value BB_0 and its segments $BB_{j,0}$, $j = 1, \ldots, n$ at $t_0 = 0$ we have

$$BB_0 = \sum_{j=1}^{n} BB_{j,0},$$

and for the ending value of the portfolio value EB_T and its segments $EB_{i,T}$, $j = 1, \ldots,$ n at $t_N = T$ we have

$$EB_T = \sum_{j=1}^{n} EB_{j,T}.$$

The overall cash flow of the benchmark BC_k is composed of the ***internal cash flow BIC*** and the ***external cash flow BEC***

$$BC_k = \sum_{j=1}^{n} BC_{i,k} = \sum_{j=1}^{n} \left(BIC_{j,k} + BEC_{j,k}\right).$$

The weights $W_{j,k}$ of the benchmark segment j at the beginning of period k are

$$V_{j,k} = \frac{BB_{j,k}}{\sum_{i=1}^{n} BB_{i,k}}$$

and for the benchmark return $B_{j,k}$ over the period $[k, k+1]$, $k = 0, \ldots, N-1$ of segment j , $j = 1, \ldots, n$ we have

$$B_{j,k} = \frac{BE_{j,k+1} - BB_{j,k}}{BB_{j,k}}.$$

As a result, for the market value of the benchmark at time $k+1$ in segment j we have:

$$\mathrm{BE}_{j,k+1} = \left(1 + \mathrm{B}_{j,k}\right)\mathrm{BB}_{j,k}. \tag{2.5.12}$$

Furthermore BC_j^k denotes the derived portfolio cash flow and benchmark cash flow, respectively in or out of segment j at time k. There is

$$\mathrm{PC}_{j,k} = \mathrm{PB}_{j,k} - \mathrm{PE}_{j,k} \quad \text{and} \quad \mathrm{BC}_{j,k} = \mathrm{BB}_{j,k} - \mathrm{BE}_{j,k}. \tag{2.5.13}$$

Definition 2.32 Starting with the portfolio or the benchmark

$$\mathrm{EV}_{j,k}(\bullet,\bullet), \quad \mathrm{EV}_{P,k}(\bullet,\bullet), \quad k = 1, 2, \ldots, N, \quad \text{resp.} \tag{2.5.14a}$$

denotes ***the ending value EV***, using R (return of segment in the Portfolio) or B (return of a segment in the Benchmark) for the first argument and W (weights of a segment in the Portfolio) or V (weights of segment in the Benchmark) for the second argument.

We note

$$\sum_{j=1}^{n} \mathrm{EV}_{j,T}(\bullet,\bullet) = \mathrm{EV}_{P,T}(\bullet,\bullet), \quad k = 1, 2, \ldots, N. \tag{2.5.14b}$$

Remark 2.17 With the notation (14) we have

$$\mathrm{EV}_{P,T}(R, W) = \mathrm{PE}_T,$$
$$\mathrm{EV}_{P,T}(B, V) = \mathrm{PB}_T.$$

We introduce the following four versions of the IR and the corresponding Profit and Loss:

1. **(Portfolio)**

$$\begin{aligned} \mathrm{IR}_j(R, W) &= \mathrm{IR}\left(R_{j,k}, W_{j,k},\ k = 0, \ldots, N-1\right) = \\ &\mathrm{IR}\left(\mathrm{PB}_{j,0}, \mathrm{PC}_{j,k},\ k = 1, \ldots, N-1,\ \mathrm{EV}_{j,T}(R, W)\right) \end{aligned} \tag{2.5.15a}$$

is the IR_i using the returns and the weight of the portfolio. (5) and (15a) are the same. In addition we have

$$\mathrm{PL}_j(R, W) = \mathrm{PB}_{j,0} - \sum_{k=1}^{K-1} \mathrm{PC}_{j,k} - \mathrm{EV}_{j,T}(R, W) \tag{2.5.15b}$$

where PL_j in (6) is the same as the left-hand side of (14b).

2. **(Notional Portfolio)**

$$\begin{aligned} &\mathrm{IR}_{\mathrm{j}}(\mathrm{R},\mathrm{V}) = \mathrm{IR}\big(\mathrm{R}_{\mathrm{j,k}},\mathrm{V}_{\mathrm{,k}},\ \mathrm{k}=0,\ \ldots,\mathrm{N}-1\big) = \\ &\mathrm{IR}\big(\mathrm{PB}_{\mathrm{j,0}},\mathrm{BC}_{\mathrm{j,k}},\ \mathrm{k}=1,\ \ldots,\mathrm{N}-1,\ \mathrm{PE}_{\mathrm{j,T}}(\mathrm{R},\mathrm{V})\big) \end{aligned} \tag{2.5.15c}$$

is the IR using the returns of the portfolio and the weight of the benchmark and

$$\mathrm{PL}_{\mathrm{j}}(\mathrm{R},\mathrm{V}) = \mathrm{PB}_{\mathrm{j,0}} - \sum_{\mathrm{k}=1}^{\mathrm{N}-1} \mathrm{BC}_{\mathrm{j,k}} - \mathrm{PE}_{\mathrm{j,T}}(\mathrm{R},\mathrm{V}), \tag{2.5.15d}$$

3. **(Notional Portfolio)**

$$\begin{aligned} &\mathrm{IR}_{\mathrm{j}}(\mathrm{B},\mathrm{W}) = \mathrm{IR}\big(\mathrm{B}_{\mathrm{j,k}},\mathrm{W}_{\mathrm{j,k}},\ \mathrm{k}=0,\ \ldots,\ \mathrm{N}-1\big) = \\ &\mathrm{IR}\big(\mathrm{PB}_{\mathrm{j,0}},\mathrm{PC}_{\mathrm{j,k}},\ \mathrm{k}=0,\ \ldots,\mathrm{N}-1,\ \mathrm{PE}_{\mathrm{j,T}}(\mathrm{B},\mathrm{W})\big) \end{aligned} \tag{2.5.15e}$$

is the IR using the returns of the benchmark and the weight of the portfolio and

$$\mathrm{PL}_{\mathrm{j}}(\mathrm{B},\ \mathrm{W}) = \mathrm{PVB}_{\mathrm{i,0}} - \sum_{\mathrm{k}=1}^{\mathrm{N}-1} \mathrm{PC}_{\mathrm{j,k}} - \mathrm{EV}_{\mathrm{j,T}}(\mathrm{B},\mathrm{W}) \tag{2.5.15f}$$

4. **(Benchmark)**

$$\begin{aligned} &\mathrm{IR}_{\mathrm{j}}(\mathrm{B},\ \mathrm{V}) = \mathrm{IR}\big(\mathrm{B}_{\mathrm{j,k}},\mathrm{V}_{\mathrm{j,k}},\ \mathrm{k}=0,\ \ldots,\ \mathrm{N}-1\big) = \\ &\mathrm{IR}\big(\mathrm{BB}_{\mathrm{j,0}},\ \mathrm{BC}_{\mathrm{j,k}},\ \mathrm{k}=0,\ \ldots,\mathrm{N}-1,\ \mathrm{EV}_{\mathrm{j,T}}(\mathrm{B},\ \mathrm{V})\big) \end{aligned} \tag{2.5.15g}$$

is the IR using the returns of the benchmark and the weight of the portfolio and

$$\mathrm{PL}_{\mathrm{j}}(\mathrm{B},\ \mathrm{V}) = \mathrm{PVB}_{\mathrm{j,0}} - \sum_{\mathrm{k}=1}^{\mathrm{N}-1} \mathrm{PC}_{\mathrm{j,k}}\ - \mathrm{EV}_{\mathrm{j,T}}(\mathrm{B},\mathrm{V}). \tag{2.5.15h}$$

By (2), (4) and (15a) we have $\mathrm{IR_P} = \mathrm{IR_P}(\mathrm{R},\ \mathrm{W})$ and by the left-hand side in (6) we have $\mathrm{PL_P} = \mathrm{PL_P}(\mathrm{R},\ \mathrm{W})$. Similarly to (15c)–(15h) we introduce the notation $\mathrm{IR_P}(\mathrm{R},\ \mathrm{W})$, $\mathrm{PL_P}(\mathrm{R},\ \mathrm{W})$, $\mathrm{IR_P}(\mathrm{R},\ \mathrm{V})$, $\mathrm{PL_P}(\mathrm{R},\ \mathrm{V})$, $\mathrm{IR_P}(\mathrm{B},\ \mathrm{W})$, $\mathrm{PL_P}(\mathrm{B},\ \mathrm{W})$, $\mathrm{IR_P}(\mathrm{B},\ \mathrm{V})$ and $\mathrm{PL_P}(\mathrm{B},\ \mathrm{V})$.

Definition 2.33 Referring to the definition of the argument in Definition 2.31, we define ***the average invested capital*** $AIC_j(\bullet,\bullet)$ for the first argument R or B and for the second argument W or V by

$$\mathrm{AIC}_{\mathrm{j}}(\bullet,\bullet) = \frac{\mathrm{PL}_{\mathrm{j}}(\bullet,\bullet)}{\mathrm{IR}_{\mathrm{j}}(\bullet,\bullet)}, \tag{2.5.16a}$$

$$\mathrm{AIC_P}(\bullet,\bullet)=\frac{\mathrm{PL_P}(\bullet,\bullet)}{\mathrm{IR_P}(\bullet,\bullet)},\quad \text{resp}. \tag{2.5.16b}$$

Based on the Brinson-Hood-Beebower method essentially expressed in (2.2.11) we proceed with the identity

$$\begin{gathered} X-Y= \\ \mathrm{Z-Y+V-Y+X-\ Z-V+Y}, \\ \forall\mathrm{X}\in\mathbf{R}^1,\forall\mathrm{Y}\in\mathbf{R}^1,\forall\mathrm{Z}\in\mathbf{R}^1,\forall\mathrm{V}\in\mathbf{R}^1. \end{gathered} \tag{2.5.17}$$

This identity can be applied to

$$\begin{gathered} \mathrm{AIC_j(R,W)-AIC_j(B,V)=} \\ \mathrm{AIC_j(B,W)-AIC_j(B,V)+AIC_j(R,V)-AIC_j(B,V)+} \\ \mathrm{AIC_j(R,W)-AIC_j(B,W)-AIC_j(R,V)+AIC_j(B,V),} \end{gathered}$$

or

$$\begin{gathered} \mathrm{AIC_P(R,W)-AIC_P(B,V)=} \\ \mathrm{AIC_P(B,W)-AIC_P(B,V)+AIC_P(R,V)-AIC_P(B,V)+} \\ \mathrm{AIC_P(R,W)-AIC_P(B,W)-AIC_P(R,V)+AIC_P(B,V).} \end{gathered}$$

By using the notation for IR in (15) we consider

$$\begin{gathered} \frac{\mathrm{AIC_j(R,\ W)}}{\mathrm{AIC_P(R,W)}}\mathrm{IR_j(R,W)}-\frac{\mathrm{AIC_j(B,\ V)}}{\mathrm{AIC_P(BV)}}\mathrm{IR_j(B,V)}= \\ \frac{\mathrm{AIC_j(B,W)}}{\mathrm{AIC_P(B,W)}}\mathrm{IR_j(B,W)}-\frac{\mathrm{AIC_j(B,V)}}{\mathrm{AIC_P(BV)}}\mathrm{IR_j(B,V)}+ \\ \frac{\mathrm{AIC_j(R,V)}}{\mathrm{AIC_P(R,V)}}\mathrm{IR_j(R,V)}-\frac{\mathrm{AIC_j(B,V)}}{\mathrm{AIC_P(BV)}}\mathrm{IR_j(B,V)}+ \\ \frac{\mathrm{AIC_j(R,W)}}{\mathrm{AIC_P(R,W)}}\mathrm{IR_j(R,W)}-\frac{\mathrm{AIC_j(B,W)}}{\mathrm{AIC_P(B,W)}}\mathrm{IR_j(B,W)}- \\ \frac{\mathrm{AIC_j(R,V)}}{\mathrm{AIC_P(R,V)}}\mathrm{IR_j(R,V)}+\frac{\mathrm{AIC_j(B,V)}}{\mathrm{AIC_P(BV)}}\mathrm{IR_j(B,V)}. \end{gathered} \tag{2.5.18}$$

We proceed with

$$\mathrm{PL_P}(\bullet,\bullet)=\sum_{\mathrm{j=1}}^{\mathrm{n}}\mathrm{PL_j}(\bullet,\bullet). \tag{2.5.19}$$

Hence

$$\mathrm{PL_P(R, W) - PL_P(B, V)} = \sum_{j=1}^{n} \left(\mathrm{PL}_j(\mathrm{R, W}) - \mathrm{PL}_j(\mathrm{BV})\right) \tag{2.5.20}$$

Furthermore with (16b) we find

$$\mathrm{IR_P(R, W) - IR_P(B, V)} = \frac{\mathrm{PL_P(R, W)}}{\mathrm{AIC_P(R, W)}} - \frac{\mathrm{PL_P(BV)}}{\mathrm{AIC_P(BV)}}.$$

Hence with (20)

$$\mathrm{IR_P(R, W) - IR_P(B, V)} = \sum_{j=1}^{n} \left(\frac{\mathrm{PL}_j(\mathrm{R, W})}{\mathrm{AIC_P(R, W)}} - \frac{\mathrm{PL}_j(\mathrm{BV})}{\mathrm{AIC_P(BV)}}\right)$$

and (16a)

$$\mathrm{IR_P(R, W) - IR_P(B, V)} = \sum_{j=1}^{n} \left(\frac{\mathrm{AIC}_j(\mathrm{R, W})}{\mathrm{AIC}_P(\mathrm{R, W})}\mathrm{IR}_j(\mathrm{R, W}) - \frac{\mathrm{AIC}_j(\mathrm{BV})}{\mathrm{AIC_P(BV)}}\mathrm{IR}_j(\mathrm{BV})\right)$$

and with (18) we have

$$\begin{aligned}
&\mathrm{IR_P(R,\ W) - IR_P(B,\ V)} = \\
&\sum_{j=1}^{n} \left(\frac{\mathrm{AIC}_j(\mathrm{B, W})}{\mathrm{AIC_P(B, W)}}\mathrm{IR}_j(\mathrm{B, W}) - \frac{\mathrm{AIC}_j(\mathrm{B, V})}{\mathrm{AIC_{tot}(BV)}}\mathrm{IR}_j(\mathrm{B, V}) + \right. \\
&\frac{\mathrm{AIC}_j(\mathrm{R, V})}{\mathrm{AIC_P(R, V)}}\mathrm{IR}_j(\mathrm{R, V}) - \frac{\mathrm{AIC}_j(\mathrm{B, V})}{\mathrm{AIC_P(BV)}}\mathrm{IR}_j(\mathrm{B, V}) + \\
&\frac{\mathrm{AIC}_j(\mathrm{R, W})}{\mathrm{AIC_P(R, W)}}\mathrm{IR}_j(\mathrm{R, W}) - \frac{\mathrm{AIC}_j(\mathrm{B, W})}{\mathrm{AIC_P(B, W)}}\mathrm{IR}_j(\mathrm{B, W}) - \\
&\left.\frac{\mathrm{AIC}_j(\mathrm{R, V})}{\mathrm{AIC_P(R, V)}}\mathrm{IR}_j(\mathrm{R, V}) + \frac{\mathrm{AIC}_j(\mathrm{B, V})}{\mathrm{AIC_P(BV)}}\mathrm{IR}_j(\mathrm{B, V})\right).
\end{aligned} \tag{2.5.21}$$

Definition 2.34 ***The return contribution*** $RC_j(\bullet, \bullet)$ is defined by

$$\mathrm{RC}_j(\bullet, \bullet) = \frac{\mathrm{AIC}_j^{\mathrm{IR}}(\bullet, \bullet)}{\mathrm{AIC}_P^{\mathrm{IR}}(\bullet, \bullet)} * \mathrm{IR}_j(\bullet, \bullet). \tag{2.5.22}$$

Theorem 2.5 As described at the beginning of this section, with (1) we consider a portfolio P over the time interval [0, T] with n segments. Then the decomposition of the internal rate of relative return based on the return contribution (see Definition 2.30)

$$\mathrm{IR_P(R,W) - IR_P(B,V) = A_P + S_P + I_P} \tag{2.5.23a}$$

is consistent with the PL of the portfolio and the segments. The asset allocation A_P effect is given by

$$\mathrm{A_P} = \sum_{j=1}^{n} \mathrm{A_j} = \sum_{j=1}^{n} \left[\mathrm{RC_j(B,W) - RC_j(B,V)}\right], \tag{2.5.23b}$$

the stock picking effect S_{tot} is given by

$$\mathrm{S_P} = \sum_{j=1}^{n} \mathrm{S_j} = \sum_{j=1}^{n} \left[\mathrm{RC_j(R,V) - RC_i(B,V)}\right], \tag{2.5.23c}$$

and the interaction effect I_P is given by

$$\begin{aligned} \mathrm{I_P} &= \sum_{i=1}^{n} \mathrm{I_i} = \\ &\sum_{j=1}^{n} \left[\mathrm{RC_j(R,W) - RC_i(R,V)}\right] - \left[\mathrm{RC_j(B,W) - RC_i(B,V)}\right]. \end{aligned} \tag{2.5.23d}$$

Proof The relationships in (23) follow from (21) and (22).

◇

1. **External cash flow**
   External cash flows are cash flows that are withdrawn or transferred from the portfolio manager to the client. If we have external cash flows the value of the IR is not equal to the TWR and the return from the client's perspective is not equal to the return achieved by the portfolio manager.
2. **Internal cash flow**
   This cash flow might be typically caused by a change to a different asset allocation on the part of the portfolio manager or due to the rebalancing of the benchmark.

   We pursue different strategies:

1. If we have no external cash flows in the portfolio, benchmark, resp.

$$\sum_{i=0}^{n} \mathrm{IPC_i^k} = 0, \quad \sum_{i=0}^{n} \mathrm{IBC_i^k} = 0, \quad \text{resp.,} \quad \mathrm{k} = 1, \ldots, \mathrm{N} - 1$$

2. Buy and hold in portfolio, benchmark, resp. $k = 1,\ldots,N - 1$

$$\mathrm{IPC}_i^k = 0, \quad \mathrm{IBC}_i^k = 0, \quad \text{resp}.$$

We note that (1) follows (2).

Example 2.26 (continued) We consider a **Benchmark** with initial value $BB_0 = \$100$ over 2 years. The benchmark consists of two segments. The first one has a value of $BB_{1,0} = \$20$ and the second one of $BB_{2,0} = \$80$. In the first period the returns of the first and second segments are

$$B_{1,0} = 150\%, \quad B_{2,0} = 50\%, \tag{2.5.24a}$$

resp. and in the second period the returns are

$$B_{1,1} = 20\%, \quad B_{2,1} = 120\%, \quad \text{resp.} \tag{2.5.24b}$$

There is no external cash flow, i.e., $BEC_1 = 0$, $BEC_2 = 0$ and there is an internal cash flow of

$$BIC_1 = \$40, \quad BIC_2 = -\$40 \tag{2.5.24c}$$

in the middle of the 2-year period.

We discuss the relative return of the portfolio versus the benchmark and proceed first with the calculation for the benchmark. Based on (12) and (24) we have

$$\begin{aligned}
EV_{1,1}(B,V) &= BE_{1,1} = \$20 \cdot (1 + 1.50) = \$50, \\
EV_{2,1}(B,V) &= BE_{2,1} = \$80 \cdot (1 + 0.50) = \$120, \\
EV_1(B,V) &= BE_1 = BE_{1,1} + BE_{2,1} = \$170.
\end{aligned}$$

Furthermore by (13)

$$\begin{aligned}
BB_{1,1} &= \$50 + \$40 = \$90, \\
BB_{2,1} &= \$120 - \$40 = \$80.
\end{aligned}$$

Again with (12) and (22) we have

$$\begin{aligned}
EV_{1,2}(B,V) &= BE_{1,2} = \$90 \cdot (1 + 0.2) = \$108, \\
EV_{2,2}(B,V) &= BE_{2,2} = \$80 \cdot (1 + 1.2) = \$176, \\
EV_2(B,V) &= BE_2 = BE_{1,2} + BE_{2,2} = \$284.
\end{aligned}$$

With (6) we have

$$\begin{aligned}
PL_1 &= -\$20 + \$40 + \$108 = \$128, \\
PL_2 &= -\$80 - \$40 - \$176 = \$56, \\
PL_P &= \$284 - \$100 = \$184,
\end{aligned}$$

$$\mathrm{IR}_1 = 252.9822\%, \mathrm{IR}_2 = 2.5416\%, \mathrm{IR}_P = 68.52\%.$$

Adapting the notation (14) and (20)

$$\mathrm{AIC}_1^{\mathrm{IR}}(\mathrm{BV}) = \frac{\$128}{252.98} = \$0.5059,$$
$$\mathrm{AIC}_2^{\mathrm{IR}}(\mathrm{B,V}) = \frac{\$56}{2.54} = \$2.2032,$$
$$\mathrm{AIC}_P^{\mathrm{IR}}(\mathrm{B,V}) = \frac{\$184}{68.52} = \$2.6865.$$
$$\mathrm{RC}_1(\mathrm{B,\ V}) = \frac{\mathrm{AIC}_1^{\mathrm{IR}}(\mathrm{BV})}{\mathrm{AIC}_P^{\mathrm{IR}}(\mathrm{BV})}\mathrm{IR}_1(\mathrm{B,\ V}) = \frac{128{*}68.52\%}{184} = 47.6681\%,$$
$$\mathrm{RC}_2(\mathrm{B,\ V}) = \frac{\mathrm{AIC}_2^{\mathrm{IR}}(\mathrm{BV})}{\mathrm{AIC}_P^{\mathrm{IR}}(\mathrm{BV})}\mathrm{IR}_2(\mathrm{B,\ V}) = \frac{56{*}68.52\%}{184} = 20.8548\%,$$
$$\mathrm{RC}_1(\mathrm{B,\ V}) + \mathrm{RC}_2(\mathrm{B,V}) = 68.5230\%.$$

The argument (•, •) yields the following numerical results:

(R, W) in (15a) and (15b)
(R, V) in (15c) and (15d)
(B, W) in (15e) and (15f)
(B, V) in (15g) and (15h)

We note that the IRR in the Tables 2.11, 2.12, 2.13 and 2.14 are non-weighted. The relative consideration of portfolios versus benchmarks yields the following results. For the PL we find according to Tables 2.11 and 2.14

Table 2.11 Portfolio

i	Cash flow			PL(R, W)	IR(R, W)
1	− 60	−15	120	45	29.5
2	− 40	15	98	73	76.4
tot	−100	0	218	118	47.6

Table 2.12 Notional Portfolio

i	Cash flow			PL(R, W)	IR(R, W)
1	−20	40	130	150	273.86
2	−80	−40	98	−22	−11.5
tot	−100	0	228	128	50.9

Table 2.13 Notional Portfolio

i	Cash flow			PL(R, V)	IR(R, V)
1	−60	−15	162	87	52.3
2	−40	15	165	140	122.7
tot	−100	0	327	227	80.8

Table 2.14 Benchmark

i	Cash flow			PL(B,V)	IR(B,V)
1	−20	40	108	128	253.0
2	−80	−40	176	56	25.4
tot	−100	0	284	184	68.5

Table 2.15 Management effects

	Asset allocation	Stock picking	Interaction effect	Total
1	22	−41	−64	−83
2	−78	84	11	17
tot	−56	43	−53	−66

Table 2.16 Management effects

	Asset allocation	Stock picking	Interaction effect	Total
1	0.1209	−0.1668	−0.2490	−0.2949
2	−0.2961	−0.2899	0.0924	0.0862
tot	−0.1752	0.1230	−0.1565	−0.2087

$$\begin{aligned} \$45 - \$128 &= -\$83, \\ \$73 - \$56 &= \quad \$17, \\ \$118 - \$184 &= -\$66, \end{aligned}$$

which are the numbers in the last column of Table 2.15. The decomposition is based on (19) and (20).

In keeping with the figures in Tables 2.11, 2.14 and 2.16 we find for the difference of the IRR for the portfolio and benchmark

$$-20.87\% = -29.49\% + 8.62\%.$$

For the effects, according to (21)–(23) we find

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Chapter 3
Risk Measurement

It is the task of every performance measurement department to calculate return figures. It goes without saying that this is important information for the investor, as the return reflects the change in a portfolio's value. However, a return of a specific portfolio is just based on one realization of the portfolio in the past and it is up to risk measurement to investigate different behaviors of the portfolio under different market conditions. Risk considerations are called for. It may very well be that two portfolios have the same returns but the accompanying risks are significantly different. Return calculations are deterministic and risk relates to randomness. Risk calculation uses methods from statistics and from probability theory. Generally speaking, returns are the result of precise calculations, while risk figures are rather estimations that are based on models. We see that different risks can assume different characteristics and forms.

In this chapter we focus on risk and in Chaps. 4 and 5 we discuss return and risk simultaneously, i.e., the performance is discussed.

The term "risk" has a long tradition. For instance the following originates from Thomas Aquinas (1225–1274): "if a lender gives a trader or craftsman money for his business, the lender shares the risk. As such, he may be entitled to demand a part of the resulting profit as it is were from his own."

In the sixteenth century, the term "risk" was used by Italian merchants to represent dangers or hazards. The expression refers to potential losses and damages in the context of a company or an enterprise. Risks are pitfalls that have to be avoided.

The notion of risk is used in many areas and contexts. A general definition can be as follows: Risk is the potential for an event to have an undesired negative consequence. In the American Heritage Dictionary, risk is defined as the possibility of suffering losses or losses due to an event that will quite probably occur.

In financial markets, distinctions are made between different risks. In the following, we proceed by describing some main types of risk.

With **market risk** we refer to risk resulting from movements in market prices, in particular, changes in interest rates, foreign exchange rates, and equity and

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commodity prices. The value of investments may decline over a given time period simply because of economic changes or other events that impact large portions of the market. Asset allocation and diversification can protect against market risk because different portions of the market tend to underperform at different times.

Credit risk is usually an important issue in bond investments. It reflects the possibility that a bond issuer can default by failing to repay the principal and/or interest in a timely manner. Bonds issued by a government or an authority, for the most part, are safe from default, since governments can simply print more money. Bonds issued by corporations are more likely to be defaulted on, since companies certainly can go bankrupt. Credit risk is also called **default risk**.

Liquidity risk refers to the fact that an investor can have difficulties selling an asset. Unfortunately, an insufficient secondary market may prevent the liquidation or limit the funds that can be generated from an asset. Some assets are highly liquid and have low liquidity risk (such as stock in a publicly traded company), while other assets are highly illiquid and are highly risky (such as for instance a house).

The **call risk** is reflected by the cash flow resulting from the possibility that a callable bond will be redeemed before maturity. Callable bonds can be called by the company that issued them, meaning that bonds have to be redeemed to the bondholder, usually so that the issuer can issue new bonds at a lower interest rate. This forces the investor to reinvest the principal sooner than expected, usually at a lower interest rate.

Political risk is the risk of changes in a country's political structure or policies caused by tax laws, tariffs, expropriation of an asset or restrictions on the repatriation of profits; for example, a company may suffer from losses in the case of expropriation or more stringent foreign exchange repatriation rules, or from increased credit risk if the government changes policies, making it difficult to pay creditors.

Legal risk is a description of the potential for loss arising from the uncertainty of legal proceedings, such as bankruptcy and other potential legal proceedings.

The **currency risk** refers to the fact that business operations and the value of investment are affected by changes in currency rates. The risk usually influences business but it can also affect individual investors who make international investments. It is also called **exchange rate risk**.

Inflation risk reflects the fact that the purchasing power of the currency shrinks the value of assets. Inflation causes money to decrease in value at a certain rate, and does so whether the money is invested or not.

The **financial risk** is the sum total of the risks described above. Nonfinancial risks are for instance the risks relating to operations or technology. In financial markets a security whose return is likely to change considerably is said to be *risky* and a security whose return most likely won't change much is said to carry *little risk*. Generally we can say that equity has higher risk than a money market instrument. In the following we discuss the risk of a portfolio in view of some of the financial risks discussed above.

3.1 Absolute Risk

3.1.1 The Risk of a Portfolio

We now introduce some notions from descriptive statistics, the overall idea being to merge a series of data into one value. We start with a return series of the portfolio P

$$r_{P,k}, k = 1, 2, \ldots, N \tag{3.1.1a}$$

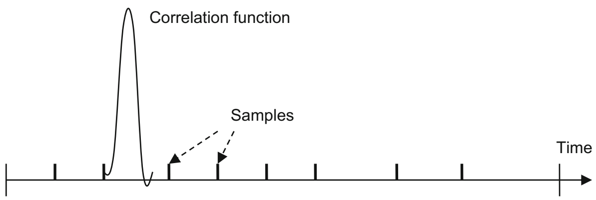
of periodic returns, i.e., $r_{P,k}$ is the return between $t_{k-1} = k - 1$, $t_k = k$, $k = 1, \ldots,$ $N = T$. In addition we define for $\beta \in \mathbf{R^1}$, $\gamma \in \mathbf{R^1}$ the notation $\beta \cdot P + \gamma$ by means of the series

$$\beta \cdot r_{P,k} + \gamma, k = 1, 2, \ldots, N. \tag{3.1.1b}$$

Remark 3.1 (on collecting representative statistical samples) The different "camps" of statisticians have been arguing about this problem for a long time. Ideally, we need a uniform and representative range of samples. Assuming that the statistical model we chose is right and adequate for the process in question, then the method for collecting samples will be defined by the chosen statistical model. Note that we always make some assumptions, explicit or implicit, about the nature of our process when we choose methods for collecting samples. Some people may not realize this when they collect samples, but it is in fact impossible to pick out the method for collecting samples without such assumptions. A typical assumption is that the sample is ***independent*** and ***identically distributed***. The correlation or comovement time is assumed inferior to the periodic time of the sample. It dictates the periodicity of the sample (Fig. 3.1). Further, the assumption of equidistant knots underpins the independence of the sample. So, the method for collecting samples is defined by several factors: the nature of the process or phenomenon we are studying, our model, our measurement tools and our purposes.

Average is a generic term for statistics describing the middle of the data set. The ***geometric mean return*** defined in Definition 2.16 and the ***arithmetic mean return*** defined in Definition 2.17 are examples of ***averages***.

Fig. 3.1 Equidistant knots



Definition 3.1 A ***population*** is defined as all members of a specified group.

Definition 3.2 The ***variance var(P)*** of a portfolio P is the sum of the squared deviations from the arithmetic mean $\bar{r}_P$ defined in (2.3.4) divided by the number of returns in (1):

$$\mathrm{var(P)} = \frac{1}{N}\sum_{k=1}^{N}(r_{P,k} - \bar{r}_P)^2 (\text{population variance}). \tag{3.1.2}$$

Definition 3.3 The ***standard deviation std(P)*** of a portfolio P is defined by

$$\mathrm{std(P)} = \sqrt{\mathrm{var(P)}}. \tag{3.1.3}$$

Often the term ***absolute risk*** is used similarly with standard deviation or particularly in finance with ***volatility***.

Remark 3.2 (intuition) The standard deviation is a measure of how widely the actual returns are dispersed from the arithmetic mean return. $\bar{r}_P$ does not mean much if the variance is large. On the other hand, if var(P) = 0 the mean is equal to the return series (1). In [27] these two cases (small and large var(P)) are called Mediocristan and Extremistan.

Remark 3.3 (units) The units used for the standard deviation and the return are percentages or decimals. The standard deviation has the same unit as the return series.

Example 3.1 We consider N = 1, 2, 3, ….

$$r_{P,k} = 5\%, k = 1, 3, \ldots, 2N - 1,$$
$$r_{P,k} = 15\%, k = 2, 4, \ldots, 2N.$$

For the partial sum in (2.3.4) we then have 5, 10, $8\frac{1}{3}$, 10, 9, 10,…. and find for $N \to \infty$

$$\bar{r}_P = 10\%$$

and also for $N \to \infty$ we have

$$\mathrm{std(P)} = \sqrt{\frac{1}{N}\sum_{n=1}^{N}(5\%)^2} = 5\%,$$

i.e., by assuming Remark 2.11 the values for std(P) are independent for N in the natural numbers **N**.

◇

Remark 3.4 (annualizing) The variance is additive, i.e., if we have $f, f = 0, 1, 2, \ldots$ variances per year that are independent and identically distributed, we can multiply by f but the standard deviation is not additive. The variance is annualized by f, whereas the risk and volatility is multiplied by $\sqrt{f}$.

As in science, the first attempt to analyze data in finance like the return series (1) is ***the normal distribution***. It is the benchmark in the theory of distributions. It assumes that N in (1) and (2) is large and tends or converges to the mean μ and the standard deviation σ. The normal distribution has the probability density function

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \forall x \in \mathbf{R^1}. \tag{3.1.4}$$

It can be characterized by returns that are equally distant from the mean return and have the same relative frequency of observations. Most of the returns are close to the average mean return and there are relatively few extremely high and low returns. Sometimes, the normal distribution is referred to by its nickname, the bell curve. Statistically, the following statements hold:

- About 50 % of the observed returns will be within ±2/3 standard deviation above or below the mean return.
- About 68.3 % of the observed returns will be within ±1 standard deviation above or below the mean return.
- About 90 % of the observed returns will be within ±1.65 standard deviations.
- About 95.4 % of the observed returns will be within ± 2.0 standard deviations.
- About 99.7 %, i.e., almost all of the observed returns will be within ± 3.0 standard deviations.

Modern portfolio theory (see Sect. 4.2) is based on the assumption that the asset can be described by return and variance and follows a normal distribution.

We proceed with a ***log normal distribution*** using the probability density function

$$f(x) = \frac{1}{x\sqrt{2\pi\sigma^2}} e^{-\frac{\ln(x)-\mu}{2\sigma^2}}, \forall x > 0.$$

The mean is

$$\mu_L = e^{\left(\mu + 0.5\sigma^2\right)}.$$

If returns ${}_k r_{k+1}$ are normally distributed and linked to asset prices ${}_k P_{k+1}, k = 1, 2, \ldots, N$ we have

$$\frac{P_{k+1}}{P_k} = 1 + {}_k r_{k+1}.$$

The arithmetic return ${}_k R_{k+1}$ and the continuous return ${}_k r_{k+1}$ are connected by the natural logarithm

$${}_k r_{k+1} = \ln\left(\frac{P_{k+1}}{P_k}\right) = \ln(1 + {}_k R_{k+1}).$$

We see that the

$${}_0 r_N = {}_0 r_1 + {}_1 r_2 + \ldots + {}_{N-1} r_N.$$

If ${}_k r_{k+1} = \mu, k = 1, 2, \ldots, N$ we have

$${}_0 r_N = \mu T.$$

In order to annualize monthly returns we use

$${}_0 r_{12} = 12\mu T$$

and for the variance we have

$$\sigma^2(r_{0,N}) = \sigma^2 T.$$

In order to annualize monthly variance we use

$$\sigma^2(r_{0,12}) = \sigma^2 12.$$

In summary, continuously compounded, normally distributed returns and log normally distributed asset prices are consistent.

Definition 3.4 A ***sample*** is a subset of a population.

We estimate statistically figures like variance, etc. by means of a sample of a population, i.e., a specific aspect of a population is estimated. The estimation of the historical mean is unbiased, i.e., the expected value of the estimation is the mean $\bar{r}_P$ (For the expected value see Sect. 3.5). If the mean $\bar{r}_P$ is unknown and consequently has to be estimated, it can be shown that the estimator for the variance (2) is biased, meaning there is a systematic error in the estimation. The unbiased estimator for $K << N$ of samples for the **empirical or sample variance** is

$$\mathrm{Evar}(P) = \frac{1}{K\text{-}1} \sum_{k=1}^{K} (r_{P,k} - \bar{r}_P)^2. \tag{3.1.5}$$

The corresponding **empirical or sample standard variation** is

$$\mathrm{Estd(P)} = \sqrt{\mathrm{Evar(P)}}. \tag{3.1.6}$$

However, if the arithmetic mean $\bar{r}_P$ is known the estimator Estd(P) is equal to (2). We consider the following easy example.

Example 3.1 (continued) With the known arithmetic mean $\bar{r}_P = 10\%$ we have

$$\mathrm{Evar(P)} = \frac{1}{K}\sum_{k=1}^{K}(5\%)^2 = (5\%)^2$$

$$\mathrm{Estd(P)} = 5\%.$$

If $\bar{r}_P$ is unknown, numerical experiments show that Evar converges to 5 %. (5) approximates 5 % better than (2).

◇

More precisely (2), (5), resp. are called the ***centered variance*** and ***standard deviation***, resp. In passive management it is often assumed that the averages in (2) and (5) vanish. Then we consider the ***uncentered variance*** and ***standard deviation***, resp. (see Sect. 2.5).

Referring to (4), the normal distribution is important in the theory of probabilities. For an introduction to the theory of probabilities, see Sect. 3.5.

We proceed by examining the concept of the value-at-risk and start with a brief historical review.

3.1.2 Value at Risk

In 1992 J.P. Morgan launched the RiskMetrics methodology on the marketplace in order to enhance the assessment of risk for firms. RiskMetrics set an industry-wide standard. In a highly dynamic world with round-the-clock market activities, the need for instant market valuation of trading positions (known as marking-to-market) became a necessity. ***Value at risk*** **(VaR)** is probably the most widely used risk measure at institutions. Value at risk translates the percentage of the loss into the base currency of the portfolio. Moreover, it is based on a generalization or extension from maximum loss of the portfolio. Maximum loss does not take probabilities into account and the idea is simply to replace 'maximum loss' by 'maximum loss which is not exceeded with a given high probability,' the so-called ***confidence level***.

Definition 3.5 ***Quantile*** is a generic term for grouping data when sampling in descriptive statistics. Examples of quantiles are median = 2, quartiles = 4, quintiles = 5, deciles = 10 or percentiles = 100.

Example 3.2 Let us consider in increasing order the five data points 2, 4, 6, 8 and 10. The *first quartile* is 6/4 = 1.5. The quartile ends in 1.5. The fact that the linear interpolation of 2 and 4 is 3 shows us that one data point is to the left of 3, i.e., 20 % of the points are to the left of 3. Reflecting the linear interpolation, the *second quartile* starts with 3 and ends with 6 in terms of the data points; thus 4 belongs certainly to the second quartile. As 6 is the middle of the second and the third quartiles, it can be counted in either the second or the third quartile; thus whether 6 belongs to the second or the third quartile is a matter of definition.

◇

In probabilistic terms, VaR refers to a quantile of the loss distribution. When we need to communicate the risk associated with a portfolio or an investment, it is useful to convert the standard deviation in percentage into units of the currency in which the portfolio is expressed.

Definition 3.6 ***Value at risk*** $\boldsymbol{VaR_\alpha}$ of a portfolio at $t = 0$ with portfolio value P_0 is the estimate of maximizing dollar loss we could expect to experience, over the time period, with a stated level of confidence α.

Thus the input parameters for calculating value at risk figures are

- The level of confidence α
- A set of returns (1) with a frequency, typically days or months

Typical value are $\alpha = 0.90$, $\alpha = 0.95$ or $\alpha = 0.99$ in market risk management. Usually the time periods are

- 1 or 10 days in credit management
- 1 year in operational risk management.

Remark 3.5 The level of confidence is also used in conjunction with the test of a hypothesis. When a hypothesis is statistically tested, two types of errors can be made. The first one is that we reject the null hypothesis while it is actually true; this is referred to as a type I error. The second one, a type II error, is when the null hypothesis is not rejected, but the alternative is true. The probability of a type I error is directly controlled by researchers through their choice of the significance level α. When a test is performed at the 5 % level, the probability of rejecting the null hypothesis while it is true is 5 %. A confidence interval α means

- A probability of $1 - \alpha$ for an error of the first kind.
- A probability of α for an error of the second kind.

So we see that there is a tradeoff between the two errors.

Remark 3.6 VaR has been fundamentally criticized as a risk measure on the grounds that it is has poor aggregation properties.

We proceed by describing three types of methodology for calculating value at risk figures:

3.1.2.1 Historical VaR

We start with a frequency distribution or a histogram and select the $\alpha\%$ worst returns. $(100 - \alpha)\%$ of the returns do not fall under the value of the maximum of these worst returns.

Example 3.3 We assume that we have in (1) $N = 100$, i.e., we consider 100 returns of a portfolio with portfolio value P_0 and

$$r_{1,P} < r_{2,P} < r_{3,P} < \ldots < r_{100,P}.$$

Then we have

$$VaR^{H}_{5\%} = r_{5,P} \cdot P_0$$

and

$$VaR^{H}_{1\%} = r_{1,P} \cdot P_0.$$

◇

We see that the general case shows some minor complications like the equal sign or the fact that the number of return points is not divisible by the confidence level. Furthermore the number of data points has to exceed a minimal number of points for calculating VaR^{H}_{α}.

This methodology is solely empirical and does not use a model (is model-independent). Thus it is non-parametric. The following VaR in b2, however, is model-dependent because it uses a function with certain parameters.

3.1.2.2 Parametric VaR

We start by assuming that the loss function is normal distributed (see Fig. 3.2). The shortfall probability under the normality assumption is

$$F(z) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{z} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx.$$

Then $VaR^{P}_{z\%}$ is

$$VaR^{P}_{z\%} = -(\mu + z\sigma) \cdot P_0.$$

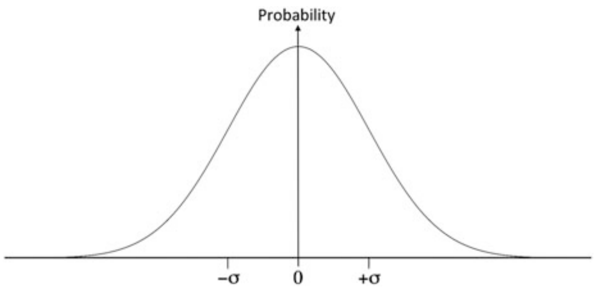


Fig. 3.2 The normal distribution

For $z_1 = -1.64448$ we have $F(z_1) = 0.05$ with

$$\mathrm{VaR}^{P}_{5\%} = -(\mu + z_1\sigma) \cdot P_0$$

and for $z_2 = -2.3263$ we have $F(z_1) = 0.01$ with

$$\mathrm{VaR}^{P}_{1\%} = -(\mu + z_2\sigma) \cdot P_0.$$

μ, σ, resp. is estimated by (2.3.4), (5), resp.

3.1.2.3 Monte Carlo Simulation

If the behavior of a portfolio cannot be described analytically, portfolio analytics falls back on simulation methods. Monte Carlo simulation is a computer-assisted method and solves problems statistically, at least approximately, by calculating probability distributions. Generally speaking, it is applied to complex financial instruments and portfolios; pension funds also apply simulations to the liabilities versus their benchmarks. The idea is simple: transforming market conditions into changes in portfolios. The calculations are intensive and depend on the distribution of the randomly applied market conditions. This approach is often well suited for portfolios that have strongly non-linear payoffs.

3.2 Relative Risk

3.2.1 The Relative Risk of Two Portfolios

Referring to (3.1.1a), we proceed by considering three periodic return series

$$r_{P\,j,k}, k = 1, 2, \ldots, N, \; j = 1, 2, 3 \tag{3.2.1}$$

of three portfolios $P_j \cdot \bar{r}_{P\,j}$ denotes the arithmetic mean (see Definition 2.18 and Remark 2.11).

Definition 3.7 The ***covariance*** $\boldsymbol{cov}\left(\boldsymbol{P}, \hat{\boldsymbol{P}}\right)$ of the portfolios P and $\hat{P}$ is the sum of the products of the return (3.1.2) of P and the return (1) of $\hat{P}$ divided by the number N of returns:

$$\mathrm{cov}\left(P, \hat{P}\right) = \frac{1}{N}\sum_{k=1}^{N} \left(r_{P,k} - \bar{r}_P\right)\left(r_{\hat{P},k} - \bar{r}_{\hat{P}}\right). \tag{3.2.2}$$

The linearity properties of the covariance are shown in the following lemma:

Lemma 3.1 For $\beta \in \mathbf{R^1}$, $\gamma \in \mathbf{R^1}$ we have

$$\mathrm{cov}(\beta \cdot P_1 + \gamma \cdot P_2, P_3) = \beta \ \mathrm{cov}(P_1, P_3) + \gamma \ \mathrm{cov}(P_2, P_3)$$
$$\mathrm{cov}(P_1, \beta \cdot P_2 + \gamma \cdot P_3) = \beta \ \mathrm{cov}(P_1, P_2) + \gamma \ \mathrm{cov}(P_1, P_3).$$

Proof: We consider the definition of the arithmetic mean

$$\bar{r}_P = \frac{1}{N}\sum_{k=1}^{N} r_{P,k} = \sum_{k=1}^{N} \frac{r_{P,k}}{N}.$$

The assertion follows from algebraically rearranging the sum in the covariance: Firstly β drops and secondly the sum can be written with γ before or after the sum.

◇

Corollary 3.1 For $\beta \in \mathbf{R^1}$, $\gamma \in \mathbf{R^1}$ we have

$$\beta^2 \mathrm{var}(P_1) = \mathrm{var}(\beta \cdot P_1 + \gamma, \beta \cdot P_1 + \gamma).$$

Proof: The assertion follows from Lemma 3.1 by identifying the argument in the covariance.

◇

Covariance is a statistical measure for assessing the tendency of two data series moving together. It measures the direction and the degree of association of the returns of the portfolios P and $\hat{P}$. It is difficult to interpret as anything other than the average product of the difference between the deviations of the portfolio returns from the arithmetic mean in (2). Furthermore, it is also difficult to use for portfolio comparison, because it is influenced by the absolute size of the returns.

Referring to the empirical or sample variance in (3.1.5) with K < <N, the same is valid for the ***empirical or sample covariance***

$$\mathrm{Ecov}\left(P,\hat{P}\right) = \frac{1}{K-1}\sum_{k=1}^{K}\left(r_{P,k} - \bar{r}_P\right)\left(\hat{r}_{P,k} - \hat{\bar{r}}_P\right). \tag{3.2.3}$$

Definition 3.8 The ***correlation corr*** $\left(\boldsymbol{P},\hat{\boldsymbol{P}}\right)$ of the portfolios P and $\hat{P}$ is defined by

$$\mathrm{corr}\left(P,\hat{P}\right) = \frac{\mathrm{cov}\left(P,\hat{P}\right)}{\sqrt{\mathrm{var}(P)}\sqrt{\mathrm{var}\left(\hat{P}\right)}}. \tag{3.2.4}$$

In the following example we illustrate the difference between the covariance and the correlation by means of two different pairs of time series.

Example 3.4 We consider for $N = 1, 2, 3, \ldots$.

$$\begin{aligned} r_{P_1,k} &= 1\%, k = 1, 3, \ldots, 2N-1, \\ r_{P_1,k} &= -1\%, k = 2, 4, \ldots, 2N. \end{aligned}$$

Then we have

$$\bar{r}_{P_1} = 0 \text{ and } \mathrm{std}(P_1) = 1\%, N = 1, 2, 3, \ldots.$$

We consider a second portfolio for $N = 1, 2, 3, \ldots$.

$$\begin{aligned} r_{P_2,k} &= 1\%, k = 1, 3, \ldots, 2N-1, \\ r_{P_2,k} &= -1\%, k = 2, 4, \ldots, 2N. \end{aligned}$$

Then we have

$$\bar{r}_{P_2} = 0\% \text{ and } \mathrm{std}(P_2) = 1\%, N = 1, 2, 3, \ldots,$$

and

$$\mathrm{cov}(P_1, P_2) = 1 \text{ and } \mathrm{corr}(P_1, P_2) = 1.$$

We consider a third portfolio for $N = 1, 2, 3, \ldots$.

$$\begin{aligned} r_{P_3,k} &= 3\%, k = 1, 3, \ldots, 2N-1, \\ r_{P_3,k} &= -3\%, k = 2, 4, \ldots, 2N. \end{aligned}$$

Then we have

$$\bar{r}_{P_3} = 0\% \text{ and } \mathrm{std}(P_3) = 1\%, N = 1, 2, 3, \ldots,$$

and

$$\mathrm{cov}(\mathrm{P}_1, \mathrm{P}_3) = 3 \text{ and } \mathrm{corr}(\mathrm{P}_1, \mathrm{P}_3) = 1.$$

We see that the correlation detects the similar behavior of portfolios 2 and 3 and yields the same value. Mathematically, the difference between series 2 and 3 is their difference in the standard deviation and also affects their covariance to series 1. However, the division by the standard deviation eliminates this difference in the covariances and leads to the same value for the correlation of series 1 versus series 2 and series 1 versus series 3.

◇

Then we have

Remark 3.7 We note that the covariance and variances used in (4) are not influenced by the choice between sample and population in (3.1.2), (3.1.5), resp. and (2), (3), resp. for substituting identical data series in the sample and population version.

Remark 3.8 By means of the Cauchy Schwarz inequality [13] it can be shown that

$$-1 \leq \mathrm{corr}\left(\mathrm{P}, \hat{\mathrm{P}}\right) \leq 1.$$

This mathematical property is valid for any two series as introduced in (3.1.1a) and (1).

Remark 3.9 The numerations of the entries in the formulae do not influence the numerical values, i.e., any permutation of the measurement leaves the numerical value unchanged. There are no dependencies between the measurements.

Correlation normalizes the covariance to the interval $[-1, 1]$ and is used for the direct comparison of different portfolios. It has no units and is a scalar. A correlation close to 1 or -1, respectively indicates that the two time series move together or in the opposite direction, respectively. A correlation close to 0, however, indicates that they are out of sync. A correlation equal to 1 does not mean that any two series of (3.1.1a) and (1) are the same.

Definition 3.9 The ***coefficient of determination $R(P_1, P_2)$—squared or $R^2\left(P, \hat{P}\right)$*** is defined by

$$\mathrm{R}^2\left(\mathrm{P}, \hat{\mathrm{P}}\right) = \left(\mathrm{Corr}\left(\mathrm{P}, \hat{\mathrm{P}}\right)\right)^2.$$

$\mathrm{R}^2\left(\mathrm{P}, \hat{\mathrm{P}}\right)$ is the proportion of variability in the returns of the portfolio P that relates to the variability of the returns of portfolio $\hat{\mathrm{P}}$. It measures the degree of association of the portfolios P and $\hat{\mathrm{P}}$. A high $\mathrm{R}^2\left(\mathrm{P}, \hat{\mathrm{P}}\right)$ indicates that the portfolios P and $\hat{\mathrm{P}}$ are probably exposed to similar risk exposure that are driving return.

3.2.2 The Tracking Error

We proceed by presenting two definitions for the tracking error based on historical returns of the portfolio and the benchmark. Thus they introduce two ***ex post risk measurements***. The tracking error is a measure of the relative risk of a portfolio versus the benchmark. We consider a benchmark portfolio B with the periodic historical return series

$$r_{B,k}, k = 1, 2, \ldots, N.$$

We proceed with two definitions of the tracking error. The *first definition 3.10* is only dependent on the input data and is thus ***model-independent***. In *the second definition 3.11* the model assumption is that the portfolio returns are a linear function of the benchmark returns (***model-dependent***).

Definition 3.10 (model-independent) The ***tracking error TE(1)*** is defined as follows

$$\mathrm{TE}(1) = \sqrt{\frac{1}{N}\sum_{k=1}^{N} d_k^2} \tag{3.2.5a}$$

where

$$d_k = r_{P,k} - \bar{r}_P - r_{B,k} + \bar{r}_B, k = 1, 2, \ldots, N. \tag{3.2.5b}$$

The following lemma shows that the portfolio returns and the benchmark returns can fluctuate arbitrarily, even though the tracking error vanishes. There is a family of portfolios that yields a tracking error of zero. There is no relation of inequality or even equality between the absolute and the relative return. For instance the absolute error may be extremely large, while the tracking error is almost zero.

Lemma 3.2 We assume that the portfolio and the benchmark with returns (5) has a tracking error TE(1) zero. We introduce

$$\begin{aligned} \hat{r}_{P,k} &= r_{P,k} + c_k, k = 1, 2, \ldots, N, \\ \hat{r}_{B,k} &= r_{B,k} + c_k, k = 1, 2, \ldots, N \end{aligned} \tag{3.2.6}$$

where the series $c_k \in \mathbf{R^1}$ satisfies

$$\sum_{k=1}^{N} c_k = 0.$$

Then the tracking error TE(1) of $\hat{r}_{P,k}$ versus $\hat{r}_{B,k}$ is zero.

Proof: It follows that the arithmetic means of series (5) and (6) are equal. Then we have for the difference $\hat{d}_k$ in (6)

$$\hat{d}_k = \hat{r}_{P,k} - \hat{r}_{B,k} - \bar{r}_P + \bar{r}_B =$$
$$r_{P,k} + c_k - r_{P,k} - c_k - \bar{r}_P + \bar{r}_B = r_{P,k} - r_{P,k} - \bar{r}_P + \bar{r}_B.$$

Thus by summing up and by assumption of the lemma follows

$$\sqrt{\frac{1}{N}\sum_{k=1}^{N}\widehat{d_k^2}} = \sqrt{\frac{1}{N}\sum_{k=1}^{N} d_k^2} = 0.$$

◇

We proceed by illustrating the computation of the tracking error. In Table 3.1 it is assumed that the number of portfolio values and the benchmark values considered are the same and that the periodic return of portfolio value and the benchmark values are taken at the same time points.

The returns d_k in (5) can be calculated in different ways. We consider four versions by using simple returns and continuous returns.

(1a)

$$r_{P,k} = \frac{P_k - P_{k-1}}{P_{k-1}}, r_{B,k} = \frac{B_k - B_{k-1}}{B_{k-1}}, k = 1, \ldots, N, \tag{3.2.7a}$$

(1b)

$$r_{P,k} = \ln\frac{P_k}{P_{k-1}}, r_{B,k} = \ln\frac{B_k}{B_{k-1}}, k = 1, \ldots, N \tag{3.2.7b}$$

for using (5). We proceed with

$$Q_k = \frac{P_k}{B_k}, k = 1, \ldots, N, \tag{3.2.7c}$$

Table 3.1 Input to the ex post tracking error

Time	Portfolio value	Benchmark value
t_0	P_0	B_0
t_1	P_1	B_1
t_2	P_2	B_2
t_3	P_3	B_3
⋮	⋮	⋮
t_N	P_N	B_N

and consider

(2a)

$$r_{P,k} - r_{B,k} = \frac{Q_k - Q_{k-1}}{Q_{k-1}}, k = 1, \ldots, N, \tag{3.2.7d}$$

(2b)

$$r_{P,k} - r_{B,k} = \ln\frac{Q_k}{Q_{k-1}}, k = 1, \ldots, N, \tag{3.2.7e}$$

for using (5). By calculating the difference $r_{P,k} - r_{B,k}$ with (7b) and comparing it to the difference in (7d) we see then that they are algebraically different. If the volatility is significant, they can differ substantially. However, the following lemma shows a helpful property of continuous compounding:

Lemma 3.3 Cases (7b) and (7e) are the same and the return between two time points does not depend on measurements in between the time points.

Proof: We have two assertions. Firstly based on the properties of the logarithm function

$$\ln\frac{Q_k}{Q_{k-1}} = \ln\ Q_k - \ln\ Q_{k-1} = \ln\frac{P_k}{B_k} - \ln\frac{P_{k-1}}{B_{k-1}} =$$
$$\ln\ P_k - \ln\ B_k - \ln\ P_{k-1} + \ln\ B_{k-1} = \ln\frac{P_k}{P_{k-1}} - \ln\frac{B_k}{B_{k-1}}, k = 1, \ldots, N.$$

Secondly for $1 \leq k_1 \leq k \leq k_2 \leq N$ we have again with the properties of the logarithm function

$$\ln\frac{Q_{k_1}}{Q_{k_2}} = \ln\frac{Q_{k_1}}{Q_{k_1+1}} \cdot \ldots \cdot \ln\frac{Q_k}{Q_{k+1}} \cdot \ldots \cdot \ln\frac{Q_{k_2-1}}{Q_{k_2}}.$$

◇

We proceed with a second definition of the tracking error:

Definition 3.11 (regression portfolio versus benchmark, model-dependent) The ***tracking error TE(2)*** is defined by

$$TE(2) = std(P)\sqrt{1 - \rho^2(P, B)}. \tag{3.2.8}$$

Theorem 3.1 The tracking error TE(2) is equal to the sum of the distances to the regression line, i.e., the line

$$\hat{r}_{P,k} = \gamma + \beta\ r_{B,k}, k = 1, \ldots, N, N \geq 2 \tag{3.2.9}$$

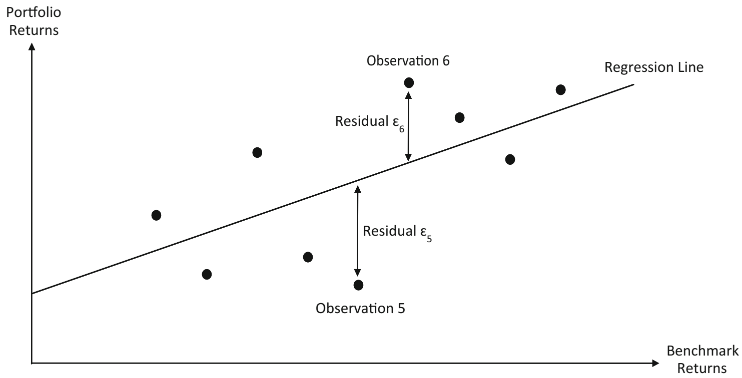


Fig. 3.3 The regression

with (see Fig. 3.3)

$$\gamma = \bar{r}_P - \beta \bar{r}_B \tag{3.2.10a}$$

and

$$\beta = \frac{\mathrm{cov}(P, B)}{\mathrm{var}(B)} \tag{3.2.10b}$$

that minimizes the sum of the square distance from the portfolio returns $r_{P,k}$ and the model returns $\hat{r}_{P,k}$ defined by (9).

Proof: We consider the error by using (10a)

$$\varepsilon_k = r_{P,\ k} - \gamma - \beta\ r_{B,\ k} = r_{P,\ k} - \bar{r}_P - (r_{B,\ k} - \bar{r}_B)$$

and find

$$\sum_{k=1}^{N} \varepsilon_k^2 = \sum_{k=1}^{n} (r_{P,k} - \bar{r}_P - \beta(r_{B,k} - \bar{r}_B))^2 =$$
$$\sum_{k=1}^{N} \left((r_{P,k} - \bar{r}_P)^2 - 2\beta(r_{B,k} - \bar{r}_B)(r_{P,k} - \bar{r}_P) + \beta^2 (r_{B,k} - \bar{r}_B)^2 \right).$$

With (10b) we find

$$2\beta \sum_{k=1}^{N} (r_{B,k} - \bar{r}_B)\ (r_{P,k} - \bar{r}_P) = 2\beta^2 \sum_{k=1}^{N} (r_{B,k} - \bar{r}_B)^2$$

thus

$$\sum_{k=1}^{N} \varepsilon_k^2 = \sum_{k=1}^{N} \left((r_{P,k} - \bar{r}_P)^2 - \beta^2 (r_{B,k} - \bar{r}_B)^2 \right)$$

thus by definition (8) and (10b) we have

$$\sqrt{\sum_{k=1}^{N} \varepsilon_k^2} = \mathrm{std}(P)\sqrt{1 - \beta^2 \frac{\mathrm{var}(B)}{\mathrm{var}(P)}} = \mathrm{std}(P)\sqrt{1 - \rho^2(P, B)} = \mathrm{TE}(2).$$

i.e., we have shown the assertion of the lemma.

◇

Remark 3.10 Theorem 3.1 is based on time series, i.e., we model over time (time series analysis). By referring to (3.4.1) in Sect. 3.4 we consider one time unit; more specifically we model the different returns of the security in an investment universe over one time unit (*cross-sectional analysis*).

Lemma 3.4 If the portfolio returns can be expressed as a linear function of the benchmark return, then $\mathrm{TE}(2) = 0$.

Proof: This follows from the fact that $\varepsilon_k = 0$ in (9).

◇

Lemma 3.5 TE(1) and TE(2) satisfy the inequality:

$$\mathrm{TE}(2) \leq \mathrm{TE}(1).$$

Proof: Based on Theorem 3.1 we see that TE(2) minimizes all linear combination and following the proof of Lemma 3.1 TE(1) is such a specific linear combination and as a consequence error $\varepsilon_k = 0$.

◇

The notion of tracking error stems from control theory and is defined as a measure of the difference between the desired output and the measured output. Originally the aim was to achieve a tracking error equal to zero. In passive asset allocation the investor wants to have a tracking error that dwindles to zero, a goal which is in line with the original definition of the tracking error. As the name implies, in finance the error between the return of the portfolio and the benchmark is measured. Today, however, the tracking error is also used in active asset allocation.

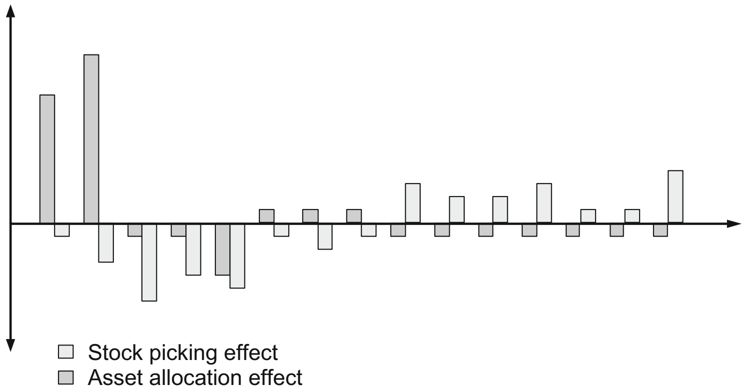


Fig. 3.4 Do the management effects vary over time?

A tracking error of significant size means that a portfolio manager's portfolio deviates considerably from the benchmark.

When using ex post risk, certain fundamental questions arise when using time series:

- What is the optimal length of a particular time series to be used for the performance analysis of a given portfolio?
- What data frequency (daily, monthly etc.) should be used?

In order to assess the quality of an asset manager, we need to average the asset allocation and stock picking effects over the total observation period. Normally the performance attribution calculates a total figure for the sum of the different management effects over the whole reporting period and does not show the management effects over time, for example on a monthly basis. In Fig. 3.4 we compare the management effects for the whole period with those on a monthly basis. The total figures indicate that the portfolio manager was a fairly poor stock picker but a good asset allocator during the reporting period. But this is the wrong conclusion, because the positive asset allocation effect was mainly generated in the first 2 months of the reporting period and the monthly asset allocation effect over the last 7 months was consistently negative. With respect to the stock picking effect there is a similar situation but vice versa. The monthly stock picking effect was negative over the first 8 months but consistently positive over the last 7 months. It will now require further analysis to gain deeper insights into the figures and to come up with the "right" conclusions.

3.3 Risk Decomposition at the Segment Level

On the one hand, an investor could argue that there is no ex post risk because there were no unprecedented events in the past; on the other, another investor could claim that there is no risk in the future, as no real event has materialized. We conclude that risk has to do with probabilities, introduced in Sect. 3.5.

The last section is based on the historical return series of a portfolio. For instance a shift from equities to bonds a year ago is reflected in the analysis. The exposition in the last section is thus called ***ex post risk*** analysis. In this section we break down the risk of a portfolio into segments (Definition 2.5) or individual investments. The weights of the portfolio are current as of today, which is why the approach pursued in this section is called ***ex ante risk*** analysis. No historical data of the portfolio is reflected. In addition we presume that the market conditions prevail in the (near) future. However, the returns of the investments or segments are examples of historical data. The time series of these returns can provide estimations for the different variances and covariances. We note that these estimations are objective and do not depend on the portfolio holdings.

We start with some concepts from a special discipline in mathematics called linear algebra.

A real matrix A is defined by

$$A = \begin{bmatrix} a_{1,1} & . & . & a_{1,n} \\ . & & & . \\ a_{N,1} & . & . & a_{N,n} \end{bmatrix} \tag{3.3.1}$$

where $a_{i,\,j}, i = 1, \ldots, N, j = 1, \ldots, n$ are real numbers, i.e., for short A is an element in $\mathbf{R}^{\mathbf{N}\times\mathbf{n}}$.

A real vector v is defined by

$$v = \begin{bmatrix} v_1 \\ . \\ . \\ v_n \end{bmatrix} \tag{3.3.2}$$

where $v_j, j = 1, \ldots, n$ are real numbers, i.e., for short, v is an element in $\mathbf{R}^{\mathbf{n}\times\mathbf{1}} = \mathbf{R}^{\mathbf{n}}$. The null vector v is defined by

$$v = \begin{bmatrix} 0 \\ . \\ . \\ 0 \end{bmatrix}.$$

The transpose $\boldsymbol{A^T}$ of A is defined by

$$A^T = \begin{bmatrix} a_{1,1} & a_{1,N} \\ a_{n,1} & a_{n,N} \end{bmatrix},$$

for short, A^T is an element in $\mathbf{R}^{\mathbf{n} \times \mathbf{N}}$. ***The transpose*** $\boldsymbol{v^T}$ of v is defined by

$$v^T = [v_1, \ldots, v_n],$$

for short, and v^T is an element in $\mathbf{R}^{\mathbf{1} \times \mathbf{n}} = \mathbf{R}^{\mathbf{n}}$.

We proceed by introducing the matrix multiplication, i.e., the multiplication of two matrices

$$A = \begin{bmatrix} a_{1,1} & . & . & a_{1,n} \\ . & & & . \\ a_{N,1} & . & . & a_{m,n} \end{bmatrix}$$

and

$$B = \begin{bmatrix} b_{1,1} & . & . & b_{1,M} \\ . & & & . \\ b_{n,1} & . & . & b_{n,m} \end{bmatrix}.$$

We consider the matrix C, defined by

$$C = \begin{bmatrix} c_{1,1} & . & . & c_{1,M} \\ . & & & . \\ c_{N,1} & . & . & c_{n,M} \end{bmatrix}.$$

Then the product or the matrix multiplication $C = A \cdot B$ is defined by the elements

$$c_{i,\,j} = \sum_{k=1}^{n} a_{i,k} b_{k,\,j},\ 1 \leq i \leq N,\ 1 \leq j \leq M.$$

Example 3.5 We consider the matrices

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$$

and calculate the product $C = A \cdot B$, which yields

$$C = \begin{bmatrix} 14 & 8 & 3 \\ 8 & 5 & 2 \\ 3 & 2 & 1 \end{bmatrix}.$$

◇

Definition 3.12 $A \in \mathbf{R}^{nxn}$ is called ***positive definite*** if

$$\sum_{j=1}^{n}\sum_{i=1}^{n} a_{i,j} v_j v_i = 0 \quad \text{for only} \quad v_1 = 0, \ldots, v_n = 0$$

and otherwise

$$\sum_{j=1}^{n}\sum_{i=1}^{n} a_{i,j} v_j v_i > 0, \forall v_j \in \mathbf{R}^1, \forall v_i \in \mathbf{R}^1 \tag{3.3.3a}$$

is satisfied.

Furthermore $A \in \mathbf{R}^{nxn}$ is called ***positive semi-definite*** if

$$\sum_{j=1}^{n}\sum_{i=1}^{n} a_{i,j} v_j v_i \geq 0, \forall v_j \in \mathbf{R}^1, \forall v_i \in \mathbf{R}^1 \tag{3.3.3b}$$

is satisfied.

The elements of the matrices (1) are variances, standard deviations or correlations of the constituents of a portfolio or a benchmark. The components of a vector (2) are weights (holdings) of the portfolio, benchmark, resp. or the returns of an investment or a segment of the portfolio, benchmark, resp.

In the following we consider a portfolio P and a benchmark B with n constituents $C_j, 1 \geq j \geq n$ with returns

$$r_{P,k}, r_{B,k}, r_{j,k}, 1 \leq j \leq n, 1 \leq k \leq N \tag{3.3.4}$$

at time t_k. The constituents $C_j, 1 \geq j \geq n$ can be segments or investments like stocks or bonds. Here the underlying frequency of the time points is not fixed. In most types of performance software, however, a monthly or daily data frequency is used. Similarly to (3.1.2) and (2) we introduce

$$\mathrm{var}(C_i) = \frac{1}{N}\sum_{k=1}^{N} (r_{i,k} - \bar{r}_i)^2, i = 1, \ldots, n \tag{3.3.5a}$$

and

$$\mathrm{cov}\left(C_i, C_j\right) = \frac{1}{N}\sum_{k=1}^{N}\left(r_{i,k} - \bar{r}_i\right)\left(r_{j,k} - \bar{r}_j\right), 1 \le i, j \le n, i \ne j. \tag{3.3.5b}$$

The calculation of ex ante risk figures are based on these two inputs. Ex ante risk considerations can be perceived as risk management by applying different weight schemes. With

$$\bar{r}_j = \frac{1}{N}\sum_{k=1}^{N} r_{j,k}, 1 \le j \le N$$

the input data is given

$$R = \begin{bmatrix} r_{1,1} - \bar{r}_1 & . & . & . & r_{1,N} - \bar{r}_n \\ . & & & & . \\ . & & & & . \\ . & & & & . \\ . & & & & . \\ . & & & & . \\ r_{n,1} - \bar{r}_1 & . & . & . & r_{n,N} - \bar{r}_n \end{bmatrix}. \tag{3.3.6a}$$

whereas

$$R^T = \begin{bmatrix} r_{1,1} - \bar{r}_1 & . & . & . & . & . & . & r_{n,1} - \bar{r}_1 \\ . & & & & & & & . \\ . & & & & & & & . \\ . & & & & & & & . \\ r_{1,N} - \bar{r}_n & . & . & . & . & . & . & r_{n,N} - \bar{r}_n \end{bmatrix}. \tag{3.3.6b}$$

Lemma 3.6 (variance covariance matrix based on return data) Considering (4) and (6), for the matrix

$$S = \begin{bmatrix} s_{1,1} & . & . & s_{1,n} \\ . & & & . \\ . & & & . \\ s_{n,1} & . & . & s_{n,n} \end{bmatrix}$$

defined by

$$S = \frac{1}{N} R^T R \tag{3.3.7}$$

we have

(a) $$s_{jj} = \mathrm{var}\big(C_j, C_j\big), 1 \le j \le n,$$

$$s_{ij} = \mathrm{cov}\big(C_i, C_j\big), i \ne j, 1 \le i, j \le n.$$

(b) S is positive semi-definite, i.e., S satisfies (3b).

Proof: The first assertion (a) follows from (3) and (5) and we proceed by demonstrating assertion (b).

(b) We multiply S by $w^T \in \mathbf{R^n}$ and $w \in \mathbf{R^n}$ and find

$$w^T S\, w \in \mathbf{R^1}, \forall w \in \mathbf{R^n}.$$

With (7) and $v \in \mathbf{R^n}$

$$v = w\ \ R$$

we have

$$w^T S\, w = \frac{1}{N} w^T R^T R w = \frac{1}{N} v^T v, \forall w \in \mathbf{R^n}.$$

By components

$$\begin{bmatrix} v_1 & . & . & v_n \end{bmatrix} \begin{bmatrix} v_1 \\ . \\ . \\ v_n \end{bmatrix}$$

the right-hand side is

$$\frac{1}{N}\big(v_1^2 + \ldots + v_n^2\big) \ge 0.$$

Thus the assertion (3b)

$$w^T S\, w \ge 0, \forall w \in \mathbf{R^n}$$

is shown.

◇

Remark 3.11 S is called the ***risk matrix*** and is the starting point for the portfolio optimization theory (Chap. 4).

The following remarks are an attempt to investigate the structure of the risk matrix. They go in the direction of a scenario analysis of the risk matrix. Which matrices can and which matrices cannot stem from the return data of a portfolio, benchmark, resp.? For instance in Example 3.7 we have a matrix which cannot be a variance covariance matrix.

Remark 3.12 Lemma 3.14 essentially states that a data matrix (6) leads to a positive semi-definite risk matrix (7). Discussing positive semi-definite matrices in terms of the underlying data series is the inverse problem. To the best of our knowledge, this problem has not been widely discussed. A first result in this direction is based on a well known theorem from linear algebra: The so-called ***Cholesky*** decomposition of a matrix states that a positive definite matrix can be decomposed in the form (7) such that the matrix R in (6) is quadratic, i.e., $N = n$ in (6) and is upper triangular, i.e., the elements under diagonal are zero. In addition, if the elements in the diagonal are assumed to be positive the decomposition (7) is unique.

Example 3.6 We consider the matrix

$$S = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}$$

and find

$$R = \begin{bmatrix} 2\sqrt{2} & \frac{1}{\sqrt{2}} \\ 0 & \sqrt{\frac{15}{2}} \end{bmatrix}$$

such that (7) is satisfied with $N = 2$.

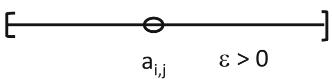
◇

Remark 3.13 The definition of positive definitiveness for a quadratic matrix (see Definition 3.12) consists an inequality sign. In mathematics it is known that if we replace an element $a_{i,j}$ of a matrix (1) with $1 \le i \le n, 1 \le j \le n$ by a number in a sufficiently small circle with radius $\varepsilon > 0, \varepsilon \in \mathbf{R}^1$, the inequality survives. We refer to this as perturbation of the element. In positive semi-definite matrices, the perturbation is generally not possible (Fig. 3.5).

Example 3.7 Assuming a portfolio with three data points (Variable N) and n segments (Variable n) we choose the vector w of return

$$w = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Fig. 3.5 Perturbation



and calculate $w^T w$

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 14.$$

Then $\frac{1}{N} \mathrm{w}^{\mathrm{T}} \mathrm{w} = \frac{14}{3}$ is the uncentered variance of an investment. Assuming one data point (Variable N) and three segments (Variable n) we choose the vector w of return

$$\mathrm{w} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

and calculate w w^{T}

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}.$$

Evaluating for instance the non-null vector

$$\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$$

yields

$$\begin{bmatrix} 2 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = 0,$$

i.e., we see that the matrix is only positive semi-definite. Then w w^{T} is the variance covariance matrix of a portfolio with three segments.

◇

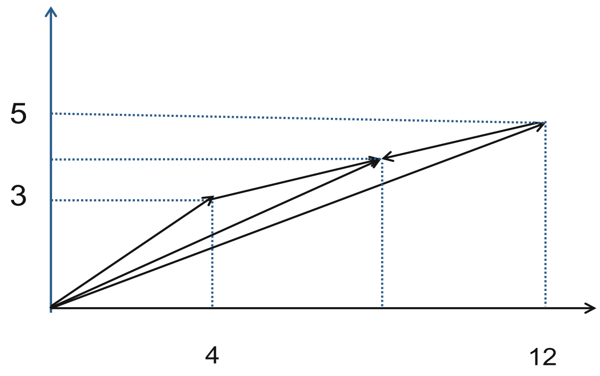
Example 3.8 (degree of freedom) Referring to Fig. 3.6 we consider the vector

$$\mathrm{v}_1 = \begin{bmatrix} \mathrm{v}_{11} \\ \mathrm{v}_{12} \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

with length 5 and the vector

$$\mathrm{v}_2 = \begin{bmatrix} \mathrm{v}_{21} \\ \mathrm{v}_{22} \end{bmatrix} = \begin{bmatrix} 12 \\ 5 \end{bmatrix}$$

Fig. 3.6 Degrees of freedom



with length 13. Then for the arithmetic mean we have

$$\frac{v_1 + v_2}{2} = \begin{bmatrix} 8 \\ 4 \end{bmatrix}.$$

For deviation from the arithmetic mean we have:

$$v_1 - \frac{v_1 + v_2}{2} = -v_2 + \frac{v_1 + v_2}{2} = \begin{pmatrix} -4 \\ -1 \end{pmatrix},$$

the deviation of v_1 as vector from the arithmetic mean is opposite to the deviation of v_2 from the arithmetic mean. This is valid for any two points in the plane, i.e., the distance of any two points to the arithmetic mean is the same for the two points. As easy application, we consider the empirical variance of one constituent C for two data points

$$\mathrm{var}(C) = \left[v_1 - \left(\tfrac{v_1+v_2}{2}\right)\right]^2 + \left[v_2 - \left(\tfrac{v_2+v_1}{2}\right)\right]^2 = \left(\tfrac{v_1-v_2}{2}\right)^2 + \left(\tfrac{v_2-v_1}{2}\right)^2.$$

The squares of the two deviations are equal, i.e., we only have one variable in the deviation that can be given. With linear algebra it can be shown that n variables have only $n-1$ free parameters. See also population variation (3.1.2) versus empirical variation (3.1.5).

◇

In the following theorem we discuss the risk of a portfolio and the tracking error denoted by TE(3). As they are solely based on actual weights of the portfolio, they are called ex ante figures. The tracking error defined by TE(1), TE(2), resp. defined in (3.2.5), (3.2.8), resp. are ex post tracking errors.

Theorem 3.2 We consider a portfolio P and benchmark B with n constituents $C_j, 1 \le j \le n$ with return history (4) and the vector w for the weights

w_j for the portfolio and the vector b for the weights b_j of the benchmark, $1 \leq j \leq n$. Then the following holds:

(a) For the variance of the portfolio P we have

$$\mathrm{Var}(P) = w^T S w =$$
$$\sum_{j=1}^{n}\sum_{i=1}^{n} s_{i,j} w_j w_i =$$
$$\sum_{j=1}^{n}\sum_{\substack{i=1 \\ i\neq j}}^{n} w_j w_i \mathrm{cov}(C_j, C_i) + \sum_{j=1}^{n} (w_j)^2 \mathrm{var}(C_j) =$$
$$\begin{bmatrix} w_1 & . & . & w_n \end{bmatrix}^T \begin{bmatrix} s_{1,1} & . & . & s_{1,n} \\ . & & & . \\ . & & & . \\ s_{n,1} & . & . & s_{n,n} \end{bmatrix} \begin{bmatrix} w_1 \\ . \\ . \\ w_n \end{bmatrix}$$

and the corresponding risk is

$$\mathrm{std}(P) = \sqrt{w^T S w} = \sqrt{\sum_{j=1}^{n}\sum_{i=1}^{n} s_{i,j} w_j w_i}. \tag{3.3.8}$$

(b) For the ex ante tracking error denoted by TE(3) we have

$$\begin{aligned} \mathrm{TE}(3) &= \sqrt{(w-b)^T S (w-b)} = \\ &\sqrt{\sum_{j=1}^{n}\sum_{i=1}^{n} s_{i,j} (w_j - b_j)(w_i - b_i)} = \\ &\begin{bmatrix} w_1 - b_1 & . & . & w_n - b_n \end{bmatrix}^T \begin{bmatrix} s_{1,1} & . & . & s_{1,n} \\ . & & & . \\ . & & & . \\ s_{n,1} & . & . & s_{n,n} \end{bmatrix} \begin{bmatrix} w_1 - b_1 \\ . \\ . \\ w_n - b_n \end{bmatrix}. \end{aligned} \tag{3.3.9}$$

Proof: From (2.2.4) and (3.1.2) we have

$$\mathrm{var}(P) = \frac{1}{N}\sum_{k=1}^{N} (r_{P,k} - \bar{r}_P)^2 = \frac{1}{N}\sum_{k=1}^{N} \left(\sum_{j=1}^{n} w_j (r_{j,k} - \bar{r}_j) \right)^2 =$$
$$\frac{1}{N}\sum_{k=1}^{N} \left(\sum_{j=1}^{n}\sum_{i=1}^{n} w_j w_i (r_{j,k} - \bar{r}_j)(r_{i,k} - \bar{r}_i) \right) =$$
$$\sum_{j=1}^{n}\sum_{i=1}^{n} w_j w_i \frac{1}{N}\sum_{k=1}^{N} (r_{j,k} - \bar{r}_j)(r_{i,k} - \bar{r}_i).$$

By Lemma 3.3 (a) we have

$$\operatorname{var}(P) = \sum_{j=1}^{n} \sum_{\substack{i=1 \\ i \neq j}}^{n} w_j w_i \operatorname{cov}(C_j, C_i) + \sum_{j=1}^{n} w_j^2 \operatorname{var}(C_j)$$

and conclude the assertion (6) in (a). The proof of (b) followings by substituting (3.2.4) in the proof of (a).

◇

Furthermore the ***correlation matrix*** is defined by

$$\rho = \begin{bmatrix} 1 & . & . & \rho_{1,n} \\ . & & & . \\ . & & & . \\ \rho_{n,1} & . & . & 1 \end{bmatrix} \tag{3.3.10a}$$

where

$$\rho_{i,j} = \frac{s_{i,j}}{\sqrt{s_{i,i}}\sqrt{s_{j,j}}}. \tag{3.3.10b}$$

Lemma 3.7 If the covariance variance matrix is positive semi-definite, positive definite, resp. the correlation matrix based on (7) with (5) is positive semi-definite, positive definitive resp.

Proof: If S is positive semi-definite we have

$$\sum_{j=1}^{n} \sum_{i=1}^{n} s_{i,j} v_j v_i \geq 0, \forall v_j \in \mathbf{R^1}, \forall v_i \in \mathbf{R^1}.$$

We consider

$$v_i = \frac{\hat{v}_i}{\sqrt{s_{i,i}}},$$

hence

$$\sum_{j=1}^{n} \frac{s_{i,j}}{\sqrt{s_i}\sqrt{s_j}} \hat{v}_j \hat{v}_i \geq 0, \forall \hat{v}_j \in \mathbf{R^1}, \forall \hat{v}_i \in \mathbf{R^1}$$

and conclude that the correlation matrix is also semi-positive definite. The positive definite case follows analogously.

◇

In the following we investigate the structure of the risk matrix and the correlation matrix.

Example 3.9 We consider a portfolio P with investments or segments C_1 and C_2 and risk

$$S = \begin{pmatrix} \mathrm{var}(C_1) & \mathrm{cov}(C_1, C_2) \\ \mathrm{cov}(C_1, C_2) & \mathrm{var}(C_2) \end{pmatrix}.$$

With matrices the risk of the portfolio is

$$\mathrm{var}(P) = \begin{pmatrix} w_1 & w_2 \end{pmatrix} \begin{pmatrix} \mathrm{var}(C_1) & \mathrm{cov}(C_1, C_2) \\ \mathrm{cov}(C_1, C_2) & \mathrm{var}(C_2) \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix},$$

i.e.,

$$\mathrm{var}(P) = (w_1)^2 \mathrm{var}(C_1) + 2 w_1 w_2 \mathrm{cov}(C_1, C_2) + (w_2)^2 \mathrm{var}(C_2).$$

We consider three special cases:

1. $\mathrm{cov}(C_1, C_2) = \mathrm{var}(C_1)^* \mathrm{var}(C_2)\ \ (\rho = \ 1)$

$$\mathrm{var}(P) = \mathrm{var}(C_1 + C_2)$$

   i.e., the variances are additive and the angle in Fig. 1.2 is 180°.
2. $\mathrm{cov}(C_1, C_2) = 0 (\rho = 0)$

$$\mathrm{var}(P) = \mathrm{var}(C_1) + \mathrm{var}(C_2),$$

   i.e., the angle in Fig. 1.2 is 90°.
3. $\mathrm{cov}(C_1, C_2) = -\mathrm{var}(C_1)^* \mathrm{var}(C_2)\ (\rho = -1)$

$$\mathrm{var}(P) = \mathrm{var}(C_1 - C_2)$$

   i.e., the total variance is equal to the difference $C_1 - C_2$, i.e., the angle in Fig. 1.2 is −180°.

   The example illustrates the Pythagorean theorem in Fig. 1.2.

◇

An ***eigenvalue*** $\lambda \in \mathbf{R}^1$ and an ***eigenvector*** $v \in \mathbf{R}^n$ of a quadratic matrix $A \in \mathbf{R}^{nxn}$ are defined by

$$Av = \lambda v. \tag{3.3.11}$$

In order to calculate the eigenvalue and eigenvector we refer to a text in linear algebra, see for e.g. [17].

Lemma 3.8 The matrix $A \in \mathbf{R}^{nxn}$ is positive definite if and only if the eigenvalues of the matrix are positive.

Proof: [17].

◇

The following theorem yields a further criterion for checking that a matrix is positive definite [28].

Lemma 3.9 (Gerschgorin) Let $A \in \mathbf{R}^{nxn}$ be an arbitrary quadratic matrix and we consider

$$\alpha_j = \sum_{\substack{i=1 \\ i \neq j}}^{n} \left| a_{i,j} \right| \tag{3.3.12}$$

then there is an eigenvalue λ of A and an index j_0 such that

$$\left| \lambda - a_{j_0, j_0} \right| \leq \alpha_{j_0}.$$

Proof: We consider (7) in components

$$\left(\lambda - a_{jj} \right) v_j = \sum_{\substack{i=1 \\ i \neq j}}^{n} a_{i,j} v_i.$$

Then there exist j_0 such that

$$\left| v_{j_0} \right| \geq \left| v_i \right|, \forall i$$

and

$$\left| v_{j_0} \right| \neq 0,$$

i.e., by (8)

$$\left| \lambda - a_{i,i} \right| \left| v_i \right| \leq \sum_{\substack{i=1 \\ i \neq j}}^{n} \left| a_{i,j} \right| \left| v_j \right| = \alpha_{j_0} \left| v_{j_0} \right|,$$

hence

$$\frac{\left| \lambda - a_{i,i} \right| \left| v_i \right|}{\left| v_{j_0} \right|} \leq \alpha_{j_0},$$

As this relationship is valid for all i we choose $i = j_0$, which yields the assertion.

◇

Example 3.10 Let

$$S = \begin{bmatrix} 1 & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & 1 & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & 1 \end{bmatrix}.$$

Based on Gerschgorin the matrix is positive definite, thus S represents a correlation matrix. This matrix can be perturbed (see Remark 3.13 and Fig. 3.5) without losing the positive definiteness.

◇

Example 3.11 We consider the matrix

$$S = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}.$$

For the following we refer to [17]. The calculation of the eigenvalues is based on the ***characteristic polynomial P(λ)***.

With

$$ID = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

we find

$$P(\lambda) = \det(A - \lambda\ ID) = \begin{bmatrix} 1-\lambda & -1 & 1 \\ -1 & 1-\lambda & -1 \\ 1 & -1 & 1-\lambda \end{bmatrix} =$$

$$(1-\lambda)\ \det\begin{bmatrix} 1-\lambda & -1 \\ -1 & 1-\lambda \end{bmatrix} + \det\begin{bmatrix} -1 & 1 \\ -1 & 1-\lambda \end{bmatrix}$$

$$+\det\begin{bmatrix} -1 & 1 \\ 1-\lambda & -1 \end{bmatrix} = 0,$$

where det stands for the determinant defined by

$$\det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - cd.$$

Hence

$$P(\lambda) = \lambda^2(\lambda - 3) = 0,$$

thus S is positive semi-definite and as a result has the properties of a correlation matrix.

For $\lambda = 3$ we have

$$v_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

and for $\lambda = 0$ we have

$$v_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

for the eigenvectors.

◇

Example 3.12 With

$$S = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix}$$

the vector

$$v = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

yields

$$v^T S \ v = -3,$$

i.e., the matrix is not positive semi-definite. It can be seen that three assets cannot have mutual correlations of -1. In other words: For a group of three assets, it is not possible that each asset has a correlation of -1 with the other two assets.

◇

3.4 Risk Decomposition with Factor Analysis

3.4.1 Introduction

Factor models are a well-accepted way of reducing the number of variables and analyzing the return and risk of financial investments. If we have a portfolio of n assets the number of covariances, correlations, resp. is

$$\frac{n(n-1)}{2},$$

i.e., there is quadratic growth with n.

We start by illustrating the concept of the factor analysis by supposing a set of given data and we attempt to find a factor that allows the reproduction of data 1 and data 2. In Fig. 3.7 data 1 (data 2) is approximately 1.5 (2.0) times the factor.

In factor analysis, the aim is to assess the main drivers of the risk and to discuss the risk of a portfolio in term of sensitivity. If we invest e.g. in Royal Dutch Shell, a factor analysis would potentially include the oil price as a factor and asses the sensitivity of the portfolio with respect to oil prices. We proceed by introducing the following notations for $1 \leq i \leq N$ and $1 \leq p \leq M$:

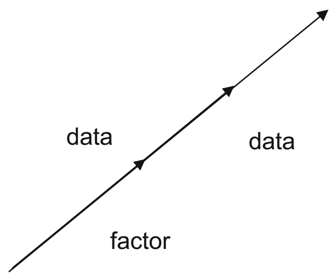
$r_{i,t}$ is the observed return of the security i at time t (independent variable);
$\beta_{i,p,t}$ is the exposure or sensitivity of the security i to the factor p at time t (dependent variable);
$f_{p,t}$ are the factors to be estimated (slopes);
b is a vector with returns that are known a priori (deterministic), like currencies or the risk free rate.

There are different types of factor like

- Macroeconomic factors
- Fundamental factors
- Statistically significant independent factors

Here we do not describe the process of identification for a set of factors and proceed by considering two types of factor analysis.

Fig. 3.7 Basic concept of factor analysis



3.4.2 Cross-Sectional Model (Regression Over Securities)

The following basic linear relationship is only over a time period k and has no relation to the prior time periods. It decomposes for $T = 1, 2, 3, \ldots, N, 1 \le k \le T$ the return $r_{i,k}$, $1 \le i \le n$ of securities of a pre-specified universe. The characteristics of the securities are assumed to be assigned *linearly* with exposure $\beta_{i,p,k}$ to the different factors $f_{p,k}$:

$$r_{i,k} = \beta_{i,1,k} f_{1,k} + \beta_{i,2,k} f_{2,k} + \ldots + \beta_{i,M,k} f_{M,k} + b + \varepsilon_{i,k}. \tag{3.4.1}$$

We assume

$$\beta_{i,\,p,k} = \beta_{p,k}$$

and obviously the relationship (1) for

$$\beta_{p,k}, 1 \le p \le M$$

cannot be satisfied exactly as $M \le n$ in realistic situations, however, there are techniques for minimizing the error ε with an appropriate measurement. It explains security returns via measured sensitivities to systematic factors and estimated returns associated with those factors. Changes in security characteristics are reflected, which relates well to portfolio management. We consider

$$r_{i,k} = \sum_{p=1}^{M} \beta_{p,k} \cdot f_{m,k} + b.$$

We consider

$$F = \begin{bmatrix} f_{1,1} - \overline{f}_1 & . & . & . & f_{1,N} - \overline{f}_N \\ . & & & & . \\ . & & & & . \\ . & & & & . \\ . & & & & . \\ . & & & & . \\ f_{M,1} - \overline{f}_1 & . & . & . & f_{M,N} - \overline{f}_N \end{bmatrix} \tag{3.4.2}$$

and consider the covariance matrix

$$T = \begin{bmatrix} t_{1,1} & . & . & t_{1,M} \\ . & & & . \\ . & & & . \\ t_{M,1} & . & . & t_{M,M} \end{bmatrix}$$

defined by

$$T = \frac{1}{N} F^T F.$$

In keeping with Lemma 3.1, T is positive semi-definite. Furthermore the components $\beta_{i,p}$ of the vector

$$\beta_i = \begin{bmatrix} \beta_{i1} \\ \\ \beta_{iM} \end{bmatrix}$$

measure the impact of the factor $f_{i,\,p}, 1 \leq p \leq M, 1 \leq i \leq n$ on the overall return $r_{i,\,k}$ of the asset i; for short, $\beta_{i,p}$ is called the exposure of asset i to the factor $f_{i,p}$. In keeping with Theorem 3.2 the risk of an asset is the

$$\operatorname{std}(C_i) = \sqrt{\beta_i{}^T T \beta_i} = \sqrt{\sum_{p=1}^{M} \sum_{q=1}^{M} t_{p,q} \beta_{i,\,p} \beta_{i,q}}$$

and covariance of

$$\operatorname{cov}(C_i, C_j) = \sqrt{\beta_j^T T \beta_i} = \sqrt{\sum_{p=1}^{M} \sum_{q=1}^{M} t_{p,q} \beta_{j,\,p} \beta_{i,q}}$$

$$\beta = [\beta_1 \quad . \quad . \quad . \quad \beta_n] = \begin{bmatrix} \beta_{1,1} & . & . & . & \beta_{1,M} \\ . & & & & . \\ . & & & & . \\ . & & & & . \\ . & & & & . \\ . & & & & . \\ \beta_{n,1} & . & . & . & \beta_{n,M} \end{bmatrix}$$

Theorem 3.3 We consider a portfolio P with a return series (3.1.1a) and n constituents $C_j, 1 \leq j \leq n$ with return history (3.3.3b) and weights $w_j, 1 \leq j \leq n$. Then the following holds:

(a) for the variance of the portfolio P we have

$$\operatorname{var}(P) = w^T \beta^T T \;\; \beta \;\; w =$$
$$\sum_{j=1}^{n} \sum_{i=1}^{n} w_j w_i \sum_{p=1}^{M} \sum_{q=1}^{M} t_{p,q} \beta_{i,q} \beta_{j,q} =$$
$$\sum_{j=1}^{n} \sum_{\substack{i=1 \\ i \neq j}}^{n} w_j w_i \operatorname{cov}(C_j, C_i) + \sum_{j=1}^{n} w_j \operatorname{var}(C_j) =$$

$$
\begin{bmatrix} w_1 & . & . & w_n \end{bmatrix}^T \begin{bmatrix} \beta_{1,1} & . & \beta_{1,M} \\ . & & . \\ . & & . \\ \beta_{n,1} & . & \beta_{n,M} \end{bmatrix}^T \begin{bmatrix} t_{1,1} & . & t_{M,1} \\ . & & . \\ t_{1,M} & . & t_{M,M} \end{bmatrix} \begin{bmatrix} \beta_{1,1} & . & . & \beta_{1,n} \\ . & & & . \\ \beta_{M,1} & . & . & \beta_{M,n} \end{bmatrix} \begin{bmatrix} w_1 \\ . \\ . \\ w_n \end{bmatrix}
$$

and the corresponding risk is

$$
\mathrm{std}(P) = \sqrt{w^T \beta^T T \ \beta \ w} = \sqrt{\sum_{j=1}^{n} \sum_{i=1}^{n} w_j w_i \sum_{p=1}^{M} \sum_{q=1}^{M} t_{p,q} \beta_{i,q} \beta_{j,q}}. \tag{3.4.3a}
$$

(b) For the tracking error TE(3) we have

$$
\mathrm{TE}(3) = \sqrt{(w-b)^T \beta^T T \ \beta \ (w-b)} =
$$

$$
\sum_{j=1}^{n} \sum_{i=1}^{n} (w_j - b_j)(w_i - b_i) \sum_{p=1}^{M} \sum_{q=1}^{M} t_{p,q} \beta_{i,q} \beta_{j,q} \tag{3.4.3b}
$$

$$
\begin{bmatrix} w_1 - b_1 & w_n - b_n \end{bmatrix}^T \begin{bmatrix} \beta_{1,1} & \beta_{1,M} \\ \beta_{n,1} & \beta_{n,M} \end{bmatrix}^T \begin{bmatrix} t_{1,1} & t_{M,1} \\ t_{1,M} & t_{M,M} \end{bmatrix} \begin{bmatrix} \beta_{1,1} & \beta_{1,n} \\ \beta_{M,1} & \beta_{M,n} \end{bmatrix} \begin{bmatrix} w_1 - b_1 \\ w_n - b_n \end{bmatrix}.
$$

◇

The risk measures in (3.4.3) are ***called ex ante*** because they only depend on the weights $w_j, 1 \le j \le n$ of the portfolio and the benchmarks $b_j, 1 \le j \le n$ as of today. These weights are subjective and the variance and the covariance in (3.4.3) are objective, i.e., they are only dependent on the markets. One approach is to estimate them using statistical techniques, but they can also be derived on the basis of purely economic considerations.

3.4.3 Time Series Model (Regression Over Time)

Unlike in (1), here we assume

$$
\beta_{i,\,p} = \beta_{i,\,p,t,}\, 1 \le p \le M
$$

i.e., the factors do not depend on t, thus we consider

$$r_{i,k} = \beta_{i,1}\, f_{i,k} + \beta_{i,2}\, f_{2,k} + \ldots + \beta_{i,n}\, f_{n,k} + b = \sum_{p=1}^{M} \beta_{i,\,p} \cdot f_{p,k} + b,$$

i.e., we vary over time whereas in section b we vary in the universe at a point in time or over a unit of time. Here we explain security return *over time* via the measured return associated with systemic factors and estimated sensitivities to those factors. The classic example is the Capital Asset Pricing Model (CAPM), a one-factor model that we will discuss in Chap. 4.

3.4.4 The Principal Component Analysis

Contrary to the factor analysis described in a, this analysis is not based on pre-specified factors but aims at identifying the most influential factors. It is based on an eigenvalue analysis (see Sect. 3.3 and in particular Definition (3.3.11) and Example 3.11) and seeks to find a small number factor that describes most of the variation in a large number of correlated factors. To the best of our knowledge this analysis is mostly used in fixed attribution. It can be shown that a fixed income portfolio is governed by three factors referred to as shift, twist and butterfly.

We proceed by illustrating the factor analysis using two typical investment processes in portfolio management. It is essential that the performance measurement mirrors the investment process. In factor analysis we assume an investment process in which all investment decisions are made simultaneously and independently.

Example 3.13 We illustrate a typical ***equity investment* process** (see Fig. 3.8). We invest first in regions, then in sectors, which is followed by stock selection.

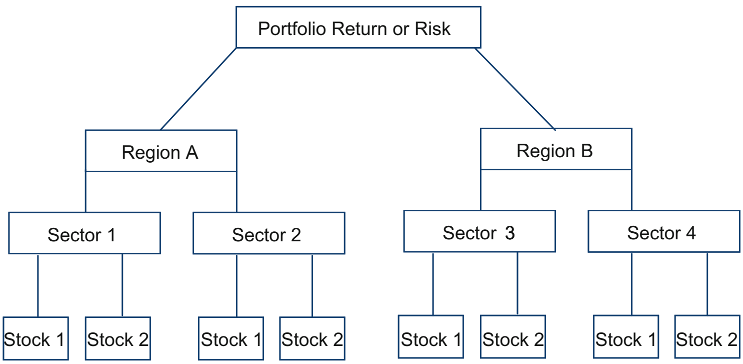


Fig. 3.8 The equity investment process

For an equity universe, (1) can assume the following form

$$r_{i,k}^{loc} - r_{f,k} = \sum_{p=1}^{M} e_{i,\,p,k} \cdot f_{i,k} + \sum_{ind/\,sec} \delta_{ind/\,sec} \cdot f_{ind/\,sec\,,k} + \sum_{region} \delta_{region} \cdot f_{region,k} + \sum_{country} \delta_{country,k} \cdot \beta_{i,k} \cdot f_{country,k} + \varepsilon_{i,k}$$

where $r_{i,k}^{loc}$ is the local return of the ith security, $r_{f,k}$ is the risk-free rate, $e_{i,j,t}$ is the exposure of the ith security to the jth factor, $f_{j,k}$ is the factor return, $\beta_{j,k}$ is the exposure of the ith security to its country factor, and $\varepsilon_{j,k}$ is the residual of the ith security with expected value zero.

Exposure to region/country/industry group assumes either 0 or 1, depending on whether or not the security is found in the region, country or industry group.

Table 3.2 shows a specification of the factors.

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Example 3.14 We illustrate a typical ***fixed income process*** (see Fig. 3.9). We first invest in currencies, followed by durations and then specific issues.

At $t = 0$ the price $P_{i,0}$ of the ith bond in local currency with time to maturity $t_{i,N}$ is given by the total sum of the discounted cash flow:

$$P_{i,0} = \sum_{k=0}^{N} CF_{i,k} e^{-[r(t_k)+s(t_k)]\ \ t_k}$$

Here r is the spot rate and s is a spread to the spot rate at time $t_k, 0 \le k \le N$. To calculate a return on this bond, it is necessary to express the price of the bond at some later small or instantaneous time t

$$P_{i,t} = \sum_{k=0}^{N} CF_{i,k} e^{-\left[r(t)+dr(t)+\left(s(t)+ds(t)\right]\,(t_k-t)\right.}.$$

Following the work done by Nelson Siegel on representing the shape of the yield curve and the fact that the dynamic of the yield curve can be described using three basic movements, it is proposed that dr(t) can be seen in an expansion of three factors around $dr(t) = 0$:

$$dr(t) = x_1 + x_2\left(1 - e^{-\left(\frac{t}{7}\right)}\right) + x_3\frac{t}{7}e^{\left(1-\frac{t}{7}\right)} \qquad (3.4.4)$$

where x_1, x_2, x_3 are coefficients for the exponentials in the above expansion. We identify the factors

Table 3.2 Equity factors source: Wilshire Associates (Wilshire®)

Region	Asia	Latin America	Mediterranean/Middle East/Africa	Europe	North America	Pacific
Currency	Yes	Yes	Yes	Yes	Yes	Yes
Country	China India Indonesia Malaysia Pakistan Philippines South Korea Taiwan Thailand	Argentina Brazil Chile Colombia Mexico Peru Venezuela	Czech Republic Egypt Hungary Israel Jordan Morocco Poland Russia South Africa Turkey	Euro Zone Denmark Norway Sweden Switzerland United Kingdom	USA Canada	Australia Japan Hong Kong New Zealand Singapore
Industry	MSCI/S&P GICS Sector			MSCI/S&P GICS Industry Group		
Fundamentals	Log MCAP, E/P Ratio, B/P Ratio, Volatility, Momentum					

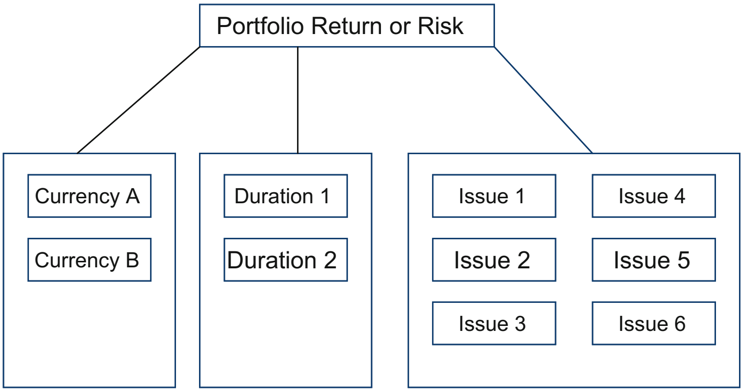


Fig. 3.9 A fixed income investment process

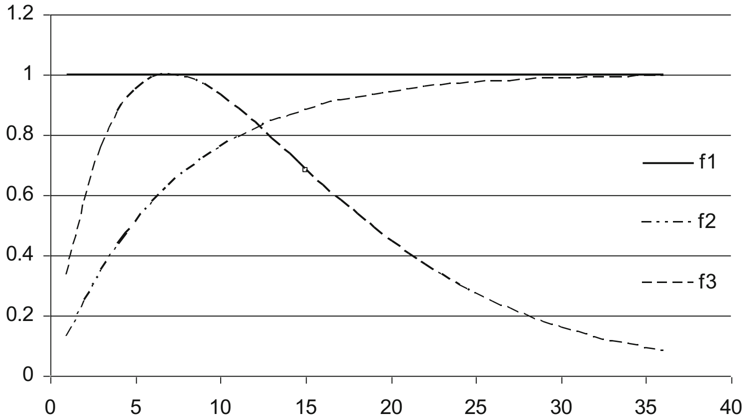


Fig. 3.10 Structure of yield curve factors

$$f_1(t) = 1,\ f_2(t) = \left(1 - e^{-\left(\frac{t}{7}\right)}\right), \quad f_3(t) = \frac{t}{7} e^{\left(1-\frac{t}{7}\right)}$$

(see Fig. 3.10). We see that f_2 and f_3 are zero at $t = 0$. At large t, $f_3(t)$ approaches 0 and, relative to f_1, f_2 dominates the yield, whereas at intermediate t ($t = 7$), f_3 dominates f_2 relative to f_1. Consequently f_1 dominates the short part of the curves, f_2 affects the long end and f_3 impacts the middles part of the yield curve.

We decompose the change of the spread ds(t) linearly

$$ds_i = \sum_j \chi_{i,\,j} \lambda_j$$

where λ_j are the spread factors and $\chi_{i,j}$ is the identity function which takes the value 0 or 1, depending on whether or not the security is found in the spread bucket.

With (4) we find

$$P_{i,t} = \sum_{k=0}^{N} CF_{i,k} e^{-\left(r(t_k)+x_1+x_2\left(1-e^{-\left(\frac{t_k}{7}\right)}\right)+x_3\frac{t}{7}e^{\left(1_k-\frac{t}{7}\right)}+s(t_k)+\sum_j \chi^i_j \frac{\partial P}{\partial \lambda_j} d\lambda_j+\varepsilon_i\right)t_k}.$$

Over time P is a function of $P = P(x_{1,}\ x_{2,}\ x_{3,}\ \lambda_{j,}\ t)$. The differential is

$$\frac{1}{P_i} dP_i = \frac{1}{P_i}\frac{\partial P_i}{\partial t} dt + \frac{1}{P_i}\frac{\partial P_i}{\partial x_1} dx_1 + \frac{1}{P_i}\frac{\partial P_i}{\partial x_2} dx_2 + \frac{1}{P_i}\frac{\partial P_i}{\partial x_3} dx_3 + \frac{1}{P_i}\sum_j \frac{\partial P_i}{\partial \lambda_j} d\lambda_j + \varepsilon_i.$$

We introduce the local return $r^{loc}_{i,k}$ of bond i at time k and the yield $r^{loc}_{i,k}$ by holding the bond over time t

$$r^{loc}_{i,k} - r^{yield}_{i,k} = \frac{1}{P_i} dP_i - \frac{1}{P_i}\frac{\partial P_i}{\partial t} dt_i.$$

The exposures are denoted by $D1_{i,k}$, $D2_{i,k}$, $D2_{i,k}$ and the factor returns are $f_{k,i}$, $k = 1, 2, 3$.

$$r^{loc}_{i,k} - r^{f}_{t} = D1_{i,k} f_{k,1} + D2_{i,k} f_{k,2} + D3_{i,k} f_{k,3} + \sum_j \lambda_j f_{\lambda j} + \varepsilon_i$$

$$D1_{i,t} = \frac{1}{P_i}\frac{\partial P_i}{\partial x_1}\bigg|_{x_1=\lambda_1=0} = \frac{1}{P_i}\sum_{k=1}^{N} t_k CF_{i,t_k} e^{-(r(t_k)+s(t_k))t_k},$$

$$D2_{i,t} = \frac{1}{P_i}\frac{\partial P_i}{\partial x_2}\bigg|_{x_2=\lambda_2=0} =$$

$$\frac{1}{P_i}\sum_{k=1}^{N} t_k \left(1 - e^{-\left(\frac{t_k}{7}\right)}\right) CF_{i,t_k} e^{-(r(t_k)+s(t_k))t_k},$$

$$D3_{i,t} = \frac{1}{P_i}\frac{\partial P_i}{\partial x_3}\bigg|_{x_3=\lambda_3=0} = \frac{1}{P_i}\sum_{k=1}^{N} t_k \left(\frac{t_k}{7} e^{\left(1-\frac{t_k}{7}\right)}\right) CF_{i,t_k} e^{-(r(t_k)+s(t_k))t_k}.$$

Similarly for the spread factors

$$SD1_{i,k,\,j} = \frac{1}{P_i}\frac{\partial P_i}{\partial \lambda_j}\bigg|_{x_1=\lambda_j=0} = \chi_{i,\,j}\sum_{k=1}^{N} t_k CF_{i,t_k} e^{-(r(t_k)+s(t_k))t_k}.$$

The factor returns are classified according to Table 3.3 and a function (1) for a bond universe can assume the form

Table 3.3 Fixed income factors

	North/Latin America			Europe/Middle East			Africa	Asia Pacific		
Currency Market	Canada	Brazil Chile Colombia Mexico Peru	United States	Euro Zone	Czech Republic Denmark Hungary Norway Poland Russia Slovakia Sweden Turkey	Switzerland United Kingdom	South Africa	Australia Hong Kong Japan	India Indonesia New Zealand Philippines Taiwan	China Malaysia Singapore South Korea Thailand
Currency	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes
D1, D2, D3	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes
Sector	Agency Bank/Finance Corp/Industrial Inflation-linked Supranational		Agency Bank/Finance FHLMC MBS FNMA MBS GNMA MBS Industrial Inflation-linked Supranational Telecom Utilities-Electric Utilities-Other Yankee	Agency Bank/Finance Corp/Industrial Inflation-linked Supranational Mortgage		Agency* Bank/Finance Corp/Industrial Inflation-linked* Mortgage Supranational		Agency Bank/Finance Corp/Industrial Mortgage** Supranational		Agency
Quality	Aa A Baa		Aa A Baa Ba B Caa	Aa A Baa		Aa A Baa		Aa A Baa		Partial coverage***
Other Spread			15 Year MBS Balloon MBS FHMLC Prepay FNMA Prepay GNMA Prepay Term Spread Volatility	Austria Belgium Finland France Greece Ireland Italy Luxembourg Netherlands Portugal Slovakia Spain						

*United Kingdom only. **Australia only. ***China: A, Baa and below; Malaysia: A; Singapore: AA, A; South Korea: AA, A; and Thailand: A, Baa and below.

Source: Wilshire Associates (Wilshire®)

$$
\begin{aligned}
r^{loc}_{i,k} - r_{f,k} = {} & \chi_{i,c\notin \mathrm{Euro}} \sum_{c \in \mathrm{countries}} \chi_{i,c} (D1_{i,k}\, f_{1,k,c} + D2_{i,k}\, f_{2,k,c} + D3_{i,k}\, f_{3,k,c}) + \\
& \chi_{i,c\in \mathrm{Euro}} \left(D1_{i,k}\, f_{1,k,\mathrm{Euro}} + D2_{i,k}\, f_{2,k,\mathrm{Euro}} + D3_{i,k}\, f_{3,k,\mathrm{Euro}} + \sum_{\substack{c \,\in\, \mathrm{Euro} \\ \mathrm{Countries}}} D1_{i,k}\, f_{c,k,\mathrm{Euro}} \right) + \\
& + \sum_{\substack{s \,\in\, \mathrm{Sector} \\ c \,\in\, \mathrm{Currency}}} \chi_{i,s,c} SD1_{i,k}\, f_{s,k,c} + \sum_{\substack{q \,\in\, \mathrm{qualities} \\ c \,\in\, \mathrm{Currency}}} \chi_{i,q,c} SD1_{i,k}\, f_{q,k,c} + \\
& + \chi^{i}_{c=\mathrm{USA}} \sum_{a \in \text{ other factors}} A_{a,i,k}\, f_{a,k} + \varepsilon_{i,k}.
\end{aligned}
$$

◇

We proceed with a few concluding remarks on the material presented in this chapter.

1. A comparison or evaluation of ex post risk figures (see e.g. TE(1) in (3.2.5) and TE(2) in (3.2.8)) versus ex ante risk figures (see e.g. TE(3) in (3.3.8)) is left to further research. It is seen from Examples 3.11 and 3.12 that ex ante absolute risk

and ex ante tracking error are based on the choice of factors in the factor analysis.

2. It is seen that this chapter is based on Definition 3.1 for variance and Definition 3.5 for covariance. This definition is fairly basic and therefore often criticized in the literature because for instance

   - The center around which standard deviation is calculated is the arithmetical average historical return. However, prevailing market conditions can be substantially different and are thus also differentially forecasted from historical indications.
   - Standard deviation assigns equal weight to positive and negative return deviations from the mean. However, as standard deviation is perceived as a risk, only negative deviations should be counted, as only they represent adverse market movements for the investor.

Downside risk is an asymmetrical type of risk. A possible starting point is the decomposition of the variance

$$\frac{1}{n}\sum_{j=1}^{n}\left(r_{j_i}-\bar{r}\right)^2=\frac{1}{n}\sum_{\substack{j=1\\ r_j\leq r}}^{n}\left(r_j-\bar{r}\right)^2+\frac{1}{n}\sum_{\substack{j=1\\ r_j>\bar{r}}}^{n}\left(r_j-\bar{r}\right)^2. \tag{3.4.5}$$

Further exposition of downside risk lies beyond the scope of this work and we refer to the available literature [9]. However, to the best of our knowledge there have been no studies on the superior performance of downside risk in comparison to symmetrical risk measures.

3.5 Probability Concepts

We already started discussing the transition from an ex post to an ex ante analysis of a portfolio (see also Fig. 1.1). The considerations in Sects. 3.3 and 3.4 are called ex ante because they depend only on the holdings of the portfolio as of today and not on the history of the portfolio. Probabilities are a further concept for an ex ante analysis of an investment, a segment or a portfolio. There are two prominent examples where past price data is not available. Firstly in the area of equities an Initial Public Offering (IPO) has no price history and an investigation based on probability might be appropriate. Secondly for a brand new issue on the bond market there will be no historical data for this specific issue. Here the issue on the basis of historical prices is especially precarious because there are many specific features of different fixed income instruments and this allows no comparison to similar instruments in the past. In the following, we introduce some basic definitions:

Definition 3.13 (random variable) A ***random variable*** is a quantity the outcome of which is uncertain.

In the following we consider one time period, i.e., with (2.3.1) we have $N = 1$, $t_0 = 0$, $t_1 = T$ and consider the return of an investment or the portfolio a random variable.

Definition 3.14 (event) An ***event*** is a set of outcomes.

Definition 3.15 (probability) A probability p_j, $j = 0, 1, 2, \ldots, n$ of an event is a real number satisfying

$$0 \le p_j \le 1$$

and for mutually exclusive and exhaustive events we have

$$\sum_{j=1}^{n} p_j = 1.$$

We distinguish between different probabilities:

Empirical probabilities are based on the estimation of historical data. They are considered to be objective and do not depend on an individual opinion.

Subjective probabilities are often used in economic research. Scenarios are then formulated and probabilities are assigned. In [21] different scenarios are discussed in portfolio optimization. There the probabilities are computed by means of an optimization process.

A priori probabilities are derived on logical considerations of the problem. For example for a die the probabilities are known a priori.

In this book we do not discuss forecast models. The simplest forecast model, however, is the assumption that the asset prices remain unchanged. Every forecast model has to be checked against 'tomorrow equal to today'. The concept population versus sample (Sect. 3.1) is similar to forecasting versus empirical or historical information, as it is the transition from considerations of a model to empirical estimations.

Definition 3.16 (Expected value) The ***expected value E(X)*** of a random variable X is the weighted average of the possible outcome of the random variable

$$E(X) = p_1X_1 + p_2X_2 + \ldots + p_nX_n. \tag{3.5.1}$$

Definition 3.17 (Variance) The ***centered variance*** $\boldsymbol{\sigma^2}$ of a random value X is the expected value (the probability-weighted average) of squared deviations from the random variable's expected value:

$$\sigma^2(X) = E\{[(X - E(X)]^2\}. \tag{3.5.2a}$$

Remark 3.14 (2a) is the same as

$$\sigma^2(X) = E(X^T X) - E(X^T)\, E(X). \tag{3.5.2b}$$

If it is assumed that E(X) = 0 then (2) is called the ***uncentered variance σ^2***.

Definition 3.18 (Standard deviation) The ***centered standard deviation std*** is the positive square root of the variance:

$$\mathrm{std}(X) = \sqrt{E\{[(X - E(X)]^2}. \tag{3.5.3}$$

If it is assumed that E(X) = 0 then (3) is called the ***uncentered standard deviation***.

Definition 3.19 (covariance) Given two random variables X_i and Y_j the ***covariance*** between them is defined by

$$\mathrm{Cov}(X_i, Y_j) = E\{[X_i - E(X)] \cdot [Y_j - E(Y)]\}, \quad 1 \le i, j \le n. \tag{3.5.4}$$

Lemma 3.10 For 3 random variables X, Y, Z we have for scalars $\alpha \in \mathbf{R^1}$ and $\beta \in \mathbf{R^1}$

$$\alpha E(X) + \beta E(Y) = E(\alpha X + \beta Y) \tag{3.5.5a}$$

$$\alpha \mathrm{Cov}(X, Z) + \beta\, \mathrm{Cov}(Y, Z) = \mathrm{Cov}(\alpha X + \beta Y, Z) \tag{3.5.5b}$$

$$\alpha\, \mathrm{Cov}(X, Y) + \beta \mathrm{Cov}(X, Z) = \mathrm{Cov}(X, \alpha Y + \beta Z) \tag{3.5.5c}$$

$$\sigma^2(\alpha X) = \alpha^2 \sigma^2(X) \tag{3.5.5d}$$

$$\sigma^2(X) = E(X^T X) - E(X^T)\, E(X). \tag{3.5.5e}$$

Proof: (5a) follows from the definition of the expected value in (1)

$$\alpha \sum_{i=1}^{n} p_j^X X_j + \beta \sum_{j=1}^{n} p_j^y Y_j = \sum_{j=1}^{n} \left(\alpha p_j^X X_j + \beta p_j^y Y_j\right).$$

The left-hand side of (5b) is the same as by using (5a) we have

$$\alpha E\{[X - E(X)] \cdot [Z - E(Z)]\} + \beta E\{[Y - E(Y)] \cdot [Z - E(Z)]\} =$$
$$E\{\alpha[X - E(X)] \cdot [Z - E(Z)]\} + E\{\beta[Y - E(Y)] \cdot [Z - E(Z)]\}.$$

Again by using (5a)

$$\alpha E\{[X-E(X)]\cdot[Z-E(Z)]\}+\beta E\{[Y-E(Y)]\cdot[Z-E(Z)]\}=$$
$$E\{\alpha[X-E(X)]\cdot[Z-E(Z)]+\beta E[Y-E(Y)]\cdot[Z-E(Z)]\}.$$

By using definition (4) we find the right-hand side of (5b).

(5c) is the same as (5b) for the second argument in the covariance. (5d) follows from (5c) and (5d) by identifying the arguments X and Y with $Z=0$.

◇

We proceed by applying the probability concept to a portfolio. We consider (2.2.4):

$$R_P=\sum_{j=1}^{n} w_j R_j = w_1R_1+w_2R_2+\ldots+w_nR_n$$

and consider R_P, R_1,. . ..., R_n as a random variable in one period. For the expected value r_P of the portfolio we find by (5a)

$$E(r_P)=E(w_1R_1+\ldots\ldots+w_nR_n)=w_1E(R_1)+-\ldots\ldots..+w_n(R_n), \quad (3.5.6)$$

i.e., to assess the expected value of the portfolio we assess the expected value of the assets.

Remark 3.15 Referring to Example 3.6 we can express (6) with vectors

$$[w_1\,w_2\,w_3]\begin{bmatrix}E(R_1)\\E(R_2)\\E(R_3)\end{bmatrix}=E\left(r_p\right).$$

Given two random variables R_i and R_j the covariance between the two is defined by

$$\mathrm{Cov}\left(R_i,R_j\right)=E\left\{[R_i-E(R_i)]\left[R_j-E\left(R_j\right)\right]\right\}1\le i,j\le n \qquad (3.5.7a)$$

and in particular for $i=j$ we have

$$\mathrm{Var}(R_i)=E\{[R_i-E(R_i)]\}^2. \qquad (3.5.7b)$$

For the variance of a portfolio we have by (2) and (3)

$$\begin{aligned}\mathrm{Var}(R_P) &= \mathrm{Var}(w_1R_1 + w_2R_2 + \ldots + w_nR_n) = \\ &\sum_{i=1,\ j=1}^{n} w_i w_j \mathrm{Cov}\left(R_i, R_j\right) = \\ &\sum_{i=1,\ j=1,\ j\neq i}^{n} w_i w_j \mathrm{Cov}\left(R_i, R_j\right) + \sum_{j=1}^{n} w_i^2 \mathrm{Var}(R_i).\end{aligned} \tag{3.5.8}$$

Definition 3.20 (correlation) By using (2) the correlation is defined by

$$\rho\left(R_i, R_j\right) = \frac{\mathrm{Cov}\left(R_i, R_j\right)}{\sigma(R_i)\sigma\left(R_j\right)}. \tag{3.5.9}$$

Remark 3.16 For $i = j$ we have

$$\rho\left(R_i, R_j\right) = 1.$$

We intend to calculate the covariance by means of the probability concept. We consider m scenarios and the scenarios are denoted by upper indices. We assume that the probabilities denoted by $p^{k,m}$, $1 \leq k$, $m \leq n$ are independent, then

$$p^{k,m} = p^k \cdot p^m.$$

The input for the elements of the covariance matrix is

$$\mathrm{Var}\left(R_j\right) = \sum_{j=1}^{n}\sum_{p=1}^{m}\left(p_j^p \cdot \left(R_j^p - E\left(R_j\right)\right)\right)^2 \tag{3.5.10}$$

$$\mathrm{Cov}\left(R_i, R_j\right) = \sum_{\substack{j=1\\ i=1\\ i\neq j}} n \sum_{q=1}^{m}\sum_{p=1}^{m}\left(p_j^p \cdot \left(R_j^p - E\left(R_j\right)\right)\right)\left(p_i^q \cdot \left(R_i^q - E(R_i)\right)\right).$$

Example 3.15 We consider a portfolio that invests 75 % of the portfolio in a mutual fund A and 25 % in a mutual fund B. The expected return of fund A is 20 % and the expected return of fund B is 12 %. Furthermore the assumed covariance is in Table 3.4.

By (7) the correlation matrix is in Table 3.5. By (1) and (5) the expected return is

$$E\left(r_p\right) = 0.75 \cdot 20\% + 0.25 \cdot 12\% = 18\%.$$

By (2) and (6) the variance of the portfolio is

Table 3.4 Risk matrix

Fund	A	B
A	625	120
B	120	196

Table 3.5 Correlation matrix

Fund	A	B
A	1	0.342857
B	0.342857	1

$$\mathrm{Var}(\mathrm{E}(\mathrm{r}_{\mathrm{p}})) =$$
$$(0.75)^2 \cdot (25\%)^2 + 2 \cdot 0.75 \cdot 0.25120(\%)^2 + (0.25)^2 \cdot (14\%)^2 =$$
$$408.8125(\%)^2$$

and the standard deviation (3) of the portfolio is

$$\mathrm{std}(\mathrm{E}(\mathrm{r}_{\mathrm{p}})) = \sqrt{\mathrm{Var}(\mathrm{r}_{\mathrm{P}}))} = 20.2191\%.$$

◇

Example 3.16 (centered variance) As in the previous example, here we consider a portfolio that invests 75 % of the portfolio in a mutual fund A and 25 % in a mutual fund B. We consider the two scenarios (Table 3.6).

The values in Table 3.7 are computed by (10). By (9) for the correlation matrix we find in Table 3.8. By (1) the expected return is 20 %. By (2) and (6) the variance of the portfolio is found to be $0.78125(\%)^2$ and the standard deviation (3) of the portfolio is 0.883883 %. As the correlation is -1 there is a portfolio with vanishing variation (see Sect. 4.2)

and with our chosen weights we conjecture that we are 'close' to the minimum variance portfolio.

◇

Example 3.17 (uncentered variance) We consider the portfolio in Example 3.17 and assume that the average is zero. The values in Table 3.9 are computed by (10):

By (9) the correlation matrix is given in Table 3.10.

By (1) the expected return is again 20 %. By (2) and (6) the variance of the portfolio is found to be $150.7813(\%)^2$ and the standard deviation (3) of the portfolio is 12.279 %. As the mean is substantially different from zero the results are also very different from those in Example 3.16.

◇

Table 3.6 Assumed scenario

Probabilities of 3 scenarios	Return Fund A	Return Fund B
0.25	15	25
0.5	20	20
0.25	25	15

Table 3.7 Risk matrix (centered)

Fund	A	B
A	3.125	−3.125
B	−3.125	3.125

Table 3.8 Correlation matrix (centered)

Fund	A	B
A	1	−1
B	−1	1

Table 3.9 Risk matrix (uncentered)

Fund	A	B
A	153.125	146.875
B	146.875	153.125

Table 3.10 Correlation matrix (uncentered)

Fund	A	B
A	1.0000	0.9591
B	0.9591	1.0000

Example 3.18 (randomly perturbed Matrix) The following example shows that a given positive definite Matrix is invariant under perturbation with normal distributed random numbers with

$$\sigma = 1 \tag{3.5.11}$$

in (3.1.4). A possible application of the calculus would be the investigation of the returns observed in the market place versus the assumption that the returns are normal distributed. We consider the Cholesky Factorization (see Remark 3.12) of the known covariance Matrix S (see (3.3.7))

$$S = L^T L. \tag{3.5.12}$$

We consider the product of the random matrix R and L

$$Z = LR$$

and by using (2d) calculate the covariance Matrix V of Z:

$$\mathrm{V}(\mathrm{LR}) = \mathrm{E}\Big((\mathrm{LR})^{\mathrm{T}}(\mathrm{LR})\Big) - \mathrm{E}\Big((\mathrm{LR})^{\mathrm{T}}\Big)\,\mathrm{E}(\mathrm{LR}).$$

This is the same as

$$\mathrm{V}(\mathrm{LR}) = \mathrm{E}\big(\mathrm{R}^{\mathrm{T}}\mathrm{L}^{\mathrm{T}}\mathrm{LR}\big) - \mathrm{E}\big(\mathrm{R}^{\mathrm{T}}\mathrm{L}^{\mathrm{T}}\big)\,\mathrm{E}(\mathrm{LR}).$$

By (5a) as L is a constant matrix we have

$$\mathrm{V}(\mathrm{LR}) = \mathrm{L}^{\mathrm{T}}\big(\big(\mathrm{E}\big(\mathrm{R}^{\mathrm{T}}\mathrm{R}\big) - \mathrm{E}\big(\mathrm{R}^{\mathrm{T}}\big)\,\mathrm{E}(\mathrm{R})\big)\mathrm{L} = \mathrm{L}^{\mathrm{T}}\mathrm{E}(\mathrm{R})\mathrm{L}.$$

Thus by (11) and (12)

$$\mathrm{V}(\mathrm{LR}) = \mathrm{S},$$

i.e., the covariance matrix of the perturbed Matrix is the same as the original matrix.
◇

Chapter 4
Performance Measurement

4.1 Investment Process and Portfolio Construction

We start by considering the investment process depicted in Fig. 4.1. The discussion of return and risk in the previous Chaps. 2 and 3 is relevant for all parts of the investment process. We proceed by focusing on the different steps in the investment process. The topics of this chapter are the ***asset allocation***, ***portfolio construction*** and ***rebalancing***. The evaluation of the investment strategy will follow in Chap. 5.

The investment process is a cycle and the portfolio manager enters this cycle when he or she creates a portfolio with new money to invest.

Definition 4.1 An ***asset class*** is a group of securities that exhibit similar characteristics, behave similarly on the marketplace and are subject to the same laws and regulations. The three main asset classes are equities (stocks), fixed-incomes (bonds) and cash equivalents (money market instruments) (cf. www.investopedia.com).

The ***asset allocation process*** is the decision-oriented process of investing in the different asset classes.

A typical ***investment strategy*** is reflected by the asset allocation process followed by the portfolio construction. In Example 2.7 the asset allocation process is the decision-making process between equities markets in different countries, whereas the portfolio construction identifies individual equities. In most situations the investment process is much more complex and consists of several steps. In this section we describe some approaches to the decision-making process of the investment strategy pursued, which is based on ***actual market data and forecasts***.

The definition of the investment universe (Definition 2.4) and applicable constraints on the portfolio and its constituents are:

- Liquidity needs
- Expected cash flow
- Investable funds (i.e., assets and liabilities)
- Time horizon

W. Marty, *Portfolio Analytics*, Springer Texts in Business and Economics,
DOI 10.1007/978-3-319-19812-5_4

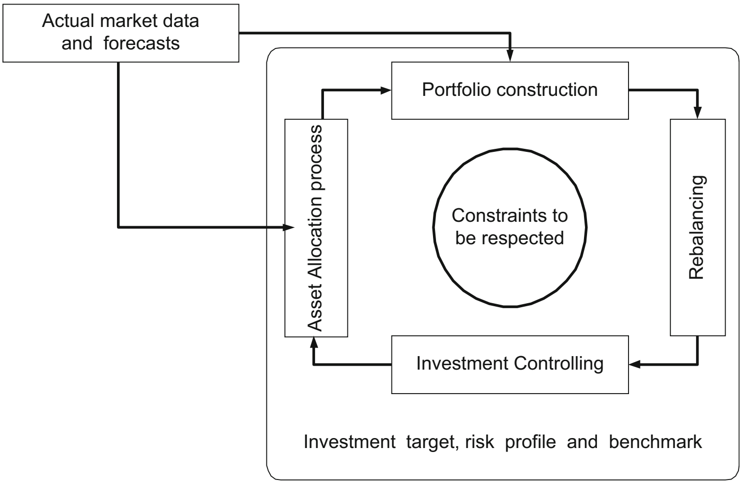


Fig. 4.1 The investment process

- Tax considered
- Regulatory and legal circumstances
- Investor preferences and applicable prohibitions, circumstances and unique needs
- Proxy voting responsibility and guidelines

With the help of a performance attribution analysis of a specific portfolio, the impact on a restriction can be investigated. By applying a performance attribution with or without a constraint, the contribution of such a constraint can be calculated. We consider two realistic examples of performance management:

- If there are restrictions such that the asset manager is not able to invest according to the passive investment alternative—the benchmark—then the calculated management effects may be misleading to the extent that limiting an investment may have a positive or negative impact on the excess return.
- There are equity indices that are dominated by a specific security—assuming for instance 60 % of the index consists of this security—where a maximum weight limit of 10 % is an implicit bet on this specific security by predefining the minimum underweight of this security of 50 %. The question is who is responsible for the return contribution due to this investment restriction. The same is true if there are minimum limits for specific securities.

We proceed with the following definition:

Definition 4.2 The objective of an ***active (passive) investment strategy*** is to outperform (mimic) the benchmark. Portfolios that try to outperform (mimic) the benchmark are said to be ***actively (passively) managed***.

4.1.1 *Active Investment Strategy*

An active investment process requires research into financial markets, currencies, individual securities, yield curves etc. and is thus much more costly than a passive investment process. An active investment strategy attempts to take advantage of changes in the risk premium and consequently to tactically respond to changing market conditions. We proceed by describing approaches to analyzing and forecasting different investments like bonds, equities, commodities or particular financial markets.

In the following we give an overview of the different disciplines for assessing financial markets. For more extensive and in-depth information we recommend [18].

Referring to [18] the Firm-Foundation Theory and the Castle-in-the-Air Theory are the starting point of the following two sections on fundamental and technical research (Fig. 4.2).

4.1.1.1 Fundamental Research

Introduction and Terminology

For those investing in a company's stock, it is vital to know the 'right value' of the common stock. Fundamental research strives to be relatively immune to the optimism and pessimism of the crowd and makes a sharp distinction between a stock's market price and a theoretical price. There might be many such theoretical prices and in the literature they are often called the inner or intrinsic value of a common stock. Fundamental research is based on the idea that in the long term the market price adjusts itself to ***its intrinsic or true value***, which depends on the company and economic data.

In estimating the firm-foundation value of a security, the fundamental task is to estimate e.g. the firm's future stream of earnings, dividends, sales level, operating costs, corporate tax rate, depreciation policies and the costs of its capital requirements.

Current deviations from the intrinsic value can be used profitably, as the market will correct the valuation. Market participants trade rationally and research is

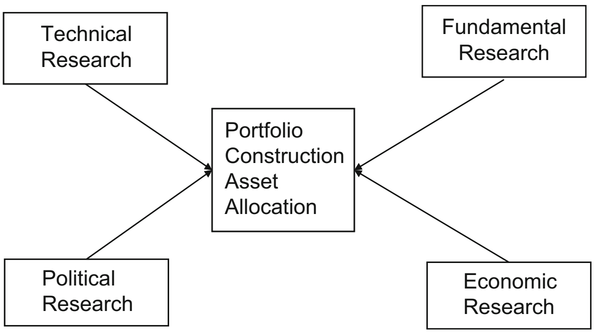


Fig. 4.2 Some different areas of financial market research

worthwhile. Overall fundamental research is mostly corporate research, which is obviously closely connected to the industry or the sector the company belongs to.

Evaluating a Stock: Some First Approaches

The most basic and easy to understand model for evaluating a common stock is ***the discounted cash flow model***. It is based on the arbitrage condition that the value of a common stock today is the same as the present value of the future dividends C_1 and the value of the stock PV_1 discounted to today, hence we have

$$PV_0 = \frac{C_1}{1+r} + \frac{PV_1}{1+r}.$$

By further assuming that the dividends C_j, $j = 1, \ldots, N$ are known, iteration to N periods yields

$$PV_0 = \sum_{j=1}^{N} \frac{C_j}{(1+r)^j} + PV_N$$

where PV_N is the value of the stock after N periods. We see that this is a version of the IRR equation considered in (2.4.1). By assuming that the dividends are known for all future periods we have

$$PV_0 = \sum_{j=1}^{\infty} \frac{C_j}{(1+r)^j}. \tag{4.1.1}$$

A first question is whether instead of the dividends, the earnings or earnings plus non-cash earnings should be discounted in (1) ([4], Chapter 15) [18, 20].

As (2.1.1) is used in various versions, (1) is mainly applied in two ways.

Firstly, an intrinsic PV_0 is computed by using estimates on the dividends and an appropriate discount rate r. An appropriate r can be calculated by using the security market line and the risk premium of the stock.

Secondly, the idea is to use (1) to calculate the discount rate r from the input parameters PV_0 and C_j, $j = 1, \ldots, N, \ldots$

Here are some realizations of the growth of C_j, $j = 1, \ldots, N, \ldots$ in (1):

1. Constant growth over an infinite amount of time
2. Growth for a finite number of years at a constant rate, then growth at the same rate as a typical firm in the economy from that point on
3. Growth for a finite number of years at a constant rate, followed by a period during which growth declines to a steady-state level over a second period of years. Growth is then assumed to continue at the steady-state level into the indefinite future

4.1.1.2 Technical Research

Introduction and Terminology

Technical research is based on the graphical representation of financial data. The value of technical analysis can be expressed by the old proverb "A picture is worth a thousand words." Technical research provides an abstract form of access to the financial market, as every investment or particular financial market can be studied by means of graphical representations. Examples of these investments include gold, currencies markets or equities markets. For instance a graph can contain the price, return or other market data measured vertically in the graph and quantified by the units on the vertical axis. The horizontal axis usually shows the time scale of the graph.

A chartist or ***a technical analyst*** seeks to identify price patterns and trends in financial markets and to exploit those patterns. Although the study of price charts is primary, technicians also use various further methods and approaches for investing in financial markets. These are called ***technical indicators***. The ***moving averages MA(N)*** for a return defined by

$$\mathrm{MA(N)} = \frac{1}{\mathrm{N}}\sum_{\mathrm{k}=1}^{\mathrm{N}} \mathrm{r}_{\mathrm{P,k}}.$$

are a typical example of such an indicator. Referring to Fig. 2.4 we use N data in the past and produce a forecast. The time horizon is $t_{N+1} = N + 1$ and MA(N) is a forecast at time N + 1. Furthermore for N = 1 we have the simplest forecast model, namely "tomorrow is the same as today."

The idea is that the forecast of the future development depends exclusively on the historical direction of prices and the volume of trading. Most chartists believe that the market movement is only 10 % logical and 90 % psychological. Essentially it is assumed that financial markets have a memory and that it is possible to conclude their future behavior on the basis of past behavior.

Some properties of technical analysis are

- Based on strict rules and guidelines
- Provides objectivity in an investment process
- The theoretical framework can be applied immediately
- Applicable for any time horizon and market

Description of Some Important Concepts

We proceed with some basic technical indicators (see Fig. 4.3):

A ***trend*** is a line that connects at least two points. In an uptrend, low points are connected; in a downtrend, high points are connected. A trend, however, is more significant the more points can be connected. "The trend is my friend until it

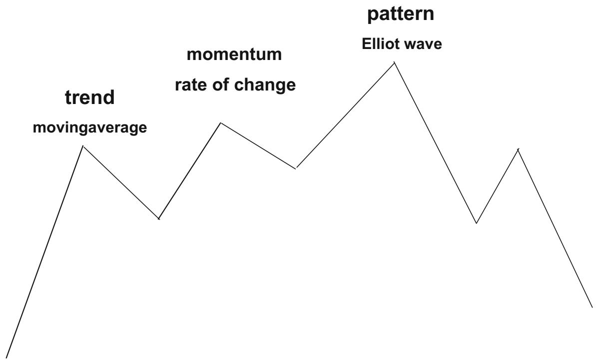


Fig. 4.3 Base concepts and their schematic representation

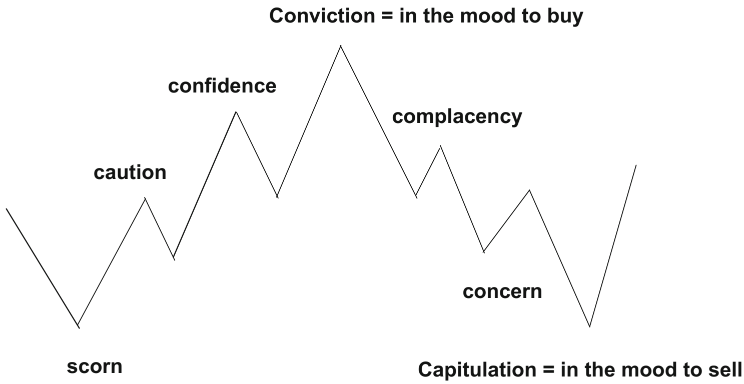


Fig. 4.4 Selling and buying behavior

breaks" is a common saying from technical analysis because future movements of the financial markets are anticipated and invested in accordingly.

In physics, ***momentum*** is an important notion, as it assesses the increase or the decrease in the speed of an object. A momentum is a ***rate of change***. Momentum indicators give a first signal before a trend starts changing; they show whether a trend is accelerating or decelerating.

The Elliot wave principle is a form of technical analysis that investors use to forecast trends on the financial market by identifying extremes in investor psychology. Ralph Nelson Elliott (1871–1948) developed the concept in the 1930s. He proposed that market prices unfold in specific ***patterns***, which practitioners called ***Elliot waves*** or simply waves.

The wave principle posits that collective investor psychology moves from optimism to pessimism and back again in a natural sequence.

Figure 4.4 shows the buy and sell behavior of an investor. Experience shows that the investor is also too late when he buys or sells. He buys when an uptrend is already on the way and he sells when a downtrend has already partially manifested on the marketplace. An optimist is ready to buy, while a pessimist is in the mood to sell.

> The investor must learn when to buy when he is pessimistic and is anxious and when to sell he is optimistic and euphoric.

The most important goal of technical analysis is to detect buy and sell signals, i.e., to notice when there is a turning point.

4.1.1.3 Economic Research

We distinguish between microeconomic and macroeconomic research. Fundamental research is an area that can be perceived as microeconomics, whereas macroeconomics deals with the economics of a country or a region. Economic projections have an important impact on the performance of a portfolio. Someone with a balanced portfolio might switch from one asset class to the other depending on the changing economic situation. In a booming phase an investor should invest in equities. In declining markets, bonds are preferred, and in a recession or an economic crisis, commodities and cash or money market-like instruments are generally recommended (see Fig. 4.5).

Here are some definitions of macroeconomic variables (see www.investorwords.com).

Inflation is the rate at which the general level of prices for goods and services is rising, and, subsequently, purchasing power is falling. Central banks attempt to stop severe inflation, along with severe deflation, in an attempt to keep the excessive growth of prices to a minimum.

Stagflation occurs when the economy is not growing but prices are increasing. This is not a good situation for a country. This happened to a great extent during the 1970s, when world oil prices rose dramatically, fuelling sharp inflation in developed countries. For these countries, including the U.S., stagnation increased the inflationary effects.

A general decline in prices is called ***deflation***, which is often caused by a reduction in the supply of money or credit. Deflation can also be caused by a decrease in government, personal or investment spending. As the opposite of inflation, deflation has the side effect of increased unemployment since there is a lower level of demand in the economy, which can lead to an economic depression.

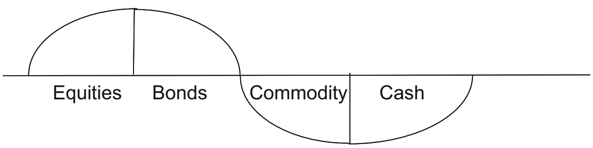


Fig. 4.5 The investment cycle

Central banks attempt to stop severe deflation, along with severe inflation, in an attempt to keep the excessive drop in prices to a minimum.

Extremely rapid or out of control inflation is called ***hyperinflation***. There is no precise numerical definition for hyperinflation. Hyperinflation is a situation in which the price increases are so out of control that the concept of inflation becomes meaningless.

The Consumer Price Index (CPI) is a measure that examines the weighted average of prices of a pre-specified set of consumer goods and services that are representative from an economic prospective. Examples include transportation, food and medical care. The CPI is calculated using average price changes for each good and service considered; in addition, they are weighted according to their importance. Changes in the CPI are used to assess price changes associated with the cost of living. Sometimes this is referred to as ***headline inflation***.

The ***interest rate*** is the amount charged, expressed as a percentage of the principal, by a lender to a borrower for the use of assets. Interest rates are typically calculated on an annual basis, known as the annual percentage rate. The assets borrowed could include, cash, consumer goods, or large assets such as a vehicle or building. Interest is essentially a rental or leasing charge to the borrower in return for their being allowed to use the asset. In the case of a large asset, like a vehicle or building, the interest rate is sometimes known as the lease rate.

We consider the geometric decomposition of the ***nominal interest rate r***

$$1 + r = (1 + r_f)(1 + r_{in})(1 + r_e)$$

in the risk-free rate r_f, the expected inflation rate r_{in} and the risk premium r_e.

Example 4.1 (impact of a macroeconomic variable) An inflation-linked bond protects the bond investor against the risk of raising inflation.

◇

The Gross Domestic Product (GDP) is the monetary value of all the finished goods and services produced within a country's borders in a specific time period, though GDP is usually calculated on an annual basis. It includes all private and public consumption, government outlays, investments and exports less imports that occur within a defined territory. We have

$$\mathrm{GDP} = C + G + I + NX$$

where:

C is equal to all private consumption, or consumer spending, in a nation's economy.

G is the sum of government spending.

I is the sum of all spending on business and private investments.

NX is the nation's total net exports, calculated as total exports minus total imports (NX = Exports – Imports).

The Gross National Product (GNP) is an economic statistic that includes GDP, plus any income earned by residents from overseas investments, minus income earned within the domestic economy by overseas residents.

The economic variables can be measured in the past, but economists also publish ***forecasts*** for them on a regular basis (see Fig. 1.1). Every major financial institution provides forecasts and formulates its own view, called an economic outlook. The forecasts are debated in the media, but also applied to the portfolios of the clients. A forecast is tied to a time horizon. We distinguish between short-term and long-term forecasts. Usually the maximum time horizon is 1 year. In many cases, the accuracy of the forecast is not tested. Providing periodic forecasts is, however, one of the main tasks of economic research.

A further area of economic research is the assessment of dependency between economic variables.

There is a substantial interest in ***business cycles*** (Figs. 4.4 and 4.5) in economics [10, 14]. With t as the time variable, S as the savings function, I as the investment function, Y as the economic activity or real income of companies, K as the capital stock, $\alpha > 0$ and $\delta > 0$, the ***Kaldor Model*** [10, 14] is represented by the ordinary differential equations

$$\begin{aligned} \frac{dY}{dt} &= \alpha(I(Y,K) - S(Y,K)) \\ \frac{dK}{dt} &= I(Y,K) - \delta(Y,K) \end{aligned} \tag{4.1.2}$$

and is a prototype of a dynamical system which generates business cycles.

In the first equation of (2), economic activity is seen to tend towards a level where savings and investments are equal. A discrepancy between investments and savings induces a change in the level of economic activity, which continues until the discrepancy is eliminated. If savings and investments were linear functions of the level of activity, the economic system would lead to hyperinflation with full employment, or a state of complete collapse with zero employment [10]. As economic systems in general do not behave in this manner, we see that savings and investment cannot be realistically characterised in terms of linear functions. They must be assumed to be nonlinear.

The second equation of (2) represents the accumulation of capital stock. If there is no investment it is seen that the capital stock depreciates by a factor δ.

By using the Poincaré-Bendixson theorem it is shown in [14] that (2) exhibits business cycles. The numerical computation of business cycles is left to future research.

4.1.1.4 Political Research

The investment decision also depends on political considerations. For instance, a cautious portfolio manager will only invest in politically stable countries. Often an investor avoids a certain country because its government is not stable enough. A buyer who purchases such a government bond is at risk of the bond defaulting or changing its conditions (“haircut”). Similarly, before an election in a country, a cautious portfolio manager might choose a portfolio close to the benchmark. An international investor will probably prefer a country that claims to have free financial markets. Low regulations are often conducive to free financial markets.

In a more recent development, some investors will only invest in governments that favor sustainable assets. These might be governments that seek to reduce pollution and the accompanying emissions.

4.1.1.5 Quantitative Research

Generally, data offers important information for investors. The extraction of relevant data is the task of quantitative research, applying the full range of mathematical techniques (see for e.g. for further reading [7]), especially from statistics and the theory of probability. Qualitative research describes crucial facts from the financial market and is often accompanied by quantitative research. Often a conjecture from qualitative research is confirmed by quantitative research.

Quantitative research also involves the development of models. Modern Portfolio Theory belongs in particular to quantitative research, and quantifies financial markets. Modern Portfolio Theory needs both return and risk as input. Deciding on the asset allocation in a portfolio and portfolio construction are forward-looking processes because they need forecasts. With the help of fundamental and technical research, forecasts can be generated.

There are some participants in the financial markets who believe that security prices cannot be forecasted such that a benchmark can be consistently beaten, or in other words that the research produced by quantitative research is useless. The ***random walk*** or ***efficient market*** theory refers to the fact that security prices fully reflect all available information. This is a very strong hypothesis. The efficient market hypothesis has historically been broken into three categories, each of them dealing with a different type of information:

The ***weak information efficiency*** of a market refers to the fact that future stock prices cannot be predicted on the basis of past stock prices.

The ***semi-strong information efficiency*** of a market indicates that, even when available published information is used, future prices are not predicable.

The ***strong information efficiency*** of a market refers to the fact that no information—not even unpublished developments—can be of use for predicting future prices, i.e., the current market price also reflects the relevant non-public (insider) information.

The stronger the information efficiency in the capital market is, the less interesting and the less informative it is to put effort in financial market research.

4.1.2 Passive Investment Strategy

A passive manager uses a range of different approaches. Under ***full replication*** we understand the investment in all securities with the corresponding weight of the benchmark. In ***stratified sampling*** the investor considers the constituents of the benchmarks that influence the benchmark most. More precisely, it allows the portfolio to match the basic characteristics of the index without requiring full replication. Characteristics of a fixed income benchmark include e.g. the maturity profile or the capitalization profile. Risk considerations such as investigating correlation structures do not enter in the portfolio construction. ***Optimization techniques*** are used to find a tracking portfolio with fewer securities than the benchmark but which has similar or even the same characteristics as the benchmark. In many cases the number of securities is fixed and a portfolio with low tracking error and other desirable properties is selected.

Index swaps are also used to track a benchmark. By definition an index swap is a hedging arrangement in which one party exchanges one cash flow for another party's cash flow on specified dates for a specified period. This swap essentially exchanges the return of the index for the value development of a portfolio, i.e., the tracking problem is transferred to another party. The set of equities might include e.g. German or French equities, even though the swap guarantees a Swiss Equity Market Index.

The overall aim is to track the benchmark with a tracking error equal or almost equal to zero. The passive investment style needs no forecasts and is therefore objective.

4.2 Portfolio Optimization

Portfolio optimization is an approach for constructing portfolios. The risks as introduced and explained in Chap. 3 is not discussed in Sect. 4.1. Portfolio optimization reflects the interaction between different investments or segments.

Harry Markowitz published his seminal work on portfolio theory in the 1950s. Together with William F. Sharpe he conducted a scientific study of portfolios invested in the US equity market in 1952. Harry Markowitz, William F. Sharpe and Merton H. Miller received the Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel in 1990.

In financial markets many greedy investors wanted to earn more money with less risk. The abnormality of a particular strategy can be measured by Modern Portfolio Theory (MPT).

Portfolio optimization is a part of MPT and in Sect. 4.3 we continue with MPT. The basic ideas of MPT are still regularly discussed in the media, see e.g. [19] and in literature, e.g. [4] and [27]. For a comprehensive introduction to MPT we refer to the textbooks [8] and [23].

We proceed by introducing the assets considered in this section.

Remark 4.1 We distinguish between ***risk-less assets*** and ***risky assets***. Risk-less assets are introduced in Definition 2.6. As described in Sect. 3 an asset can bear different risks. In the following we consider risk to be the degree of fluctuation of the return expressed in volatility (see Definition 3.2). Risky assets can involve significant volatility, i.e., the investor can experience substantial gain and loss and they do ***not*** have a volatility 'close' to zero.

Typical risky assets are equities and MPT originates from research into equities.

The problem of portfolio optimization under appropriate conditions is solved theoretically and optimal portfolios can be numerically computed. There are many companies like Ibottson, Barra and Wilshire that offer products with portfolio optimization software. They need the forecasts of the investors as input, and the risk is estimated by means of factor analysis (see Sects. 3.3 and 3.4).

However, its practical benefit is highly dependent on the accuracy of the associated forecasts. Here lies the key, since even forecast realizations that differ only slightly from expectations can have a major impact on the portfolio.

Definition 4.3 (see [4], p. 169) The ***set of efficient or optimal portfolios*** is the set of mean-variance choices from the investment opportunity set where for a given variance (or standard deviation) no other investment opportunity offers a higher mean return or, equivalently, where no other investment opportunity offers less variance (or standard deviation) for a given mean return.

Remark 4.2 The notion of a benchmark is introduced in Definition 2.8. We see in the following that we can optimize with respect to a money market rate (absolute optimization) or with respect to a benchmark (relative optimization).

A mathematical formulation for the optimal portfolio problem is as follows. We denote (see (2.2.3a)) ***by w the vector of the weights for the portfolio***

$$\mathrm{w} = \begin{bmatrix} \mathrm{w}_1 \\ . \\ . \\ . \\ \mathrm{w}_\mathrm{n} \end{bmatrix}, \tag{4.2.1}$$

by b the vector of the weights for the benchmark

$$\mathrm{b} = \begin{bmatrix} \mathrm{b}_1 \\ . \\ . \\ . \\ \mathrm{b}_\mathrm{n} \end{bmatrix} \tag{4.2.2}$$

where

$$\sum_{i=1}^{n} b_i = 1, b_i \geq 0, \tag{4.2.3}$$

by p the vector of the weights for the positions

$$p = \begin{bmatrix} p_1 \\ \cdot \\ \cdot \\ \cdot \\ p_n \end{bmatrix} = \begin{bmatrix} w_1 \\ \cdot \\ \cdot \\ \cdot \\ w_n \end{bmatrix} - \begin{bmatrix} b_1 \\ \cdot \\ \cdot \\ \cdot \\ b_n \end{bmatrix}, \tag{4.2.4}$$

and ***by r the vector of returns*** (see (2.2.3b))

$$r = \begin{bmatrix} r_1 \\ \cdot \\ \cdot \\ \cdot \\ r_n \end{bmatrix}. \tag{4.2.5}$$

Furthermore by referring to (2.3.4) with the matrix S defined by (3.3.7) the optimal portfolios $w \in \mathbf{R^1}$ are defined as those

(a) That minimize the absolute risk $\sigma \in \mathbf{R^1}$ or the relative risk TE(3) defined by (3.3.9) for a given return $\mu \in \mathbf{R^1}$, i.e., we have for the objective function

(ABS1): **absolute optimization**

$$\sigma^2 = \min_w w^T S w.$$

(REL1): **relative optimization**

$$\sigma^2 = \min_p p^T S p$$

(b) That yield a maximum return $\mu \in \mathbf{R^1}$ for a given risk. We have to distinguish between the absolute risk $\sigma \in \mathbf{R^1}$ and the relative risk TE(3) (tracking error), i.e.,

(ABS2) **the absolute constraint is**

$$\sigma^2 = w^T S w.$$

(REL2) Referring to (3.3.9) **the relative constraint is**

$$TE(3)^2 = p^T S \ \ p.$$

Fig. 4.6 Absolute optimization

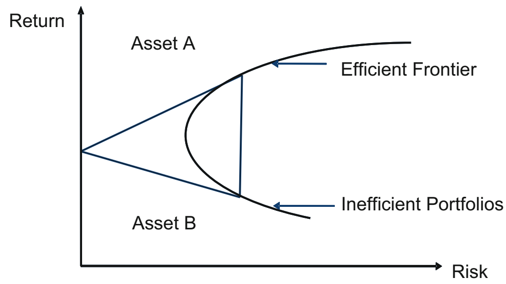
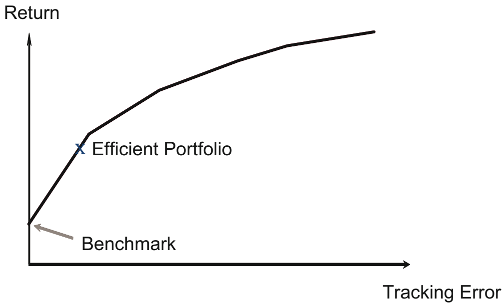


Fig. 4.7 Relative optimization



The portfolio optimization problem together with the constraints

$$\sum_{i=1}^{n} w_i = 1 \tag{4.2.6a}$$

and

$$w_i \geq 0, i = 1, 2, \ldots, n \tag{4.2.6b}$$

is called the ***(Markowitz) Standard Model***.

If only the budget constraint (6a) is assumed and no short selling constraint is considered, the optimization model is referred to as a ***Black Model*** ([2], p. 159). Optimal portfolios are hence to be found on the efficient frontier (Figs. 4.6 and 4.7).

Remark 4.3 From (3), (4) and (6a) follows for the positions

$$\sum_{i=1}^{n} p_i = 0.$$

We see that the positions can be positive or negative, whereas with (6b) we have only positive weights.

Remark 4.4 The relative optimization assumes the benchmark portfolio is considered as the portfolio with risk zero and therefore the reference portfolio. The absolute optimization assumes only vanishing risk for specified correlation structures.

Remark 4.5 Positions are discussed in Table 2.3. We consider three positions: an overweight, neutral and underweight of securities versus the benchmark.

Remark 4.6 The problem of 'maximum risk' is not tackled in the framework discussed here.

Remark 4.7 The (Markowitz) Standard Model for two assets can be solved explicitly and does not need a numerical algorithm. We refer to the following Examples 4.2 and 4.4.

Example 4.2 (absolute optimization with 2 assets) We consider a portfolio which consists of two assets, A and B, and assume for the covariance matrix (3.3.7)

$$S = \begin{bmatrix} 3 & -1.5 \\ -1.5 & 2 \end{bmatrix} \tag{4.2.7}$$

and for the return (5)

$$r = \begin{bmatrix} 0.07 \\ 0.02 \end{bmatrix}. \tag{4.2.8}$$

The return of the portfolio is

$$\mu = 0.07 \;\; w_1 + 0.02 \;\; w_2. \tag{4.2.9}$$

In the following we assume the constraints (6), i.e.,

$$w_2 = 1 - w_1, w_1 \in [0, 1]$$

which yields by (9)

$$20\mu = w_1 + 0.4, \mu \in [0.02, 0.07] \tag{4.2.10}$$

hence the parameterization of the weights is

$$\begin{aligned} w_1 &= 20\mu - 0.4, \\ w_2 &= 1.4 - 20\mu. \end{aligned} \tag{4.2.11}$$

By vector

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} -0.4 \\ 1.4 \end{bmatrix} + 20\mu \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \mu \in [0.02, 0.07].$$

These weights also comprise portfolios that are not efficient and can be found on the inefficient branch of the efficient frontier.

By (3.3.7) and (3.3.8) in Theorem 3.2 the variance of the portfolio is with (7)

$$\begin{gathered}\sigma^2(\mu) = w_1^2 \mathrm{var}(A) + 2w_1w_2\mathrm{cov}(A,B) + w_2^2\mathrm{var}(B) = \\ 3w_1^2 - 3w_1w_2 + 2w_2^2.\end{gathered} \tag{4.2.12}$$

In the risk σ return μ representation we find by (11)

$$\begin{gathered}\sigma^2(\mu) = 3\ (20\mu - 0.4)^2 - \\ 3\ (20\mu - 0.4)(1.4 - 20\mu) + 2\ (1.4 - 20\mu)^2 = \\ 3\ (800\mu^2 - 52\mu + 0.72) + 2\ (1.96 - 56\mu + 400\mu^2) = \\ 2400\mu^2 - 156\mu + 2.16 + 3.92 - 112\mu + 800\mu^2 = \\ 3200\mu^2 - 268\mu + 6.08.\end{gathered}$$

Although the above relationship has two solutions for μ, the absolute optimization problem (see (ABS1) and (ABS2)) delivers the solution with the higher return (see also Fig. 4.6).

The maximum return portfolio (ABS2) is with $\mu = 0.07$

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} -0.4 \\ 1.4 \end{bmatrix} + 1.4\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

We proceed with the minimum risk portfolio (ABS1). By (6), (7) and (12) we have

$$\mathrm{Var}(\mu) = 3w_1^2 - 3w_1(1 - w_1) + 2\ (1 - w_1)^2 = 8w_1^2 - 7w_1 + 2.$$

The first derivative with respect to w_1 is

$$\frac{\partial \mathrm{Var}(w_1)}{\partial w_1} = 16w_1 - 7.$$

By requesting

$$\frac{\partial \mathrm{Var}(w_1)}{\partial w_1} = 0$$

we find

$$w_g = \begin{bmatrix} \dfrac{7}{16} \\[2mm] \dfrac{9}{16} \end{bmatrix} = \begin{bmatrix} 0.4375 \\ 0.5625 \end{bmatrix}$$

and by (10)

$$\mu_g = 0.041875.$$

And by (12)

$$\mathrm{Var}\left(\mu_g\right) = 0.46875.$$

The efficient branch for the Standard Model, i.e., (6) is satisfied in

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} -0.4 \\ 1.4 \end{bmatrix} + 20\ \mu \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \mu \in [0.041875, 0.07]$$

and for the Black Model, i.e., only (6a) and not (6b) is satisfied in

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} -0.4 \\ 1.4 \end{bmatrix} + 20\ \mu \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \mu \in [0.041875, \infty].$$

Thus the short selling constraint (6b) does affect the minimum risk solution in this example.

◇

In order to analyze the global minimal portfolio in more detail we need the following notion from linear algebra. If a matrix $S \in \mathbf{R}^{nxn}$ has a unique matrix $S^{-1} \in \mathbf{R}^{nxn}$ with

$$S^{-1}S = SS^{-1} = I$$

then S^{-1} is called the ***inverse*** of S.

Lemma 4.1 If S is positive definite, then there exists an inverse matrix S^{-1}.

Proof The assertion follows from the property that positive definite matrices have a smallest eigenvalue that is positive. Further details see [17].

◇

The minimum variance portfolio is discussed in detail in the relevant literature [12] and offers a variety of interesting characteristics. For example, it does not depend on forecasts.

Lemma 4.2 (global minimum risk portfolio) It is assumed that S is positive definite. Furthermore the inverse of S is denoted with S^{-1}. With

$$ID = \begin{bmatrix} 1 \\ . \\ . \\ . \\ 1 \end{bmatrix}$$

the solution w_g of

$$\sigma = \min_{w} w^T S w$$

for the Black Model is

$$w_g = \frac{S^{-1}\ ID}{ID^T S^{-1} ID}. \tag{4.2.13}$$

Proof: The Lagrange function is

$$L = w^T S w - \lambda\left(ID^T w - 1\right), \forall\lambda \in \mathbf{R^1}.$$

The differentiation with respect to the weight w_j ,j = 1,....,n is

$$\frac{\partial L}{\partial w_j} = 2(Sw)_j - \lambda(ID)_j, \forall\lambda \in \mathbf{R^1}$$

and from the optimality condition

$$\frac{\partial L}{\partial w_j} = 0$$

we get the optimal weights w_g:

$$w_g = \frac{1}{2}\lambda S^{-1} ID$$

and from the budget constraint (6a) there follows

$$\lambda = 2\frac{1}{ID^T S^{-1} ID}$$

which yields (13). By considering an arbitrary portfolio and its difference from the optimal portfolio it can be shown by using (13) and the bilinearity of the risk matrix that the risk is increasing. Thus the optimal risk portfolio with weights w_g is minimal and not maximal.

◇

Example 4.2 (continuation) We evaluate (13):

$$S^{-1} = \frac{1}{8.25}\begin{bmatrix} 2 & -1.5 \\ -1.5 & 3 \end{bmatrix},$$

thus

$$S^{-1}\mathrm{ID} = \frac{1}{8.25}\begin{bmatrix} 3.5 \\ 4.5 \end{bmatrix}$$

and

$$\mathrm{IDS}^{-1}\mathrm{ID} = \frac{8}{8.25}.$$

By (13) we find again

$$w_g = \begin{bmatrix} \frac{7}{16} \\ \frac{9}{16} \end{bmatrix} = \begin{bmatrix} 0.4375 \\ 0.5625 \end{bmatrix}.$$

◇

In the following example, S is only positive semi-definite and not positive definite.

Example 4.3 (minimum risk portfolio) We consider the correlation matrix

$$S = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}.$$

As shown in Example 3.9 this matrix has an eigenvalue 3 and an eigenvalue 0 with multiplicity 2. There is a one-dimensional eigenspace with eigenvalue 3

$$v_1 = \lambda_1 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \lambda_1 \in \mathbf{R^1}$$

and a two-dimensional eigenspace for the eigenvalue 0 with the basis

$$v_2 = \lambda_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \lambda_2 \in \mathbf{R^1}, v_3 = \lambda_3 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \lambda_3 \in \mathbf{R^1}. \tag{4.2.14}$$

For the minimal risk portfolio we consider the eigenspace of the eigenvalue 0

$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \lambda_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \lambda_3 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}. \tag{4.2.15}$$

With the constraint (6a)

$$w_1 + w_2 + w_3 = 1$$

(15) yields

$$2(\lambda_2 + \lambda_3) = 1.0,$$

thus

$$\lambda_2 + \lambda_3 = 0.5$$

and

$$\lambda_3 = 0.5 - \lambda_2,$$

i.e.,

$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0.5 \\ 0.5 \end{bmatrix} + \lambda_2 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, -\infty \le \lambda_2 \le \infty$$

and with the constraint (6b) we have

$$w_1 \ge 0, w_2 \ge 0, w_3 \ge 0 \tag{4.2.16}$$

and we have in the Standard Markowitz model

$$0 \le \lambda_2 \le 0.5.$$

◇

Example 4.4 (relative optimization with two assets) We again assume the return (6) and the risk (7). We consider the benchmark weights (2) with $n = 2$

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}.$$

The return is the same as in (8) and the risk for the position p_1, p_2 is

$$\begin{gathered} \mathrm{Var}(P) = p_1^2 \mathrm{var}(A) + 2\,p_1\,p_2 \mathrm{cov}(A, B) + p_2^2 \mathrm{var}(B) = \\ 3\,p_1^2 - 3\,p_1\,p_2 + 2\,p_2^2. \end{gathered} \tag{4.2.17}$$

We first consider the investment in the benchmark:

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}, \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} 0.0 \\ 0.0 \end{bmatrix}.$$

Thus from (9) we have $\mu = 0.045$ and by (17) we find Var(P) = 0 and as a result REL1 is solved. By overweighting the asset with the return 0.07 and for $\mu = 0.07$ REL2 is solved by

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 1.0 \\ 0.0 \end{bmatrix}, \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} 0.5 \\ -0.5 \end{bmatrix}. \tag{4.2.18}$$

(17) with (18) yields Var(P) = 2.

The Standard Model, i.e., (6) is satisfied in

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} -0.4 \\ 1.4 \end{bmatrix} + 20\ \mu \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \mu \in [0.045, 0.07]$$

and the positions are

$$\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} -0.9 \\ 0.9 \end{bmatrix} + 20\mu \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \mu \in [0.045, 0.07]$$

and the Black Model, i.e., only (6a) and not (6b) is satisfied in

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} -0.4 \\ 1.4 \end{bmatrix} + 20\mu \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \mu \in [0.045, \infty]$$

and the positions are

$$\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} -0.9 \\ 0.9 \end{bmatrix} + 20\mu \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \mu \in [0.045, \infty].$$

In the risk return μ space we have

$$\begin{aligned} \mathrm{Var}(\mu) &= 3(20\mu - 0.9)^2 - \\ 3(20\mu - 0.9)&(0.9 - 20\mu) + 2(0.9 - 20\mu)^2 = \\ 8(20\mu - 0.9)^2 &= 3200\mu^2 - 288\mu + 6.48. \end{aligned}$$

◇

Asset A and Asset B represent the risk and return of two risky securities such as equities. Figure 4.6 portrays the risk and return on various portfolios comprising securities A and B.

In actual practice, however, portfolios comprising many risky equities are optimized. In such cases, the efficient frontier looks the same or at least similar, depending on the boundary constraints and the input parameters.

Alongside risk and return, an additional important parameter for portfolio optimization is correlation. This is a measure for the movement of securities relative to each other. In mathematical terms, the correlation must lie between the two extremes of –1 and +1. A and B are linked in three different ways in Fig. 4.6, corresponding to three different correlation values. For a correlation of +1, the risk-return curve is a straight line linking the two securities. For a correlation of –1, the efficient portfolios are at least partially linear, and one of the portfolios is risk-free. This is on the return axis in Fig. 4.6. And for a correlation value between –1 and +1, A and B are linked by a curve; optimization programs derive portfolios that are on the efficient frontier.

What's key is that risk declines in line with correlation. Figure 4.6 shows that only the portfolios above the minimum variance portfolio are optimal. The part of the efficient frontier below the minimum variance portfolio is described as the "inefficient branch" in the relevant jargon.

What's clear is that optimization is one approach to quantitative modeling available to the financial industry, allowing forecasts to be expressed in the form of a concrete portfolio recommendation. Critics claim that optimization can very easily be skewed by forecasting errors. If there is a benchmark, optimization can be conducted in as stable a manner as required; for, if the investor then buys that benchmark, he or she is actually unaffected by portfolio optimization and the forecasts made. The well-known approach of Black und Littermann is essentially based on this concept.

Figures 4.6 and 4.7 are extremely simplified. They merely reflect the risk and return for different portfolios and assume that the risk and return of the individual investments can be defined. But it is precisely the forecasts and the estimates of risk that are critical. It is also important to note that the optimal portfolios move in line with changing forecasts and changing market data. Hence, the optimal portfolio is constantly in a state of flux and has to be optimized continuously. "Stability is the enemy of optimality" is therefore often heard among specialists.

As mentioned before, mathematics has solved the portfolio optimization problem. However, the practical benefit of the optimized portfolio is highly dependent on the accuracy of the financial market forecasts. And it is precisely their verification that forms the subject of many empirical studies, and offers the basis for further development of Markowitz's original portfolio optimization theory [6, 21].

The basic findings of MPT can be expressed as follows:

- Higher risk is always accompanied by higher average return.
- The interaction of the different investments in the portfolio is reflected.

Although the graphs to be found in many textbooks give the impression that the efficient frontier is continuously differentiable, the following example shows that the efficient frontier is in general only continuous and not differentiable [29]. For a simple example of a function that is continuous but not differentiable, please see Fig. 4.8.

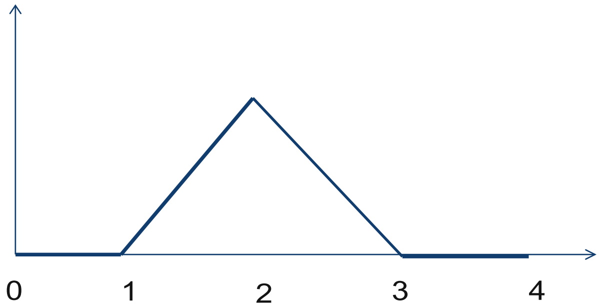


Fig. 4.8 Continuous, but not differentiable function

Example 4.5 (non-differentiability of the efficient frontier) We consider in (3.3.7)

$$S = \begin{bmatrix} 3 & 3 & -1 \\ 3 & 11 & 23 \\ -1 & 23 & 75 \end{bmatrix}.$$

and the return

$$\mu = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

An algorithm for computing optimal portfolios yields the following results.

(a) We assume constraints (6). For the global minimum variance portfolio we find

$$\mu_{\min} = \mu_G = 1.2, \sigma_G^2 = 2.8, w_G = (0.95, 0, 0.05).$$

The maximal value is $\mu_{\max} = 5$. The efficient portfolios consist of four pieces:

$$\sigma_1^2(\mu) = 5\mu^2 - 12\mu + 10, \mu \in [1.2, 1.5],$$
$$\sigma_2^2(\mu) = \mu^2 + 1, \mu \in [1.5, 2.0],$$
$$\sigma_3^2(\mu) = 2\mu^2 - 4\mu + 5, \mu \in [2.0, 3.0],$$
$$\sigma_4^2(\mu) = 10\mu^2 - 48\mu + 65, \mu \in [3.0, 5.0].$$

The efficient frontier is continuous on [1.2, 5.0]:

$$\sigma_1^2(1.5) = \sigma_2^2(1.5) = 3.25,$$
$$\sigma_2^2(2.0) = \sigma_3^2(2.0) = 5.0,$$
$$\sigma_3^2(3.0) = \sigma_4^2(3.0) = 11.0.$$

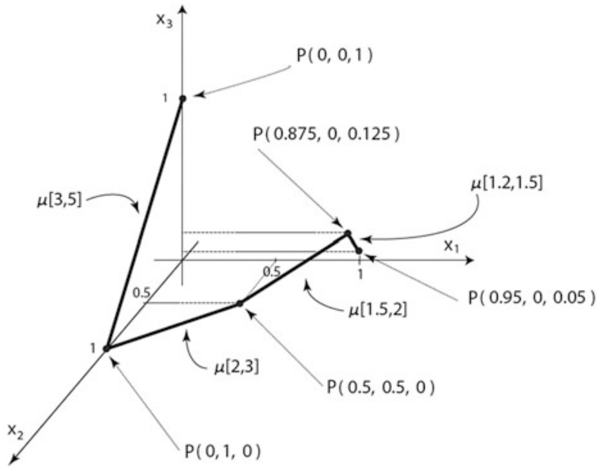


Fig. 4.9 Efficient portfolio for the Standard Model

Furthermore we find for the differentiability:

$$\left.\frac{d\sigma_1}{d\mu}\right|_{\mu=1.5} = \left.\frac{d\sigma_2}{d\mu}\right|_{\mu=1.5} = 3.0,$$
$$\left.\frac{d\sigma_2}{d\mu}\right|_{\mu=2.0} = \left.\frac{d\sigma_3}{d\mu}\right|_{\mu=2.0} = 4.0,$$
$$\left.\frac{d\sigma_3}{d\mu}\right|_{\mu^+=3.0} = 8, \left.\frac{d\sigma_4}{d\mu}\right|_{\mu^-=3.0} = 12.$$

Thus except for $\mu = 3$, $\sigma(\mu)$ is differentiable on (1.2, 5.0). $\sigma(\mu)$ has a kink at $\mu = 3$. In Fig. 4.9 we see the Standard Markowitz model.

(b) We assume only constraints (6a). For the global minimum variance portfolio we find

$$\mu_{\min} = \mu_G = 0, \sigma_G^2 = 1.0, w_G = (2.0, -1.5, 0.5) \text{ and } \mu_{\max} = \infty.$$

We have

$$\mu \in [0, \infty] : \sigma^2(\mu) = \mu^2 + 1.$$

The efficient portfolios are along the straight line

$$w(\mu) = \begin{bmatrix} 2 \\ -1.5 \\ 0.5 \end{bmatrix} + \mu \begin{bmatrix} -0.75 \\ 1.00 \\ -0.25 \end{bmatrix} = \begin{bmatrix} 2 - 0.75\mu \\ -1.5 + \mu \\ 0.5 - 0.25\mu \end{bmatrix}, \mu \geq 0.$$

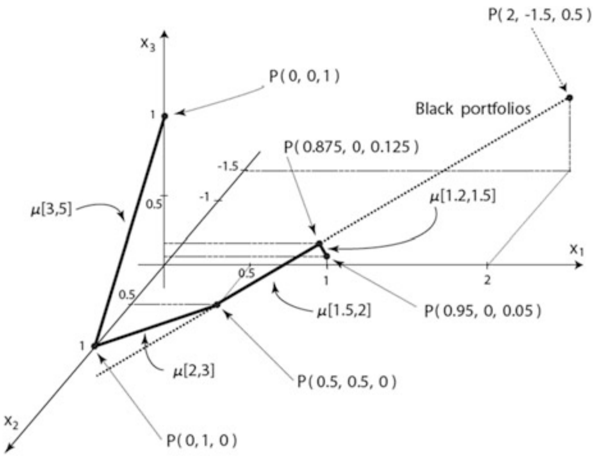


Fig. 4.10 Efficient portfolios for the Standard and Black Models

Figure 4.10 shows the numerical realization of the Standard Model and the Black Model.

With this example we have shown that the Standard Model can behave significantly differently from the Black Model.

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4.3 Absolute Risk-Adjusted Measures

One aspect of portfolio analysis is the examination of the portfolio along the time axis. The comparison of a specific portfolio to similar portfolios is another aspect of portfolio analysis, i.e., peer analysis is the focus of this section. Certainly the investor looks first at the returns and then at the risk involved. We proceed by considering measures that consider both return ***and*** risk, i.e., we introduce risk-adjusted return measures.

Definition 4.4 We assume that a return series (3.1.1a) is given. The ***coefficient of variation CVA*** is the ratio of the standard deviation (3.1.3) to the arithmetic mean return (2.3.4)

$$\mathrm{CVA(P)} = \frac{\mathrm{std(P)}}{\bar{\mathrm{r}}_{\mathrm{P}}}. \tag{4.3.1}$$

We note that the denominator and the numerator in this ratio have the same unit, i.e., the ratio is a scale-free measurement. Furthermore, the CVA reflects the basic principle that for higher return we also expect higher risk, i.e., if Portfolio Manager 1 has achieved twice as much return by taking twice as much risk as Portfolio Manager 2, they have the same CVA and thus the same level of skill.

If the return $\bar{r}_P$ of the portfolio is positive then it can be said qualitatively that a low CVA is desirable.

Example 4.6 (higher risk but lower coefficient of variation) We consider a return series (3.1.1) with arithmetic mean $\bar{r}_P$. We define for $\sigma \in \mathbf{R}^1$ and $N = 1,\ldots.$

$$r_{P,k} = \bar{r}_P - \sigma, \quad k = 1, 3, \ldots, 2N - 1,$$
$$r_{P,k} = \bar{r}_P + \sigma, \quad k = 2, 4, \ldots, 2N.$$

Then we have

$$\mathrm{std}(P) = \sigma \tag{4.3.2}$$

and

$$\mathrm{CVA}(P) = \frac{\sigma}{\bar{r}_P}.$$

And in Example 3.1 we assume $\bar{r}_P = 10\%$ and $\sigma = 5\ \%$. Thus by (1) and (2)

$$\mathrm{CVA}(P) = \frac{1}{2}.$$

With $\bar{r}_P = 10\%$ and $\sigma = 25\ \%$ we find by (1)

$$\mathrm{CVA}(P) = \frac{2}{5}.$$

We see that the CVA increases as the σ increases.

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We continue with an illustration from physics. We consider two drivers who drove from A to B, C to D, resp. We assume that both drivers drove at a constant speed. The question is:

Which driver is faster?

If the distance or time, resp. is the same, the question can be answered by the time or distance, resp. that the two drivers drove. If the distances and the times are different, we need to normalize or gauge them by the distance, by the time, resp. Speed defined as the distance divided by time has to be used instead of distance (time), resp. Here we can see the comparison to portfolio analysis, when we want to compare two portfolios with both different returns and risks.

Definition 4.5 The ***risk-adjusted return RAR*** is the inverse of the CVA, i.e.,

$$\mathrm{RAR} = \frac{\bar{r}_P}{\mathrm{std}(P)}.$$

Following Definition 2.6 and similarly to (3.1.1) we denote with

$$r_{f,k}, k = 1, 2, \ldots, N$$

the risk-free return in period k, k = 1,. . ..,N and following Definition 3.2 the arithmetic mean return $\bar{r}_f$ for the series of risk-free returns considered is

$$\bar{r}_f = \frac{1}{N}\sum_{k=0}^{N} r_{f,k}.$$

As risky investments, risk-free investments are also part of modern portfolio theory. The following risk-adjusted return measure is named in honor of Dr. William Sharpe and reflects the fact that the investor cannot earn a return above the risk-free return without taking on risk.

Definition 4.6 The ***Sharpe ratio SR*** is the difference between the annual arithmetic mean return and annual arithmetic mean risk-free return divided by the annualized deviation of the fund, i.e.,

$$\mathrm{SR} = \frac{\bar{r}_P - \bar{r}_f}{\mathrm{std}(P)} \tag{4.3.3}$$

Remark 4.8 The CVA, RAR and SR are used for classifying a set of portfolios. They map return and risk figures in the set of the real numbers. If a portfolio manager is judged on the basis of his SR, he will try to maximize his SR because he will hope to achieve a high return with as little as possible risk. Even if the returns of two different portfolios are equal, the SR can be different. If this return is positive, the portfolio manager with less risk is superior to the one with more risk. If the return is negative, the verdict is just the opposite, which ***makes no sense***. Strictly speaking the domain of applicability for the SR is limited to positive return. The same is true for the CVA and RAR.

4.4 The Capital Asset Pricing Model

The assumptions of the capital asset pricing model (CAPM) are ([4], p. 194):

- Investors evaluate portfolios by looking at the expected returns and standard deviation of the portfolios over one period of time.

- Investors are never satiated, so when given a choice between two otherwise identical portfolios they will choose the one with the higher expected return.
- Investors are risk-averse, so when given a choice between two otherwise identical portfolios they will choose the one with lower standard deviation.
- Individual assets are infinitely divisible, meaning that an investor can buy a fraction of a share if he or she so desires.
- There is a risk-free rate at which the investor may either lend or borrow money.
- Taxes and transaction costs are irrelevant.

In the following we consider one period of time $t_k = k$, $t_{k+1} = k + 1$, $k = 0, \ldots, N - 1$ and portfolios that consist of a risk-free investment and risky investments (see Remark 4.1). As risky securities, risk-free investments are also part of modern portfolio theory.

We will now revisit the return and risk considerations from the previous chapter and distinguish between active and the passive investors. In the previous chapter an active investor was considered, as he could steer his return risk preference. A passive investor does not see any opportunities on the marketplace and clearing prices prevail. Every investor will hold a portfolio that consists of a share of the market portfolio defined as follows:

Definition 4.7 The ***market portfolio MP*** with return r_{MP} is a portfolio consisting of an investment in all securities like stocks, bonds, real estate, etc. and in which the proportion to be invested in each security corresponds to its relative market value. The relative market value of a security is simply equal to the aggregate market value of the security divided by the sum of the aggregate market value of all securities.

In principle the market portfolio is a thought experiment and it is not possible to capture any single available asset in a portfolio. My and your belongings are part of the market portfolio. However, big universes like the MSCI world equity can serve as a proxy.

Based on the two-fund separation principle described in [4], p. 181, a first market equilibrium principle yields:

Theorem 4.1 (Capital market line (CML)) Each investor will have a utility maximizing portfolio that is a combination of the risk-free asset and a portfolio (or fund) of risky assets that is determined by the line drawn from the risk-free rate of return tangent to the investor's efficient set of risky assets. The portfolio that is on the tangent to the efficient frontier and on the efficient frontier is the market portfolio.

Proof: We consider a portfolio consisting of the risk-less asset RF and the market portfolio MP

$$r_P = (1 - w_1) r_f + w_1 \, r_{MP}$$

and for the standard deviation of the portfolio we have

$$\mathrm{std}(P) =$$
$$\left((1-w_1)^2\mathrm{var}(RF) + 2w_1(1w_1)\mathrm{cov}(RF,\ MP) + w_1^2\mathrm{var}(MP)\right)^{\frac{1}{2}}$$
$$w_1\mathrm{std}(MP).$$

As var(RF) = 0 and cov(RF, MP) = 0 we have

$$w_1 = \frac{\mathrm{std}(P)}{\mathrm{std}(MP)} \tag{4.4.1}$$

and

$$r_P = \left(1 - \frac{\mathrm{std}(P)}{\mathrm{std}(MP)}\right)r_f + \frac{\mathrm{std}(P)}{\mathrm{std}(MP)}r_{MP},$$

hence

$$r_P = r_f + \frac{\mathrm{std}(P)}{\mathrm{std}(MP)}(r_{MP} - r_f)$$

and in the risk return graph we have

$$r_p = r_f + \frac{r_{MP} - r_f}{\mathrm{std}(MP)}\mathrm{std}(P).$$

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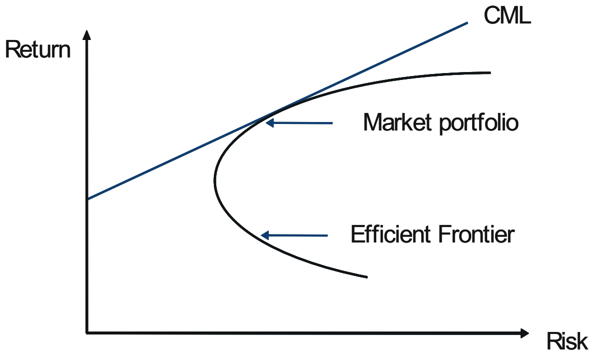
Definition 4.8 This line specified in Theorem 4.1 is called the **capital market line (CML)** (see Fig. 4.11).

Fig. 4.11 The modern portfolio theory



Remark 4.9 Except for $w_1 \geq 0$ (see (1)) the proof of Theorem 4.1 does not respect any constraints. The impact of constraints on the CML is still subject to future research.

Remark 4.10 As expected, approximations of the market portfolio with real market data are not efficient [5].

Remark 4.11 RARs are slopes on the efficient frontier and the SR is the slope of the CML. We note that the investors have the same Sharpe ratio

$$\frac{r_{MP} - r_f}{\mathrm{std}(MP)}, \tag{4.4.2}$$

i.e., the risk-return relationship is linear along the CML.

Example 4.2 (continued) We consider the risk matrix (4.2.7) and the return vector (4.2.8) and arrive at

$$SR(\mu) = \frac{\mu - r_f}{3200\mu^2 - 268\ \mu + 6.08}.$$

By considering

$$\frac{\partial SR\ (\mu)}{\partial \mu} = 0$$

we find

$$-3200\ \mu^2 + 640\ \mu\ r_f + 6.08 - 268 r_f = 0.$$

By solving the quadratic equation with the return of the global minimum portfolio with $r_f = 0.04185$ (see Example 4.3).

$$SR_{max}(\mu) = 0.053978$$

and for the weights of the market portfolio we find that

$$w_1 = 0.679561, w_2 = 0.320439.$$

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Theorem 4.2 (Capital asset pricing model (CAPM)) We consider a portfolio P that consists of a risky asset C_1 with return r_1. Together with the return of the riskless asset r_f and the market portfolio MP with return r_M we find:

$$r_1 - r_f = \beta(C_1, MP)(r_{MP} - r_f) \tag{4.4.3a}$$

where

$$\beta(C_1, MP) = \frac{cov(C_1, MP)}{var(MP)}. \tag{4.4.3b}$$

Proof: We consider a portfolio P that consists of an asset C_1 with return r_1 and a weight w_1 and the market portfolio MP considered as an asset. For the return of the portfolio we have

$$r_P = w_1 r_1 + (1 - w_1) r_{MP}$$

and for the standard deviation of the portfolio we have

$$std(P) = \left((w_1)^2 var(C_1) - 2w_1(1 - w_1)cov(C_1,\ MP) + (1 - w_1)^2 var(MP)\right)^{\frac{1}{2}}.$$

We differentiate between the return and the standard deviation of the portfolio with respect to w_1:

$$\frac{d\,r_p}{d\,w_1} = r_1 - r_{MP},$$
$$\frac{d\,std(P)}{d\,w_1} = \frac{1}{2}\left(std(P)\right)^{-1}\left(2w_1 var(C_1) + 2(1 - w_1)cov(C_1, MP) - 2w_1 cov(C_1, MP) - 2(1 - w_1)var(MP)\right).$$

For the derivative in the return risk graph in the market portfolio, i.e., $w_1 = 0$ we find

$$\left.\frac{dr_p}{dstd(P)}\right|_{w_1=0} = \left.\frac{\frac{d\,r_p}{d\,w_1}}{\frac{d\,std(P)}{d\,w_1}}\right|_{w_1=0} = \frac{r_1 \ - \ r_{MP}}{\frac{cov(C_1,MP) - var(MP)}{std(MP)}}$$

As the slope has to be same the slope (1), hence

> The excess return, i.e., the return above the risk free return of a risky investment is equal to the CAPM beta times the excess return of the market portfolio return.

$$\frac{r_1 - r_{MP}}{\frac{\mathrm{cov}(C_1,MP)-\mathrm{var}(MP)}{\mathrm{std}(MP)}} = \frac{r_{MP} - r_f}{\mathrm{std}(MP)},$$

thus

$$r_1 - r_{MP} = (r_{MP} - r_f)\left[\frac{\mathrm{cov}(C_1, MP) - \mathrm{var}(MP)}{\mathrm{var}(MP)}\right] =$$
$$r_{MP} \cdot \left[\frac{\mathrm{cov}(C_1, MP)}{\mathrm{var}(MP)} - 1\right] - r_f \cdot \left[\frac{\mathrm{cov}(C_1, MP)}{\mathrm{var}(MP)} - 1\right].$$

This is the same as

$$r_1 = r_{MP} \cdot \left[\frac{\mathrm{cov}(C_1, MP)}{\mathrm{var}(MP)}\right] - r_f \cdot \left[\frac{\mathrm{cov}(C_1, MP)}{\mathrm{var}(MP)} - 1\right]$$

which yields (2).

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Remark 4.12 The $\beta(C_1, MP)$ defined in (2) is called the ***CAPM beta of the asset C_1*** and similarly for a portfolio P we define the ***CAPM beta of a portfolio P*** by the asset C_1 and

$$\beta(P, MP) = \frac{\mathrm{cov}(P, MP)}{\mathrm{var}(MP)}. \tag{4.4.4}$$

We summarize the CAPM as follows:

We conclude that the risky assets can be classified by the CAPM beta.

The CAPM is one of the fundamental building blocks of MPT. For a qualitative description we refer to [9], p. 191 and for a more quantitative formulation we refer to [4]. The CAPM has the following properties:

Lemma 4.2 The CAPM beta (2), (4) respectively is linear, i.e., (2) is also valid for a portfolio P with return r_P consisting of segments $C_1,\ldots, C_n$ with return $r_1,\ldots, r_n$ and weights $w_1, w_2,\ldots, w_n$

$$r_p = \sum_{j=1}^{n} w_j r_j = w_1 r_1 + w_2 r_2 + \ldots\ldots + w_n\ r_n,$$

where

$$\beta(P, MP) =$$
$$w_1\beta(C_1, MP) + w_2\beta(C_2, MP) + +w_n\beta(C_n, MP).$$

Proof: The assertion follows from the proprieties of the covariance formulated in Lemma 3.1.

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- The CAPM β(P,MP) is consistent with the least square β in (3.3.7).
- For the market portfolio we have

$$\beta(\mathrm{MP}, \mathrm{MP}) = \frac{\mathrm{cov}(\mathrm{MP}, \mathrm{MP})}{\mathrm{var}(\mathrm{MP})} = \frac{\mathrm{var}(\mathrm{MP})}{\mathrm{var}(\mathrm{MP})} = 1.$$

  We note that there are also other portfolios that have $\beta(\mathrm{P, MP}) = 1$. These are tracking portfolios of MP, i.e., portfolios that are supposed to follow the market.
- An investment that has no risk will earn the risk-free return, where risk is measured by the volatility of returns.
- Two different types of risk cause the volatility in investment returns. The first is the ***market risk*** of the investment, which reflects the degree to which investment values vary when the level of prices in the underlying market changes. Volatility that affects the market as a whole is assumed to correspond to changes in the underlying factors that influence market prices in general. For example, if we assume that stock valuations respond positively to increases in the growth rate of the economy as a whole, we can say that the economic growth factor is to some degree common to all stocks. Economic growth, and other factors which influence the value of market prices as a whole, are ***systematic risk*** factors.
- Systematic risk factors are reflected in the market returns; therefore, we can isolate the influence of systematic factors on an individual asset by observing market returns.
- All investments within the market are influenced by systematic risks, but the degree of exposure to systematic risks varies from asset to asset.
- Factors specific to the asset can generate volatility, for example, a change of management within a company. This type of volatility is unique to the asset, or ***unsystematic***.
- Of the two components of risk, those risks specific to particular assets can be eliminated via portfolio diversification. We assume that the particular risks of different assets will offset each other in a diversified investment fund.
- Because the unique risk is diversifiable, the market does not reward the asset with a premium for this risk. The market rewards only the exposure to systematic risk. This is because systematic risk cannot be diversified away and is thus borne by every investor in the market.

- Investors expect a risk premium in exchange for bearing this risk

- Investments are awarded a degree of return over the risk-free rate, or a ***risk premium***, based on the degree of market risk. The reward-to-risk ratio is a linear

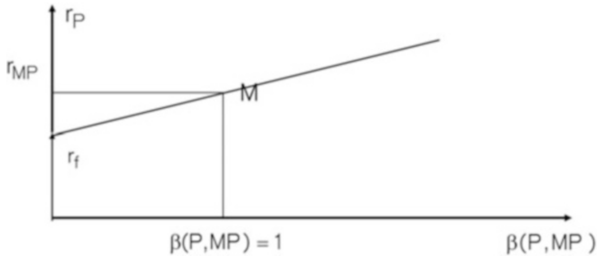

Fig. 4.12 β version

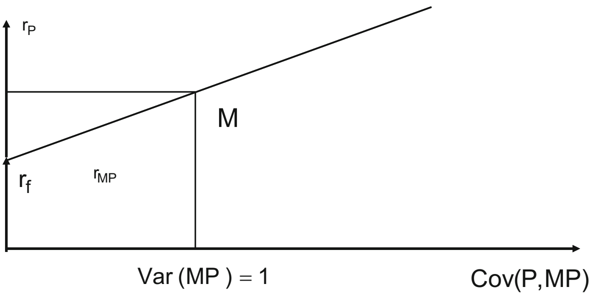

Fig. 4.13 Covariance version

function: For each unit of systematic risk the asset is exposed to on the market, the market will reward the asset with one unit of excess return.

- An investment that has risk exposure similar to that borne by the market as a whole will have returns that vary in line with the market. If an asset is less exposed to market factors, its returns will vary to a lesser degree than the market as a whole; if the investment has greater exposure to systematic factors, returns will vary to a greater degree than the market returns.

The graphical representation of the CAPM in the security market line. In Fig. 4.12 we have the β version and in Fig. 4.13 the covariance version.

We will now examine the critique of the CAPM in [4], p. 217, referring to [22].

1. The only legitimate test of the CAPM is whether or not the market portfolio (which includes *all* assets) is mean-variance efficient.
2. If performance is measured relative to an index that is ex post efficient, then from the mathematics of the efficient set, no security will have abnormal performance when measured as a departure from the security market line.
3. If performance is measured relative to an ex post inefficient index, then any ranking of portfolio performance is possible, depending on which inefficient index has been chosen.

One application of the CAPM beta is the calculation of the Treynor ratio. This ratio can be used to rank the desirability of a particular asset in combination with other assets, where part of the total risk inherent in the standard deviation will be diversified. The Treynor ratio is the return in excess of the risk-free rate divided by the beta.

Definition 4.9 The ***Treynor ratio TR*** is the difference between the annual arithmetic mean return and annual arithmetic mean risk-free return divided by the annualized beta of the fund return, i.e.,

$$\mathrm{TR} = \frac{\bar{\mathrm{r}}_{\mathrm{P}} - \bar{\mathrm{r}}_{\mathrm{f}}}{\beta(\mathrm{P}, \mathrm{MP})}. \tag{4.4.5}$$

Remark 4.13 Unlike the Sharpe ratio (4.3.1), which has no dimension, the Treynor ratio has percentage as a dimension.

The CAPM can be applied to forecast the return of an asset or a portfolio, or to measure to which extent an asset C_1 or a portfolio P does not satisfy the CAPM (Theorem 4.2). Algebraically we introduce the deviation $\alpha(\mathrm{C}_1,\mathrm{MB})$, $\alpha(\mathrm{P},\mathrm{MB})$, resp. of an asset C_1, a portfolio P, resp. from the CAPM expressed by (3), by (4), resp., i.e.,

$$\mathrm{r}_1 = \alpha(\mathrm{C}_1, \mathrm{MP}) + \mathrm{r}_{\mathrm{f}} + \beta(\mathrm{C}_1, \mathrm{MP})(\mathrm{r}_1 - \mathrm{r}_{\mathrm{M}})$$

and

$$\mathrm{r}_{\mathrm{P}} = \alpha(\mathrm{P}, \mathrm{MP}) + \mathrm{r}_{\mathrm{f}} + \beta(\mathrm{P}, \mathrm{MP})(\mathrm{r}_{\mathrm{P}} - \mathrm{r}_{\mathrm{M}}).$$

This leads us to the following definition:

Definition 4.10 The ***CAPM alpha*** or ***Jensen's alpha*** α(P, MB) is the factor that reconciles actual return with those predicted by the CAPM, defined by

$$\alpha(\mathrm{P}, \mathrm{MP}) = \mathrm{r}_{\mathrm{P}} - \mathrm{r}_{\mathrm{f}} - \mathrm{b}(\mathrm{P}, \mathrm{MP})(\mathrm{r}_{\mathrm{P}} - \mathrm{r}_{\mathrm{M}}).$$

Remark 4.14 Unlike the Sharpe ratio (4.3.3) and Treynor ratio (5), the Jensen's alpha of a portfolio need no comparison to another portfolio. In this regression, r_{M} is the independent variable and r_{P} is the dependent variable. If α is greater than zero, the fund has a return higher than expected by the CAPM. A negative α indicates that the fund performed worse than predicted, given the market risk taken.

4.5 Relative Risk-Adjusted Measures

Tracking error and downside risk is useful when measuring the active risk. In this section we present two measures that relate relative return to relative risk. The following ratio starts from a target return that can be induced by the benchmark of the portfolio.

Definition 4.11 We denote the target return by $\mathrm{T} \in \mathbf{R}^{\mathbf{1}}$ and assume that a return series (3.1.1) of a portfolio P is given. Furthermore we consider ***the annualized downside deviation*** or ***semi-standard deviation semistd***, defined by

$$\text{semistd}\ (P,T,N) = \sqrt{\frac{1}{N}\sum_{\substack{k=1\\ r_k < T}}^{N} (r_k - T)^2 \cdot f}. \qquad (4.5.1a)$$

The ***Sortino ratio SoR(P,T)*** of a portfolio P is defined by the ratio between the annual average difference of the portfolio $\bar{r}_P$ and T and the annualized downside deviation, i.e.,

$$\text{SoR}(P,T,N) = \frac{(\bar{r}_P - T)\cdot f}{\text{semistd}(P,T)} \qquad (4.5.1b)$$

where f is the periodicity of the data per year.

Remark 4.15 (1) is the first part of the sum on the left side in (3.3.7) for $\bar{r} = 1$.

Remark 4.16 The explicit expression of (1) is

$$\text{SoR}(P,T,N) = \frac{(\bar{r}_P - T)}{\sqrt{\frac{1}{N}\sum_{\substack{k=1\\ r_k < T}}^{N} (r_k - T)^2}}\sqrt{f}.$$

The following example shows that the Sortino ratio is of the same type as the Sharpe ratio (See also Remark 4.7).

Example 4.7 With T = 0 in (1) we consider a first portfolio for

$$\begin{aligned} r_{P_1,k} &= -\sqrt{6}\%, k = 1,3,\ldots..,N \ \text{ odd} \\ r_{P_1,k} &= 2\% + \sqrt{6}\%, k = 2,4,\ldots.,N \ \text{ even} \end{aligned} \qquad (4.5.2a)$$

and assume that this is monthly data, i.e., f = 12 in (1). In the following we evaluate (1). For $N \to \infty$ the arithmetic average is

$$\bar{r}_{P_1,k} = 1\%.$$

An evaluation of (1) for N = 1, 3, 5, 7,... yields

$$\text{semistd}\ (P_1,0) = \sqrt{\frac{1}{N}\sum_{\substack{k=1\\ r_k<0}}^{N}\left(\sqrt{6}\right)^2 \cdot 12\%} = \sqrt{\frac{1}{2N}\sum_{k=1}^{N+1}\left(\sqrt{6}\right)^2 \cdot 12\%} =$$

$$\sqrt{\frac{N+1}{2N}\left(\sqrt{6}\right)^2 12\%} = \sqrt{\left(\frac{1}{2}+\frac{1}{2N}\right)2\cdot 6\%}$$

and an evaluation of (1) for N = 2, 4, 6, 8,... yields

$$\text{semistd}(P_1, 0) = \sqrt{\frac{1}{N}\sum_{\substack{k=1\\ r_k<0}}^{N}\left(\sqrt{6}\right)^2 \cdot 12\%} = \sqrt{\frac{1}{2N}\sum_{k=1}^{N}\left(\sqrt{6}\right)^2 \cdot 12\%} = \\ = \sqrt{\left(\sqrt{3}\right)^2 12\%} = 6\%.$$

Hence for $N \to \infty$

$$\text{semistd}(P_1, 0) = 6\%$$

and

$$\text{SoR}(P_1, 0) = \frac{12\%}{6\%} = 2. \tag{4.5.2b}$$

We consider a second portfolio for N = 1, 2, 3, ...

$$\begin{aligned} r_{P_2,k} &= -2\sqrt{6}\%, k = 1, 3, \ldots.., N \ \text{ odd}, \\ r_{P_2,k} &= 2\% + 2\sqrt{6}\%, k = 2, 4, \ldots., N \ \text{ even}. \end{aligned} \tag{4.5.3a}$$

Then we have for $N \to \infty$

$$\bar{r}_{P_2,k} = 1\%.$$

An evaluation of (1) for N = 1, 3, 5, 7,... yields

$$\text{semistd}(P_2, 0) = \sqrt{\frac{1}{N}\sum_{\substack{k=1\\ r_k<0}}^{N}\left(2\sqrt{6}\right)^2 \cdot 12\%} = \sqrt{\frac{N+1}{2N}\left(2\sqrt{6}\right)^2 12\%} = \\ \sqrt{\left(\frac{1}{2} + \frac{1}{2N}\right)2 \cdot 12\%}$$

and an evaluation of (1) for N = 2, 4, 6, 8,... yields

$$\text{semistd}(P_2, 0) = \sqrt{\frac{1}{N}\sum_{\substack{k=1\\ r_k<0}}^{N}\left(2\sqrt{6}\right)^2 \cdot 12\%} = \sqrt{\frac{1}{2N}\sum_{k=1}^{N}\left(2\sqrt{6}\right)^2 \cdot 12\%} = \\ = \sqrt{\left(\sqrt{3}\right)^2 12\%} = 12\%.$$

Then we have for $N \to \infty$

$$\mathrm{semistd}(P_2, T) = 12\%$$

and

$$\mathrm{SoR}(P_2, 0) = 1, N = 1, 2, 3, \ldots.. \tag{4.5.3b}$$

We consider a third portfolio for N = 1, 2, 3, ….

$$\begin{aligned} r_{P_3,k} &= -4\sqrt{6}\%, k = 1, 3, \ldots, N \text{ odd}, \\ r_{P_3,k} &= 2\% + 4\sqrt{6}\%, k = 2, 4, \ldots, N \text{ even}. \end{aligned} \tag{4.5.4a}$$

Then we have for $N \to \infty$

$$\bar{r}_{P_3,k} = 1$$

An evaluation of (1) for N = 1, 3, 5, 7,… yields

$$\mathrm{semistd}(P_3, 0) = \sqrt{\frac{1}{N}\sum_{\substack{k=1 \\ r_k<0}}^{N}\left(4\sqrt{6}\right)^2 \cdot 12\%} = \sqrt{\frac{N+1}{2N}\left(4\sqrt{6}\right)^2 12\%} = \sqrt{\left(\frac{1}{2} + \frac{1}{2N}\right)2 \cdot 24\%}$$

and an evaluation of (1) for N = 2, 4, 6, 8,… yields

$$\mathrm{semistd}(P_3, 0) = \sqrt{\frac{1}{N}\sum_{\substack{k=1 \\ r_k<0}}^{N}\left(4\sqrt{6}\right)^2 \cdot 12\%} = \sqrt{\frac{1}{2N}\sum_{k=1}^{N}\left(4\sqrt{6}\right)^2 \cdot 12\%} = \\ = \sqrt{\left(\sqrt{3}\right)^2 12\%} = 24\ \%.$$

and

$$\mathrm{semistd}(P_3, 0) = \sqrt{\frac{1}{N}\sum_{\substack{k=1 \\ r_k<0}}^{N}\left(4\sqrt{6}\right)^2 \cdot 12\%.}$$

Then we have for $N \to \infty$

$$\mathrm{semistd}(P_3, 0) = 24\%$$

and

$$\mathrm{SoR}(P_3, 0) = \frac{12\%}{24\%} = \frac{1}{2}, N = 1, 2, 3 \ldots \tag{4.5.4b}$$

The definition of the Sortino ratio (see Definition 4.13) shows that an increase of the nominator, i.e., an increase of the outperformance of the portfolio versus the target return and a decrease of the denominator, i.e., a decrease of the semi-variance, leads to an overall increase of the Sortino ratio. The example shows that the three funds outperformed the target return $T = 0$. All three funds have the same arithmetic average $\bar{r}_{P_i,k} = 1$, $i = 1, 2, 3$, but have different semi-variances. By increasing the semi-variance in the portfolios (2), (3) and (4), the corresponding Sortino ratios decrease. The Sortino ratio has the same drawbacks as the Sharpe ratio as described in Remark 4.7, i.e., for an underperformance of the three portfolios against the target return the judgment would be just the opposite, which again makes no sense.

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Definition 4.12 The ***information ratio IR*** is a measure of the relative return (i.e., portfolio return–benchmark return) gained for taking on active risk, as expressed in the tracking error TE(1) defined by (3.2.4)

$$\mathrm{IR}(N) = \frac{\frac{1}{N}\sum_{k=1}^{N}\left(r_{P,k} - r_{B,k}\right)}{\mathrm{TE}(1)} \tag{4.2.5a}$$

and

$$\text{annualized } \mathrm{IR}(N) = \frac{\frac{1}{N}\sum_{k=1}^{N}\left(r_{P,k} - r_{B,k}\right)}{\mathrm{TE}(1)}\sqrt{f} \tag{4.2.5b}$$

where f is the periodicity of the data per year.

Example 4.8 We consider for $N = 1, 2, 3, \ldots$.

$$\begin{aligned} r_{P_1,k} &= 1\%, k = 1, 3, \ldots, N \text{ odd}, \\ r_{P_1,k} &= -1\%, k = 2, 4, \ldots, N \text{ even} \end{aligned} \tag{4.5.6a}$$

and assume that this is monthly data, i.e., $f = 12$ in (5). We consider a second portfolio for $N = 1, 2, 3, \ldots$.

$$\begin{aligned} r_{P_2,k} &= 3\%, k = 1, 3, \ldots., N \ \text{odd}, \\ r_{P_2,k} &= -3\%, k = 2, 4, \ldots., N \ \text{even}. \end{aligned} \tag{4.5.6b}$$

Then we consider a benchmark portfolio for $N = 1, 2, 3, \ldots.$

$$\begin{aligned} r_{B,k} &= -1\%, k = 1, 3, \ldots., N \ \text{odd}, \\ r_{B,k} &= 1\%, \ \ k = 2, 4, \ldots., N \ \text{even}. \end{aligned} \tag{4.5.7}$$

For the difference we find

$$\begin{aligned} r_{P_1,k} - r_{B,k} &= 1 + 1 - 1 - 1 + 1 + 1 - \ldots.. \\ r_{P_2,k} - r_{B,k} &= 3 + 1 - 3 - 1 + 3 + 1 - \ldots.. \end{aligned}$$

The information rate for both portfolios (6) against the benchmark (7) is

$$\begin{aligned} IR(N) &= \frac{1}{N}, \ \ N \ \text{odd} \\ IR(N) &= 0, \ \ N \ \text{even} \end{aligned}$$

and the annualized information rate is

$$\begin{aligned} \text{annualized} \ \ IR(N) &= \frac{\sqrt{12}}{N}, \ \ N \ \text{odd}, \\ \text{annualized} \ \ IR(N) &= 0, \ \ N \ \text{even}. \end{aligned}$$

We see that both portfolios have a trade-off between return and risk. If we had more trials, the information ratio would decline.

◇

The information ratio and the Sharpe ratio are similar, as they both divide return by risk. The information ratio is used for relative performance measurement, whereas the Sharpe ratio is used for absolute performance measurement.

Chapter 5
Investment Controlling

5.1 Objectives and Activities

As Fig. 4.1 shows, in this chapter we focus on investment controlling in the cycle of the investment process. Referring back to Chaps. 2, 3 and 4, investment controlling is a much younger discipline than portfolio optimization theory. In Chaps. 2 and 3 we introduced the basic concepts used and applied in investment controlling. We start this chapter with a definition for investment controlling:

Definition 5.1 ***Investment controlling*** is the independent supervision and monitoring of the quality and performance of asset management portfolios and products. Its goal is to ensure that the asset management clients get what was promised to them.

Not only asset management companies but also clients and consultants are requesting an investment controlling according to the above definition. In Switzerland, the best-known and most influential consultants are Ecofin, Complementa, Investment-Controlling AG and PPCmetrics. The companies Russel and Wyson Watt are active on international financial markets.

Setting up an investment system is a challenge for every asset management company. Figure 5.1 shows different aspects of investment controlling. Non-investment risks like operational risk are also part of investment controlling. Furthermore we have discussed the concept of performance, which encompasses return and risk. Investments also influence the finances of asset management companies, as poor performance may lead to a drop in revenues, while above-average performing investments can translate into above-average revenues. The compliance department also has to be familiar with investment controlling. As almost all portfolios have to satisfy certain constraints, one task of the compliance department is to ensure those constraints are not disregarded. This is an ongoing process, as the value of the portfolio and its components change along with market movements (see e.g. (2.2.3a)).

W. Marty, *Portfolio Analytics*, Springer Texts in Business and Economics,
DOI 10.1007/978-3-319-19812-5_5

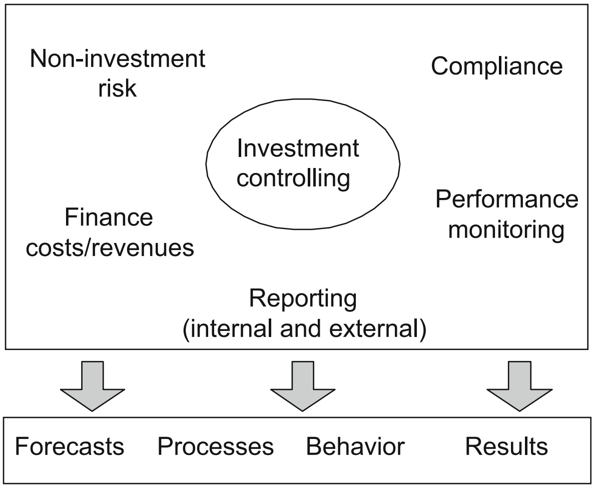


Fig. 5.1 Different aspects of investment controlling

In this chapter we first describe the various tasks of investment controlling and introduce the sequence of different steps involved in investment controlling before turning to an illustrative sample portfolio. In Sect. 5.5 we provide an overview of current research topics.

Generally speaking, investment controlling improves the visibility, transparency and credibility of any asset management company. More specifically, investment controlling aims at:

- Producing a performance analysis of the asset management portfolios or products such that the implementation of the Global Investment Performance standards (GIPS) [3] is possible.
- Enabling the thorough analysis necessary to identify the real drivers of portfolio performance on an ex post and ex ante basis. With the help of factor analysis it is possible to identify the drivers that influence the portfolio (see Sect. 3.5).
- Monitoring the risk and return of portfolios and/or products against their designated benchmarks.
- Creating or increasing the transparency of and comparability between the asset management products and/or portfolios.
- Addressing performance issues on a regular basis and if necessary making the appropriate adjustments.
- Creating a basis not only for ongoing analyses but also for structural changes in the investment process.

As can be seen in Fig. 5.1 and as illustrated in the following list, investment controlling involves many different areas of activity:

- Calculating performance attribution on an ex post as well as on an ex ante basis

- Benchmark comparisons, composite dispersion analysis and peer group analysis with respect to the return and/or risk but also to characteristics like asset class or sector weights, duration, and exposure to specific risk factors
- Calculating performance figures and statistics that reflect manager skills or the investment style and running a style analysis
- Reviewing the setup of the specific asset management portfolio with respect to e.g. the benchmark, investment guidelines, transaction costs and management fees
- Identifying actual and potential performance calculation issues and reporting the serious ones to the senior management
- Suggesting remedial action to resolve performance issues
- Analyzing and identifying all steps in the investment process
- Checking whether risk levels and limits are appropriate and respected (risk budgeting)
- Aggregated performance reporting to the senior management

In order to deliver accurate and reliable investment controlling, the following list covers some of the most necessary requirements and resources:

- The analysis has to be based on high-quality data that is available in a reasonable timeframe. The following types of data are needed:
  - Daily data on holdings and prices on securities for both the portfolio and the benchmark.
  - The forecasts and the tactical asset allocations with the appropriate transactions.
- It is paramount that an investment controller combines both general controlling expertise and asset management knowledge in theory and practice.
- Access to peer group information
- Availability of appropriate analytical tools to run performance attributions or simulations, as well as tools to maintain composites or to do the performance reporting
- Access to relevant documents, e.g. asset management agreements between the asset management company and their clients, updated prospects of mutual funds, and investment guidelines.

In practice, we rarely see that all of the above requirements are completely met. However, without the essential resources and without the relevant data, investment controlling cannot be performed satisfactorily. For example, attempting to run a risk attribution for a Swiss fixed income portfolio with credit exposure using a risk analysis tool that does not incorporate a Swiss credit model is useless from an investment controlling point of view, because it cannot accurately reflect the investment process.

5.2 Quantitative Building Blocks

The performance monitoring process can be split up as illustrated in following Fig. 5.2.

The ***first four steps*** are quantitatively oriented and are used for calculating, maintaining or storing, visualizing and analyzing different performance figures for a specific portfolio, product or composite.

The last three steps are less production-oriented and deliver qualitative statements on the investment process and its results. The performance watch list determines problematic asset management portfolios and/or products that are subsequently analyzed using portfolio analytics and are discussed in depth in the performance review. In this respect, working out proposals for improvement and pointing out possible consequences for the investment process are the primary objectives of the qualitatively oriented analyses.

In the following we illustrate the performance monitoring process, the purpose being to provide an overview of the different aspects of performance measurement and analysis and to see how to incorporate them into one general investment controlling framework.

5.2.1 Performance Measurement

This first step of the performance monitoring process deals with all aspects of return and risk measurement, i.e., the calculation of all necessary return and risk measures

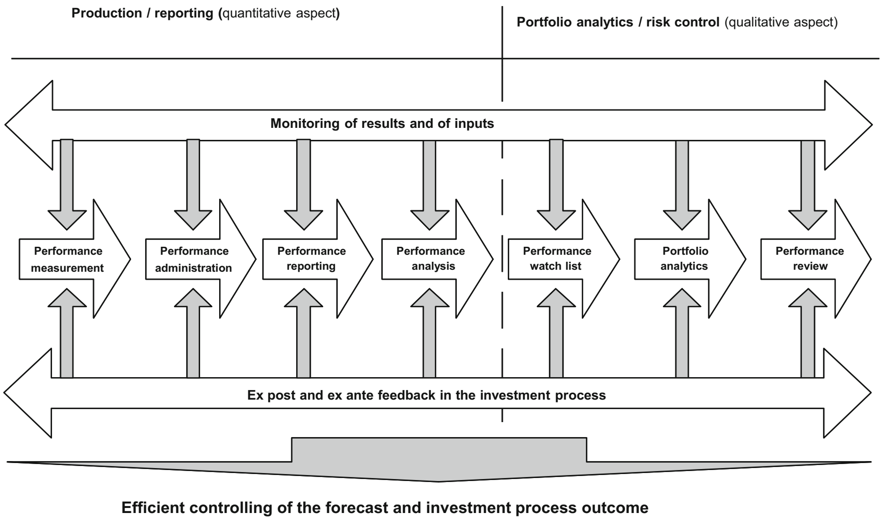
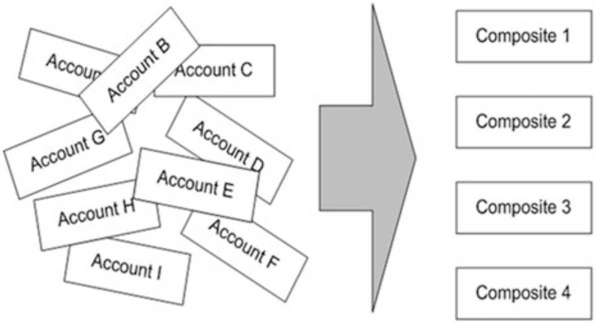


Fig. 5.2 The performance monitoring process

Fig. 5.3 Idea of composite construction



like e.g. portfolio returns, time-weighted rates of return and money-weighted rates of return, as well as risk measures like volatility or tracking error. Performance measurement usually focuses on the level of the portfolio as a whole and yields time series.

5.2.2 Performance Administration

Referring to Fig. 5.3 we start by defining a composite:

Definition 5.2 A ***composite*** is an aggregation of one or more portfolios managed according to a similar investment mandate, objective or strategy.

Performance administration consists in constructing and maintaining composites. Creating meaningful composites is essential to the fair presentation, consistency, and comparability of performance over time and between different companies. In order to assess the asset management skills of one or more asset managers or even of a whole firm, it is necessary to classify the different portfolios and assign portfolios with the same characteristics, e.g. the same benchmarks, same investment strategies and/or styles to a composite as illustrated in Fig. 5.3. ***A composite is a portfolio of portfolios***.

Furthermore, performance administration consists in calculating tailor-made and industry benchmarks.

Complying with the GIPS standards [3] means observing best practices with respect to transparency and comparability in presenting performance. It provides the basis for efficient investment controlling of an asset management company.

5.2.3 Performance Reporting

This step in the performance monitoring process includes the reporting on different performance figures, usually at the level of the portfolio as a whole. A report is

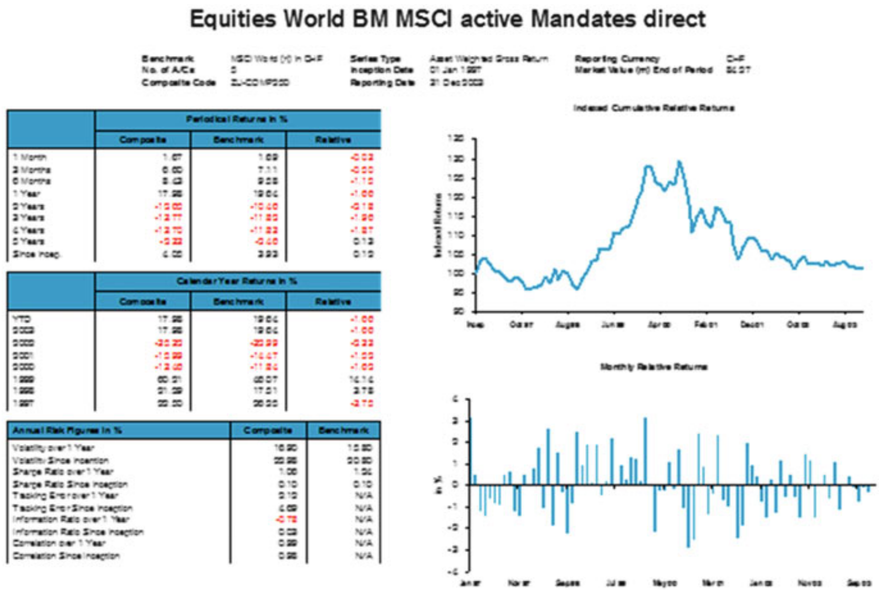


Fig. 5.4 Sample composite

based on the data from e.g. a portfolio, a product or a composite. The name of the composite refers to the investment universe, the benchmark, the investment style and the type of portfolio. The name of the composite in Fig. 5.4 is “Equity World BM MSCI active mandate direct.” The name refers to the set of mandates with direct investments in equities worldwide that are actively managed with the same benchmark, which is well known in the industry—MSCI World.

The figures in Fig. 5.4 are solely based on historical data and summarize the returns and volatility of the composite. In addition we have absolute and relative risk-adjusted figures like the Sharpe ratio, the tracking error and the information ratio.

As a rule, the report setup is user-specific. The user can be an in-house or external client of the asset management company. Depending on the size of the company, the reports are produced with the help of a spreadsheet or external software solution. With regard to external software solutions, it is often the case that the composite maintenance software and the reporting software are part of the same software solution.

The first step in setting up a report is to answer the following questions:

- Which product, composite or portfolio should be analyzed?
- Which time period should be considered?
- Which returns and risk measures should be shown?
- Should rolling and/or annual performance figures be presented?

A thorough analysis should investigate the sensitivity of

- Methodology
- Time periods
- Input data

to the figures shown in the report. Issues in this context include e.g.:

- Varying the time period, e.g. only for 1 month forward, may result in a highly underperforming account transforming into a highly outperforming one.
- The calculation of the Sharpe Ratio can use different risk-free rates. In continental Europe it is common to use the relevant total return of an alternative risk-free investment and in other countries it is more common to use the actual risk-free rate. Depending on the chosen type of risk-free rate you may come up with different conclusions.
- As shown here, we distinguish between ex post and ex ante performance analysis. A client may be interested in the past performance of his or her portfolio or the senior management may want to rank the future performance of competing products.

For more detailed information on the sources of return and risk, the analysis goes beyond the total-portfolio level into performance attribution. The development of performance attribution analyses started around 1980. In the early days, the return and risk contributions were first considered at an aggregated level—following a top-down approach. Particularly in recent years, more detailed analyses have been developed; as a result, it is increasingly possible to conduct performance attribution at the stock level and on a daily basis. Parallel to this development, the focus of performance attribution has shifted away from returns and toward risk and risk-adjusted returns. However, depending on the investment category considered and the investment instruments used, risk analysis and risk-adjusted return analysis still require considerable further development. The development of performance measurement and attribution is illustrated in Fig. 5.5.

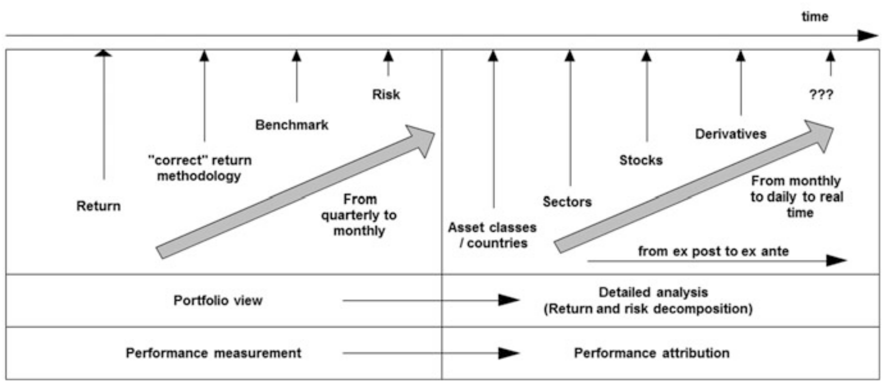


Fig. 5.5 The development of performance attribution

We proceed with an analogy from the field of science. What atoms are to physicist and chemists, securities are to portfolio managers. The prices of securities and thus prices of portfolios are comparable to phenomena in physics, and it's up to the scientists or portfolio managers to investigate them.

5.2.4 Portfolio Analysis

Identifying performance contributions and attributions (see Chaps. 2 and 3) is at the heart of portfolio analysis. This analysis is an important component of the performance monitoring process and is defined as a process that assesses the performance contributions and attributions of the individual decision-making steps within the investment process. As mentioned in previous chapters, performance attribution attempts to answer the question: "Which return and risk contributions are due to which decisions and can be attributed to which decision-makers?" It is desirable that the analyses be conducted on both an ex-post and an ex-ante basis. The results produced by this analysis are judgment-free and are thus ***objective***. In the following we present a sample performance attribution produced by the performance software provider Wilshire over the course of 1 year. In the two first columns A and B of Tables 5.1 and 5.2 we show the ***average weights*** of the portfolio and the benchmark for two different MSCI segmentations.

Remark 5.1 For the calculation in Tables 5.1 and 5.2 the segments are like securities. Differences between the weighted segments of the portfolio and the benchmark are not shown.

Weights change every day and it can be shown that the average weighting obeys also the budgeting constraints (see (2.2.5a)) and the last line of the columns A and B′ in Table 5.1 shows that the budgeting constraint is satisfied. Already illustrated for instance in Table 2.4 we find the underweights and overweights, i.e., we have the ***relative weights*** in column C. The returns are calculated on a daily basis, i.e.,

$$\mathrm{t_k = k.\ \ k = 1, \ldots, N,}$$

where N is the number of days in the performance attribution report (see Fig. 2.5). The return calculation is based on the consideration in Sect. 2.3, i.e., the time-weighted rate of return is applied. For the absolute return of the portfolio and the benchmark we refer to Sect. 2.3.1 and for the relative return and the management effects we refer to Sect. 2.3.2.

The column D, E resp. is the unweighted return of the portfolio, the benchmark resp. It thus allows us to compare the return of the two different sets of the MSCI segments considered in Tables 5.1 and 5.2.

In column F we can see whether the portfolio segments' return underperformed or outperformed the corresponding benchmark segments, and column H shows the segment return of the Benchmark versus the overall Benchmark return. The figures in column F depend on the portfolio, whereas the figures in column H are portfolio-

Table 5.1 Sectorial segmentation

MSCI sector	Average portfolio weight (%) A	Average benchmark weight (%) B	Relative weight (%) C	Absolute portfolio return (%) D	Absolute benchmark return (%) E	Difference absolute portfolio return—absolute benchmark return (%) F	Difference absolute benchmark return—total benchmark return (%) H
Consumer discretionary	12.03	12.17	−0.14	**15.58**	**23.07**	**−7.49**	**3.47**
Consumer staples	8.83	9.26	−0.43	**7.74**	**5.09**	**2.65**	**−14.51**
Energy	8.29	7.41	0.88	**10.51**	**13.30**	**−2.79**	**−6.30**
Financials	21.70	22.94	−1.24	**23.90**	**24.94**	**−1.04**	**5.34**
Health Care	11.73	12.53	−0.80	**9.60**	**7.31**	**2.29**	**−12.29**
Industrials	10.45	9.71	0.74	**34.93**	**24.08**	**10.85**	**4.48**
Information Technology	12.33	12.49	−0.16	**25.06**	**32.79**	**−7.73**	**13.19**
Materials	4.84	4.53	0.31	**23.11**	**30.33**	**−7.22**	**10.73**
Telecomm Services	6.15	5.24	0.91	**15.02**	**12.64**	**2.38**	**−6.96**
Utilities	2.44	3.72	**−1.28**	**34.09**	**15.45**	**18.64**	**−4.15**
Cash	1.21	0.00	1.21	**8.94**	**0.00**	**8.94**	**−19.60**
	100.00	100.00	0.00	**18.28**	**19.60**	**−1.32**	**0.00**

Table 5.2 Regional segmentation

MSCI sector	Average portfolio weight (%)	Average benchmark weight (%)	Relative weight (%)	Absolute portfolio return (%)	Absolute benchmark return (%)	Difference absolute portfolio return—absolute benchmark return (%)	Difference absolute benchmark return—total benchmark return (%)
	A	B	C	D	E	F	H
Asia ex Japan	1.86	3.17	−1.31	12.52	31.51	−18.99	11.91
Europa	27.62	28.92	**−1.30**	25.40	24.36	1.04	4.76
Japan	9.83	8.74	1.09	20.16	21.75	−1.59	2.15
North America	59.48	59.17	0.31	15.99	16.38	−0.39	−3.22
Cash	1.21	0.00	1.21	8.94	0.00	8.94	−19.60
	100.00	100.00	0.00	18.28	19.60	−1.32	0.00

Table 5.3 Management effect (Sector)

MSCI sector	Asset allocation (%)	Stock selection (%)	Interaction (%)	Total (%)
Consumer discretionary	0.00	−**0.86**	−0.04	−0.90
Consumer staples	0.05	0.28	0.03	0.36
Energy	0.00	−0.19	−0.02	−0.21
Financials	−0.10	−0.21	0.01	−0.30
Health Care	0.20	0.35	−0.03	0.52
Industrials	−0.04	**0.99**	0.12	1.07
Information Technology	−0.21	−**0.91**	0.01	−1.11
Materials	0.01	−0.28	−0.08	−0.35
Telecomm Services	−0.11	0.21	−0.08	0.02
Utilities	0.02	**0.74**	−**0.41**	0.35
Cash	−0.78	0.00	0.00	−**0.78**
	−0.96	0.13	−0.49	−1.32

Table 5.4 Management effect (Regional)

MSCI sector	Asset allocation (%)	Stock selection (%)	Interaction (%)	Total (%)
Asia ex Japan	−0.10	−0.56	0.30	−0.36
Europa	−0.04	0.34	−0.07	0.23
Japan	−0.09	−0.16	−0.08	−0.33
North America	0.01	−0.20	0.14	−0.05
Cash	−0.78	0.00	0.00	−**0.78**
	−1.01	−0.59	0.28	−1.32

independent. By combining F and H, the portfolio segment return versus the overall benchmark return can be derived. The last line of column F shows the relative return of the portfolio versus the benchmark, and the last line of column H shows the relative return of the benchmark versus the benchmark itself, which is obviously 0.

Unlike in Example 2.9, here we have a multi-period analysis, as we measure the return over days and produce a multi-day return analysis. The calculations in Table 2.5 can only be applied to a 1-day analysis. The figures in bold are explained in 5.3.

In the following Tables 5.3 and 5.4 we show the decomposition of the relative return in the management effects (for their definitions, see (2.3.19)).

Remark 5.2 In Tables 5.3 and 5.4 we consider weighted segments of the portfolio and the benchmark. The column 'Total' shows the sum of the effects.

We proceed with an ex ante factor analysis and show the decomposition of the ex ante risk of the portfolio and the benchmark in the factor and stock-specific security risk (Table 5.5).

Table 5.5 Absolute Risk

	Portfolio	Benchmark
Number of Securities	**57**	**1550**
Ex ante Total Risk	18.81 %	18.21 %
Factor Specific Risk	18.66 %	18.18 %
Stock Specific Risk	2.39 %	1.02 %

Table 5.6 Risk attribution (absolute versus relative)

	Portfolio (%)	Tracking error (%)
Total Risk	18.81	2.57
Factor Specific Risk	18.66	**1.50**
Region	11.50	0.18
Country	6.98	0.83
Industry	2.64	0.77
Fundamental	1.44	0.76
Currency	0.42	0.27
Covariance	0.35	0.52
Stock Specific Risk	2.39	**2.08**

In Table 5.6 we show the decomposition of the factor-specific risk of the portfolio and the tracking error.

5.3 Qualitative Building Blocks

5.3.1 Performance Watch List

The first step after the quantitative building blocks of investment controlling is the identification of portfolios that have problems, which are then placed on the watch list. There are many qualitative and quantitative reasons why a portfolio, a composite or a product is worth a closer look. Examples include underperformance in comparison to a benchmark or a peer group, a profile not suitable to a specific client, or a sheer client complaint.

The determination of the watch list follows the process illustrated in Fig. 5.6. Usually it starts with a mechanical filter focusing on the historical and forward-looking characteristics of the products and of the individual accounts, ex ante risk limits or risk budgeting constraints, as well as on the client's feedback. In the next step, the ***investment controlling committee*** decides which portfolios, products or composites seem to be problematic and should go onto the performance watch list. Afterwards, at ***the performance review meeting***, each portfolio, composite or product on the watch list is analyzed in detail, considering all kinds of information

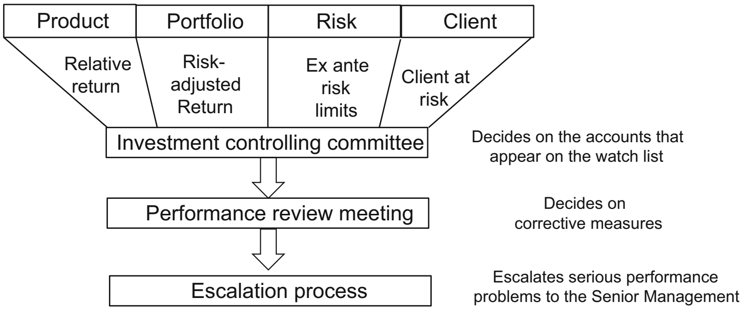


Fig. 5.6 Performance watch process

from investment guidelines up to an ex ante risk breakdown. As a result of this performance review corrective steps are then defined and implemented. If the performance does not improve over a longer period of time, serious performance problems are reported to the senior management ***(escalation process)***.

5.3.2 *Portfolio Analytics*

Portfolio analytics provide deeper insights into the portfolio or products on the watch list by taking all the available quantitative information produced in one of the preceding steps of the performance monitoring process into consideration.

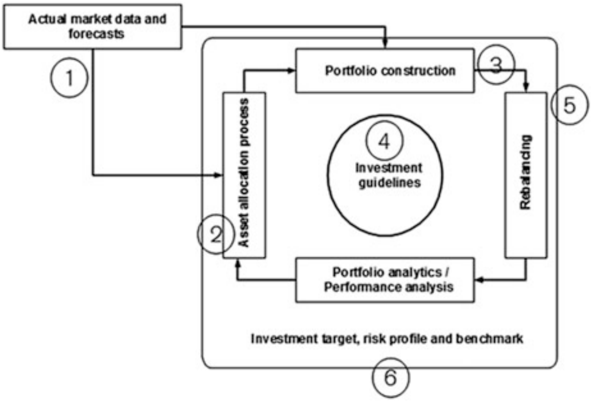
The performance attribution analysis on its own and without any qualitative judgment is nothing more than a detailed performance reporting. Yet, in interpreting the performance figures by considering all relevant circumstances, it can become a meaningful management information tool. Here we refer the reader back to the tentative definition of portfolio analytics in the introduction of this book.

Performance attribution is the quantitative and portfolio analytics the qualitative assessment for a portfolio, composite or product of an asset management firm.

5.3.3 *Performance Review*

The performance review is concerned with the portfolio, composite and products on the performance watch list and seeks to analyze where the performance problems came from and whether any corrective action is necessary or would help to get the product back on the right track. Within this step of the performance monitoring process, the portfolios or products are analyzed in detail, taking into consideration

Fig. 5.7 Aspects of the performance review



all available information. The following questions offer a possible starting point for a typical performance review (The figures in brackets refer to Fig. 5.7):

- Where does the return come from and from which decisions does it originate (1)?
- Are the risk profile and risk budget still appropriate in the asset allocation process (2) and in the portfolio construction (3)?
- Are the investment guidelines still reasonable and have they been respected (4)?
- What was the impact of the costs on the overall return (5)?
- Is the choice of benchmark still sensible? Are the general circumstances still valid or reasonable (6)?
- What was competitors' performance like?

5.4 Example of a Performance Review

This section applies the points discussed in the previous section, and we investigate the return and the accompanying risk. A performance review follows and analyzes each step in the investment process for a specific portfolio, composite or product.

We assume that the composite "Equities world BM MSCI active mandate direct" is on the watch list because of the negative relative return of −1.32 % over a particular year (see Tables 5.1 and 5.2). The investment controller was asked to run a performance review and to analyze why this composite seriously underperformed.

After analyzing the different performance reports and referring to (2) and (3) in Fig. 5.7, the investment controller arrived at the following conclusions (see Tables 5.1 and 5.2):

- The asset manager did not make big sector bets → maximum underweight 1.28 %
- The asset manager did not make big country bets → maximum underweight 1.30 %

- The asset manager made big security-specific bets by investing "only" 67 out of 1550 securities in the benchmark
- The asset manager took quite a lot of security-specific risks → stock-specific risk of 2.08 % versus factor-specific risk of 1.50 %
- **Conclusion 1:** The asset manager pursued a stock picking approach with neutral sector and country bets!
- The biggest return contributions came from
  - Overweight in cash → −0.78 %
  - Stock selection
    (a) Consumer discretionary → −0.86 %
    (b) Information technology → −0.91 %
    (c) Industrials → +0.99 %
    (d) Utilities → +0.74 %
  - Interaction utilities → −0.41 %
- The biggest relative risk contribution came from the stock-specific risk → 2.08 %
- **Conclusion 2:** Their stock picking did not pay off!

As described, the negative excess return mainly came from the cash bet. In addition, the underweight of utilities—an outperforming sector—resulted in a quite high negative interaction effect. One could conclude that the asset manager had quite bad luck with his asset allocation decisions over the last 12 months. As a result of this analysis the investment controller could conclude that no remedial action is required and could decide to wait for another period and see whether the product recovers.

5.5 Review and Concluding Remarks

Figure 5.8 illustrates the various levels of performance attribution on the ***left-hand side***. Furthermore return and risk contributions can be calculated on an absolute basis, i.e., isolated for an asset management portfolio or for a specific benchmark, or on a relative basis, i.e., for an asset management portfolio in comparison to a benchmark. The performance attribution may be focused on the past (ex post) or the future (ex ante). Performance attribution is defined as the decomposition of historical and expected, absolute or relative return and/or historical and expected, absolute or relative risk.

As seen on the ***right-hand side*** in Fig. 5.8, general performance attribution can be applied to measure the return and risk contributions of categories, sectors and instruments (e.g. asset categories, countries, currencies or securities), of decision-makers such as the client him—or herself, portfolio managers or consultants, and

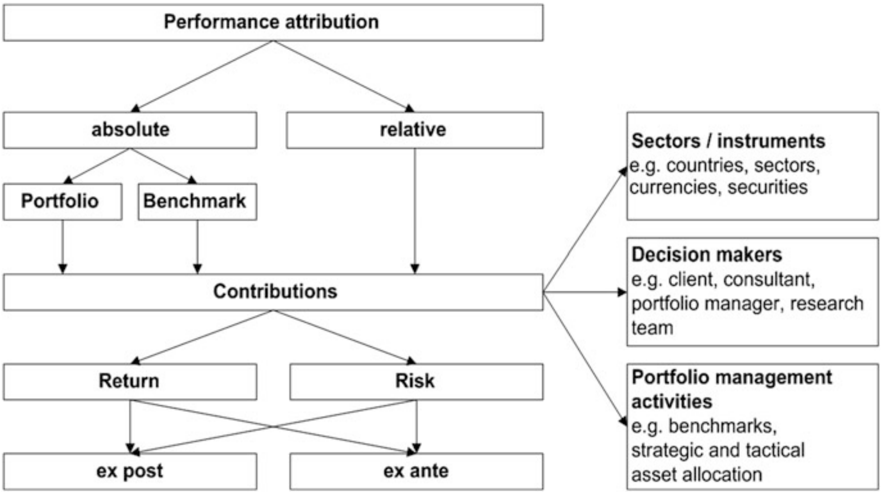


Fig. 5.8 Summary of the performance decomposition

finally of asset management activities like the definition of the benchmark, the strategic or tactical asset allocation, or the stock picking.

This book was written between the years 2011 and 2013, at the end of which time the area of performance measurement and attribution is still in development. It is nice to note that in the course of this timespan, research has made significant strides. Here we would like to mention the current research that has come to our attention; we will proceed in the order of the table of contents.

The material we have here presented in Chaps. 2 and 3 often uses the terms ex ante and ex post and discusses the concept of return and risk. By combining these ideas, we arrive at four notions:

1. ***Ex post return*** has been discussed in Chap. 2. Section 2.2.2 is concerned with geometric attribution over one time interval. Referring to Fig. 2.2 we realized that the calculations do not add up horizontally and vertically. In addition, the statement "there is no interaction effect," though often cited in the community, is still open to discussion. The full portfolio analysis return problem depicted in Fig. 2.6 is not solved here and Sect. 2.2.2 needs to be developed further.

   In (2.4.1) we considered different versions of the equation for the internal rates of return. Essentially we have two areas that pose difficulties:

   - In general, the exponents of the discount factor are not integers but real numbers. Thus the fundamental theorem of algebra, which states that the number of complex solutions is equal to the degree of the polynomial, is not directly applicable.
   - Generally speaking, the equation has more than one solution. As the Examples 2.18 and 2.19 in Sect. 2.4 indicate, the question as to whether a business-

relevant solution can be clearly identified or not is greatly dependent on the cash flows.

Section 2.4 illustrates the above points by presenting the easiest equation that avoids the first problem and discusses the second problem mentioned above. There are now a number of new publications available on this topic. We refer readers to [25] and [26].

2. ***Ex ante return*** is nothing more than forecasting. An introductory discussion on the different fields of forecasting is provided in Sect. 4.1.
3. ***Ex post risk*** is discussed in Sects. 3.1 and 3.2. This discussion is at the portfolio level and ex post attribution is still in its infancy and not yet widely researched. It essentially means that the historical weights of the portfolio are reflected in the risk figure. If an investor switches between the equity market and the bond market periodically, the risk originates from both markets. In the available literature we have only found [11].
4. ***Ex ante risk*** is discussed in Sects. 3.3 and 3.4. We have provided an introduction to risk attribution. One criticism is that factor analysis assumes that all investment decisions are made simultaneously, and that sequential chains of investment decisions are not modeled.

We proceed with the comments on Chap. 4 by discussing the relation between return models and risk models. The implementation of risk attribution is more problematic than that of return attribution because the risk attribution software available is not as flexible as the return attribution software. Usually the risk attribution software decomposes the absolute or relative risk by using a specific risk model that does not necessarily represent the actual investment process. Therefore the figures in a risk attribution should be used with caution, as the various risk factors and their contributions to the overall absolute or relative risk are not necessarily linked to the steps in the investment process or the different decisions made. Figure 5.9 illustrates this issue by indicating that in an ideal world the risk attribution would have to be linked to the return attribution.

	Portfolio	Tracking Error
Total Risk	18.81%	2.57%
Factor Specific Risk	18.66%	1.50%
Region	11.50%	0.18%
Country	6.98%	0.83%
Industry	2.64%	0.77%
Fundamental	1.44%	0.76%
Currency	0.42%	0.27%
Covariance	0.35%	0.52%
Stock Specific Risk	2.39%	**2.08%**

MSCI Sector	Allocation	Selection	Interaction	Total
Consumer discretionary	0.00%	-0.86%	-0.04%	-0.90%
Consumer staples	0.05%	0.28%	0.03%	0.36%
Energy	0.00%	-0.19%	-0.02%	-0.21%
Financials	-0.10%	-0.21%	0.01%	-0.30%
Health Care	0.20%	0.35%	-0.03%	0.52%
Industrials	-0.04%	0.99%	0.12%	1.07%
Information Technology	-0.21%	-0.91%	0.01%	-1.11%
Materials	0.01%	-0.28%	-0.08%	-0.35%
Telecom Services	-0.11%	0.21%	-0.08%	0.02%
Utilities	0.02%	0.74%	-0.41%	0.35%
Cash	-0.78%	0.00%	0.00%	-0.78%
	-0.96%	0.13%	-0.49%	-1.32%

Fig. 5.9 Do return and risk model correspond?

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About the Author

Dr. Wolfgang Marty is Senior Investment Strategist at AgaNola, Pfaeffikon, SZ, Switzerland. Between 1998 and 2015 he was working with Credit Suisse. He joined Credit Suisse Asset Management in 1998 as Head Product Engineering. He specializes in Performance Attribution, Portfolio Optimization and Fixed Income in general.

Prior to joining Credit Suisse Asset Management, Marty worked for UBS AG in Chicago, London and Zurich. He started his career as an assistant for applied mathematics at the Swiss Federal Institute of Technology.

Marty holds a university degree in Mathematics from the Swiss Federal Institute of Technology in Zurich and a doctorate from the University of Zurich. He chairs the method and measure subcommittee of the European Bond Commission (EBC) and is president of the Swiss Bond Commission (OKS). Furthermore he is a member of the Fixed Income Index Commission at the Swiss Stock Exchange and a member of the Index team that monitors the Liquid Swiss Index (LSI).

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