Convergence analysis of proximal symmetric alternating direction method of multipliers

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Outline

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2. Main result
3. Convergence analysis
4. Numerical result
5. Conclusion

1 Introduction

2 Main result

3 Convergence analysis

4 Numerical result

5 Conclusion

Introduction

Consider the convex optimization problem:

$$\min\{\theta_1(x) + \theta_2(y) \mid Ax + By = b,\ x \in \mathcal{X}, y \in \mathcal{Y}\},$$

- $\theta_1 : \Re^{n_1} \to \Re$ and $\theta_2 : \Re^{n_2} \to \Re$ are continuous convex function
- $A \in \Re^{m \times n_1}, B \in \Re^{m \times n_2}, b \in \Re^m$
- $\mathcal{X} \subset \Re^{n_1}, \mathcal{Y} \subset \Re^{n_2}$ are nonempty closed convex sets

The augmented Lagrangian function:

$$\mathcal{L}_\beta(x, y, \lambda) = \theta_1(x) + \theta_2(y) - \lambda^T(Ax + By - b) + \frac{\beta}{2}\|Ax + By - b\|^2$$

Example

Basis pursuit:

$$\min_{x \in \Re^n} \|x\|_1 + \frac{1}{2\mu}\|Ax - b\|_2^2$$

Example

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$$\min_{x \in \Re^n} \|x\|_1 + \frac{1}{2\mu}\|Ax - b\|_2^2$$

$$\downarrow$$

$$\min \left\{ \frac{1}{2\mu}\|Ax - b\|_2^2 + \|y\|_1 | x - y = 0, x, y \in \Re^n \right\}$$

ADMM

Glowinski and Marroco(1975), Gabay and Mercier(1976)

$$\begin{cases} x^{k+1} = \arg\min\{\mathcal{L}_\beta(x, y^k, \lambda^k) \mid x \in \mathcal{X}\}, \\ y^{k+1} = \arg\min\{\mathcal{L}_\beta(x^{k+1}, y, \lambda^k) \mid y \in \mathcal{Y}\}, \\ \lambda^{k+1} = \lambda^k - \beta(Ax^{k+1} + By^{k+1} - b). \end{cases}$$

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Glowinski(1984)

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- $s \in (0, \frac{1+\sqrt{5}}{2})$

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- $s \in (0, \frac{1+\sqrt{5}}{2})$
- This larger step size is crucial to getting better numerical results.

Symmetric ADMM

Peaceman-Rachford(1955), Lions and Mercier(1979)

$$\begin{cases} x^{k+1} = \arg\min\{\mathcal{L}_\beta(x, y^k, \lambda^k) \mid x \in \mathcal{X}\}, \\ \lambda^{k+\frac{1}{2}} = \lambda^k - \beta(Ax^{k+1} + By^k - b), \\ y^{k+1} = \arg\min\{\mathcal{L}_\beta(x^{k+1}, y, \lambda^{k+\frac{1}{2}}) \mid y \in \mathcal{Y}\}, \\ \lambda^{k+1} = \lambda^{k+\frac{1}{2}} - \beta(Ax^{k+1} + By^{k+1} - b). \end{cases}$$

- It may not converge.
- He, Liu, Wang, Yuan (2014) introduce a parameter $\alpha \in (0, 1)$ to shrink the step size in updating the Lagrange multiplier steps.

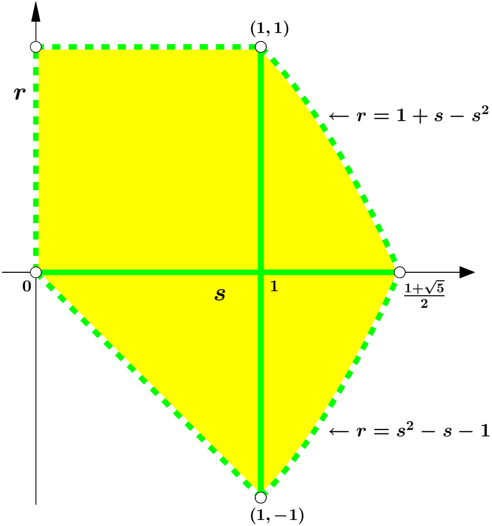
He, Ma, Yuan (2015)

$$\begin{cases} x^{k+1} = \arg\min\{\mathcal{L}_\beta(x, y^k, \lambda^k) \mid x \in \mathcal{X}\}, \\ \lambda^{k+\frac{1}{2}} = \lambda^k - r\beta(Ax^{k+1} + By^k - b), \\ y^{k+1} = \arg\min\{\mathcal{L}_\beta(x^{k+1}, y, \lambda^{k+\frac{1}{2}}) \mid y \in \mathcal{Y}\}, \\ \lambda^{k+1} = \lambda^{k+\frac{1}{2}} - s\beta(Ax^{k+1} + By^{k+1} - b), \end{cases}$$

The parameters r and s are restricted into the domain

$$\mathcal{D} = \{(r, s) \mid s \in \left(0, \tfrac{1+\sqrt{5}}{2}\right), r \in (-1, 1),\ r + s > 0,\ |r| < 1 + s - s^2\}.$$

Domain $\mathcal{D}$



Main work

Proximal Symmetric ADMM

$$\begin{cases} x^{k+1} = \arg\min\{\mathcal{L}_\beta(x, y^k, \lambda^k) + \frac{1}{2}\|x - x^k\|_R^2 \mid x \in \mathcal{X}\}, \\ \lambda^{k+\frac{1}{2}} = \lambda^k - r\beta(Ax^{k+1} + By^k - b), \\ y^{k+1} = \arg\min\{\mathcal{L}_\beta(x^{k+1}, y, \lambda^{k+\frac{1}{2}}) + \frac{1}{2}\|y - y^k\|_T^2 \mid y \in \mathcal{Y}\}, \\ \lambda^{k+1} = \lambda^{k+\frac{1}{2}} - s\beta(Ax^{k+1} + By^{k+1} - b), \end{cases}$$

- R and T are positive semidefinite matrices
- The parameters r and s are restricted into the domain

$$\mathcal{D} = \{(r, s) \mid s \in \left(0, \frac{1+\sqrt{5}}{2}\right), r \in (-1, 1),\ r + s > 0,\ |r| < 1 + s - s^2\}.$$

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Optimality condition

Proposition 3.1

Let $\mathcal{X} \subset \Re^n$ *be a closed convex set,* $\theta(x) : \Re^n \to \Re$ *and* $f(x) : \Re^n \to \Re$ *be convex functions. In addition,* $f(x)$ *is differentiable. We assume that the solution set of the minimization problem* $\min\{\theta(x) + f(x) \,|\, x \in \mathcal{X}\}$ *is nonempty. Then,*

$$x^* = \arg\min\{\theta(x) + f(x) \,|\, x \in \mathcal{X}\},$$

if and only if

$$x^* \in \mathcal{X}, \quad \theta(x) - \theta(x^*) + (x - x^*)^T \nabla f(x^*) \geqslant 0, \quad \forall\, x \in \mathcal{X}.$$

Denote:

$$w = \begin{pmatrix} x \\ y \\ \lambda \end{pmatrix}, \; u = \begin{pmatrix} x \\ y \end{pmatrix}, \; F(w) := \begin{pmatrix} -A^T\lambda \\ -B^T\lambda \\ Ax + By - b \end{pmatrix},$$

$$\theta(u) = \theta_1(x) + \theta_2(y), \; \tilde{x}^k = x^{k+1}, \; \tilde{y}^k = y^{k+1},$$

$$\tilde{\lambda}^k = \lambda^k - \beta(Ax^{k+1} + By^k - b).$$

prediction step

Lemma 1

Let w^{k+1} be generated by the proximal symmetric ADMM. Then, we have

$$\tilde{w}^k \in \Omega, \ \ \theta(u)-\theta(\tilde{u}^k)+(w-\tilde{w}^k)^T F(\tilde{w}^k) \geqslant (w-\tilde{w}^k)^T Q(w^k-\tilde{w}^k), \ \ \forall w \in \Omega,$$

where

$$Q = \begin{pmatrix} R & 0 & 0 \\ 0 & \beta B^T B + T & -rB^T \\ 0 & -B & \frac{1}{\beta} I_m \end{pmatrix}.$$

correction step

Lemma 2

Let w^{k+1} be generated by the proximal symmetric ADMM. Then, we have

$$w^{k+1} = w^k - M(w^k - \tilde{w}^k),$$

where

$$M = \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & -s\beta B & (r+s)I_m \end{pmatrix}.$$

correction step

Lemma 2

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$$(w - \tilde{w}^k)^T Q(w^k - \tilde{w}^k) = (w - \tilde{w}^k)^T H(w^k - w^{k+1})$$

correction step

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$$(w - \tilde{w}^k)^T Q(w^k - \tilde{w}^k) = (w - \tilde{w}^k)^T H(w^k - w^{k+1})$$

where $H = QM^{-1}$

Main result

Theorem 3

For the sequence $\{w^k\}$ *generated by the proximal symmetric ADMM, we have*

$$\begin{aligned}&\theta(u) - \theta(\tilde{u}^k) + (w - \tilde{w}^k)^T F(w) \\ \geqslant\ & \frac{1}{2}\left(\|w - w^{k+1}\|_H^2 - \|w - w^k\|_H^2\right) + \frac{1}{2}\|w^k - \tilde{w}^k\|_G^2, \ \ \forall w \in \Omega,\end{aligned}$$

where $H = \begin{pmatrix} R & 0 & 0 \\ 0 & (1 - \frac{rs}{r+s})\beta B^T B + T & -\frac{r}{r+s} B^T \\ 0 & -\frac{r}{r+s} B & \frac{1}{(r+s)\beta} I_m \end{pmatrix}$ *and*

$$G = \begin{pmatrix} R & 0 & 0 \\ 0 & (1-s)\beta B^T B + T & -(1-s)B^T \\ 0 & -(1-s)B & \frac{1}{\beta}(2 - (r+s)) I_m \end{pmatrix}$$

Using optimality condition, we can obtain:

$$\|w^{k+1} - w^*\|_H^2 \leqslant \|w^k - w^*\|_H^2 - \|w^k - \tilde{w}^k\|_G^2, \quad \forall w^* \in \Omega^*$$

- H is a positive semidefinite matrix for $(s, r) \in \mathcal{D}$
- G is a positive semidefinite matrix if $s < 1$
- G may not be a positive semidefinite matrix if $s > 1$

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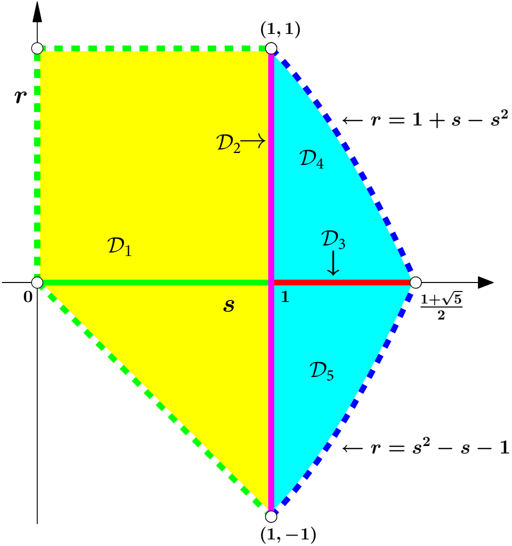
Convergence analysis

$$
\begin{cases}
\mathcal{D}_1 & = & \big\{(s,r)\,|\,s\in(0,1),\ r\in(-1,1),\ r+s>0\big\},\\
\mathcal{D}_2 & = & \big\{(s,r)\,|\,s=1,\ r\in(-1,1)\big\},\\
\mathcal{D}_3 & = & \big\{(s,r)\,|\,s\in\big(1,\frac{1+\sqrt{5}}{2}\big),\ r=0\big\},\\
\mathcal{D}_4 & = & \big\{(s,r)\,|\,s\in\big(1,\frac{1+\sqrt{5}}{2}\big),\ r\in(0,1),\ r<1+s-s^2\big\},\\
\mathcal{D}_5 & = & \big\{(s,r)\,|\,s\in\big(1,\frac{1+\sqrt{5}}{2}\big),\ r\in(-1,0),\ -r<1+s-s^2\big\}.
\end{cases}
$$



(1, 1)
r
$\leftarrow r = 1 + s - s^2$
$\mathcal{D}_2 \rightarrow$
$\mathcal{D}_4$
$\mathcal{D}_3$
$\mathcal{D}_1$
0
s
1
$\frac{1+\sqrt{5}}{2}$
$\mathcal{D}_5$
$\leftarrow r = s^2 - s - 1$
(1, −1)

Theorem 4

Let $\{w^k\}$ be generated by the proximal symmetric ADMM. Then, there exist constants C_i, $i = 0, 1, 2, 3$, such that

$$\begin{array}{rcl} \|w^k - \tilde{w}^k\|_G^2 & \geqslant & C_0\beta\big(\|Ax^{k+1} + By^{k+1} - b\|^2 - \|Ax^k + By^k - b\|^2\big) \\ & + & C_1\beta\|B(y^k - y^{k+1})\|^2 + C_2\beta\|Ax^{k+1} + By^{k+1} - b\|^2 \\ & + & C_3(\|y^k - y^{k+1}\|_T^2 - \|y^{k-1} - y^k\|_T^2), \end{array}$$

where

- $C_1, C_2 > 0$, $C_0 = C_3 = 0$ *if* $(r, s) \in \mathcal{D}_1$,
- $C_0 = 0$, C_1, C_2, $C_3 > 0$ *if* $(r, s) \in \mathcal{D}_2$, *and*
- $C_0, C_1, C_2, C_3 > 0$ *if* $(r, s) \in \mathcal{D}_3 \cup \mathcal{D}_4 \cup \mathcal{D}_5$.

Global convergence

Theorem 5

For the sequence $\{w^k\}$ *generated by the proximal symmetric ADMM, we have*

$$\lim_{k\to\infty}\left(\|B(y^k-y^{k+1})\|^2+\|Ax^{k+1}+By^{k+1}-b\|^2\right)=0.$$

Moreover, if the matrices A, B *are assumed to be full column rank, then the sequence* $\{w^k\}$ *converges to a solution point* $w^\infty\in\Omega^*$.

- stopping criterion:

$$\max\{\|B(y^k-y^{k+1})\|^2,\|Ax^{k+1}+By^{k+1}-b\|^2\}\leqslant\varepsilon$$

which $\varepsilon>0$ is the tolerance specified by the user.

Convergence rate

Worse-case $O(1/t)$ convergence rate in the ergodic sense:

Theorem 6

Let the sequence $\{w^k\}$ *be generated by the proximal symmetric ADMM. Then, for* $(s, r) \in \mathcal{D}$ *and any integer number* $t > 0$*, we have*

$$\begin{aligned}&\theta(\tilde{u}_t) - \theta(u) + (\tilde{w}_t - w)^T F(w)\\ \leqslant&\frac{1}{2t}\left(\|w - w^1\|_H^2 + C_0\|Ax^1 + By^1 - b\|^2 + C_3\|y^0 - y^1\|_T^2\right) \quad \forall w \in \Omega,\end{aligned}$$

where

$$\tilde{w}_t = \frac{1}{t}\sum_{k=1}^{t} \tilde{w}^k.$$

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Numerical result

Basis pursuit

$$\min\left\{\frac{1}{2\mu}\|Ax-b\|_2^2+\|y\|_1 | x-y=0, x,y\in\Re^n\right\}$$

- $A\in\Re^{m\times n}$: a random Gaussian matrix,
- x^*: original signal, has p nonzero elements whose positions are arranged randomly
- b: sampling signal with white noise

Numerical result

Basis pursuit

$$\min\left\{\frac{1}{2\mu}\|Ax-b\|_2^2+\|y\|_1|x-y=0,x,y\in\Re^n\right\}$$

- $A\in\Re^{m\times n}$: a random Gaussian matrix,
- x^*: original signal, has p nonzero elements whose positions are arranged randomly
- b: sampling signal with white noise

iterative scheme:

$$\begin{cases} x^{k+1}=(A^TA/\mu+\beta I+R)^{-1}(A^Tb/\mu+\lambda^k+\beta y^k+Rx^k),\\ \lambda^{k+\frac{1}{2}}=\lambda^k-r\beta(x^{k+1}-y^k),\\ y^{k+1}=\mathcal{S}_{\frac{1}{\beta}}(x^{k+1}-\lambda^{k+\frac{1}{2}}/\beta),\\ \lambda^{k+1}=\lambda^{k+\frac{1}{2}}-s\beta(x^{k+1}-y^{k+1}), \end{cases}$$

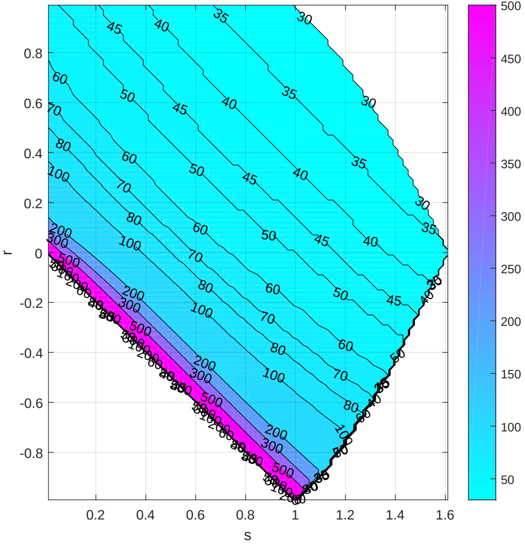

Figure: $R = \left(0.01\beta + \frac{1.01}{2}\lambda_{\max}(A^T A)\right) \cdot I - \frac{A^T A}{2}$

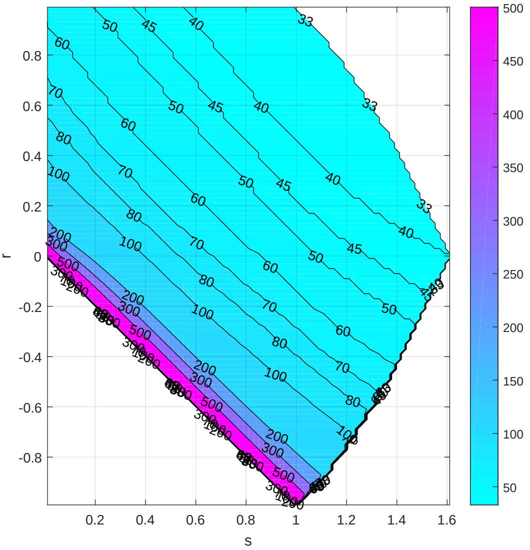

Figure: $R = \lambda_{\max}(A^T A) \cdot I - A^T A,\ T = 0$

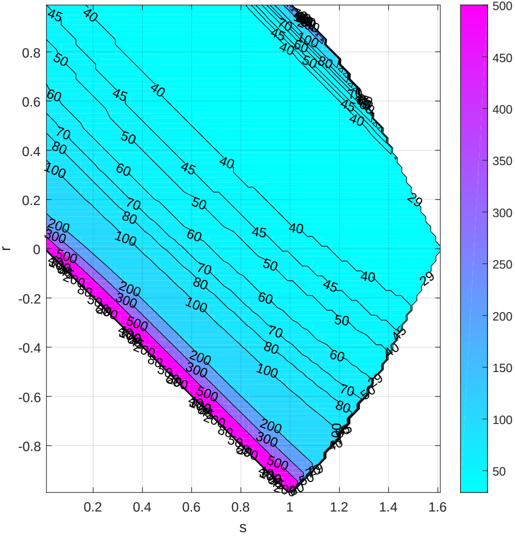
r
s
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
500
450
400
350
300
250
200
150
100
50

Figure: $R = 0, \ T = 0$

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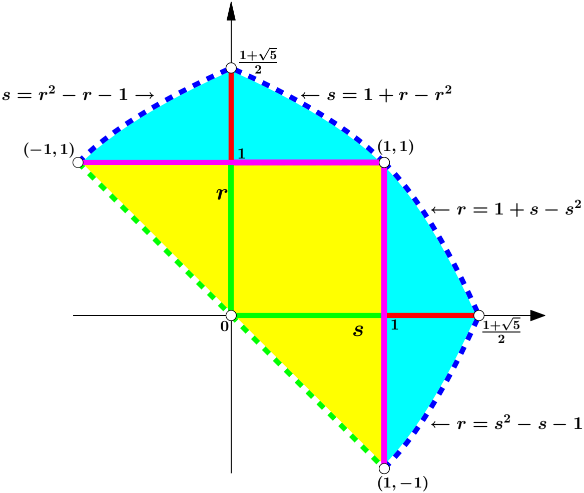
Extended domain

Noticing that the parameter r and s play the equal role in proximal symmetric ADMM scheme, clearly, the scheme of proximal symmetric ADMM can be written as:

$$\begin{cases} y^{k+1} = \arg\min\{\mathcal{L}_\beta(x^k, y, \lambda^k) + \frac{1}{2}\|y - y^k\|_T^2 \mid y \in \mathcal{Y}\}, \\ \lambda^{k+\frac{1}{2}} = \lambda^k - s\beta(Ax^k + By^{k+1} - b), \\ x^{k+1} = \arg\min\{\mathcal{L}_\beta(x, y^{k+1}, \lambda^{k+\frac{1}{2}}) + \frac{1}{2}\|x - x^k\|_R^2 \mid x \in \mathcal{X}\}, \\ \lambda^{k+1} = \lambda^{k+\frac{1}{2}} - r\beta(Ax^{k+1} + By^{k+1} - b), \end{cases}$$

The domain of the step sizes can be extended as:

$$\mathcal{D}^{\mathsf{sym}} = \{(r,s) \mid r+s > 0,\ |r| < 1+s-s^2\} \cup \{(r,s) \mid r+s > 0,\ |s| < 1+r-r^2\}$$


$\frac{1+\sqrt{5}}{2}$
$s = r^2 - r - 1 \rightarrow$
$\leftarrow s = 1 + r - r^2$
$(-1, 1)$
$(1, 1)$
1
r
$\leftarrow r = 1 + s - s^2$
0
s
1
$\frac{1+\sqrt{5}}{2}$
$\leftarrow r = s^2 - s - 1$
$(1, -1)$

Thank you!