Notes from Homework 4

Subset - $\subseteq$ - vs. Element - $\in$

$\{1\} \subseteq \{1, 2, 3, 4\}$

$\{1\} \nsubseteq \{\{1\}, 2, 3, 4\}$

For $\subseteq$, peel away a layer of the brackets on the left, and compare the two.

$\{1\} \notin \{1, 2, 3, 4\}$

$\{1\} \in \{\{1\}, 2, 3, 4\}$

$1 \in \{1, 2, 3, 4\}$

For $\in$, leave the brackets (if there are any brackets), and compare the two.

Remember that $\emptyset$ is a subset of every set.

Also remember that every set is a subset of itself, even $\emptyset$.

So, the power set, the set of all subsets, of $\emptyset$ is the singleton set $\{\emptyset\}$

Why? The only subset of $\emptyset$ is $\emptyset$. The set containing $\emptyset$ is then $\{\emptyset\}$.

The power set of $\{\emptyset\}$ would then be $\{\emptyset, \{\emptyset\}\}$

Quiz 5

1. Prove that $(q \longrightarrow (q \vee r)) \wedge p \wedge (p \longrightarrow q) \Longrightarrow q \vee r$ using the logical implication method.

Note that:

A
$A \longrightarrow B$
$\therefore$ B

by modus ponens

1) p (premise)
2) (p $\longrightarrow$ q) (premise)
3) q (modus ponens using 1 and 2)

Note that at this point, you're done, since we've proved that the RHS is true. But, continuing on anyway:

4) (q $\longrightarrow$ (q $\vee$ r)) (premise)
5) q $\vee$ r (modus ponens using 3 and 4)

2. Let A = **N**, the set of natural numbers = {0,1,2,3,...} Let B = **Z**, the set of integers = {...,-2,-1,0,1,2,...} Find

a) $A \cup B = B$

b) $A \cap B = A$

c) $B - A = \{..., -2, -1\}$

Homework 5

2. Determine whether f is a function from $\mathbf{Z}$ to $\mathbf{R}$ if

A function from set A to set B is an assignment of exactly one element from B to each element of A.

A is the domain, and B is the codomain.

a) $f(n) = + - n$.

This is not a function since a single input will give two resulting values. Does $f(3) = 3$ or -3?

b) $f(n) = \sqrt{n^2 + 1}$.

For all integers n, $f(n)$ gives a single, well-defined real value. So yes, its a function, and it does map $\mathbf{Z}$ to $\mathbf{R}$.

c) $f(n) = 1/(n^2 - 4)$.

This is a function, but not with domain $\mathbf{Z}$. It's undefined for $n = 2$ and $n = -2$. So, no.

14. Determine whether $f : \mathbf{Z}\ x\ \mathbf{Z} \longrightarrow \mathbf{Z}$ is onto if

For an onto function, the range and the codomain are the same - the function must map to all possible values in the codomain set.

Since the codomain is the set of integers **Z**), an onto function $f(m, n)$ must result in each and every integer. If there's a single integer that cannot result, the function is not onto.

a) $f(m, n) = 2m - n$.

$f(0, -n) = n$ for all integers n. So, it's onto.

b) $f(m, n) = m^2 - n^2$.

Not onto, since 2, for example, is not in the range.

c) $f(m, n) = m + n + 1$.

$f(0, n - 1) = n$ for all integers n. So, it's onto.

d) $f(m, n) = |m| - |n|$.

For all integers n, $f(0, n) =$ **-Z** and for all integers m, $f(m, 0) = +$**Z**. So, it's onto.

e) $f(m, n) = m^2 - 4$.

Not onto, since the range of f is a subset of **Z**.

16. Given an example of a function from $\mathbf{N}$ to $\mathbf{N}$ that is

a) one-to-one but not onto.

This function must map each natural number to a separate natural number (one-to-one),

but cannot map to every natural number (not onto).

$f(n) = n + 1$ (no 0).

b) onto but not one-to-one.

This function must map to all natural numbers (onto),

but the same natural number must result from at least two different input values (not one-to-one).

$f(n) = \lceil n/2 \rceil$.

c) both onto and one-to-one (don't use $f(n) = n$).

If n is even, $f(n) = n + 1$. If n is odd, $f(n) = n - 1$.

d) neither one-to-one nor onto.

$f(n) = 17$

28. Find fog and gof where $f(x) = x^2 + 1$ and $g(x) = x + 2$ are functions from $\mathbf{R}$ to $\mathbf{R}$.

$(fog)(x) = f(g(x)) = f(x + 2)$

$f(x + 2) = (x + 2)^2 + 1 = x^2 + 4x + 5$

$(gof)(x) = g(f(x)) = g(x^2)$

$g(x^2) = (x^2 + 1) + 2 = x^2 + 3$

Note that for these two functions, $fog \neq gof$.

30. Let $f(x) = ax + b$ and $g(x) = cx + d$ where a, b, c, and d are constants. Determine for which constants a, b, c, and d it is true that $fog = gof$.

$f(g(x)) = f(cx + d) = a(cx + d) + b = acx + ad + b$

$g(f(x) = g(ax + b) = c(ax + b) + d = acx + cb + d$

$fog = gof$ if $ad + b = cb + d$.

Equality holds for all combinations of (a,b,c,d) where $ad + b = cb + d$.

34. Let f be the function from $\mathbf{R}$ to $\mathbf{R}$ defined by $f(x) = x^2$. Find

a) $f^{-1}(\{1\})$.

What is the set of real number(s) that, when squared, equal 1?

$\{-1, 1\}$.

b) $f^{-1}(\{x|0 < x < 1\})$.

What is the set of real number(s) that, when squared, are between 0 and 1?

Anything between -1 and 1, besides 0.

$\{x| - 1 < x < 0 \vee 0 < x < 1\}$

c) $f^{-1}(\{x|x > 4\})$.

What is the set of real number(s) that, when squared, are greater than 4?

Anything larger than 2 or smaller than -2.

$\{x|x > 2 \vee x < -2\}$.

42. $x \in \mathbf{R}$
$m \in \mathbf{Z}$. Show that
$\lceil x + m \rceil = \lceil x \rceil + m$.

Directly:

1) As a real number, x can be deconstructed as

$x = n - \varepsilon$ where

n is an integer and

$0 \leq \varepsilon < 1$.

2) The result of $\lceil x \rceil$ will be n.

3) Then $\lceil x + m \rceil = \lceil n - \varepsilon + m \rceil = n + m$.

4) By 2, this equals $\lceil x \rceil + m$.

Or you could simply reference property (4b) in Table 1 on page 107.

66. Prove or disprove each of these statements about the floor and ceiling functions.

a) $\lfloor \lceil x \rceil \rfloor = \lceil x \rceil$ for all real numbers x.

$\lceil x \rceil$ is an integer, so this is true.

b) $\lfloor x + y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor$ for all real numbers x and y.

Disproved by counterexample: $x = y = 3/4$.

$1 \neq 0$

c) $\lceil \lceil x/2 \rceil /2 \rceil = \lceil x/4 \rceil$ for all real numbers x.

This is true.

Direct proof by parts:

Let x be defined as $4n + k$ where n is an integer and $0 \leq k < 4$.

Why? Since we divide by four on the right side, we'll define x in terms of its "multiple of fourness". So if $k = 0$, x is a multiple of 4.

The first part: If $k = 0$, both sides equal n.

The second part: If $0 < k \leq 2$.

$\lceil x/2 \rceil = 2n+1$, so the LHS is $\lceil n+1/2 \rceil = n+1$. And the RHS is $n+1$ as well.

The third part: If $2 < k < 4$.

$\lceil x/2 \rceil = 2n+2$, so the LHS is $\lceil n+1 \rceil = n+1$. And the RHS is $n+1$ as well.

Proved for all cases.

d) $\lfloor\sqrt{\lceil x\rceil}\rfloor = \lfloor\sqrt{x}\rfloor$ for all real numbers x.

Disproved by counterexample: $x = 8.5$

$3 \neq 2$

e) $\lfloor x\rfloor + \lfloor y\rfloor + \lfloor x+y\rfloor \leq \lfloor 2x\rfloor + \lfloor 2y\rfloor$ for all real numbers x and y.

This is true.

Write $x = n + \epsilon$ and $y = m + \delta$, where n and m are integers, and ϵ and δ are nonnegative real numbers less than 1.

Possibilities for the LHS:

$n + m + (n + m) = 2n + 2m$

or

$n + m + (n + m + 1) = 2n + 2m + 1$ if $\epsilon + \delta \geq 1$

Possibilities for the RHS:

$\lceil 2x\rceil$ becomes $2n$ if $\epsilon < 1/2$ or $2n + 1$ otherwise.

$\lceil 2y\rceil$ becomes $2m$ if $\delta < 1/2$ or $2m + 1$ otherwise.

Now: we are trying to prove that the LHS is always less than or equal to the RHS. Let's construct a proof by contradiction and assume the LHS is greater than the RHS.

So the LHS must be $2n + 2m + 1$ and the RHS must be $2n + 2m$.

But then $\epsilon + \delta \geq 1$, so at least one of them must be equal to or greater than $1/2$.

So the RHS cannot equal $2n + 2m$.

Yay!