Numerical methods of conservation law equation

有限差分法求解二维涡量流函数方程

Outline

I. 从N-S方程推导涡量流函数方程

II. 物理问题描述

III. 离散形式以及代数方程组

IV. 数值结果与文献结果的比较

V. Fortran codes

Vorticity—Stream function form

(I) 涡量流函数方程的推导

无量纲N-S方程：

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{\partial p}{\partial x}+\frac{1}{Re}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial x^2}\right) \quad (1)$$

$$\frac{\partial v}{\partial v}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{\partial p}{\partial y}+\frac{1}{Re}\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial x^2}\right) \quad (2)$$

将方程(1,2)分别对y, x求偏导：

$$\frac{\partial}{\partial y}\left\{\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{\partial p}{\partial x}+\frac{1}{Re}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial x^2}\right)\right\}$$

$$\frac{\partial}{\partial x}\left\{\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{\partial p}{\partial x}+\frac{1}{Re}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial x^2}\right)\right\}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

$$\omega=\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}$$

$$u=\frac{\partial \varphi}{\partial y}, v=-\frac{\partial \varphi}{\partial x}$$

涡量流函数的方程(非稳态对流扩散方程+泊松方程):

$$\frac{\partial^2 \varphi}{\partial x^2}+\frac{\partial^2 \varphi}{\partial y^2}=-\omega$$

$$\frac{\partial \omega}{\partial t}+\frac{\partial \varphi}{\partial y}\frac{\partial \omega}{\partial x}-\frac{\partial \varphi}{\partial x}\frac{\partial \omega}{\partial y}=\frac{1}{Re}\left(\frac{\partial^2 \omega}{\partial x^2}+\frac{\partial^2 \omega}{\partial x^2}\right)$$

非稳态对流扩散方程:

$$\frac{\partial f}{\partial t}+u\frac{\partial f}{\partial x}+v\frac{\partial f}{\partial y}=D\frac{\partial^2 f}{\partial x^2}+D\frac{\partial^2 f}{\partial y^2}$$

Vorticity—Stream function form

(I) 涡量流函数方程的推导

$$\frac{\partial f}{\partial t}+u\frac{\partial f}{\partial x}+v\frac{\partial f}{\partial y}=D\frac{\partial^2 f}{\partial x^2}+D\frac{\partial^2 f}{\partial y^2}$$

$$\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}=-\omega$$

$$\frac{\partial\omega}{\partial t}+\frac{\partial\varphi}{\partial y}\frac{\partial\omega}{\partial x}-\frac{\partial\varphi}{\partial x}\frac{\partial\omega}{\partial y}=\frac{1}{Re}\left(\frac{\partial^2\omega}{\partial x^2}+\frac{\partial^2\omega}{\partial x^2}\right)$$

本教程采用这个方程，不保留速度

$$\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}=-\omega$$

$$\frac{\partial\omega}{\partial t}+u\frac{\partial\omega}{\partial x}+v\frac{\partial\omega}{\partial y}=\frac{1}{Re}\left(\frac{\partial^2\omega}{\partial x^2}+\frac{\partial^2\omega}{\partial x^2}\right)$$

$$u=\frac{\partial\varphi}{\partial y},v=-\frac{\partial\varphi}{\partial x}$$

非守恒型的流函数方程
在关于涡量的对流扩散方程中保留速度，只要在求完流函数的泊松方程之后，通过流函数的定义来求得速度即可

$$\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}=-\omega$$

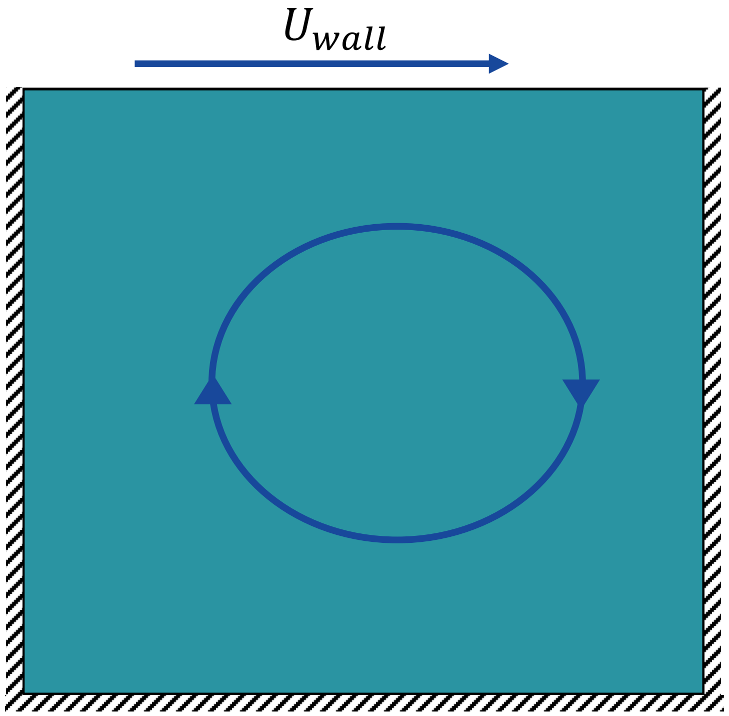
$$\frac{\partial\omega}{\partial t}+\frac{\partial(u\omega)}{\partial x}+\frac{\partial(v\omega)}{\partial y}=\frac{1}{Re}\left(\frac{\partial^2\omega}{\partial x^2}+\frac{\partial^2\omega}{\partial x^2}\right)$$

$$u=\frac{\partial\varphi}{\partial y},v=-\frac{\partial\varphi}{\partial x}$$

守恒型的流函数方程

Vorticity—Stream function form

(II) 物理问题描述以及边界条件: (1) 顶盖驱动流



左右壁面流函数满足： $u = \dfrac{\partial \varphi}{\partial y} = 0$ $\Rightarrow$ $\varphi = const$

上下壁面流函数满足： $v = -\dfrac{\partial \varphi}{\partial x} = 0$ $\Rightarrow$ $\varphi = const$

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = -\omega$$

左右壁面涡量边界条件：

$$\omega = \frac{\partial v}{\partial x} - \cancel{\frac{\partial u}{\partial y}} = -\frac{\partial^2 \varphi}{\partial x^2} - \cancel{\frac{\partial^2 \varphi}{\partial y^2}} \Rightarrow \omega_{wall} = -\frac{\partial^2 \varphi}{\partial x^2}$$

上下壁面涡量边界条件：

涡量来源于流场存在速度梯度，平板边界层内，速度梯度就是旋涡！！！

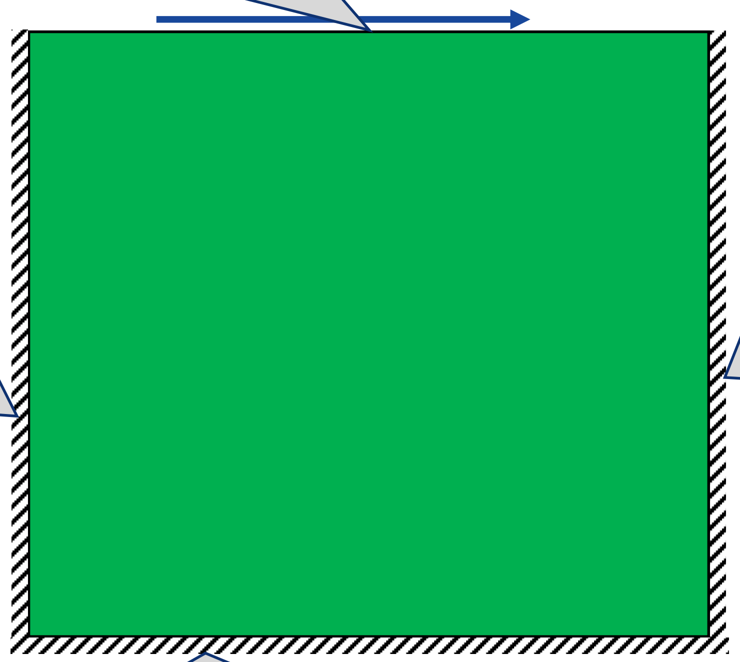
《涡运动理论》

$$\omega = \cancel{\frac{\partial v}{\partial x}} - \frac{\partial u}{\partial y} = -\cancel{\frac{\partial^2 \varphi}{\partial x^2}} - \frac{\partial^2 \varphi}{\partial y^2} \Rightarrow \omega_{wall} = -\frac{\partial^2 \varphi}{\partial y^2}$$

Vorticity—Stream function form

(II) 物理问题描述以及边界条件: (1) 顶盖驱动流

$$\frac{\partial \varphi}{\partial y} = U_{wall}, \omega_{wall} = -\frac{\partial^2 \varphi}{\partial y^2}$$



$$\frac{\partial \varphi}{\partial x} = 0$$

$$\omega_{wall} = -\frac{\partial^2 \varphi}{\partial x^2}$$

$$\frac{\partial \varphi}{\partial x} = 0$$

$$\omega_{wall} = -\frac{\partial^2 \varphi}{\partial x^2}$$

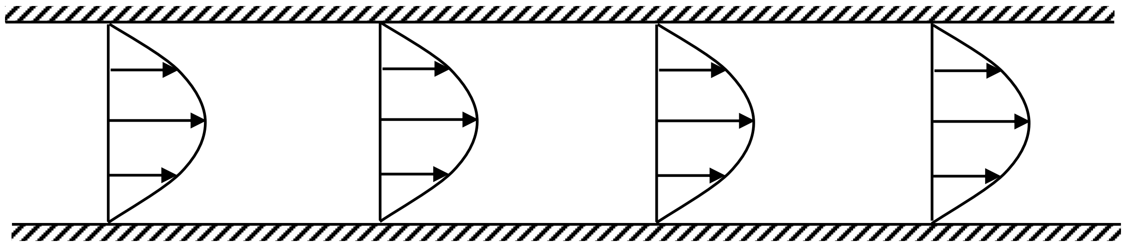
$$\frac{\partial \varphi}{\partial y} = 0, \omega_{wall} = -\frac{\partial^2 \varphi}{\partial y^2}$$

Vorticity—Stream function form

(II) 物理问题描述以及边界条件: (2) 管道泊肃叶流动

入口处均匀来流： $U_{inlet} = const$

出口处充分发展： $\dfrac{\partial U_{outlet}}{\partial x} = 0$



Fully-developed laminar duct flow

Vorticity—Stream function form

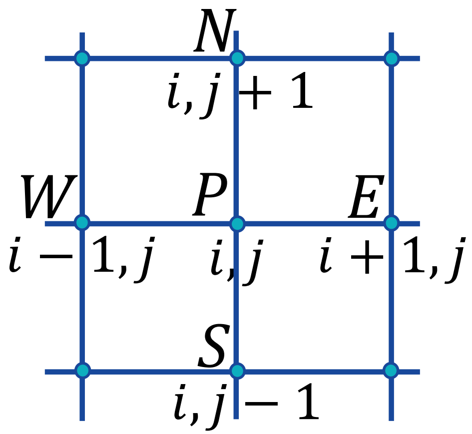
(III) 离散形式

泊松方程：迭代求解

非稳态对流扩散方程：龙格库塔显格式时间推进

$$\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}=-\omega$$

$$\frac{\partial\omega}{\partial t}+\frac{\partial\varphi}{\partial y}\frac{\partial\omega}{\partial x}-\frac{\partial\varphi}{\partial x}\frac{\partial\omega}{\partial y}=\frac{1}{Re}\left(\frac{\partial^2\omega}{\partial x^2}+\frac{\partial^2\omega}{\partial x^2}\right)$$



结构网格记号

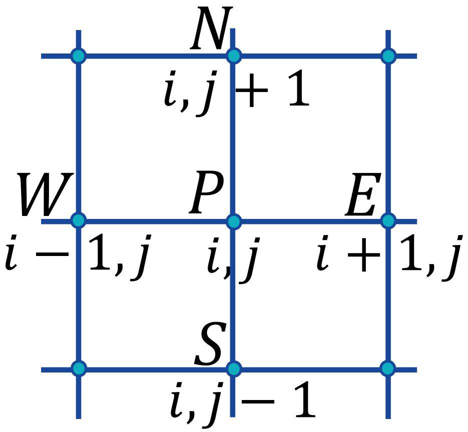
中心差分离散对流项和扩散项

时间推进采用具有TVD性质的三阶龙格库塔格式(RK-3)

Vorticity—Stream function form

(III) 泊松方程在网格点(i,j)处的离散形式(中心差分)

$$\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}=-\omega$$



网格记号

1. 二阶导数项的一般离散形式(适用于非均匀网格):

$$\left(\frac{\partial^2\varphi}{\partial x^2}\right)_{i,j}=\frac{\partial}{\partial x}\left(\frac{\partial\varphi}{\partial x}\right)_{i,j}=\frac{\left(\frac{\partial\varphi}{\partial x}\right)_{i+\frac{1}{2},j}-\left(\frac{\partial\varphi}{\partial x}\right)_{i-\frac{1}{2},j}}{\frac{1}{2}\left(x_{i+1}-x_{i-1}\right)}=\frac{\left(\frac{\varphi_{i+1,j}-\varphi_{i,j}}{x_{i+1}-x_i}\right)-\left(\frac{\varphi_{i,j}-\varphi_{i-1,j}}{x_i-x_{i-1}}\right)}{\frac{1}{2}\left(x_{i+1}-x_{i-1}\right)}$$

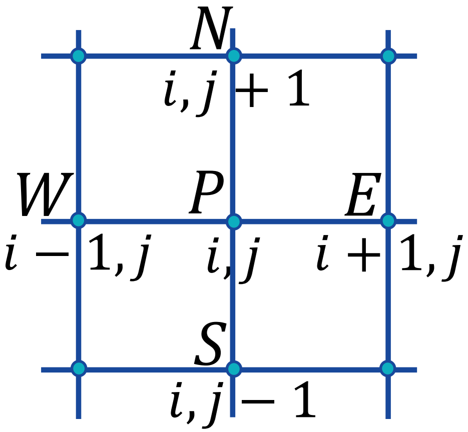
2. 泊松方程的一般离散形式(适用于非均匀网格):

$$\frac{\left(\frac{\varphi^n_{i+1,j}-\varphi^n_{i,j}}{x_{i+1}-x_i}\right)-\left(\frac{\varphi^n_{i,j}-\varphi^n_{i-1,j}}{x_i-x_{i-1}}\right)}{\frac{1}{2}\left(x_{i+1}-x_{i-1}\right)}+\frac{\left(\frac{\varphi^n_{i,j+1}-\varphi^n_{i,j}}{y_{j+1}-y_j}\right)-\left(\frac{\varphi^n_{i,j}-\varphi^n_{i,j-1}}{y_j-y_{j-1}}\right)}{\frac{1}{2}\left(y_{j+1}-y_{j-1}\right)}=-\omega^n_{i,j}$$

Vorticity—Stream function form

(III) 泊松方程在网格点(i,j)处的离散形式(中心差分)

$$\frac{\left(\dfrac{\varphi_{i+1,j}^{n}-\varphi_{i,j}^{n}}{x_{i+1}-x_{i}}\right)-\left(\dfrac{\varphi_{i,j}^{n}-\varphi_{i-1,j}^{n}}{x_{i}-x_{i-1}}\right)}{\frac{1}{2}\left(x_{i+1}-x_{i-1}\right)}+\frac{\left(\dfrac{\varphi_{i,j+1}^{n}-\varphi_{i,j}^{n}}{y_{j+1}-y_{j}}\right)-\left(\dfrac{\varphi_{i,j}^{n}-\varphi_{i,j-1}^{n}}{y_{j}-y_{j-1}}\right)}{\frac{1}{2}\left(y_{j+1}-y_{j-1}\right)}=-\omega_{i,j}^{n}$$



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$$\boldsymbol{AP*\varphi_P+AE*\varphi_E+AW*\varphi_W+AN*\varphi_N+AS*\varphi_S=Q}$$
$$\boldsymbol{AP=-(AE+AW+AN+AS)}$$

$$AE=\frac{2}{\left(x_{i+1}-x_{i-1}\right)\left(x_{i+1}-x_{i}\right)}$$

$$AW=\frac{2}{\left(x_{i+1}-x_{i-1}\right)\left(x_{i}-x_{i-1}\right)}$$

$$AN=\frac{2}{\left(y_{j+1}-y_{j-1}\right)\left(y_{j+1}-y_{j}\right)}$$

$$AS=\frac{2}{\left(y_{j+1}-y_{j-1}\right)\left(y_{j}-y_{j-1}\right)}$$

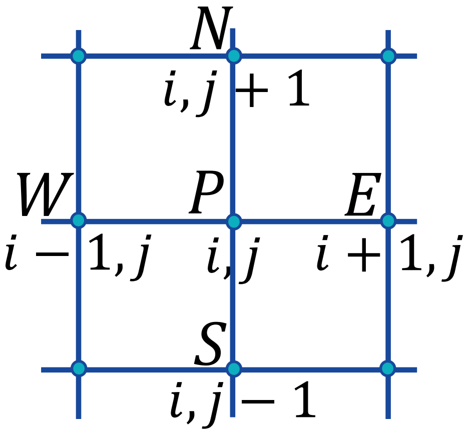
$$Q=-\omega_{i,j}^{n}$$
$$AP=-\left(AE+AW+AN+AS\right)$$

$$\phi_P^{n+1}=\frac{\beta}{AP}\left\{Q^n-AE\cdot\phi_E^n-AW\cdot\phi_W^{n+1}-AN\cdot\phi_N^n-AS\cdot\phi_S^{n+1}\right\}+\left(1-\beta\right)\phi_P^n$$

Vorticity—Stream function form

(III) 涡量方程的离散形式(中心差分，显格式时间推进)

$$\frac{\partial\omega}{\partial t}+\frac{\partial\varphi}{\partial y}\frac{\partial\omega}{\partial x}-\frac{\partial\varphi}{\partial x}\frac{\partial\omega}{\partial y}=\frac{1}{Re}\left(\frac{\partial^2\omega}{\partial x^2}+\frac{\partial^2\omega}{\partial x^2}\right)$$



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3阶显式R-K格式：

$$\frac{\partial f}{\partial t}=L(f)$$

$$f^{(1)}=f^{n}+\Delta tL(f^{n})$$

$$f^{(2)}=\frac{3}{4}f^{n}+\frac{1}{4}\left[f^{(1)}+\Delta tL(f^{(1)})\right]$$

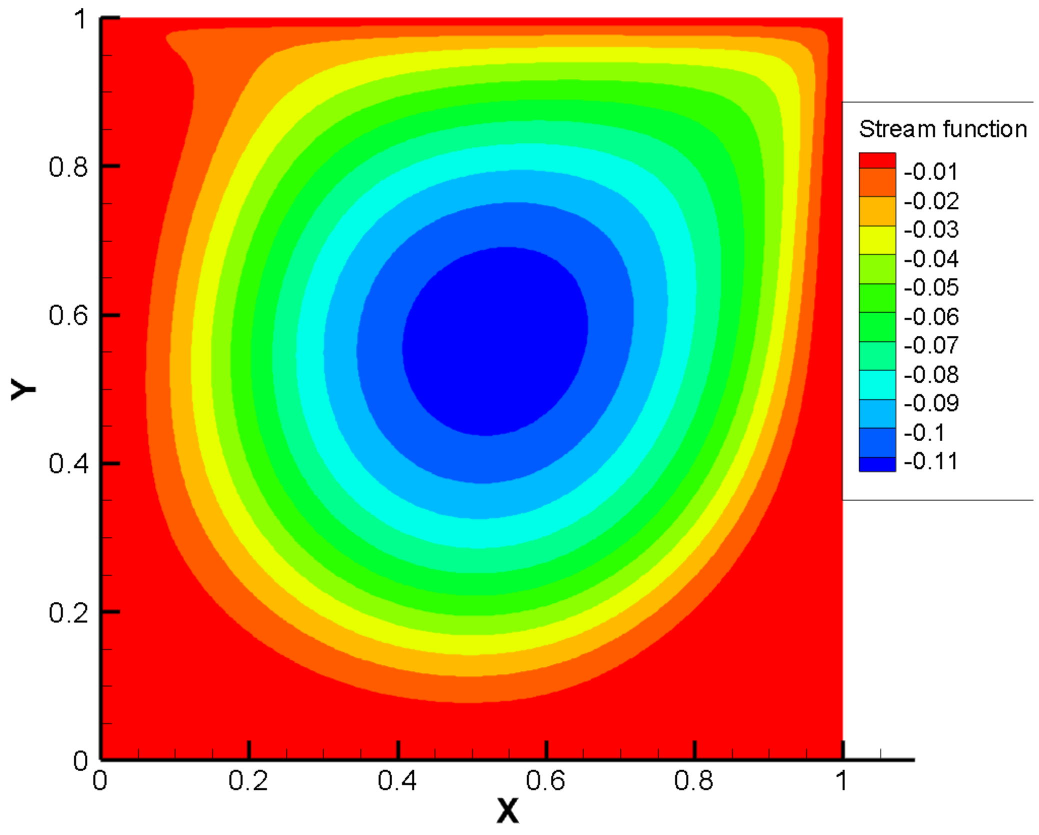
$$f^{n+1}=\frac{1}{3}f^{n}+\frac{2}{3}\left[f^{(2)}+\Delta tL(f^{(2)})\right]$$

RK-3, Step-1对应的离散方程(适用于非均匀网格):

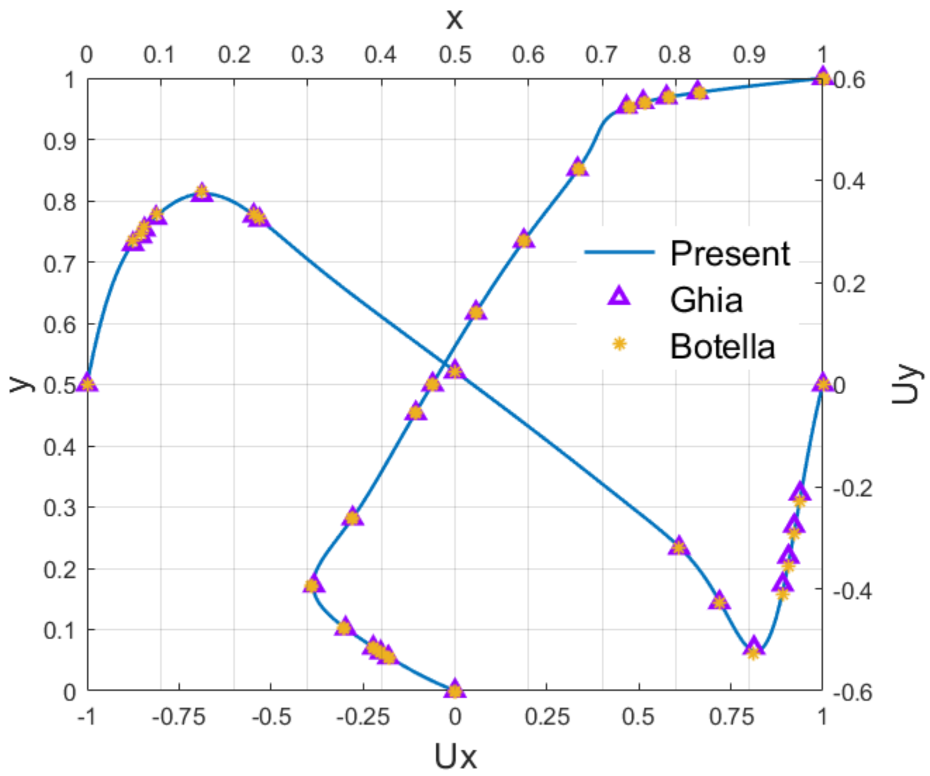
$$\frac{\omega^{*}_{i,j}-\omega^{n}_{i,j}}{\Delta t}+\left(\frac{\varphi^{n}_{i,j+1}-\varphi^{n}_{i,j-1}}{y_{j+1}-y_{j-1}}\right)\left(\frac{\omega^{n}_{i+1,j}-\omega^{n}_{i-1,j}}{x_{i+1}-x_{i-1}}\right)-\left(\frac{\varphi^{n}_{i+1,j}-\varphi^{n}_{i-1,j}}{x_{i+1}-x_{i-1}}\right)\left(\frac{\omega^{n}_{i,j+1}-\omega^{n}_{i,j-1}}{y_{j+1}-y_{j-1}}\right)$$

$$=\frac{1}{\mathrm{Re}}\left\{\frac{\left(\frac{\omega^{n}_{i+1,j}-\omega^{n}_{i,j}}{x_{i+1}-x_{i}}\right)-\left(\frac{\omega^{n}_{i,j}-\omega^{n}_{i-1,j}}{x_{i}-x_{i-1}}\right)}{\frac{1}{2}\left(x_{i+1}-x_{i-1}\right)}+\frac{\left(\frac{\omega^{n}_{i,j+1}-\omega^{n}_{i,j}}{y_{j+1}-y_{j}}\right)-\left(\frac{\omega^{n}_{i,j}-\omega^{n}_{i,j-1}}{y_{j}-y_{j-1}}\right)}{\frac{1}{2}\left(y_{j+1}-y_{j-1}\right)}\right\}$$

Vorticity—Stream function form



数值解



计算结果与文献结果的对比,Re=1000