Numerical methods of conservation law equation

有限差分法求解二维常系数非稳态对流扩散方程

Outline

I. 二维常(变)系数非稳态对流扩散方程

II. 问题描述

III. 离散形式以及代数方程组

IV. 数值解与解析解的比较

V. Fortran codes

Two-dimensional model equation

(I) The Advection-Diffusion Equation (最基本的模型方程)

$$\frac{\partial f}{\partial t} + U\frac{\partial f}{\partial x} + V\frac{\partial f}{\partial y} = D\frac{\partial^2 f}{\partial x^2} + D\frac{\partial^2 f}{\partial y^2}$$

非稳态项 → 时间项的离散

对流项 → 对流项的离散

扩散项 → 扩散项的离散

物理方程

FTCS
RK-2
RK-3
CNCS

有限差分的离散过程：
基于泰勒展开，
用差商代替微商

代数方程组 → **编程求解**

Two-dimensional model equation

(II) 问题1:二维常系数非稳态对流扩散方程

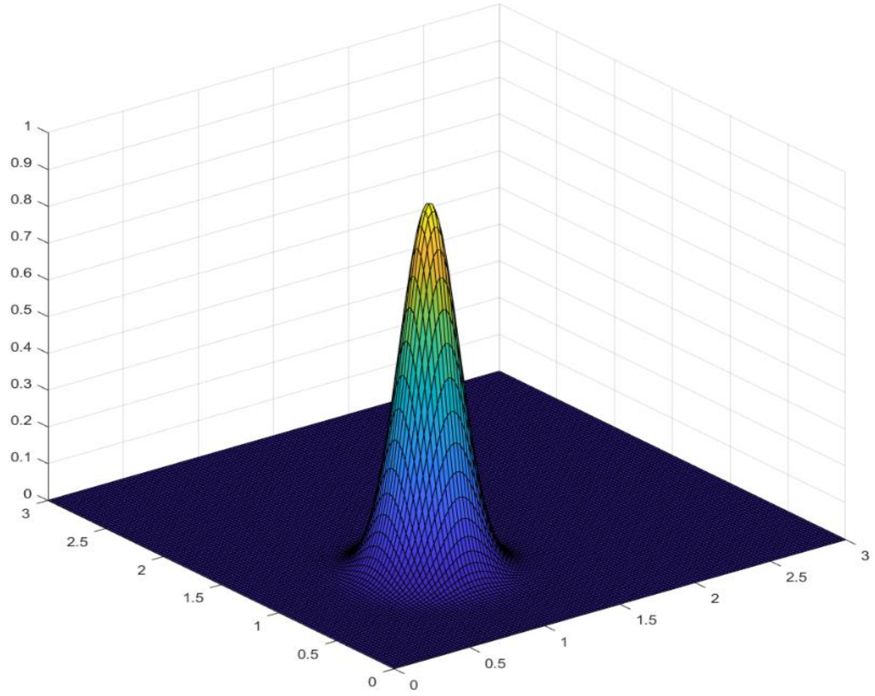
$$\frac{\partial f}{\partial t}+U\frac{\partial f}{\partial x}+V\frac{\partial f}{\partial y}=D\left(\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}\right)$$

对于二维常系数非稳态线性对流扩散方程，一个特解如下：

$$f\left(x,y,t\right)=\frac{1}{4t+1}\exp\left(-\frac{\left(x-Ut-1\right)^2+\left(y-Vt-1\right)^2}{D\left(4t+1\right)}\right)$$

当$U=V=1.0, D=0.05, x\in[0,9], y\in[0,9]$时，求该二维常系数非稳态对流扩散方程的数值解，并与$t=2.5$时的解析解进行对比

初始条件和边界条件均由解析表达式给定



$f(x,y,t=0)$的函数图像

Two-dimensional model equation

(II) 问题2:二维变系数非稳态对流扩散方程

对于二维变系数非稳态线性对流扩散方程，当U,V给定如下时

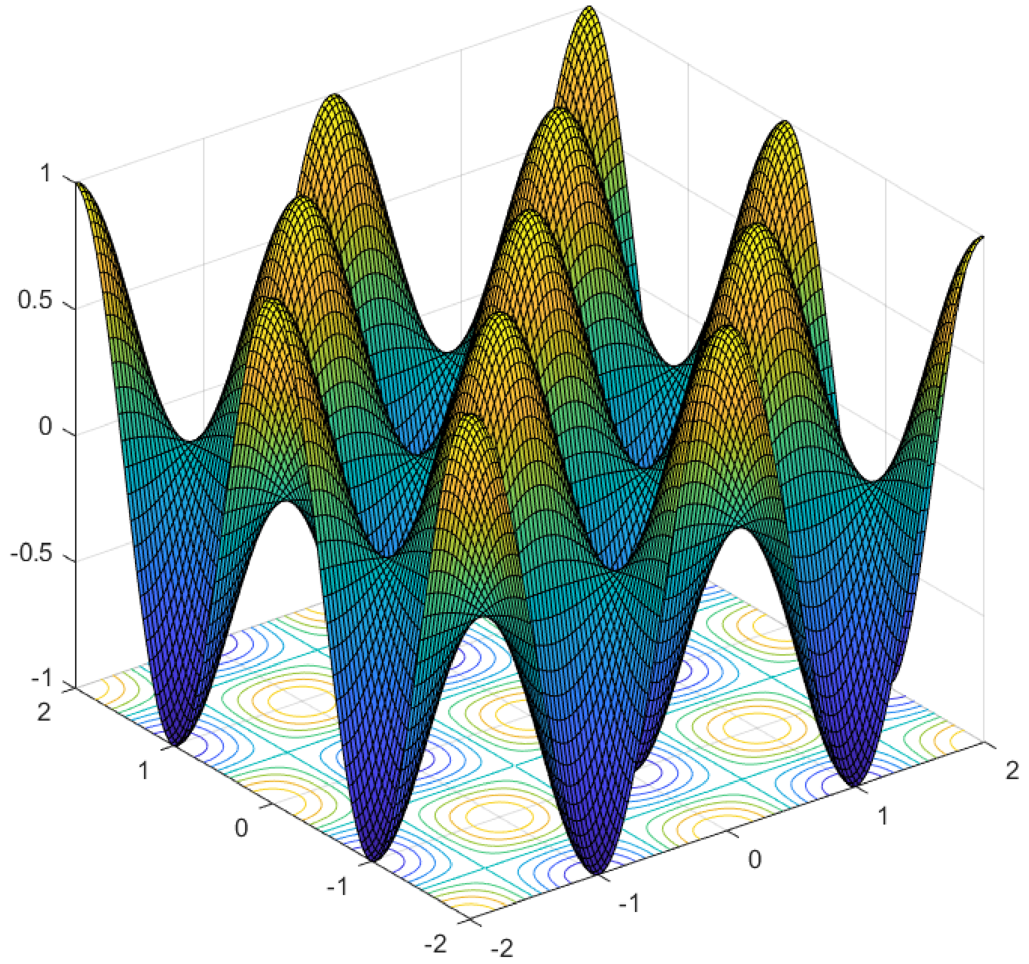
$$U = -\pi\cos(\pi x)\sin(\pi y)\exp(-2D\pi^2 t)$$

$$V = \pi\sin(\pi x)\cos(\pi y)\exp(-2D\pi^2 t)$$

其解析解为

$$f(x,y,t) = \cos(\pi x)\cos(\pi y)\exp(-2D\pi^2 t)$$

当$x \in [-2,2], y \in [-2,2], D = 0.001$时，求该二维变系数非稳态对流扩散方程的数值解，并与$t = 2.5$时的解析解进行对比，初始条件和边界条件均由解析表达式给定



$f(x,y,t=0)$的函数图像

Two-dimensional model equation

(II) 问题2:二维变系数非稳态对流扩散方程

非守恒形式：

$$\frac{\partial f}{\partial t} + U\frac{\partial f}{\partial x} + V\frac{\partial f}{\partial y} = D\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right)$$

为什么要写成守恒形式？

数学上等价的表达式在离散之后不等价！

守恒形式：

$$\frac{\partial f}{\partial t} + \frac{\partial (Uf)}{\partial x} + \frac{\partial (Vf)}{\partial y} = D\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right)$$

为什么可以写成守恒形式？

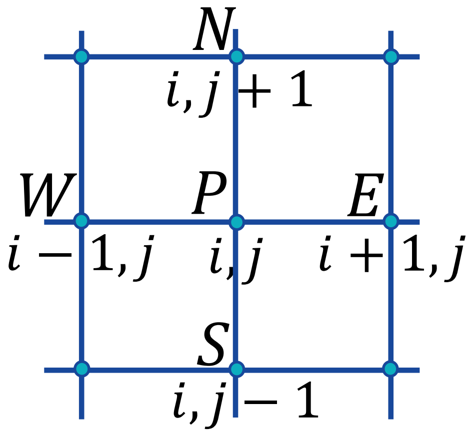
$$U = -\pi\cos(\pi x)\sin(\pi y)\exp(-2D\pi^2 t)$$

$$V = \pi\sin(\pi x)\cos(\pi y)\exp(-2D\pi^2 t)$$

$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0$ (无散度条件)

Two-dimensional model equation

(II) 问题2:二维变系数非稳态对流扩散方程



$$\frac{\partial f}{\partial t}+\frac{\partial (uf)}{\partial x}+\frac{\partial (vf)}{\partial y}=D\left(\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}\right)$$

FTCS格式对应的离散方程:

$$\frac{f_{i,j}^{n+1}-f_{i,j}^{n}}{\Delta t}+\frac{u_{i+1,j}^{n}f_{i+1,j}^{n}-u_{i-1,j}^{n}f_{i-1,j}^{n}}{2\cdot\Delta x}+\frac{v_{i,j+1}^{n}f_{i,j+1}^{n}-v_{i,j-1}^{n}f_{i,j-1}^{n}}{2\cdot\Delta y}=D\left\{\frac{f_{i+1,j}^{n}-2f_{i,j}^{n}+f_{i-1,j}^{n}}{\Delta x^2}+\frac{f_{i,j-1}^{n}-2f_{i,j}^{n}+f_{i,j-1}^{n}}{\Delta y^2}\right\}$$

1阶显式FTCS格式：

$$\frac{\partial f}{\partial t}=L(f)$$
$$f^{(n+1)}=f^{n}+\Delta t L(f^{n})$$

2阶显式R-K格式：

$$\frac{\partial f}{\partial t}=L(f)$$
$$f^{(1)}=f^{n}+\Delta t L(f^{n})$$
$$f^{(2)}=f^{(1)}+\Delta t L(f^{(1)})$$
$$f^{n+1}=\frac{1}{2}\left(f^{(n)}+f^{(2)}\right)$$

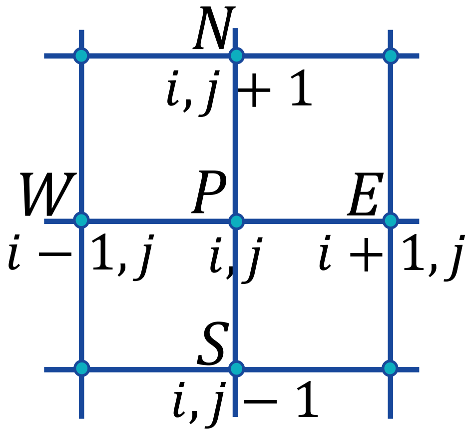
3阶显式R-K格式：

$$\frac{\partial f}{\partial t}=L(f)$$
$$f^{(1)}=f^{n}+\Delta t L(f^{n})$$
$$f^{(2)}=3/4\,f^{n}+1/4[f^{(1)}+\Delta t L(f^{(1)})]$$
$$f^{n+1}=1/3\,f^{n}+2/3[f^{(2)}+\Delta t L(f^{(2)})]$$

Two-dimensional model equation

(II) 问题2:二维变系数非稳态对流扩散方程

$$\frac{\partial f}{\partial t}+\frac{\partial (uf)}{\partial x}+\frac{\partial (vf)}{\partial y}=D\left(\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}\right)$$



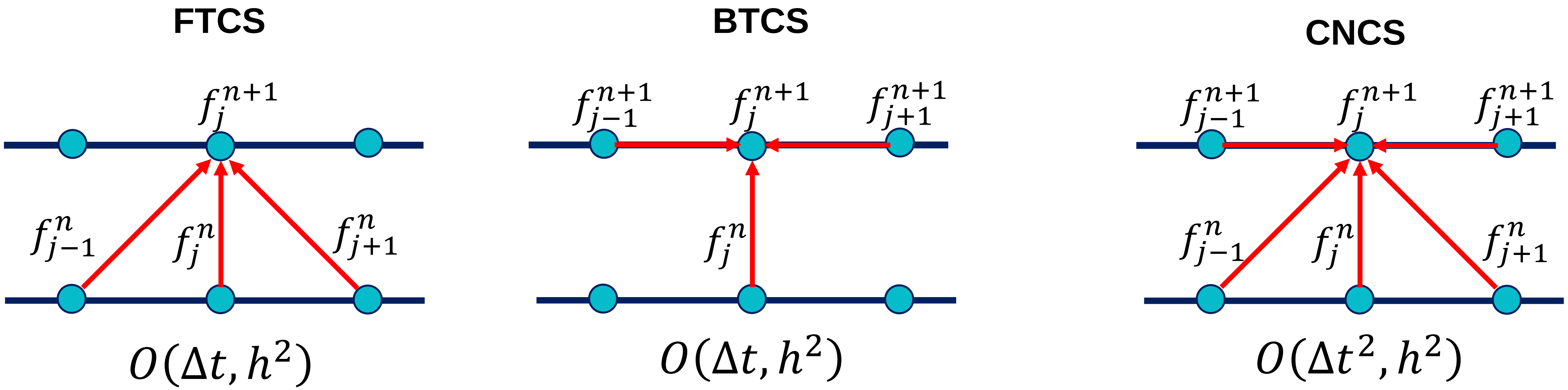
u^{n+1}, v^{n+1} 线性化成 u^{n}, v^{n}

CN格式对应的离散方程:

$$\frac{f_{i,j}^{n+1}-f_{i,j}^{n}}{\Delta t}+\frac{u_{i+1,j}^{n+1}f_{i+1,j}^{n+1}-u_{i-1,j}^{n+1}f_{i-1,j}^{n+1}}{4\cdot\Delta x}+\frac{v_{i,j+1}^{n+1}f_{i,j+1}^{n+1}-v_{i,j-1}^{n+1}f_{i,j-1}^{n+1}}{4\cdot\Delta y}-D\left\{\frac{f_{i+1,j}^{n+1}-2f_{i,j}^{n+1}+f_{i-1,j}^{n+1}}{\Delta x^2}+\frac{f_{i,j-1}^{n+1}-2f_{i,j}^{n+1}+f_{i,j-1}^{n+1}}{\Delta y^2}\right\}$$

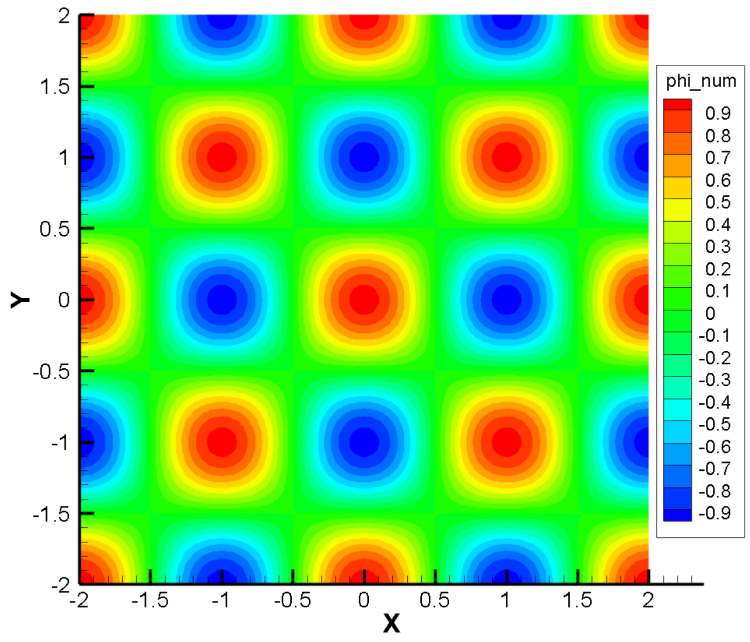
$$=-\frac{u_{i+1,j}^{n}f_{i+1,j}^{n}-u_{i-1,j}^{n}f_{i-1,j}^{n}}{4\cdot\Delta x}+\frac{v_{i,j+1}^{n}f_{i,j+1}^{n}-v_{i,j-1}^{n}f_{i,j-1}^{n}}{4\cdot\Delta y}+D\left\{\frac{f_{i+1,j}^{n}-2f_{i,j}^{n}+f_{i-1,j}^{n}}{\Delta x^2}+\frac{f_{i,j-1}^{n}-2f_{i,j}^{n}+f_{i,j-1}^{n}}{\Delta y^2}\right\}$$

Two-dimensional model equation

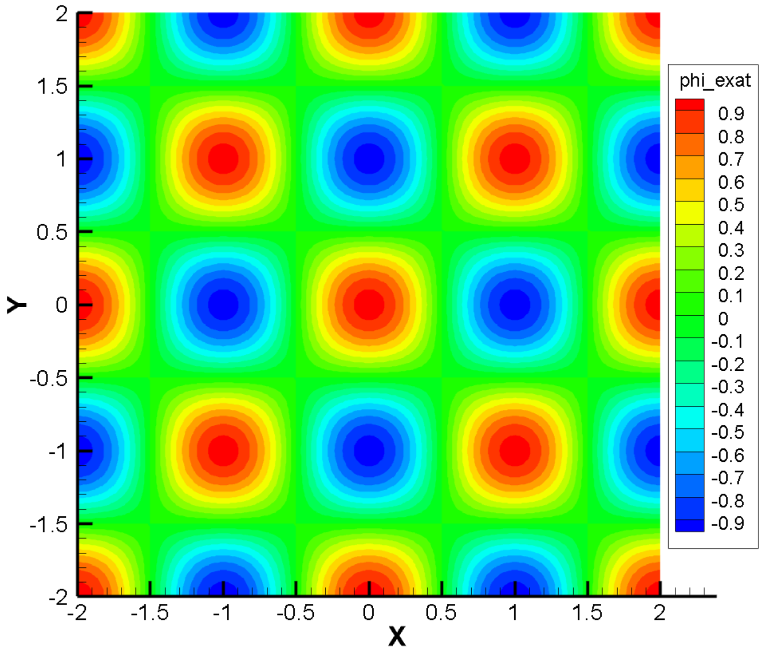


时空离散示意图

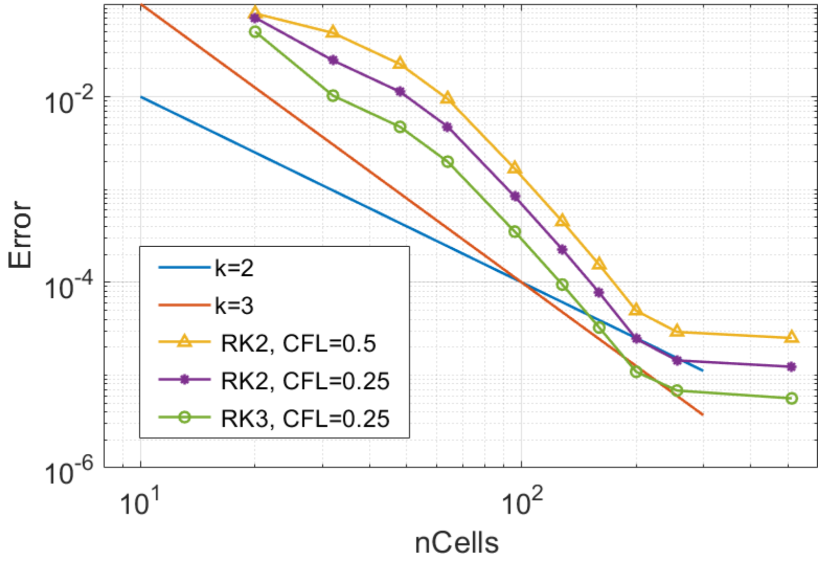
Two-dimensional model equation



数值解

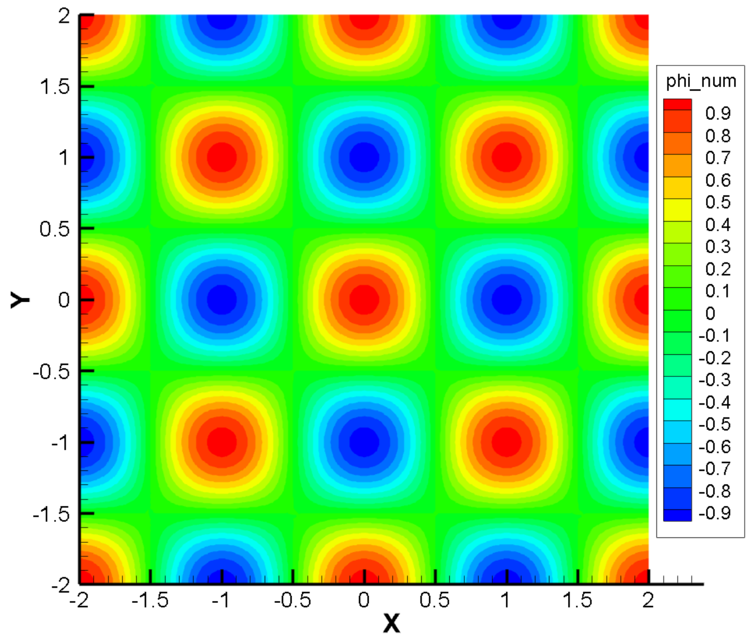


解析解

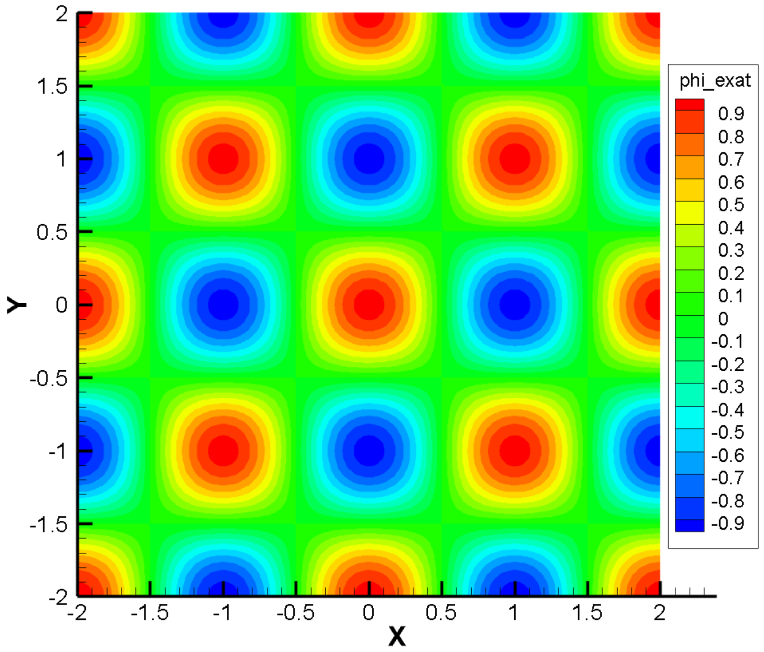


误差与网格数量关系

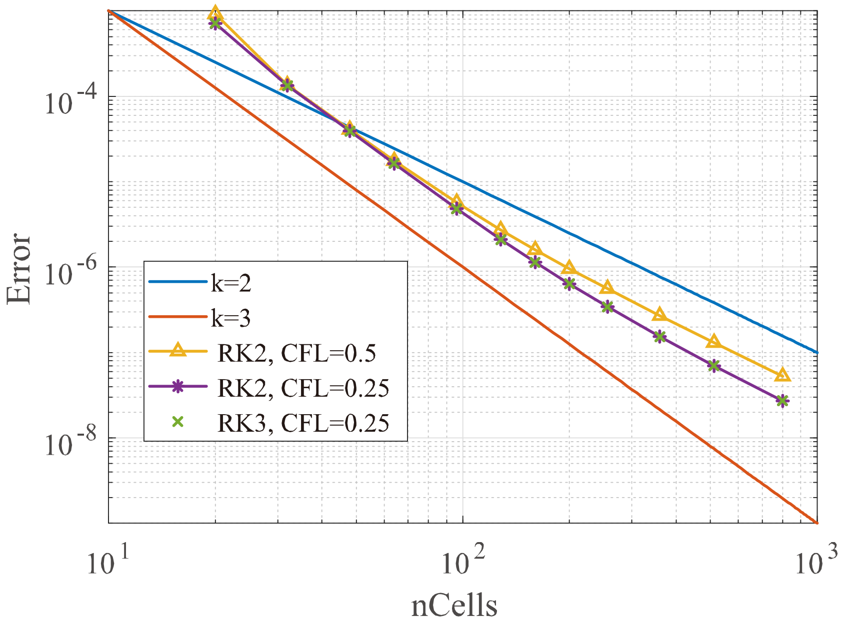
Two-dimensional model equation



数值解



解析解



误差与网格数量关系