数学分析 HW.1

14.1

1.(1) 设 $u_n=\dfrac{n}{(n+1)(n+2)(n+3)}$

$$\begin{aligned}u_n&=(1-\frac{1}{n+1})\frac{1}{(n+2)(n+3)}\\&=(\frac{1}{n+2}-\frac{1}{n+3})-\frac{1}{(n+1)(n+2)(n+3)}\\&=(\frac{1}{n+2}-\frac{1}{n+3})-\frac{1}{2}[\frac{1}{(n+1)(n+2)}-\frac{1}{(n+2)(n+3)}]\end{aligned}$$

$$\sum_{n=1}^{\infty}u_n=\frac{1}{3}-\frac{1}{12}=\frac{1}{4}$$

1.(2) 设 $u_n=\dfrac{2n-1}{2^n}$,我们有$u_n=\dfrac{2n+1}{2^{n-1}}-\dfrac{2n+3}{2^n}$,而$\dfrac{2n+3}{2^n}\to 0$，当$n\to\infty$时，那么$\sum\limits_{n=1}^{\infty}u_n=3$

1.(3) 设 $u_n=\sqrt{n+2}-2\sqrt{n+1}+\sqrt{n}$，则$u_n=\dfrac{1}{\sqrt{n+2}+\sqrt{n+1}}-\dfrac{1}{\sqrt{n+1}+\sqrt{n}}$，而$\dfrac{1}{\sqrt{n+2}+\sqrt{n+1}}\to 0,\ n\to\infty$

$$\sum_{n=1}^{\infty}u_n=0-\frac{1}{\sqrt{2}+1}=1-\sqrt{2}$$

14.2

(1)

$$\sum_{n=1}^{\infty} \frac{\prod_{i=1}^{n}(3i-1)}{\prod_{i=1}^{n}(4i-3)}$$

Solution

$$u_n = \frac{\prod_{i=1}^{n}(3i-1)}{\prod_{i=1}^{n}(4i-3)}, \frac{u_{n+1}}{u_n} = \frac{3n+2}{4n+1} < \frac{8}{9},\ n > 2.$$

所以原级数收敛。 (2)

$$\sum_{n=1}^{\infty} \frac{3^n}{n!}$$

Solution

$$u_n = \frac{3^n}{n!} \implies \frac{u_{n+1}}{u_n} = \frac{3n}{n+1} \to 3 > 1,\ n \to \infty$$

所以原级数发散

(3)

$$\sum_{n=1}^{\infty} \frac{3^n n!}{(2n)!}$$

Solution

$$u_n = \frac{3^n n!}{(2n)!} \implies \frac{u_{n+1}}{u_n} = \frac{3(n+1)}{(2n+1)(2n+2)} \to 0,\ n \to \infty.$$

原级数收敛。

(4)

$$\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$$

Solution

$$u_n = \frac{(n!)^2}{(2n)!} \implies \frac{u_{n+1}}{u_n} = \frac{(n+1)^2}{(2n+1)(2n+2)} \to \frac{1}{4},\ n \to \infty$$

原级数收敛。 (5)

$$\sum_{n=2}^{\infty}\frac{1}{(\ln n)^n}$$

Solution

$[(\ln n)^n]^{\frac{1}{n}} = \ln n > 1, n > 3$

所以原级数发散。 (6)

$$\sum_{n=1}^{\infty}(\frac{n+1}{2n+1})^n$$

Solution

$$u_n = (\frac{n+1}{2n+1})^n \implies \sqrt[n]{u_n} = \frac{n+1}{2n+1} \to \frac{1}{2} < 1,\ n \to \infty$$

原级数收敛。

(7)

$$\sum_{n=1}^{\infty}\frac{3n^3+1}{2^n}$$

Solution

$(\frac{3n^3+1}{2^n})^{\frac{1}{n}} = \frac{(3n^3+1)^{\frac{1}{n}}}{2} \to \frac{1}{2}$

原级数收敛。

(8)

$$\sum_{n=1}^{\infty}\frac{n^3[\sqrt{2}+(-1)^n]^n}{3^n}$$

Solution

$$u_n = \frac{n^3[\sqrt{2}+(-1)^n]^n}{3^n} \implies \sqrt[n]{u_n} = \sqrt[n]{n^3}\frac{\sqrt{2}+(-1)^n}{3} < 1,\ n \to \infty$$

所以原级数收敛。 (9)

$$\sum_{n=1}^{\infty} \sin^2 \frac{1}{n}$$

Solution

$\sin^2 \frac{1}{n} \sim \frac{1}{n^2}, n \to 0$

原级数收敛。

(10)

$$\sum_{n=1}^{\infty} 2^n \sin \frac{x}{3^n}$$

Solution

$$u_n = 2^n \sin \frac{x}{3^n},\ n \to \infty,\ \sqrt[n]{u_n} = 2\sqrt[n]{\sin \frac{x}{3^n}} \sim \frac{2}{3}\sqrt[n]{x} \to \frac{2}{3}$$

(11)

$$\sum_{n=1}^{\infty} \frac{1}{(n^2-1)^{\frac{1}{3}}}$$

Solution

$\frac{1}{(n^2-1)^{\frac{1}{3}}} \sim \frac{1}{n^{\frac{2}{3}}}, n \to \infty$

原级数发散。 (12)

$$\sum_{n=1}^{\infty} \frac{1}{n\sqrt[n]{n}}$$

Solution 由于$\sqrt[n]{n} < \ln n,\ n \to \infty$

$$u_n = \frac{1}{n\sqrt[n]{n}} > \frac{1}{n \ln n},\ n \to \infty$$

由比较判别法可知原级数发散

(13)

$$\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$$

Solution

$$u_n = \frac{\ln n}{n^2} < \frac{1}{n^{\frac{3}{2}}},\ n \to \infty$$

原级数收敛 (14)

$$\sum_{n=1}^{\infty} \frac{n^{n-1}}{(2n^2+n+1)^{\frac{n-1}{2}}}$$

Solution

$$u_n = \frac{n^{n-1}}{(2n^2+n+1)^{\frac{n-1}{2}}} = (\frac{1}{2+\frac{2}{n}+\frac{1}{n^2}})^{\frac{n-1}{2}} < \frac{1}{(\sqrt{2})^{n-1}}$$

由比较判别法可知原级数收敛。

(15)

$$\sum_{n=2}^{\infty} \frac{1}{(\ln n)^k}$$

Solution

$\frac{1}{\ln n} > \frac{1}{n^{\frac{1}{k}}}$, 当$n$足够大的时候. $\frac{1}{(\ln n)^k} > \frac{1}{n}$

原级数发散。 (16)

$$\sum_{n=1}^{\infty} \frac{n^{n+\frac{1}{n}}}{(n+\frac{1}{n})^n}$$

Solution

$$
\begin{aligned}
u_n &= \frac{n^{n+\frac{1}{n}}}{(n+\frac{1}{n})^n} \\
&= \exp((n+\frac{1}{n})\ln n - n\ln(n+\frac{1}{n})) \\
&= \exp\{(n^2+1)[\frac{\ln n}{n} - \frac{\ln(n+\frac{1}{n})}{n+\frac{1}{n}}]\} \\
&= \exp\{(n^2+1)[\frac{\ln n}{n} - (\frac{\ln n}{n} + \frac{1}{n} \times \frac{1-\ln n}{n^2} + o(\frac{\ln n}{n^4}))]\} \\
&= \exp[(n^2+1) \times \frac{\ln n - 1}{n^3} + o(\frac{\ln n}{n^2})] \\
&= \exp(\frac{\ln n - 1}{n} + o(\frac{\ln}{n^2})) \to 1,\ n \to \infty
\end{aligned}
$$

所以原级数发散。 (17)

$$\sum(1-\cos\frac{x}{n})\ (x \in \mathrm{R})$$

Solution

$$1-\cos\frac{x}{n} \sim \frac{(\frac{x}{n})^2}{2}$$

原级数收敛。 (18)

$$\sum_{n=1}^{\infty}(\frac{1}{n} - \ln\frac{n+1}{n})$$

Solution

$$u_n = \frac{1}{n} - \ln\frac{n+1}{n} = \frac{1}{n} - \frac{1}{n} + \frac{1}{2n^2} + o(\frac{1}{n^2})$$

原级数收敛。 (19)

$$\sum(\frac{1}{\sqrt{n}} - \sqrt{\ln(1+\frac{1}{n})})$$

Solution

$\frac{1}{\sqrt{n}} - \sqrt{\ln(1+\frac{1}{n})} = \frac{\frac{1}{n} - \ln(1+\frac{1}{n})}{\frac{1}{\sqrt{n}} + \sqrt{\ln(1+\frac{1}{n})}}$,考虑 *Taylor Formula*:

$$\frac{1}{n}-\ln(1+\frac{1}{n})=\frac{1}{n}-(\frac{1}{n}-\frac{1}{2n^2}+o(\frac{1}{n^2}))=\frac{1}{2n^2}+o(\frac{1}{n^2})$$

$$\sqrt{\frac{1}{n}}+\sqrt{\ln(1+\frac{1}{n})}>2\sqrt{\ln(1+\frac{1}{n})}$$

所以：

$$\frac{\frac{1}{n}-\ln(1+\frac{1}{n})}{\frac{1}{\sqrt{n}}+\sqrt{\ln(1+\frac{1}{n})}}<\frac{\frac{1}{2n^2}}{2\sqrt{\ln(1+\frac{1}{n})}}\sim\frac{1}{n^{\frac{5}{2}}}$$

原级数收敛。 (20)

$$\sum_{n=3}^{\infty}\frac{1}{n\ln n\ln\ln n}$$

Solution $u_n=\frac{1}{n\ln n\ln\ln n}$, $f(x)=\frac{1}{x\ln x\ln x}$

$$\int_3^N f(x)\mathrm{d}x=\int_3^N\frac{1}{\ln x\ln\ln x}\mathrm{d}\ln x=\int_{\ln 3}^{\ln N}\frac{1}{t\ln t}\mathrm{d}t=\int_{\ln\ln 3}^{\ln\ln N}\frac{1}{s}\mathrm{d}s\to\infty,\ N\to\infty$$

(21)

$$\sum_{n=2}^{\infty}\frac{1}{\ln(n!)}$$

Solution

$\ln(n!)=\sum_{i=1}^{n}\ln i$。根据积分不等式可以得到以下估计：

$$\int_1^n\ln x\mathrm{d}x<\sum_{i=1}^{n}\ln i<\int_1^{n+1}\ln x\mathrm{d}x \tag{2.1}$$

或者直接使用放缩：

$$\ln(n!)<n\ln n$$

$$\frac{1}{\ln(n!)}>\frac{1}{n\ln n}$$

$$\sum\frac{1}{n(\ln n)^p}\Rightarrow\begin{cases}\text{covergent}, p>1\\ \text{divergent}, 0<p\le 1\end{cases} \tag{2.2}$$

原级数发散。

14.3

1 设$u_n = \left|\frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n+1)}{n!}\right|$.

$$\frac{u_n}{u_{n+1}} = \left|\frac{n+1}{\alpha-n}\right|$$

当 n 足够大的时候，$\alpha - n < 0$

$$\frac{u_n}{u_{n+1}} = \frac{n+1}{n-\alpha}$$

$$R_n = n(\frac{u_n}{u_{n+1}} - 1) = \frac{n}{n-\alpha}(\alpha+1) \to \alpha+1 > 1,\ n \to \infty$$

由*Raabe*判别法，原级数收敛。

2 (1) $u_n = \dfrac{\sqrt{n!}}{(a+1)(a+\sqrt{2})\cdots(a+\sqrt{n})}$

$$\frac{u_n}{u_{n+1}} = \frac{a+\sqrt{n+1}}{\sqrt{n+1}}$$

$$R_n = n(\frac{u_n}{u_{n+1}} - 1) = n\frac{a}{\sqrt{n+1}} \to \infty,\ n \to \infty$$

所以原级数收敛。

2 (2) $u_n = \dfrac{n!n^{-p}}{q(q+1)\cdots(q+n)}$

$$
\begin{aligned}
\frac{u_n}{u_{n+1}} &= \frac{q+n+1}{n+1}\left(1+\frac{1}{n}\right)^p \\
&= \left(1+\frac{q}{n+1}\right)\left(1+\frac{1}{n}\right)^p \\
&= \left(1+\frac{q}{n+1}\right)\left(1+\frac{p}{n}+\frac{p(p-1)}{2n^2}+o\left(\frac{1}{n^2}\right)\right) \\
&= 1+\frac{q}{n+1}+\frac{p}{n}+\frac{pq}{n(n+1)}+\frac{p(p-1)}{2n^2}+o\left(\frac{1}{n^2}\right) \\
&= 1+\frac{p}{n}+\frac{q}{n}-\frac{q}{n(n+1)}+o\left(\frac{1}{n\ln n}\right) \\
&= 1+\frac{p+q}{n}+o\left(\frac{1}{n\ln n}\right)
\end{aligned}
$$

由 *Raabe* 判别法：

$$n\left(\frac{u_n}{u_{n+1}}-1\right) \to p+q,\ n\to\infty$$

1. 当$p+q<1$时，原级数发散
2. 当$p+q>1$时，原级数收敛
3. 当$p+q=1$时，由高斯判别法：$\frac{u_n}{u_{n+1}}=\lambda+\frac{\mu}{n}+\frac{v}{n\ln n}+o\left(\frac{1}{n\ln n}\right)$ 中的$\lambda=\mu=1, v<1$可知级数发散。