练习14.2

（奇数题号）2024/9/3

(1)

$$\sum_{n=1}^{\infty}\frac{\prod_{i=1}^{n}(3i-1)}{\prod_{i=1}^{n}(4i-3)}$$

Solution $u_n=\frac{\prod_{i=1}^{n}(3i-1)}{\prod_{i=1}^{n}(4i-3)},\frac{u_{n+1}}{u_n}=\frac{3n+2}{4n+1}<\frac{8}{9}, when\ n>2.$

所以原级数收敛。

(5)

$$\sum_{n=2}^{\infty}\frac{1}{(\ln n)^n}$$

Solution $[(\ln n)^n]^{\frac{1}{n}}=\ln n>1,n>3$

所以原级数发散。

(7)

$$\sum_{n=1}^{\infty}\frac{3n^3+1}{2^n}$$

Solution $(\frac{3n^3+1}{2^n})^{\frac{1}{n}}=\frac{(3n^3+1)^{\frac{1}{n}}}{2}\to\frac{1}{2}$

原级数收敛。

(9)

$$\sum_{n=1}^{\infty}\sin^2\frac{1}{n}$$

Solution $\sin^2\frac{1}{n}\sim\frac{1}{n^2},n\to 0$

原级数收敛。

(11)

$$\sum_{n=1}^{\infty}\frac{1}{(n^2-1)^{\frac{1}{3}}}$$

Solution $\frac{1}{(n^2-1)^{\frac{1}{3}}}\sim\frac{1}{n^{\frac{2}{3}}}$

原级数发散。

(15)

$$\sum_{n=2}^{\infty}\frac{1}{(\ln n)^k}$$

Solution $\frac{1}{\ln n}>\frac{1}{n^{\frac{1}{k}}}$,when n is large enough. $\frac{1}{(\ln n)^k}>\frac{1}{n}$

原级数发散。

(17)

$$\sum(1-\cos\frac{x}{n})\ (x\in\mathbb{R})$$

Solution $1-\cos\frac{x}{n}\sim\frac{(\frac{x}{n})^2}{2}$

原级数收敛。

(19)

$$\sum(\frac{1}{\sqrt{n}}-\sqrt{\ln(1+\frac{1}{n})})$$

Solution $\frac{1}{\sqrt{n}}-\sqrt{\ln(1+\frac{1}{n})}=\frac{\frac{1}{n}-\ln(1+\frac{1}{n})}{\frac{1}{\sqrt{n}}+\sqrt{\ln(1+\frac{1}{n})}}$,consider *Taylor Formula*:

$$\frac{1}{n}-\ln(1+\frac{1}{n})=\frac{1}{n}-(\frac{1}{n}-\frac{1}{2n^2}+o(\frac{1}{n^2}))=\frac{1}{2n^2}+o(\frac{1}{n^2})$$

$$\sqrt{\frac{1}{n}}+\sqrt{\ln(1+\frac{1}{n})}>2\sqrt{\ln(1+\frac{1}{n})}$$

Then

$$\frac{\frac{1}{n}-\ln(1+\frac{1}{n})}{\frac{1}{\sqrt{n}}+\sqrt{\ln(1+\frac{1}{n})}}<\frac{\frac{1}{2n^2}}{2\sqrt{\ln(1+\frac{1}{n})}}\sim\frac{1}{n^{\frac{5}{2}}}$$

原级数收敛。
(21)

$$\sum_{n=2}^{\infty}\frac{1}{\ln(n!)}$$

Solution $\ln(n!)=\sum_{i=1}^{n}\ln i$.According integral inequlity, the estimate of this value is:

$$\int_1^n\ln x\mathrm{d}x<\sum_{i=1}^{n}\ln i<\int_1^{n+1}\ln x\mathrm{d}x \tag{2.1}$$

or use scaling method:

$$\ln(n!)<n\ln n$$

$$\frac{1}{\ln(n!)}>\frac{1}{n\ln n}$$

$$\sum\frac{1}{n(\ln n)^p}\Rightarrow\begin{cases}\text{covergent},p>1\\ \text{divergent},0<p\leq 1\end{cases} \tag{2.2}$$

原级数发散

大概前几天用到达朗贝尔和柯西判别法，后面的几题都是估计，牢记几个不等式和*Taylor Formula*。

2024/9/4

(14 A)

1.$\{a_n\}$递减并且级数$\sum_{n=1}^{\infty}a_n$收敛，求证：$\lim_{n\to\infty}na_n=0$

Proof $\forall\varepsilon>0,\exists N,when\ n>N,\forall p,\sum_{k=n+1}^{n+p}a_k<\varepsilon$,when $p=n$,the form is

$$\sum_{k=n+1}^{2n}a_k<\varepsilon$$

我们这样思考，na_{2n}是n个a_{2n}相加，consider

$$na_{2n} < \sum_{k=n+1}^{2n} a_k < \varepsilon$$

namely,

$$\lim_{n\to\infty} 2na_{2n} = 0$$

Similarly,consider odd terms:

$$\lim_{n\to\infty} (2n+1)a_{2n+1} = 0$$

2. (4).判断收敛发散

$$\sum_{n=3}^{\infty} \frac{1}{n(\ln n)^p(\ln\ln n)^q}$$

Solution

First there is a fact:

$$1 < \ln\ln n < (\ln n)^s, \forall\ 0 < s < 1, n \to \infty$$

这样便可以得出$p > 1$或$0 < p < 1$时的敛散性。

3. (组合)

$\{a_n\}$递减，且非负，求证：$\sum_{n=1}^{\infty} a_n$和$\sum_{k=1}^{\infty} 2^k a_{2^k}$同敛散性

4. (柯西收敛定理)

设$\{a_n\}$递增正数数列,求证：$\sum_{n=1}^{\infty}(1-\frac{a_n}{a_{n+1}})$在$\{a_n\}$有界时收敛，无界时发散。

Proof

(1) when $\{a_n\}$ is bounded,

$$(1-\frac{a_n}{a_{n+1}}) = \frac{a_{n+1}-a_n}{a_{n+1}}$$

note the sum as S_n

$$S_n = \frac{a_2-a_1}{a_2} + \frac{a_3-a_2}{a_3} + \cdots + \frac{a_{n+1}-a_n}{a_{n+1}} \le \frac{a_{n+1}-a_1}{a_2}$$

i.e. reduce all demominators to a_2 ,and then sum numerators. $\le$ 右边的数是有界量，原级数收敛。

(2) when a_n is unbounded:
用柯西收敛原理证明原级数不收敛。首先，我们由$\{a_n\}$发散可以得到一个结论：

$$\forall n \in \mathbf{N}, M \in \mathbf{R}^+, \exists p > 0, s.t.\ \ a_{n+p} = Ma_n$$

where p is depend on M, n
每次以上次的a_{n+p}赋值为M，我们就可以得到一个数列 $\{p_m\}$以及$\{\frac{a_{n+p_m}-a_n}{a_{n+p_m}}\} \to 1^- > 0$
对于一个给定的n

$$\sum_{k=n+1}^{n+p}(1-\frac{a_n}{a_{n+1}}) > \frac{a_{n+p}-a_n}{a_{n+p}}$$

依次在$\{p_m\}$取p，又因为上述的数列收敛到1,即：$\exists \varepsilon_0 > 0 \forall n > 0, \exists, p > 0$，$\sum_{k=n+1}^{n+p}(1-\frac{a_n}{a_{n+p}}) > \frac{a_{n+p}-a_n}{a_{n+p}}$ (可以取到在1附近的值) $\geq \varepsilon_0$

所以原级数发散

5. (ln n的阶估计)

$\sum_{n=2}^{\infty}\frac{1}{(\ln n)^{\ln n}}, \sum_{n=3}^{\infty}\frac{1}{(\ln n)^{\ln\ln n}}, \sum_{n=2}^{\infty}\frac{1}{(\ln\ln n)^{\ln n}}$

Solution

(1).

$$\ln n > 3, n > e^3 \Rightarrow \frac{1}{(\ln n)^{\ln n}} < \frac{1}{3^{\ln n}} = \frac{1}{n^{\ln 3}} \ (\sum^{\infty}\frac{1}{n^{\ln 3}} \text{ is convegent})$$

(2). $(\ln n)^{\ln\ln n} = e^{(\ln\ln n)^2} < e^{[(\ln n)^{\frac{1}{2}}]^2} = n$,所以$\sum_{n=3}^{\infty}\frac{1}{(\ln n)^{\ln\ln n}} > \sum_{n=3}^{\infty}\frac{1}{n}$

6.

7. (裂项相消) (integral inequality)

$u_n > 0, S_n = \sum_{n=1}^{\infty}$ is divergent.Argue that $\sum_{n=1}^{\infty}\frac{u_n}{S_n}$ is convegent,$\sum_{n=1}^{\infty}\frac{u_n}{S_n^2}$ is divergent, and $\sum_{n=1}^{\infty}\frac{u_n}{S^{1+\sigma}}$ is convegent.

Solution

(1). $u_n = S_n - S_{n-1} \implies \sum_{n=1}^{\infty}\frac{u_n}{S_n} = \sum_{n=1}^{\infty}(\frac{S_n - S_{n-1}}{S_n})$. According 4. , $\sum_{n=1}^{\infty}\frac{u_n}{S_n}$ is divergent.

(2). $\sum_{n=1}^{\infty}\frac{u_n}{S_n^2} < \sum_{n=1}^{\infty}\frac{S_n - S_{n-1}}{S_n S_{n-1}} = \sum_{n=1}^{\infty}(\frac{1}{S_{n-1}} - \frac{1}{S_n}) = \sum_{n=1}^{\infty}(\frac{1}{S_1} - \frac{1}{S_n}) = \frac{1}{S_1}$

(3). consider

$$\frac{u_n}{S^{1+\sigma}} = \frac{S_n - S_{n-1}}{S^{1+\sigma}} < \int_{S_{n-1}}^{S_n}\frac{1}{x^{1+\sigma}}\mathrm{d}x \tag{A.1}$$

,

$$\sum_{n=1}^{\infty}\frac{u_n}{S^{1+\sigma}} < \lim_{n\to\infty}\int_{S_1}^{S_n}\frac{1}{x^{1+\sigma}}\mathrm{d}x = \lim_{n\to\infty} -\frac{1}{\sigma}\frac{1}{x^{\sigma}}\Big|_{S_1}^{S_n} = \frac{1}{\sigma S_1^{\sigma}}$$