Chapter I：Combinatorial Analysis

Counting

Basic Counting Principles

Theorem 1.1 (Pigeonhole principle. Dirichlet, 1834) if n items are to be put into m containers, with $n > m$,then at least one container must contain more than one item.

Corollary 1.2: if n items are to be put into m containers, with $n > m$,then at least one container must contain at least $\lceil \frac{n}{m} \rceil$ item.

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example:you go to a restaurant. That day, they propose 4 starters, 4 coursesand 3 desserts. You choose one starter, one course and one dessert to forminto your meal set. How many different sets are possible?

answer: $4 \times 4 \times 3 = 48$.

thus, we can get the following Proposition.

Proposition 1.3: If r experiments are to be performed sequentially(按顺序) and the first experiment can be performed in n_1 ways, . . . , the rth experiment in n_r ways, then there are$\prod_{i=1}^{n} n_i$ ways to perform the r experiments.

example bis:Still for the meal selection, you can either choose one starter andone course, or one course and one dessert. That is, you cannot take a starter,a course and a dessert. How many different sets are possible? **answer**: $4\times 4+4\times 3=28$.

Permutations

let's begin with an example:How many different ranking orders are possible for 10 tennis players? A_{10}^{10}.

Definition 1.4 (Permutation): An ordered ranking of $n\in N^*$ distinct elements is called a **permutation**.

Proposition 1.5: There are $n(n-1)\cdots 1$ permutations of $n\in N^*$ distinct elements.

$$P_1E_1P_2P_3E_2R=\frac{A_6^6}{A_2^2\times A_3^3}$$

组合：

Combinations

Multinomials

consider:

$$(x_1+x_2+\cdots+x_r)^n$$

for $x_1^{n_1}x_2^{n_2}\cdots x_r^{n_r}$, we choose n_1's x_1 in n item, and n_2's x_2 in $n-n_1$ item.

Prop:$n_1+n_2+n_3+\cdots+n_r=n$ have $\binom{n+r-1}{r-1}$ distinct nonnegative integer_valued vectors $(n_1,n_2,\cdots,n_r)$

Proof: We can see this problem as the form following:

$$y_1+y_2+\cdots+y_r=n+r$$

where $y_i=n_i+1,y_i\geq 1$.Specifically,what we need to anwser is how many methods of inserting $r-1$ spacers in $n+r-1$ gaps. **Ex:** $n=1,r=3$

$$\bigcirc|\bigcirc|\bigcirc\bigcirc:(0,0,1)$$

$$\bigcirc|\bigcirc\bigcirc|\bigcirc:(0,1,0)$$

$$\bigcirc\bigcirc|\bigcirc|\bigcirc:(1,0,0)$$

i.e. $\binom{1+3-1}{3-1}=\binom{3}{2}=3$

Chapter.II Axioms of Probability

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Set

Definition 2.1

1. random experiment
2. sample space (is denoted by S)
3. sample point

Remark 2.2 S can be finite or **infinite** (conutable or uncountable)

Definition 2.2 subset/superset

Definition 2.3 event

Remark 2.3 Sure set: S, impossible set : $\emptyset$

Relation

1. Union
2. Intersection
3. Countable union/intersection
4. Mutually exclusive: $E \cap F = \emptyset$
5. Complement: E^c
6. Difference: $E \setminus F$
7. Symmertric difference: $E \triangle F = \{\omega | \omega \in E \setminus F \text{ or } \omega \in \mathrm{F} \setminus \mathrm{E}\}$

Proposition 2.5 (*De Morgan's Law*)

$$(\bigcup_{i=1}^{n} E_i)^c = \bigcap_{i=1}^{n} E_i$$
$$(\bigcap_{i=1}^{n} E_i)^c = \bigcup_{i=1}^{n} E_i$$

Axioms of Probability

Some Properties

$$\mathrm{P}(\emptyset) = 0$$

Proof If we consider a sequence $\{E_i\}$ where $2E_1 = S, E_i = \emptyset, \text{for } i \geq 2$,then $S = \bigcap E_i$.Hence,

$$\mathrm{P}(S) = \mathrm{P}(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} \mathrm{P}(E_i) = \mathrm{P}(S) + \sum_{i=2}^{\infty} \mathrm{P}(E_i) \implies \mathrm{P}(\emptyset) = 0$$

$$\mathrm{P}\,(\bigcup_{i=1}^{n} E_i) = \sum_{i=1}^{n} \mathrm{P}(E_i)$$

where $E_i.E_j, i \,\text{⊟}\, j$ are mutually exclusive

Proof Similar to **Proof.1**

$$\mathrm{P}(E) \leq \mathrm{P}(F)$$

where $E \subseteq F \subseteq \mathrm{A}$

Proof $\mathrm{P}(F) = \mathrm{P}(E + F \setminus E) = \mathrm{P}(E) + \mathrm{P}(F \setminus E)$

4. (inclusion and exclusion indentity) For any two events E, F

$$\mathrm{P}(E \cup F) = \mathrm{P}(E) + \mathrm{P}(F) - \mathrm{P}(E \cap F)$$

$$\mathrm{P}\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{k=1}^{n} (-1)^{k-1} \sum_{1 \leq i_1 \leq i_2 \leq \cdots \leq i_k \leq n} \mathrm{P}(E_{i_1} \cap E_{i_2} \cdots \cap E_{i_k}) \tag{2.1}$$

Proof $\mathrm{P}(E \cup F) = \mathrm{P}(E \cup E^c F) = \mathrm{P}(E) + \mathrm{P}(E^c F)$,

consider $EF + E^c F = F, EF \cap E^c F = \emptyset \implies \mathrm{P}(E) + \mathrm{P}(E^c F) = \mathrm{P}(E) + \mathrm{P}(F) - \mathrm{P}(EF)$

Ex $n = 4$:

$$\begin{aligned}
&\mathrm{P}\left(\bigcup_{i=1}^{4} E_i\right) \\
&= \mathrm{P}(E_1) + \mathrm{P}(E_2) + \mathrm{P}(E_3) + \mathrm{P}(E_4) \\
&- \mathrm{P}(E_1E_2) - \mathrm{P}(E_2E_3) - \mathrm{P}(E_2E_4) - \mathrm{P}(E_1E_4) - \mathrm{P}(E_1E_3) - \mathrm{P}(E_3E_4) \\
&+ P(E_1E_2E_3) + \mathrm{P}(E_1E_3E_4) + \mathrm{P}(E_1E_2E_4) + \mathrm{P}(E_2E_3E_4) \\
&- \mathrm{P}(E_1E_2E_3E_4)
\end{aligned}$$

$$\mathrm{P}(E \cup F) \leq \mathrm{P}(E) + \mathrm{P}(F)$$

(A gerneralization) For a finite sequence of events $E_1, E_2, \ldots, E_n$

$$\mathrm{P}\left(\bigcup_{i=1}^{n} E_i\right) \leq \mathrm{P}(E_1) + \mathrm{P}(E_2) + \cdots \mathrm{P}(E_n) \tag{2.2}$$

(Infinite) *Bool's inequality* : For a countably infinite sequence of events $\{E_i\}_{i \geq 1}$

$$\mathrm{P}\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} \mathrm{P}(E_i)$$

Proof (2.2) Note the identity:

$$\bigcup_{i=1}^{n} E_i = E_1 + E_1^c E_2 + \cdots + E_1^c E_2^c \cdots E_{n-1}^c E_n$$

or this form:

$$\begin{gathered}
F_1 = E_1 \\
F_2 = E_2 \setminus E_1 \\
\vdots \\
F_k = E_k \setminus \bigcup_{i=1}^{k} E_i
\end{gathered}$$

$$\mathrm{P}\,(\bigcup_{i=2}^{n} E_i) = \mathrm{P}(E_1) + \sum_{i=1}^{n} \mathrm{P}(E_1^c E_2^c \cdots E_{i-1}^c E_i)$$

denote $E_1^c E_2^c \cdots E_{i-1}^c E_i = B_i$, where $\mathrm{P}(B_i) \le \mathrm{P}(E_i)$.Thus,

$$\mathrm{P}\,(\bigcup_{i=1}^{n} E_i) \le \mathrm{P}(E_1) + \mathrm{P}(E_2) + \cdots \mathrm{P}(E_n)$$

Ex 2.1

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Continue Properties

Definition Increasing/Decreasing sequence $\{E_n\}$, we ***define a new event*** $\lim_{n\to\infty} E_n$ by $\lim_{n\to\infty} E_n = \bigcup_{n=1}^{\infty} E_n$

5. For decreasing/increasing sequence E_n

$$\lim_{n\to\infty} \mathrm{P}(E_n) = \mathrm{P}(\lim_{n\to\infty} E_n)$$

We prove the case for increasing sequence $\{E_n\}$

$$RHS = \mathrm{P}(\bigcup_{n=1}^{\infty} E_n) = \mathrm{P}(\bigcup_{n=1}^{\infty} F_n) = \sum_{n=1}^{\infty} \mathrm{P}(F_n) = \lim_{n\to\infty} \sum_{k=1}^{n} \mathrm{P}(F_n) = \lim_{n\to\infty} \mathrm{P}(E_n) = LHS$$

Ex 2.2

Suppose we have an infinitely large urn, and an infinite collection of balls labled as number 1,2,3,4,...

- At 1 min to 12p.m., balls numbered 1 through 10 are placed in the urn and a ball is randomly selected and withdrawn.
- At $\frac{1}{2}$ min to 12p.m., balls numbered 11 through 20 are placed in the urn and a ball is randomly selected and withdrawn.
- At $\frac{1}{4}$ min to 12p.m., balls numbered 21 through 30 are placed in the urn and a ball is randomly selected and withdrawn.

and so on. How many balls are there in the urn at 12pm.

Proof Consider 1^{th} ball, denote event $\{1^{th} \text{ ball is still in the urn at } \frac{1}{2^k} \text{ min to 12pm}\}$ as E_k, and event $\{1^{th} \text{ ball is in the urn at 12pm}\}$ as E apparently,

$$E_n \subseteq \cdots E_2 \subseteq E_1$$

$$
\begin{aligned}
P(E) &= \lim_{n\to\infty} P(\bigcup_{k=1}^{n} E_k) \\
&= \lim_{n\to\infty} P(E_n) \\
&= \lim_{k\to\infty} \frac{9}{10}\frac{18}{19}\frac{27}{28}\cdots\frac{9k}{9k+1} \\
&= \exp(\sum_{k=1}^{\infty} \ln(1-\frac{1}{9k+1})) \to 0
\end{aligned}
$$

Similarly, the event $\{i^{th} \text{ is in the urn at 12pm}\}$, denoted by F_i, $P(F_i) = 0$.

$\{\text{the urn is not empty}\} \Leftrightarrow \{\text{there is at least one ball in the urn}\}$. Finally, the urn is empty.

Uniform Probability measure on finite sample

Poker Problem

52 cards to 4 people. What is the probability that

- one of the players recieves all 13 spades. (E_1)
- each player recieves 1 ace. (E_2)

(1)

$$
P(E_1) = \frac{|E|}{|S|} = \frac{\binom{39}{13\ 13\ 13}}{\binom{52}{13\ 13\ 13\ 13}}
$$

(2)

$$
P(E_2) = \frac{\binom{4}{1\ 1\ 1\ 1}\binom{48}{12\ 12\ 12\ 12}}{\binom{52}{13\ 13\ 13\ 13}}
$$

Birthday Problem

The Matching Problem

Each of N men throw his hat into the center of room, and the hats are first mixed up. Then each man randomly selects a hat. What is the probability that none of the men selects his own hat?

Solution Denote the event 'the i^{th} man gets his hat' as E_i. According to *Inclusive&Exclusive Theorem*

$$
P(\bigcup_{i=1}^{N} E_i) = \sum_{i=1}^{N} P(E_i) - \sum_{1\le i_1,i_2\le N} P(E_{i_1}E_{i_2}) + \cdots + (-1)^{k-1} \sum_{1\le i_1,i_2,\ldots,i_k\le N} P(E_{i_1}E_{i_2}\cdots E_{i_k}) + \cdots + (-1)^{N-1} P(E_1E_2\cdots E_N
$$

note that

$$
P(E_{i_1}E_{i_2}\cdots E_{i_k}) = \frac{(N-k)!}{N!}, \binom{N}{k}\frac{(N-k)!}{N!} = \frac{1}{k!}
$$

then

$$P(\bigcup_{i=1}^{N} E_i) = 1 - \frac{1}{2!} + \frac{1}{3!} + \cdots + (-1)^{N-1}\frac{1}{N!}$$

The event "none of men selects his own hat" and E is complemenatary.

Charpter III

Conditional Probability