Chapter I：Combinatorial Analysis

Counting

Basic Counting Principles

Theorem 1.1 (Pigeonhole principle. Dirichlet, 1834) if n items are to be put into m containers, with $n > m$,then at least one container must contain more than one item.

Corollary 1.2: if n items are to be put into m containers, with $n > m$,then at least one container must contain at least $[\frac{n}{m}]$ item.

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example:you go to a restaurant. That day, they propose 4 starters, 4 coursesand 3 desserts. You choose one starter, one course and one dessert to forminto your meal set. How many different sets are possible?

answer: $4 \times 4 \times 3 = 48$.

thus, we can get the following Proposition.

Proposition 1.3: If r experiments are to be performed sequentially(按顺序) and the first experiment can be performed in n_1 ways, . . . , the rth experiment in n_r ways, then there are$\prod_{i=1}^{n} n_i$ ways to perform the r experiments.

example bis:Still for the meal selection, you can either choose one starter andone course, or one course and one dessert. That is, you cannot take a starter,a course and a dessert. How many different sets are possible? **answer**: $4 \times 4 + 4 \times 3 = 28$.

Permutations

let's begin with an example:How many different ranking orders are possible for 10 tennis players? A_{10}^{10}.

Definition 1.4 (Permutation): An ordered ranking of $n \in N^*$ distinct elements is called a **permutation**.

Proposition 1.5: There are $n(n-1)\cdots 1$ permutations of $n \in N^*$ distinct elements.

$$P_1E_1P_2P_3E_2R = \frac{A_6^6}{A_2^2 \times A_3^3}$$

组合：

Combinations

Multinomials

consider:

$$(x_1 + x_2 + \cdots + x_r)^n$$

for $x_1^{n_1}x_2^{n_2}\cdots x_r^{n_r},\quad$ we choose n_1's x_1 in n item, and n_2's x_2 in $n - n_1$ item.

Prop:$n_1 + n_2 + n_3 + \cdots + n_r = n$ have $\binom{n+r-1}{r-1}$ distinct nonnegative integer_valued vectors $(n_1, n_2, \cdots, n_r)$

Proof: We can see this problem as the form following:

$$y_1 + y_2 + \cdots + y_r = n + r$$

where $y_i = n_i + 1, y_i \geq 1$.Specifically,what we need to anwser is how many methods of inserting $r - 1$ spacers in $n + r - 1$ gaps. **Ex:** $n = 1, r = 3$

$$\bigcirc | \bigcirc | \bigcirc \bigcirc : (0,0,1)$$

$$\bigcirc | \bigcirc \bigcirc | \bigcirc : (0,1,0)$$

$$\bigcirc \bigcirc | \bigcirc | \bigcirc : (1,0,0)$$

i.e. $\binom{1+3-1}{3-1} = \binom{3}{2} = 3$

Chapter.II Axioms of Probability

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Set

Definition 2.1

1. random experiment
2. sample space (is denoted by S)
3. sample point

Remark 2.2 S can be finite or **infinite** (conutable or uncountable)

Definition 2.2 subset/superset

Definition 2.3 event

Remark 2.3 Sure set: S, impossible set : $\emptyset$

Relation

1. Union
2. Intersection
3. Countable union/intersection
4. Mutually exclusive: $E \cap F = \emptyset$
5. Complement: E^c
6. Difference: $E \setminus F$
7. Symmertric difference: $E \triangle F = \{\omega | \omega \in E \setminus F \text{ or } \omega \in \mathrm{F} \setminus \mathrm{E}\}$

Proposition 2.5 (*De Morgan's Law*)

$$(\bigcup_{i=1}^{n} E_i)^c = \bigcap_{i=1}^{n} E_i$$
$$(\bigcap_{i=1}^{n} E_i)^c = \bigcup_{i=1}^{n} E_i$$

Axioms of Probability

Some Properties

$$\mathrm{P}(\emptyset) = 0$$

Proof If we consider a sequence $\{E_i\}$ where $2E_1 = S, E_i = \emptyset, \text{for } i \geq 2$,then $S = \bigcap E_i$.Hence,

$$\mathrm{P}(S)=\mathrm{P}(\bigcup_{i=1}^{\infty} E_i)=\sum_{i=1}^{\infty}\mathrm{P}(E_i)=\mathrm{P}(S)+\sum_{i=2}^{\infty}\mathrm{P}(E_i) \implies \mathrm{P}(\emptyset)=0$$

$$\mathrm{P}\,(\bigcup_{i=1}^{n} E_i)=\sum_{i=1}^{n}\mathrm{P}(E_i)$$

where $E_i.E_j, i \neq j$ are mutually exclusive

Proof Similar to **Proof.1**

$$\mathrm{P}(E)\leq\mathrm{P}(F)$$

where $E \subseteq F \subseteq \mathrm{A}$

Proof $\mathrm{P}(F)=\mathrm{P}(E+F\setminus E)=\mathrm{P}(E)+\mathrm{P}(F\setminus E)$

4. (inclusion and exclusion indentity) For any two events E, F

$$\mathrm{P}(E\cup F)=\mathrm{P}(E)+\mathrm{P}(F)-\mathrm{P}(E\cap F)$$

$$\mathrm{P}\,(\bigcup_{i=1}^{n} E_i)=\sum_{k=1}^{n}(-1)^{k-1}\sum_{1\leq i_1\leq i_2\leq\cdots\leq i_k\leq n}\mathrm{P}(E_{i_1}\cap E_{i_2}\cdots\cap E_{i_k}) \tag{2.1}$$

Proof $\mathrm{P}(E\cup F)=\mathrm{P}(E\cup E^cF)=\mathrm{P}(E)+\mathrm{P}(E^cF)$,

consider $EF+E^cF=F, EF\cap E^cF=\emptyset \implies \mathrm{P}(E)+\mathrm{P}(E^cF)=\mathrm{P}(E)+\mathrm{P}(F)-\mathrm{P}(EF)$

Ex $n=4$:

$$\begin{aligned}
&\mathrm{P}\,(\bigcup_{i=1}^{4} E_i)\\
&=\mathrm{P}(E_1)+\mathrm{P}(E_2)+\mathrm{P}(E_3)+\mathrm{P}(E_4)\\
&-\mathrm{P}(E_1E_2)-\mathrm{P}(E_2E_3)-\mathrm{P}(E_2E_4)-\mathrm{P}(E_1E_4)-\mathrm{P}(E_1E_3)-\mathrm{P}(E_3E_4)\\
&+P\,(E_1E_2E_3)+\mathrm{P}(E_1E_3E_4)+\mathrm{P}(E_1E_2E_4)+\mathrm{P}(E_2E_3E_4)\\
&-\mathrm{P}(E_1E_2E_3E_4)
\end{aligned}$$

$$\mathrm{P}(E\cup F)\leq\mathrm{P}(E)+\mathrm{P}(F)$$

(A gerneralization) For a finite sequence of events $E_1, E_2, \ldots, E_n$

$$\mathrm{P}\,(\bigcup_{i=1}^{n} E_i)\leq\mathrm{P}(E_1)+\mathrm{P}(E_2)+\cdots\mathrm{P}(E_n) \tag{2.2}$$

(Infinite) *Bool's inequality* : For a countably infinite sequence of events $\{E_i\}_{i\geq 1}$

$$\mathrm{P}\,(\bigcup_{i=1}^{\infty} E_i)\leq\sum_{i=1}^{\infty}\mathrm{P}(E_i)$$

Proof (2.2) Note the identity:

$$\bigcup_{i=1}^{n} E_i=E_1+E_1^cE_2+\cdots+E_1^cE_2^c\cdots E_{n-1}^cE_n$$

or this form:

$$F_1 = E_1$$
$$F_2 = E_2 \setminus E_1$$
$$\vdots$$
$$F_k = E_k \setminus \bigcup_{i=1}^{k} E_i$$

$$\mathrm{P}\left(\bigcup_{i=2}^{n} E_i\right) = \mathrm{P}(E_1) + \sum_{i=1}^{n} \mathrm{P}(E_1^c E_2^c \cdots E_{i-1}^c E_i)$$

denote $E_1^c E_2^c \cdots E_{i-1}^c E_i = B_i$, where $\mathrm{P}(B_i) \leq \mathrm{P}(E_i)$.Thus,

$$\mathrm{P}\left(\bigcup_{i=1}^{n} E_i\right) \leq \mathrm{P}(E_1) + \mathrm{P}(E_2) + \cdots \mathrm{P}(E_n)$$

Ex 2.1

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Continue Properties

Definition Increasing/Decreasing sequence $\{E_n\}$, we ***define a new event*** $\lim_{n\to\infty} E_n$ by $\lim_{n\to\infty} E_n = \bigcup_{n=1}^{\infty} E_n$

5. For decreasing/increasing sequence E_n

$$\lim_{n\to\infty} \mathrm{P}(E_n) = \mathrm{P}(\lim_{n\to\infty} E_n)$$

We prove the case for increasing sequence $\{E_n\}$

$$RHS = \mathrm{P}(\bigcup_{n=1}^{\infty} E_n) = \mathrm{P}(\bigcup_{n=1}^{\infty} F_n) = \sum_{n=1}^{\infty} \mathrm{P}(F_n) = \lim_{n\to\infty} \sum_{k=1}^{n} \mathrm{P}(F_n) = \lim_{n\to\infty} \mathrm{P}(E_n) = LHS$$

Ex 2.2

Suppose we have an infinitely large urn, and an infinite collection of balls labled as number 1,2,3,4,...

- At 1 min to 12p.m., balls numbered 1 through 10 are placed in the urn and a ball is randomly selected and withdrawn.
- At $\frac{1}{2}$ min to 12p.m., balls numbered 11 through 20 are placed in the urn and a ball is randomly selected and withdrawn.
- At $\frac{1}{4}$ min to 12p.m., balls numbered 21 through 30 are placed in the urn and a ball is randomly selected and withdrawn.

and so on. How many balls are there in the urn at 12pm.

Proof Consider 1^{th} ball, denote event $\{1^{th}\text{ ball is still in the urn at }\frac{1}{2^k}\text{ min to 12pm}\}$ as E_k, and event $\{1^{th}\text{ ball is in the urn at 12pm}\}$ as E apparently,

$$E_n \subseteq \cdots E_2 \subseteq E_1$$

$$\begin{aligned}
\mathrm{P}(E) &= \lim_{n\to\infty} \mathrm{P}(\bigcup_{k=1}^{n} E_k) \\
&= \lim_{n\to\infty} \mathrm{P}(E_n) \\
&= \lim_{k\to\infty} \frac{9}{10}\frac{18}{19}\frac{27}{28}\cdots\frac{9k}{9k+1} \\
&= \exp(\sum_{k=1}^{\infty} \ln(1-\frac{1}{9k+1})) \to 0
\end{aligned}$$

Similarly, the event $\{i^{th}\text{ is in the urn at 12pm}\}$, denoted by F_i, $\mathrm{P}(F_i) = 0$.

$\{\text{the urn is not empty}\} \Leftrightarrow \{\text{there is at least one ball in the urn}\}$. Finally, the urn is empty.

Uniform Probability measure on finite sample

Poker Problem

52 cards to 4 people. What is the probability that

- one of the players recieves all 13 spades. (E_1)
- each player recieves 1 ace. (E_2)

(1)

$$\mathrm{P}(E_1) = \frac{|E|}{|S|} = \frac{\binom{39}{13\ 13\ 13}}{\binom{52}{13\ 13\ 13\ 13}}$$

(2)

$$\mathrm{P}(E_2) = \frac{\binom{4}{1\ 1\ 1\ 1}\binom{48}{12\ 12\ 12\ 12}}{\binom{52}{13\ 13\ 13\ 13}}$$

Birthday Problem

The Matching Problem

Each of N men throw his hat into the center of room, and the hats are first mixed up. Then each man randomly selects a hat. What is the probability that none of the men selects his own hat?

Solution Denote the event 'the i^{th} man gets his hat' as E_i. According to *Inclusive&Exclusive Theorem*

$$\mathrm{P}(\bigcup_{i=1}^{N} E_i) = \sum_{i=1}^{N} \mathrm{P}(E_i) - \sum_{1\le i_1,i_2\le N} \mathrm{P}(E_{i_1}E_{i_2}) + \cdots + (-1)^{k-1} \sum_{1\le i_1,i_2,\ldots,i_k\le N} \mathrm{P}(E_{i_1}E_{i_2}\cdots E_{i_k}) + \cdots + (-1)^{N-1}\mathrm{P}(E_1E_2\cdots E_N)$$

note that

$$\mathrm{P}(E_{i_1}E_{i_2}\cdots E_{i_k}) = \frac{(N-k)!}{N!}, \binom{N}{k}\frac{(N-k)!}{N!} = \frac{1}{k!}$$

then

$$\mathrm{P}(\bigcup_{i=1}^{N} E_i) = 1 - \frac{1}{2!} + \frac{1}{3!} + \cdots + (-1)^{N-1}\frac{1}{N!}$$

The event "none of men selects his own hat" and E is complemenatary.

Charpter III

Conditional Probability (obeserve and predict)

$$\mathrm{P}(E|F) = \frac{\mathrm{P}(E \cap F)}{\mathrm{P}(F)}$$

$P(\cdot|F)$ as a probability measure

Use three axioms to prove

Prop 3.3 The function mapping $\mathrm{P}_F : A \to \mathrm{R}$ defined by $\mathrm{P}_F(E) = \mathrm{P}(E|F)$ is a probability measure on (S, A)

Prop 3.4 Let F be an event with $\mathrm{P}(F) > 0$, the conditional probability of event $E \in A$ given that F has occurred can be computed as:

$$\mathrm{P}(E|F) = \frac{|E \cap F|}{|F|}$$

Prop 3.5 (*Mutiplication Rule*) Let $E_1, E_2, \ldots, E_n$ be a sequence of $n \in \mathrm{N}, n \geq 1$ events. Then we have:

$$\mathrm{P}(\bigcap_{i=1}^{n} E_i) = \mathrm{P}(E_1)\mathrm{P}(E_2|E_1)\mathrm{P}(E_3|E_1 \cap E_2)\cdots \mathrm{P}(E_n|\bigcap_{i=1}^{n-1} E_i)$$

Proof Expand RHS and calcel out each other

Example

Example 3.1 (*Pocke game revisied*) 52 cards to 4 piles of 13 cards each. Compute the probability that each pile has exactly 1 ace.

Solution

Let E_i be the event that ith pile have 1 ace

$$\mathrm{P}(E_1E_2E_3E_4) = \mathrm{P}(E_1)\mathrm{P}(E_2|E_1)\mathrm{P}(E_3|E_1 \cap E_2)\mathrm{P}(E_4|E_1 \cap E_2 \cap E_3)$$

•

Example 3.2 (*The matching problem revisited*) In matching problem, the probability that no matches occur when N people select from N mixed-up hats, denoted by P_N, is given by:

$$P_N = \sum_{i=0}^{N} (-1)^i \frac{1}{i!}$$

What is the probabiliy that exactly k of the N people have matches?

Solution

Let E be the event that $1, 2, \ldots, k$ have matches, and G be the event that no matches occur among people $k+1, \ldots N$

$$\mathrm{P}(G|E) = P_{N-k}$$

The probability that k of N people have matches is given by :

$$\binom{N}{k} \mathrm{P}(E \cap G)$$

E is the event that the first k people get their hats, which means $N-k$ hats permutate in $N-k$ people, i.e.

$$|E| = (N-k)!$$

$$\mathrm{P}(E) = \frac{|E|}{N!} = \frac{(N-k)!}{N!}$$

$$\binom{N}{k} \mathrm{P}(E \cap G) = \mathrm{P}(E)\mathrm{P}(G|E) = \frac{N!}{(N-k)!k!} \frac{(N-k)!}{N!} P_{N-k} = \frac{1}{k!} P_{N-k}$$

Baye's Formula

First Baye's fomula

$$\mathrm{P}(F|E) = \frac{\mathrm{P}(E|F)\mathrm{P}(F)}{\mathrm{P}(E)}$$

Law of total probability

Let $\{F_i\}$ be a countable **partition** of the sample space S

$$\mathrm{P}(E) = \sum_{j=1}^{\infty} \mathrm{P}(E \cap F_j)$$

with the knowledge of conditional probability,

$$\mathrm{P}(E) = \sum \mathrm{P}(E|F_i)\mathrm{P}(F_i)$$

In particular,

$$P(E) = P(E|F)P(F) + P(E|F^c)(1 - P(F))$$

Example

Example 3.3

Solution

$$P(R_2) = P(R_1)P(R_2|R_1) + P(R_2|R_1^c)P(R_1^c) = \frac{r}{r+b} = P(R_1)$$

Then we prove that $P(R_n) \equiv P(R_1)$

Suppose it's true for n

$$\begin{aligned} P(R_{n+1}) &= P(R_{n+1}|R_n^c)P(\mathrm{R}_n) + P(R_{n+1}|R_n^c)P(R_n^c) \\ &= \frac{r+s}{r+b+s}\frac{r}{r+b} + \frac{r}{r+b+s}\frac{b}{r+b} \\ &= \frac{r}{r+b} \end{aligned}$$