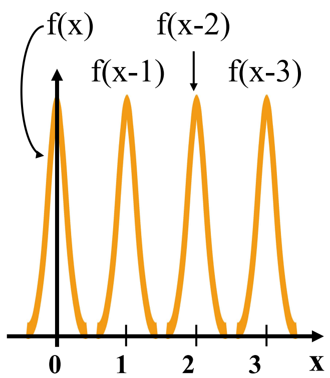
2. Diffraction

- Preparation: Review of wave
- Basic principles: Diffraction
- Basic principles: Fraunhofer and Fresnel diffraction
- Single- & Multi-slit
- The grating equation again
- Resolving power

What is waves?

A wave is anything that moves.



To displace any function f(x) to the right, just change its argument from **x** to **x-a**, where **a** is a positive number.

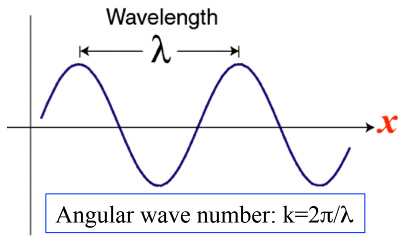
If we let **a = v t**, where **v** is positive and **t** is time, then the displacement will increase with time.

So **f(x-vt)** represents a rightward, or forward, propagating wave.

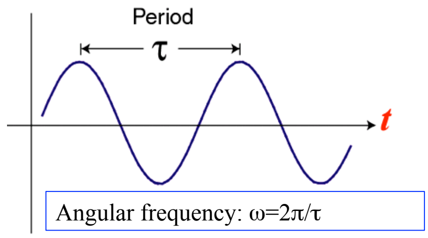
v will be the velocity of the wave.

$$E(x,t) = B\cos(kx - \omega t) + C\sin(kx - \omega t)$$
$$= A\cos(kx - \omega t - \theta)$$

Spatial quantities:

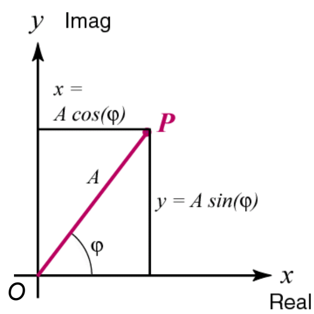


Temporal quantities:



Complex numbers

Consider a point, **P = (x,y)**, on a 2D Cartesian grid.



Instead of using an ordered pair, (x,y), we write:

$$P = x + i\,y$$
$$= A\cos(\varphi) + i\,A\sin(\varphi)$$

where $i = (-1)^{1/2}$

Euler's Formula

$$\boxed{\exp(i\varphi) = \cos(\varphi) + i\sin(\varphi)}$$

so the point, $P = A\cos(\varphi) + i\,A\sin(\varphi)$, can be written:

$$P = A\exp(i\varphi)$$

where

$$A = \text{Amplitude}, \quad \varphi = \text{Phase}$$

Waves using complex numbers

The electric field of a light wave

$$E(x,t) = A\cos(kx - \omega t - \theta)$$

can be expressed by using complex numbers.

Since $\exp(i\varphi) = \cos(\varphi) + i\sin(\varphi)$, $E(x,t)$ can be written:

$$\boxed{E(x,t) = \text{Re}\,\{\, A\exp[i(kx - \omega t - \theta)]\,\}}$$

We often leave out 'Re'.

Basic principles: Diffraction

- Consider the electromagnetic wave **E** and **H** diffracted by an opening. We define the incident electromagnetic in 3D space as

$$\psi = exp(-i\omega t) f(x,y,z)$$

and

$$f(x,y,z) = f_0 \exp[i(k_1 x + k_2 y + k_3 z)] \quad \text{for a plane wave}$$

where

$$\mathbf{k} = (k_1, k_2, k_3) \qquad (\text{where } k_1^2 + k_2^2 + k_3^2 = \omega^2/c^2)$$

Thus

$$f(x,y,z) = f_0 \exp\left[i\left(k_1 x + k_2 y + \sqrt{\omega^2/c^2 - k_1^2 - k_2^2}\, z\right)\right]$$

In general, the above equation can be written as

$$f(x,y,z) = \iint A(k_1,k_2) \exp\left[i\left(k_1 x + k_2 y + \sqrt{\omega^2/c^2 - k_1^2 - k_2^2}\, z\right)\right] dk_1 dk_2$$

At z=0

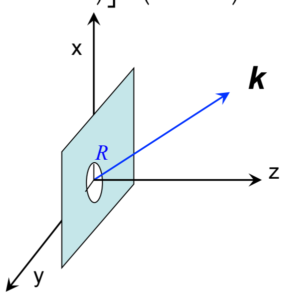
$$f(x,y,0) = \iint A(k_1,k_2) \exp[i(k_1 x + k_2 y)] dk_1 dk_2$$

Applying Fourier inversion transform for the above equation,

$$A(k_1,k_2) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} exp[-i(k_1 x + k_2 y)] f(x,y,0)\, dxdy$$

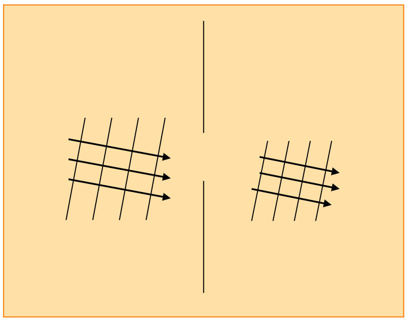
Here, let us consider an opening (see the figure below).
f(x,y,0) should be 0 outside the opening *R* when z=0. Then we have

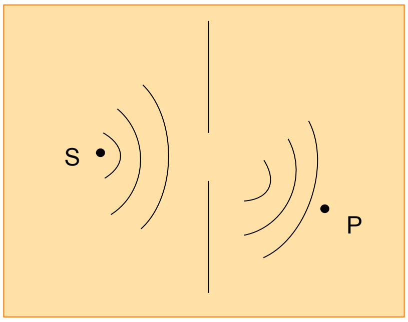
$$A(k_1,k_2) = \frac{1}{(2\pi)^2} \iint_R exp[-i(k_1 x + k_2 y)] f(x,y,0)\, dxdy$$



Fraunhofer vs. Fresnel diffraction

- In Fraunhofer diffraction, both incident and diffracted waves are considered to be plane (i.e. both S and P are at large distance)
- If either S or P are close enough that wavefront curvature is not negligible, then we have Fresnel diffraction





Fraunhofer diffraction by very long slit

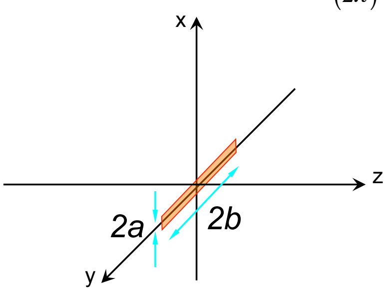
First, let's assume the rectangle slit (left figure). The length and the width of the slit are assumed to be *2b* and *2a*, respectively.

propagate along z-axis

The incident plane wave is described as $\psi = f(z)\exp[-i\omega t],\ f(z) = A\exp[ikz]$

Since $f(x,y,0)$=A, from $A(k_1,k_2) = \frac{1}{(2\pi)^2}\iint_R \exp[-i(k_1x+k_2y)]f(x,y,0)\,dxdy$

$$A(k_1,k_2) = \frac{A}{(2\pi)^2}\iint_R \exp[-i(k_1x+k_2y)]\,dxdy$$



$$A(k_1,k_2)=\frac{A}{(2\pi)^2}\iint_R \exp[-i(k_1x+k_2y)]dxdy$$

After the passage through the slit (z>0)

Assuming $k_1=k\alpha$ and $k_2=k\beta$

$$\boldsymbol{A(k_1,k_2)=\frac{A}{(2\pi)^2}\int_{-a}^{a}exp[-ik\alpha x]dx\int_{-b}^{b}exp[-ik\beta y]dy}$$

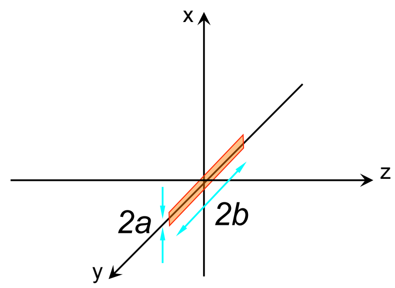
because

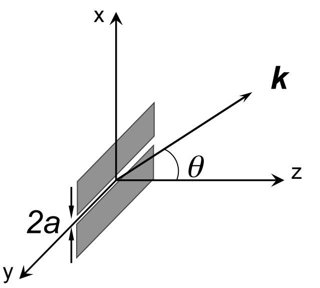
$$\sin(x)=\frac{\exp[ix]-\exp[-ix]}{2i}$$

$$\cos(x)=\frac{\exp[ix]+\exp[-ix]}{2}$$

$$=\frac{A}{(2\pi)^2}\frac{\exp[ik\alpha a]-\exp[-ik\alpha a]}{ik\alpha}\frac{\exp[ik\beta b]-\exp[-ik\beta b]}{ik\beta}$$

$$=\frac{4A}{(2\pi)^2}\frac{\sin[k\alpha a]}{k\alpha}\frac{\sin[k\beta b]}{k\beta}$$





The intensity ratio of $A(k_1,k_2)$ is therefore, ($k_1=k\alpha\rightarrow 0$, $k_2=k\beta\rightarrow 0$)

$$\frac{|A(k_1,k_2)|^2}{|A(0,0)|^2}=\left\{\frac{4A}{(2\pi)^2}\frac{\sin[k\alpha a]}{k\alpha}\frac{\sin[k\beta b]}{k\beta}\right\}^2\Big/\left\{\frac{4A}{(2\pi)^2}\frac{\sin[k\alpha a]}{k\alpha}\frac{\sin[k\beta b]}{k\beta}\right\}^2$$

$$=\left\{\frac{\sin[k\alpha a]}{k\alpha a}\right\}^2\left\{\frac{\sin[k\beta b]}{k\beta b}\right\}^2$$

$\rightarrow a$ when $k_1=k\alpha\rightarrow 0$

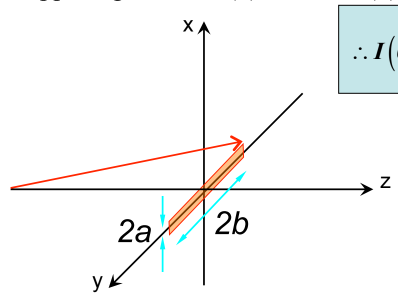
$\rightarrow b$ when $k_2=k\beta\rightarrow 0$

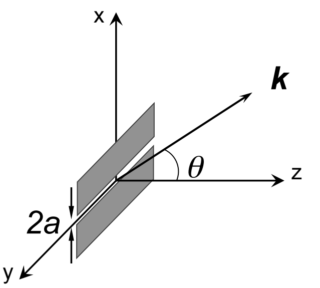
When b>>1 (long-slit), $\beta\rightarrow 0$. Accordingly,

$$\therefore I(\theta)=\left\{\frac{\sin[k\alpha]}{k\alpha}\right\}^2 I_0=\left\{\frac{\sin[ka\sin(\theta)]}{ka\sin(\theta)}\right\}^2 I_0$$

Supposing B=ka sin(θ)=2πa/λsin(θ)

$$\therefore \boldsymbol{I}(\theta)=\left\{\frac{\boldsymbol{\sin}\mathrm{B}}{\mathrm{B}}\right\}^2\boldsymbol{I_0}$$

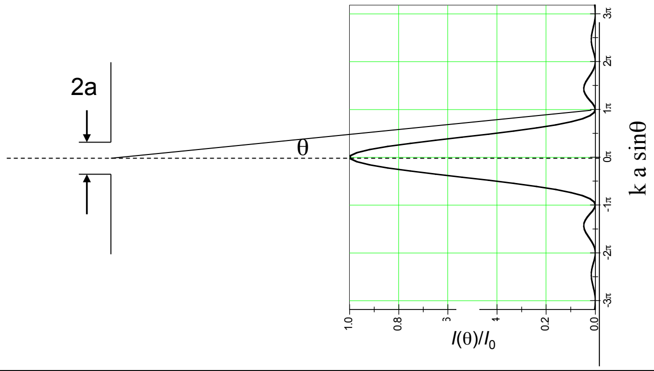




Single Slit Fraunhofer diffraction: Effect of slit width

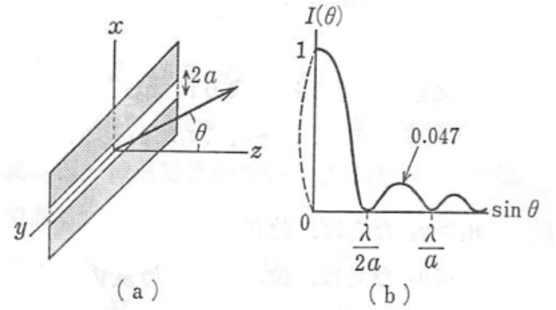
- Maximum when sin [k a sinθ] = 0
- Minimum when k a sinθ = ±pπ
- First minima at k a sinθ = ±π, namely,
  sin θ = ±λ/2a, which means…(next page)

$$I(\theta)=\left\{\frac{\sin\left[ka\sin(\theta)\right]}{ka\sin(\theta)}\right\}^2 I_0$$



Single Slit Fraunhofer diffraction: Effect of slit width

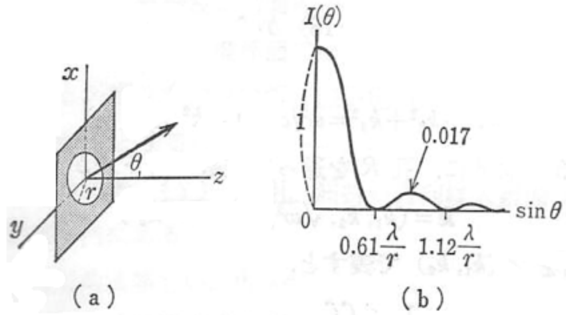
- Width of central max = λ/2a×2 = λ/a
- This relation is characteristic of Fraunhofer diffraction
- If *a* is very large $\theta \rightarrow$ 0 and a point source is imaged as a point
- If *a* is very small (~λ) θ→π/2 and light spreads out across screen



Homework

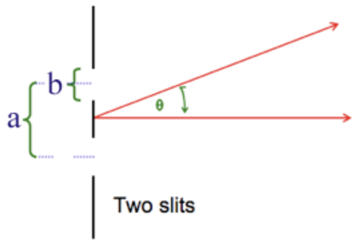
Similary, you can consider Fraunhofer diffraction through a circular aperture

$$\sin\theta = 0.61\frac{\lambda}{r},\quad 1.12\frac{\lambda}{r},\ \cdots$$



Diffraction from two slits:

- Now, let's suppose the collimated beam falls on [two slits.]

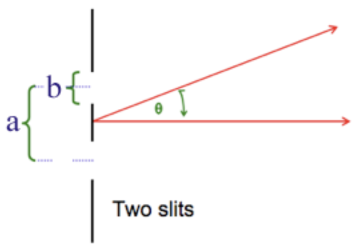


- Then we project a pattern at right: a series of bright interference fringes modulated by the envelope of the single slits

Note that, so far, we assumed the slit width = 2a

Diffraction from two slits:

- Mathematically, we can show that the bright fringes are regularly spaced, and modulated by the single slit envelope:



Same as for single slit (the envelope)

$$I_P = I_o \left(\frac{\sin\beta}{\beta}\right)^2 \cos^2\alpha$$

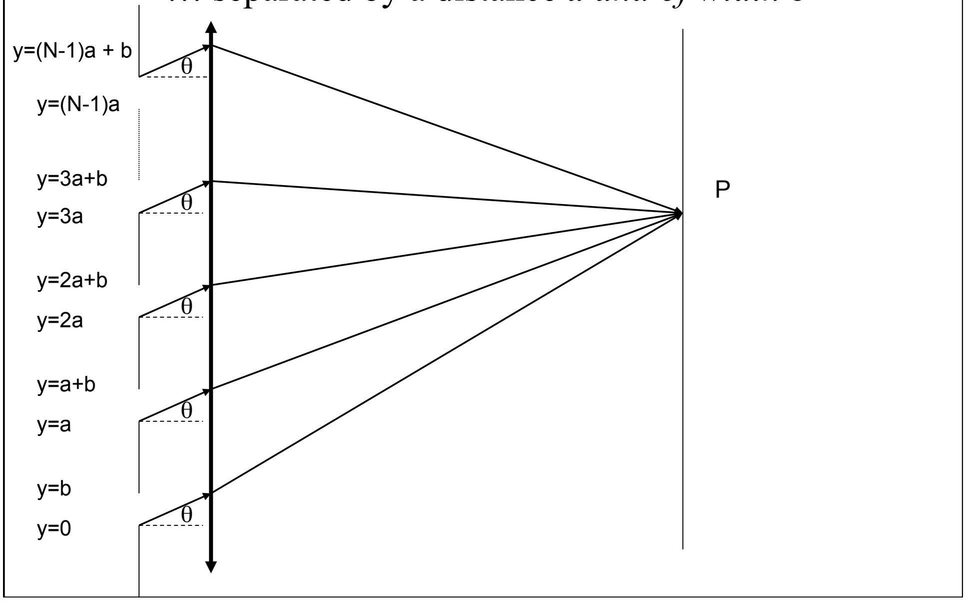
Interference term

$$\beta = k\frac{b}{2}\sin\theta$$

$$\alpha = k\frac{a}{2}\sin\theta$$

Diffraction from an array of *N* slits:

… separated by a distance *a and of width b*



Diffraction from an array of *N* slits

- It can be shown that,

Same as for single slit (the envelope)

$$I_P = I_o \left(\frac{\sin\beta}{\beta}\right)^2 \left(\frac{\sin N\alpha}{\sin\alpha}\right)^2$$

Interference term

- where,

$$\beta = k\frac{b}{2}\sin\theta \qquad \text{and} \qquad \alpha = k\frac{a}{2}\sin\theta$$

Slit width

Slit interval

$$I_P = I_o\left(\frac{\sin\beta}{\beta}\right)^2\left(\frac{\sin N\alpha}{\sin\alpha}\right)^2$$

Diffraction and interference for N slits

The diffraction term $\longrightarrow$ $\left(\frac{\sin\beta}{\beta}\right)^2$

- Minima for sin β = 0
- ⇒β = ±pπ = k(b/2)sin θ

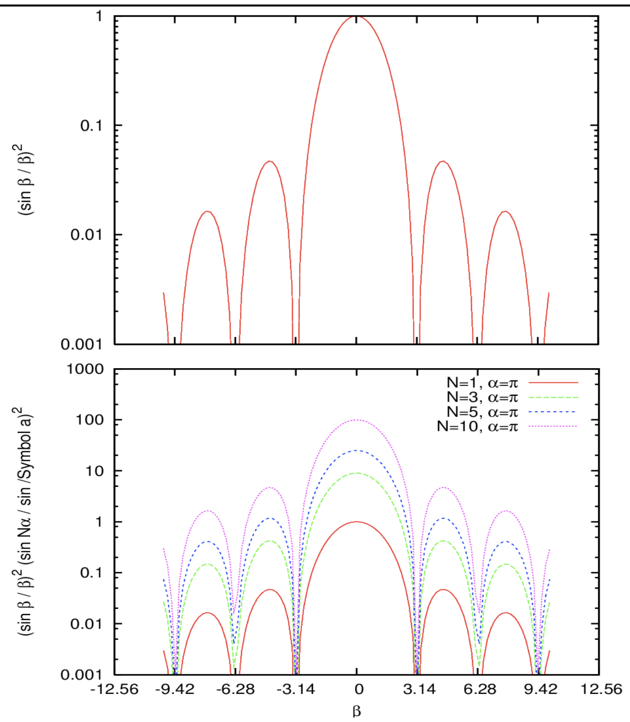
  or sinθ = ±p(λ/b) because k=2π/λ

- the same as single slit

The interference term $\longrightarrow$ $\left(\frac{\sin N\alpha}{\sin\alpha}\right)^2$

- Amplitude due to N coherent sources
- It can see this by adding N phasors that are 2α out of phase (see next page).

- Double slit when N=2
- Single slit when N=1



Diffraction and interference for N slits: Interference term

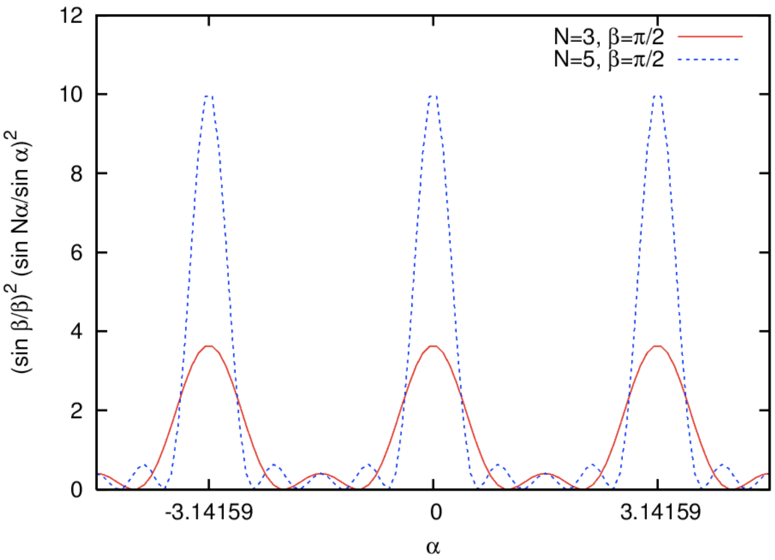
$$\boxed{I_P = I_o \left(\frac{\sin\beta}{\beta}\right)^2 \left(\frac{\sin N\alpha}{\sin\alpha}\right)^2}$$

- Maxima occur at $\alpha = \pm m\pi$ ($m = 0, 1, 2, 3,..$)
- Thus maxima occur at $\sin\theta = \pm m\lambda/a$ ← because $\alpha = k\frac{a}{2}\sin\theta$
- This is the same result we have derived for Young's double slit
- Intensity of principal maxima, $I = N^2 I_o$

Diffraction and interference for N slits: Interference term (cont.)

$$\boxed{I_P = I_o \left(\frac{\sin\beta}{\beta}\right)^2 \left(\frac{\sin N\alpha}{\sin\alpha}\right)^2}$$

- Minima occur for $\alpha = \pi/N, 2\pi/N, \ldots (N-1)\pi/N$
  Thus principal maxima have a width determined by zeros on each side
- Since $\alpha = (\pi/\lambda)a \sin\theta = \pm\pi/N$,
  the angular width is determined by $\sin\theta = \lambda/(Na)$
- Thus peaks are N times narrower than in a single slit pattern (also a > b).



Diffraction gratings

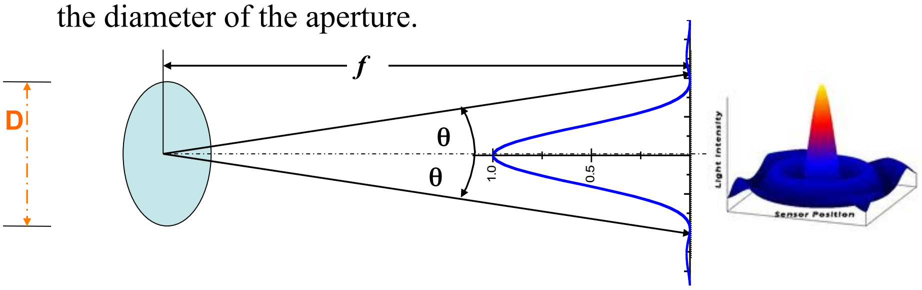
- As we learned, it is composed of systems with many slits per unit length – usually about 1000/mm
- Also it is usually used in reflection
- Thus principal maxima are vary sharp
- The width of peaks $\Delta\alpha = 2\pi/N$
- As N gets large the peak gets very narrow

Diffraction limit and Airy disks

- The diffraction pattern resulting from a uniformly-illuminated circular aperture has a bright region in the center (Airy disk) together with the series of concentric bright rings (Airy pattern).
- The angle at which the first minimum occurs, measured from the direction of incoming light, is given by the approximate formula:

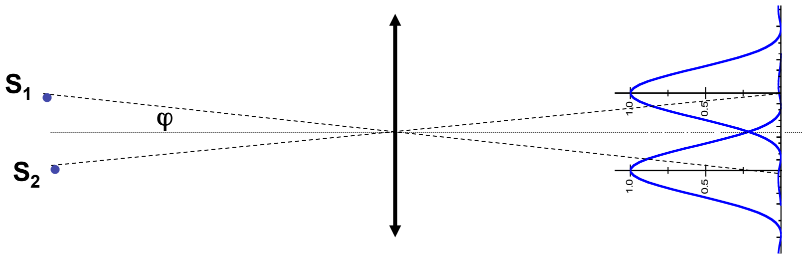
$$\sin\theta = 1.22\lambda/D \approx \theta$$

where θ is in radians and λ is the wavelength of the light and D is the diameter of the aperture.



Fraunhofer diffraction and spatial resolution

- Suppose two point sources or objects are far away (e.g. two stars)
- Imaged with some optical system
- Two Airy patterns
  - If S_1, S_2 are too close together the Airy patterns will overlap and become indistinguishable



Fraunhofer diffraction and spatial resolution

- Assume S1, S2 can just be resolved when maximum of one pattern just falls on minimum (first) of the other
- Then the angular separation at lens,

e.g. telescope D = 1 m and $\lambda = 500 \times 10^{-9}$ m

$$\phi_{min} = \frac{1.22\lambda}{D} = \frac{1.22 \times 5 \cdot 10^{-7}}{1.0} = 6.6 \times 10^{-7} \text{ rad} = 0.14''$$

e.g. human eye D ~ 7mm

$$\phi_{min} \approx \frac{1.22 \times 5 \cdot 10^{-7}}{7 \times 10^{-3}} = 8.7 \times 10^{-5} \text{ rad} = 18''$$

→ Comparable to the apparent size of Saturn