Congruences of Partitions: Rank Differences and Cubic Partition Pairs

A Project Report Submitted

in Partial Fulfilment of the Requirements

for the Degree of

BACHELOR OF TECHNOLOGY

in

Mathematics and Computing

by

Abhishek Tyagi and Seralathan V S

(Roll No. 140101004 and Roll No. 140123032)

to the

DEPARTMENT OF MATHEMATICS

INDIAN INSTITUTE OF TECHNOLOGY GUWAHATI

GUWAHATI - 781039, INDIA

April 2018

CERTIFICATE

This is to certify that the work contained in this project report entitled "**Congruences of Partitions: Rank Differences and Cubic Partition Pairs**" submitted by **Abhishek Tyagi** (**Roll No. 140101004**) and **Seralathan V S** (**Roll No. 140123032**) to Department of Mathematics, Indian Institute of Technology Guwahati and Department of towards the towards partial requirement of **Bachelor of Technology** in Mathematics and Computing has been carried out by them under my supervision.

It is also certified that this report is a survey work based on the references in the bibliography.

Turnitin Similarity: 18 %

Guwahati - 781 039 (Dr. Rupam Barman)

April 2018 Project Supervisor

ABSTRACT

This report describes the work of our B.Tech Project in the field of Partition Theory specifically in the study of Congruences of Partition Rank differences and Cubic Partition Pairs.

The celebrated Ramanujan's congruences led Dyson to conjecture the existence of the rank of a partition such that we get a combinatorial interpretation to these congruences in [10]. The notion of rank was extended to rank differences and M_2 rank differences upon which Mao conjectured several inequalities in [18, 19].

We investigated the proof to one of Mao's ten conjectures in [19] and attempted to extend the solution given by Barman and Sachdeva in [3] by using the properties of $\varphi^2(q)$, in which the coefficient of q^n counts the number of Integer solutions to $n = a^2 + b^2$ or using general proof for M_2-rank differences given in [1].

We also investigated the conjectures made by Lin in [17] before finding a proof for the same in [11].

Contents

Part 1

Chapter 1

Introduction to partitions

1.1 Background

A partition of a positive integer n is a way of writing n as the sum of positive integers. Two sums that differ only in the order of their summands are considered the same partition. The notation $\lambda \vdash n$ means that λ is a partition of n.

A summand in a partition is also called a part. The length of a partition refers to the number of parts in the partition and is denoted by $n(\lambda)$ for a partition λ. Its largest part, is denoted by $l(\lambda)$. As an example, 4 can be partitions in five distinct ways as follows

$$\begin{aligned}4 &= 4\\ &= 3+1 = 2+2\\ &= 2+1+1\\ &= 1+1+1+1.\end{aligned}$$

Where, $n((2,1,1)) = 3$ and $l((2,1,1)) = 2$.

1.2 Definitions and Terminology

1.2.1 q-Pochhammer Symbol

The q-Pochhammer symbol is defined as

$$(a)_0 := (a;q)_0 := 1, \quad (a;q)_n := \prod_{k=0}^{n-1}(1-aq^k) \quad n \geq 1,$$

$$(a)_\infty := (a;q)_\infty := \prod_{k=0}^{\infty}(1-aq^k) \quad |q| < 1.$$

If the identification of q is clear, we omit q from the notation. We state several basic identities used for simplification later as mentioned in [5].

$$(q^a;q^b)_\infty(-q^a;q^b)_\infty = (q^{2a};q^{2b})_\infty \tag{1.1}$$

$$(cq^a;q^{2b})_\infty(cq^{a+b};q^{2b})_\infty = (cq^a;q^b)_\infty \tag{1.2}$$

We will be using the following shorthand notations throughout the report and are also used in [1, 18, 19]:

$$(a_1, ..., a_k;q)_n := (a_1,q)_n...(a_k;q)_n$$

$$J_b := (q^b;q^b)_\infty$$

$$J_{a,b} := (q^a, q^{b-a}, q^b;q^b)_\infty$$

$$\chi(q) := (-q;q^2)_\infty$$

1.2.2 Ramanujan's General Theta Function

The famous Ramanujan's Theta function used in various proofs is as follows:

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \quad |ab| < 1. \tag{1.3}$$

Some shorthand related to the above generating functions will be used as [5]

$$\varphi(q) := f(q,q) = (-q,-q,q^2;q^2)_\infty,$$
$$\psi(q) := f(q,q^3) = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}.$$

1.2.3 Definitions Relating to Partitions

Partition Functions

If n is a positive integer, let $p(n)$ denote the number of unrestricted representations of n as a sum of positive integers, where representations with different orders of the same summands are not regarded as distinct. We call $p(n)$ the partition function. We further use the notation $p_m(n)$ to denote the number of partitions of n into parts that are no larger than m.

In general for a set S, $p(S,m,n)$ denotes the number of partitions of n into exactly m parts of S. Furthermore, we denote $p(m,n)$ as the number of partitions of n into exactly m parts.

We denote $Q(n)$ as the number of partitions into distinct parts. More generally, $Q(S,m,n)$ denotes the number of partitions of n into m distinct parts of S. Similarly $Q(m,n)$ is the number of partitions of n into exactly m distinct parts.

Overpartitions

An overpartition of n is a partition of n in which the first occurrence of a number may be over-lined. For example, the over partitions of 3 are:

$$\{3, \overline{3}, 2+1, 2+\overline{1}, \overline{2}+1, \overline{2}+\overline{1}, 1+1+1, \overline{1}+1+1\}$$

The number of overpartitions of n is denoted by $\overline{p}(n)$. This results in

$$\overline{p}(n) = \sum_{n_1+n_2=n} p(n_1) * Q(n_2)$$

which is a discrete convolution between p and Q, giving us $\overline{p} = p * Q$. Analogous to Dyson's rank function definitions for partitions we have $\overline{N}(s,n)$ and $\overline{N}(s,m,n)$ for overpartitions.

1.2.4 Generating Functions

The term generating function is used to describe an infinite sequence of numbers (a_n) by treating them as the coefficients of a series expansion. This infinite series is the generating function.

The generating function for the partition function $p(n)$ is represented as

$$\begin{aligned}
\sum_{n=0}^{\infty} p(n)q^n &= \sum_{n_i \geq 0} q^{1n_1+2n_2+\cdots+kn_k+\dots} \\
&= \frac{1}{1-q}\frac{1}{1-q^2}\cdots\frac{1}{1-q^k}\cdots \\
&= \frac{1}{\prod_{k=0}^{\infty}(1-qq^k)} \\
&= \frac{1}{(q;q)_\infty}
\end{aligned}$$

For general partition functions, we get

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} p(S,m,n)z^m q^n = \prod_{k\epsilon S}\frac{1}{1-zq^k},$$
$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} Q(S,m,n)z^m q^n = \prod_{k\epsilon S}(1+zq^k).$$

The generating function for overpartitions, $\sum_{n=0}^{\infty}\overline{p}(n)q^n$ is the product of general partitions and that for partitions with distinct parts as it is given by their convolution. Therefore,

$$\sum_{n=0}^{\infty}\overline{p}(n)q^n = \frac{(-q;q)_\infty}{(q;q)_\infty}$$

Chapter 2

Fundamental Properties of q-series and Theta functions

2.1 q-analogue of the Binomial Theorem

The q-analogue of the binomial theorem is a generalization involving the parameter q that returns the binomial theorem is the limit as $q \to 1$ and we replace a by q^a.

Theorem 2.1.1. *For* $|q|, |z| < 1$

$$\sum_{n=0}^{\infty} \frac{(a)_n}{(q)_n} z^n = \frac{(az)_\infty}{(z)_\infty}. \tag{2.1}$$

Proof. Note that the product on the RHS of (2.1) converges uniformly on compact subsets of $|z| < 1$ and so represents an analytic function on $|z| < 1$. Thus we may write

$$F(z) = \frac{(az)_\infty}{(z)_\infty} = \sum_{n=0}^{\infty} A_n z^n \tag{2.2}$$

From the product representation in (2.2), we can readily verify that

$$(1-z)F(z) = (1-az)F(qz). \tag{2.3}$$

Equating coefficients of z^n on both sides of (2.3), we get

$$A_n - A_{n-1} = q^n A_n - aq^{n-1} A_{n-1}$$

or

$$A_n = \frac{1-aq^{n-1}}{1-q^n} A_{n-1} \tag{2.4}$$

Using $A_0 = 1$ in (2.2), we deduce that

$$A_n = \frac{(a)_n}{(q)_n} \tag{2.5}$$

Using (2.5) in (2.2), we complete the proof for (2.1) □

2.1.1 Euler's Corollaries

$$\sum_{n=0}^{\infty} \frac{z^n}{(q)_n} = \frac{1}{(z)_\infty} \tag{2.6}$$

$$\sum_{n=0}^{\infty} \frac{(-z)^n q^{n(n-1)/2}}{(q)_n} = (z)_\infty \tag{2.7}$$

Euler discovered these corollaries by application of (2.1)

2.2 Jacobi Triple Product Identity

We present Jacobi's Triple Product Identity, which is immensely useful in the study of partitions in simplification of q-series products.

Theorem 2.2.1. *For $z \neq 0$ and $|q| < 1$*

$$\sum_{n=-\infty}^{\infty} z^n q^{n^2} = (-zq; q^2)_\infty (-q/z; q^2)_\infty (q^2; q^2)_\infty \tag{2.8}$$

Proof. In (2.7), replace q by q^2 and z by $-zq$ to get

$$\begin{aligned}(-zq; q^2)_\infty &= \sum_{n=0}^{\infty} \frac{z^n q^{n^2}}{(q^2; q^2)_n} = \frac{1}{(q^2; q^2)} \sum_{n=0}^{\infty} z^n q^{n^2} (q^{2n+2}; q^2)_\infty \\ &= \frac{1}{(q^2; q^2)} \sum_{n-\infty}^{\infty} z^n q^{n^2} (q^{2n+2}; q^2)_\infty,\end{aligned} \tag{2.9}$$

since $(q^{2n+2}; q^2)_\infty = 0$ when n is a negative integer.

Now apply (2.7), again with replacing q with q^2 and z with q^{2n+2}. Thus from (2.9),

$$\begin{aligned}(-zq; q^2)_\infty &= \frac{1}{(q^2; q^2)_\infty} \sum_{n=-\infty}^{\infty} z^n q^{n^2} \sum_{r=0}^{\infty} \frac{(-1)^r q^{(2n+2)r + r^2 - r}}{(q^2; q^2)_r} \\ &= \frac{1}{((q^2; q^2)_\infty} \sum_{r=0}^{\infty} \frac{(-1)^r z^{-r} q^r}{(q^2; q^2)_r} \sum_{n=-\infty}^{\infty} z^{n+r} q^{(n+r)^2} \\ &= \frac{1}{(q^2; q^2)_\infty} \sum_{r=0}^{\infty} \frac{(-q/z)^r}{(q^2; q^2)_r} \sum_{n=-\infty}^{\infty} z^m q^{m^2} \\ &= \frac{1}{(q^2; q^2)_\infty (-q/z; q^2)_\infty} \sum_{n=-\infty}^{\infty} z^m q^{m^2}\end{aligned}$$

by (2.6) with z replaced by $-q/z$ and q replaced by q^2, and therefore the restriction $|q/z| < 1$. Rearranging the final equation completes the proof of (2.8) for $|q/z| < 1$. However by analytic continuation, it hold's for all complex $z \neq 0$ completing the proof. $\square$

2.2.1 Corollory: Euler's Pentagonal Number Theorem

Euler's Pentagonal Number Theorem occurs as a special case of Jacobi's Triple Product Identity, which we state due to its combinatorial importance.

$$\sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n+1)/2} = (q;q)_\infty \tag{2.10}$$

Chapter 3

Ranks

3.1 Ramanujan's and Dyson's Work

The following are the most celebrated congruences due to Ramanujan,

$$p(5k+4) \equiv 0 \pmod{5}, \tag{3.1}$$

$$p(7k+5) \equiv 0 \pmod{7}, \tag{3.2}$$

$$p(11k+6) \equiv 0 \pmod{11}. \tag{3.3}$$

We present sophisticated proofs for these great results.

Proof. For (3.1)

We define $E = (q; q)$ and $J = E^3$

Now, using Euler's theorem,

$$E = \sum_{-\infty}^{\infty} (-1)^n q^{\frac{(3n^2+n)}{2}}$$

And using Jacobi triple product identity,

$$J = \sum_{n \geq 0} (-1)^n (2n+1) q^{\frac{n^2+n}{2}}$$

We notice that expnents of series of E are congruent to 0, 1 or 2 modulo 5 and exponents in series of J is congruent to 0, 1 or 3 modulo 5 but whenever the exponent is congruent to 3 modulo 5, the coefficient is divisible by 5.

We can deduce that

$$\begin{aligned} E &= E_0 + E_1 + E_2, \\ J &= J_0 + J_1. \end{aligned}$$

Now,

$$\begin{aligned} \sum_{n \geq 0} p(n) q^n &= \frac{1}{E} = \frac{E^4}{E^5} = \frac{EJ}{E^5} \\ &= \frac{(E_0 + E_1 + E_2)(J_0 + J_1)}{E^5} \end{aligned} \tag{3.4}$$

Now we see the coefficients of the form $5n+4$ and arrive at

$$\sum_{n \geq 0} p(5n+4) q^{5n+4} \equiv 0 \pmod 5$$

$\square$

A Proof on similar lines can easily be deduced for the other two congruences too.

In fact the first two congruences are a result of the following generating functions

$$\sum_{k=0}^{\infty} p(5k+4)q^k = 5\frac{(q^5;q^5)_\infty^5}{(q;q)_\infty^6}$$
$$\sum_{k=0}^{\infty} p(7k+5)q^k = 7\frac{(q^7;q^7)_\infty^3}{(q;q)_\infty^3} + 49q\frac{(q^7;q^7)_\infty^7}{(q;q)_\infty^8}$$

3.1.1 Rank of a Partition

Rank of a partition is defined as the difference of the largest part of a partition and the number of parts of a partition i.e.

$$r(\lambda) = l(\lambda) - n(\lambda) \tag{3.5}$$

Key points to notice regarding ranks of a partition λ are $r(\lambda) = -r(\lambda*)$ and that in general values of ranks of a partitions of n can only be

$$n-1, n-3, n-4, \ldots, 1, 0, -1, -2, \ldots, 3-n, 1-n$$

Dyson denoted the number of partitions of n with rank s as $N(s,n)$, and the number of partitions of n with rank s modulo m as $N(s,m,n)$. By definition,

$$N(s,m,n) = \sum_{k=-\infty}^{\infty} N(mk+s, n).$$

Ramanujan's congruences directly relate to Dyson's notion of the rank of a partition as Dyson conjectured in [4], that

$$N(s,5,5n+4) = \tfrac{p(5n+4)}{5},$$
$$N(t,7,7n+5) = \tfrac{p(7n+5)}{7}$$

The above conjectures were based on computational evidence and in an attempt to give a combinatorial interpretation of Ramanujan's congruences which follows after summing the residue classes.

The proof was given by Atkin and Swinnerton-Dyer in [1], by establishing generating function for the expression $N(s,l,ln{+}d){-}N(t,l,ln{+}d)$ for $d = 5, 7$ and $0 \le d, s, t \le l$. They obtained the difference of all values of d for $l = 5, 7$ and found it to be 0 for $l = 5$, $d = 4$ and $l = 7$, $d = 5$ which was completely in accordance with Ramanujan's congruences.

3.1.2 M_2 Rank Differences

To define M_2 rank differences we need to consider partitions without repeated odd parts which are the number of partitions of n where none of the odd parts is repeated.

M_2 rank differences is at the heart of most of the conjectures by Mao and the one that we discuss here. The M_2 rank of a partition λ without repeated odd parts is defined as

$$\left\lceil \frac{l(\lambda)}{2} \right\rceil - n(\lambda)$$

where $l(\lambda)$ and $n(\lambda)$ are defined as above. We define various notations, $N_2(s,n)$ as the number of partitions of n with no repeated odd parts and its M_2 rank equal to s and $N_2(s,m,n)$ as the number of partitions of n with distinct odd parts and M_2 rank congruent to s modulo m.

Chapter 4

Mao's Inequalities: Proofs and Conjectures

4.1 Mao's conjectures

Mao, in [18, 19] proved several inequalities using the generating function derived for Dyson's rank on partitions modulo 10 and the M_2 rank on partitions without repeated odd parts modulo 6 and 10 including,

$$N(0, 10, 5n + 1) > N(4, 10, 5n + 1)$$

$$N_2(0, 6, 3n) + N_2(1, 6, 3n) > N_2(2, 6, 3n) + N_2(3, 6, 3n)$$

Mao also left several inequalities as conjectures for which he had computational evidence.

Conjecture 4.1.1.

$$N(0,10,5n)+N(1,10,5n)>N(4,10,5n)+N(5,10,5n) \tag{4.1}$$

$$N(1,10,5n)+N(2,10,5n)>N(3,10,5n)+N(4,10,5n) \tag{4.2}$$

$$N_2(0,10,5n)+N_2(1,10,5n)>N_2(4,10,5n)+N_2(5,10,5n) \tag{4.3}$$

$$N_2(0,10,5n{+}4){+}N_2(1,10,5n{+}4)>N_2(4,10,5n{+}4){+}N_2(5,10,5n{+}4) \tag{4.4}$$

$$N_2(1,10,5n)+N_2(1,10,5n)>N_2(4,10,5n)+N_2(5,10,5n) \tag{4.5}$$

$$N_2(1,10,5n{+}2){+}N_2(2,10,5n{+}2)>N_2(3,10,5n{+}2){+}N_2(4,10,5n{+}2) \tag{4.6}$$

$$N_2(0,6,3n+2)+N_2(1,6,3n+2)>N_2(2,6,3n+2)+N_2(3,6,3n+2) \tag{4.7}$$

4.1.1 Proofs by Alwaise et al.

Alwaise et al in [1] gave the proof for (4.1), (4.2), (4.3) and (4.4). using the generating function of rank differences derived by Mao in [18] and theorems which relies on vector partitions with a given crank.

4.2 Proof of a limited version of one of Mao's Conjectures

4.2.1 Proof by Barman and Sachdeva

A limited version of (4.7) was proved by R.Barman using the generating function derived by Mao in [19] and Theorem suggested in [2].

Theorem 4.2.1. *Mao's conjecture (4.7) is true when* $3 \nmid n+1$.

To prove the above we'll take help of Mao's M_2 rank difference generating function.

Theorem 4.2.2. *(Mao [19]). We have*

$$\begin{aligned} d(n) &:= \sum_{n\geq 0} \big(N_2(0,6,n) + N_2(1,6,n) - N_2(2,6,n) - N_2(3,6,n)\big)q^n \\ &= \frac{1}{J_{9,36}} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{18n^2+9n}}{1+q^{18n+3}} + q\frac{J_{6,36}^2 J_{18,36} J_{36}^3}{J_{3,36}^2 J_{9,36} J_{15,36}^2} \\ &+ \frac{J_{6,36} J_{18,36}^2 J_{36}^3}{2q J_{9,36} J_{3,36}^2 J_{15,36}^2} - \frac{1}{J_{9,36}} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{18n^2+9n-1}}{1+q^{18n}} \end{aligned} \tag{4.8}$$

But the most important moving part of the proof was due to Barman and Baruah in [2].

Theorem 4.2.3. *We have*

$$\varphi^2(q) + \varphi^2(q^3) = 2\varphi^2(-q^6)\frac{\chi(q)\psi(-q^3)}{\chi(-q)\psi(q^3)} \tag{4.9}$$

We look at the generating function $\sum_{n\geq 0} d(3n+2)q^n$ in (4.8). Now we just take only the exponents congruent to 2 modulo 3 in (4.7) and substitute $q \hookrightarrow q^{\frac{1}{3}}$. We get the following

Proposition 4.2.4.

$$\sum_{n\geq 0} d(3n+2)q^n = \frac{1}{qJ_{3,12}}\left(\frac{J_{2,12}J_{6,12}^2J_{12}^3}{2J_{1,12}^2J_{5,12}^2} - \sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{6n^2+3n}}{1+q^{6n}}\right) \tag{4.10}$$

The following lemma is needed to tie the proof together.

Lemma 4.2.5. *We have*

$$\frac{J_{2,12}J_{6,12}^2J_{12}^3}{J_{1,12}^2J_{5,12}^2} = \frac{\varphi^2(q)+\varphi^2(q^3)}{2} \tag{4.11}$$

Proof. The proof is straight forward manipulation using the properties of q-series and using the Theorem 4.2.3. □

This now sets up to prove Theorem 4.2.1

Proof. Using Proposition 4.2.4 and Lemma 4.2.5 and noting that all the exponents in the summation inside the parenthesis is 0 (mod 3), we have

$$\begin{aligned}\sum_{n\geq 0} d(3n+2)q^n &= \frac{1}{qJ_{3,12}}\left(\frac{J_{2,12}J_{6,12}^2J_{12}^3}{2J_{1,12}^2J_{5,12}^2} - \sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{6n^2+3n}}{1+q^{6n}}\right)\\ &= \frac{1}{qJ_{3,12}}\left(\frac{\varphi^2(q)+\varphi^2(q^3)}{4} - \frac{1}{2} + \sum_{n\geq 1} a_{3n}q^{3n}\right)\end{aligned} \tag{4.12}$$

where $a_{3n} \in Z$

Now if $3 \nmid n+1$, then

$$\begin{aligned}d(3n+2)q^n &= [q^n]\frac{1}{qJ_{3,12}}\left(\frac{\varphi^2(q)+\varphi^2(q^3)}{4} - \frac{1}{2} + \sum_{n\geq 1} a_{3n}q^{3n}\right)\\ &= [q^{n+1}]\frac{\varphi^2(q)+\varphi^2(q^3)}{4J_{3,12}}\end{aligned} \tag{4.13}$$

where $[x^k]f(x)$ denotes the coefficient of x^k in $f(x)$. Now all that we need to show is that all the coefficients of $\frac{\varphi^2(q)+\varphi^2(q^3)}{4J_{3,12}}$ are positive. This can be done using the following simplification

$$\begin{aligned}\frac{\varphi^2(q)+\varphi^2(q^3)}{4J_{3,12}} &= \frac{2+4q+4q^2+\sum_{n\geq 3} b_n q^n}{(1-q^3)(q^9,q^9,q^12;q^12)_\infty} \\ &= \left(2+4q+4q^2+\sum_{n\geq 3} b_n q^n\right)\left(\sum_{n\geq 0} q^{3n}\right)\left(1+\sum_{n\geq 0} c_n q^n\right)\end{aligned}$$

where b_i and c_i are non-negative. □

4.2.2 Remarks

The result is limited to $3n+2$ when $3 \nmid n$ but computational evidence is suggested in [3] for the remaining case too.

To prove the result for the remainder of the integers would imply that the following expression has non-negative coefficients

$$\frac{1}{1-q^{12}}\left(\frac{J_{2,12}J_{6,12}^2J_{12}^3}{2J_{1,12}^2J_{5,12}^2} - \sum_{n=-\infty}^{n=\infty} \frac{(-1)^n q^{6n^2+3n}}{1+q^{6n}}\right)$$

which would also mean that it is equivalent to showing that the following has positive coefficients for all exponents of q.

$$\frac{1}{qJ_{3,12}}\left(\frac{\varphi^2(q)+\varphi^2(q^3)}{4} - \frac{1}{2} + \sum_{n\geq 1} a_{3n} q^{3n}\right)$$

Our first attempt was to use the properties of $\varphi^2(q)$, where the coefficient of q^n counts the number of integer solutions to $a^2+b^2=n$ but we need to compare this number with the other half of the above expression which does not seem to have any nice properties like this.

We could attempt a proof along the lines of the method used by Alwaise et. al. in [1] but in the general setting removing a factor with non-negative coefficient can yield a negative coefficient which is hard to convince against.

Part 2

Chapter 5

Cubic Partition Pair Congruences

5.1 Introduction

To understand Lin's work and his conjectures, we introduce some basic definitions relating to cubic partition functions.

5.1.1 Cubic Partition Function

Chan in [6, 7, 8] introduces the cubic partition function $a(n)$

Definition 5.1.1.

$$\sum_{n=0}^{\infty} a(n)q^n := \frac{1}{(q;q)_\infty (q^2;q^2)_\infty}$$

.

On investigation the analogues of cubic partition function, Kim in [14] introduced the overcubic partition function $\overline{a}(n)$

Definition 5.1.2.

$$\sum_{n=0}^{\infty} \overline{a}(n)q^n := \frac{(-q;q)_\infty(-q^2;q^2)_\infty}{(q;q)_\infty(q^2;q^2)_\infty}$$

5.1.2 Number of Cubic Partition Pairs

Zhao and Zhong in [20] then studied the partition function $b(n)$

Definition 5.1.3.

$$\sum_{n=0}^{\infty} b(n)q^n := \frac{1}{(q;q)_\infty^2(q^2;q^2)_\infty^2}$$

.

Kim established Ramanujan type congruences $b(5n+4) \equiv 0 \pmod 5$ in [15]. There, Kim referred to $b(n)$ as the number of cubic partition pairs, since it counts pair of cubic partitions.

In [16], Kim introduced the overpartition analogue for the number of cubic partition pairs $\overline{b}(n)$

Definition 5.1.4.

$$\sum_{n=0}^{\infty} \overline{b}(n)q^n := \frac{(-q;q)_\infty^2(-q^2;q^2)_\infty^2}{(q;q)_\infty^2(q^2;q^2)_\infty^2}$$

5.2 Preliminaries

Definition 5.2.1. We will be following the below mentioned notation

$$f_t := (q^t;q^t)_\infty,\ t \in \mathbb{N} \tag{5.1}$$

$$a(q) = \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2} \tag{5.2}$$

Trivially we have,

$$f_1 = \sum_{m=-\infty}^{\infty} (-1)^n q^{m(3m+1)/2}$$

using Eulers Pentagonal number theorem from Part A.

Lemma 5.2.2. *Applying the binomial theorem, we establish the following,*

$$f_1^3 \equiv f_3 \pmod 3,$$
$$f_1^9 \equiv f_3^3 \pmod 9.$$

Lemma 5.2.3. *Now using Jacobi's triple product identity, we get*

$$\varphi(-q) = \frac{f_1^2}{f_2},$$
$$\psi(q) = \frac{f_2^2}{f_1}.$$

Lemma 5.2.4.

$$\frac{1}{f_1 f_2} = a(q^6)\frac{f_9^3}{f_3^4 f_6^3} + qa(q^3)\frac{f_{18}^3}{f_3^3 f_6^4} + 3q^2\frac{f_9^3 f_{18}^3}{f_3^4 f_6^4}, \tag{5.3}$$

$$f_1 f_2 = \frac{f_6 f_9^4}{f_3 f_{18}^2} - q f_9 f_{18} - 2q^2 \frac{f_3 f_{18}^4}{f_6 f_9^2}. \tag{5.4}$$

A proof for these can be found in [20]

Theorem 5.2.5.

$$\sum_{n=0}^{\infty} b(9n+7)q^n \equiv 18 f_1 f_2 \left(\frac{f_3^7}{f_6} + q\frac{f_6^7}{f_3}\right) - 9 f_3^2 f_6^2 \pmod{27}. \tag{5.5}$$

Proof. See [17] □

Lemma 5.2.6.

$$\sum_{n=0}^{\infty} b(9n+7)q^n \equiv 18\left(\frac{f_6f_9^4}{f_3f_{18}^2} - qf_9f_{18} - 2q^2\frac{f_3f_{18}^4}{f_6f_9^2}\right)\left(\frac{f_3^7}{f_6} + q\frac{f_6^7}{f_3}\right) - 9f_3^2f_6^2 \pmod{27}. \tag{5.6}$$

Proof. This follows immediately from combining (5.4) and (5.5). □

5.3 Lin's Work

Theorem 5.3.1. *For any $n \geq 0$,*

$$b(81n+61) \equiv 0. \pmod{27} \tag{5.7}$$

Proof. Collecting the terms from (5.6) where powers of q are multiples of 3, replacing q^3 by q, we get

$$\begin{aligned}\sum_{n=0}^{\infty} b(27n+7)q^n &\equiv 18\left(\frac{f_1^7}{f_2}.\frac{f_2f_3^4}{f_1f_6^2} - 2q\frac{f_2^7}{f_1}.\frac{f_1f_6^4}{f_2f_3^2}\right) - 9f_1^2f_2^2 \pmod{27} \\ &\equiv 18\left(\frac{f_3^6}{f_6^2} - 2q\frac{f_6^6}{f_3^2}\right) - 9\frac{f_3f_6}{f_1f_2}. \pmod{27}.\end{aligned} \tag{5.8}$$

By (5.3), we find that

$$\begin{aligned}\sum_{n=0}^{\infty} b(27n+7)q^n &\equiv 18\left(\frac{f_3^6}{f_6^2} - 2q\frac{f_6^6}{f_3^2}\right) - 9f_3f_6\left(a(q^6)\frac{f_9^3}{f_3^4f_6^3} + qa(q^3)\frac{f_{18}^3}{f_3^4f_6^4} + 3q^2\frac{f_9^3f_{18}^3}{f_3^4f_6^4}\right) \pmod{27} \\ &\equiv 18\left(\frac{f_3^6}{f_6^2} - 2q\frac{f_6^6}{f_3^2}\right) - 9f_3f_6\left(a(q^6)\frac{f_9^3}{f_3^4f_6^3} + qa(q^3)\frac{f_{18}^3}{f_3^3f_6^4}\right) \pmod{27}\end{aligned} \tag{5.9}$$

Since we don't have any term of the form q^{3k+2} in the above equation, we can equate the coefficient of q^{3k+2} and get the required result.

□

Chapter 6

Proof of Lin's Conjectures

6.1 Lin's Conjectures

In [17] Lin conjectured the following:

Conjecture 6.1.1.

$$b(81n+61) \equiv 0 \pmod{243}. \tag{6.1}$$

Conjecture 6.1.2.

$$\sum_{n=0}^{\infty} b(81n+7)q^n \equiv 9\frac{(q^2;q^2)_\infty (q^3;q^3)_\infty^2}{(q^6;q^6)_\infty} \pmod{81}, \tag{6.2}$$

$$\sum_{n=0}^{\infty} b(81n+34)q^n \equiv 36\frac{(q;q)_\infty (q^6;q^6)_\infty^2}{(q^3;q^3)_\infty} \pmod{81}. \tag{6.3}$$

6.2 Gireesh and Naika's proof

In [11], Gireesh and Naika present elegant proof for general family of congruences modulo large powers of 3 for cubic partition pairs.

They use results obtained by Cooper in [9] and the "huffing" operator H to arrive at the following general congruences.

Theorem 6.2.1. *For each* $\alpha \geq 0$ *and* $n \geq 0$

$$b\left(3^{2\alpha+1}n + \frac{3^{2\alpha+1}+1}{4}\right) \equiv 0 \pmod{3^{2\alpha}} \tag{6.4}$$

$$b\left(3^{2\alpha+4}n + \frac{3^{2\alpha+5}+1}{4}\right) \equiv 0 \pmod{3^{2\alpha+5}} \tag{6.5}$$

$$b\left(3^{2\alpha+5}n + \frac{7\cdot 3^{2\alpha+4}+1}{4}\right) \equiv 0 \pmod{3^{\alpha+6}} \tag{6.6}$$

$$b\left(3^{2\alpha+5}n + \frac{11\cdot 3^{2\alpha+4}+1}{4}\right) \equiv 0 \pmod{3^{\alpha+6}} \tag{6.7}$$

Now Lin's Conjecture 6.1.1 follows directly from (6.5) by substituting $\alpha = 0$.

Proof. From [11], we elaborate on the proof of Lin's conjecture 6.1.2

Lemma 6.2.2.

$$\frac{f_2^5}{f_1^4} = \frac{f_6^6 f_9^{10}}{f_3^{10} f_{18}^5} + 4q\frac{f_6^5 f_9^7}{f_3^9 f_{18}^2} + 9q^2\frac{f_6^4 f_9^4 f_{18}}{f_3^8} + 10q^3\frac{f_6^3 f_9 f_{18}^4}{f_3^7} + 4q^4\frac{f_6^2 f_{18}^7}{f_3^6 f_9^2} \tag{6.8}$$

Proof. From [i, p. 49, Corollary],

$$\frac{f_2^2}{f_1} = \frac{f_6 f_9^2}{f_3 f_{18}} + q\frac{f_{18}^2}{f_9} \tag{6.9}$$

In [12], Hirschhorn proves,

$$\frac{f_2}{f_1^2} = \frac{f_6^4 f_9^6}{f_3^8 f_{18}^3} + 2q\frac{f_6^3 f_9^3}{f_3^7} + 4q^2\frac{f_6^2 f_{18}^2}{f_3^6} \tag{6.10}$$

By squaring (6.9), then multiplying by (6.10), we obtain (6.8) □

Lemma 6.2.3. *From [13, Eq. 5.1], we have,*

$$\frac{f_2^4 f_3^8}{f_1^8 f_6^4} + q\frac{f_2 f_6^5}{f_1^5 f_3} \equiv 1 \pmod 9 \tag{6.11}$$

Lemma 6.2.4.

$$\begin{aligned} \sum_{n=0}^{\infty} b(27n+7)q^n &\equiv 9\frac{f_3^2 f_6^2}{f_1^1 f_2^4} \\ &\equiv 9\frac{f_3^2 f_2^5}{f_1^4 f_6} \pmod{81} \end{aligned} \tag{6.12}$$

Using (6.8) and (6.12), we get,

$$\sum_{n=0}^{\infty} b(27n+7)q^n \equiv 9\frac{f_3^2}{f_6}\left(\frac{f_6^6 f_9^{10}}{f_3^{10} f_{18}^5} + 4q\frac{f_6^5 f_9^7}{f_3^9 f_{18}^2} + q^3\frac{f_6^3 f_9 f_{18}^4}{f_3^7} + 4q^4\frac{f_6^2 f_{18}^7}{f_3^6 f_9^2}\right) \pmod{81} \tag{6.13}$$

Through which we get,

$$\begin{aligned} \sum_{n=0}^{\infty} b(81n+7)q^n &\equiv 9\frac{f_1^2}{f_2}\left(\frac{f_2^6 f_3^{10}}{f_1^{10} f_6^5} + +q\frac{f_2^3 f_3 f_6^4}{f_1^7}\right) \\ &\equiv 9\frac{f_2 f_3^2}{f_6}\left(\frac{f_2^4 f_3^8}{f_1^8 f_6^4} + +q\frac{f_2 f_6^5}{f_1^5 f_3}\right) \pmod{81} \end{aligned} \tag{6.14}$$

Using (6.11) in (6.14), we arrive at (6.2)

Taking the terms from (6.13) in which the powers of q are congruent to 1 modulo 3, then dividing by q, and then finally replacing q^3 by q, we get,

$$\begin{aligned}\sum_{n=0}^{\infty} b(81n+34)q^n &\equiv 36\frac{f_1^2}{f_2}\left(\frac{f_2^5 f_3^7}{f_1^9 f_6^2} + +q\frac{f_2^2 f_3 f_6^7}{f_1^6 f_3^2}\right) \\ &\equiv 36\frac{f_1 f_6^2}{f_3}\left(\frac{f_2^4 f_3^8}{f_1^8 f_6^4} + q\frac{f_2 f_6^5}{f_1^5 f_3}\right)\end{aligned} \tag{6.15}$$

Using (6.11) in (6.15), we arrive at (6.3).

And thus we complete the demonstration of the proof for Conjecture 6.1.2

□

Bibliography

[1] E. Alwaise, E. Iannuzzi, and H. Swisher. A proof of some conjectures of Mao on partition rank inequalities . *The Ramanujan Journal, 43:3, 633-648*, 2017.

[2] R. Barman and N.D. Baruah. Certain Theta-function Identities and Ramanujan's Modular Equations of Degree 3 . *Indian Journal of Mathematics*, 48 (1):113–133, 2006.

[3] R. Barman and A. P. S. Sachdeva. Proof of a limited version of Mao's partition rank inequality using a theta function identity . *Research in Number Theory, 2:22*, 2016.

[4] B.C. Berndt. Ramanujans Notebooks. *Springer-Verlag, Part III*, 1991.

[5] B.C. Berndt. Number Theory in the Spirit of Ramanujan . *American Mathematical Society, 1-38*, 2006.

[6] H.C. Chan. Ramanujans cubic continued fraction and a generalization of his most beautiful identity. *Int. J. of Number Theory*, 6:819–834, 2010.

[7] H.C. Chan. Ramanujans cubic continued fraction and Ramanujan type congruences for a certain partition function. *Int. J. of Number Theory*, 6:819–834, 2010.

[8] H.C. Chan. Distribution of a certain partition function modulo powers of primes. *Acta Math. Sin.*, 27:625–634, 2011.

[9] S. Cooper. Series and iterations for 1/p. *Acta Arithmetica*, 141:33–58, 2010.

[10] F.J. Dyson. Some guesses in the theory of partitions . *Eureka (Cambridge) vol. 8, 10-15*, 1944.

[11] D.S. Gireesh and M.S.M. Naika. General family of congruences modulo larrge powers of 3 for cubic partition pairs. *New Zealand Journal of Mathematics*, 47:43–56, 2017.

[12] M.D. Hirschhorn and J.A. Sellers. Infinitely many congruences modulo 5 for 4-colored Frobenius partitions. *Ramanujan J.*, 40 (1):193–200, 2016.

[13] S. Hwan Son. Cubic Identities Of Theta Functions. *The Ramanujan Journal*, 2:303–316, 1998.

[14] B. Kim. The overcubic partition function mod 3. *Lecture Note Series of the Ramanujan Mathematical Society*, 14:157–163, 2010.

[15] B. Kim. Partition statistics for cubic partition pairs. *Electron. J. Combin.*, 18:128, 2011.

[16] B. Kim. On partition congruences for overcubic partition pairs. *Commun. Korean Math. Soc.*, 27:477–482, 2012.

[17] B.L.S. Lin. Congruences modulo 27 for cubic partition pairs. *Journal of Number Theory*, 171:31–42, 2017.

[18] R. Mao. Ranks of partitions modulo 10 . *J. Number Theory, 133(11):3678-3702*, 2013.

[19] R. Mao. The M2-Rank of partitions without repeated odd parts modulo 6 and 10. *Ramanujan J., 87(2):394-479*, 2015.

[20] H. Zhao and Z. Zhong. Ramanujan type congruences for a partition function. *Electron. J. Combin.*, 18:58, 2011.