Master's thesis : Option pricing in discrete time

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In this thesis we have decided to study the option pricing in discrete time, starting with the basic ideas related to the option theory in discrete time, including the fundamental theorems of asset pricing. Then, we applied these results by exposing the Cox-Ross-Rubinstein model, which provides an efficient computational method for option pricing, this allows us to price several kind of options. The last part is devoted to the study of incomplete markets within the framework of the Trinomial model which allows to highlight the problems due to this incompleteness such as the non-uniqueness of the initial price of a self-funded portfolio or the absence of a strategy to hedge perfectly the seller's position, before proposing imperfect hedging solutions. [1]

[1] I would like to thank my supervisor Stéphane Villeneuve for his guidance, patience and availability throughout this project that helped make some sense of the confusion more than once.

1 Introduction

In this section we will introduce financial markets and give important facts on it. Financial markets are at the heart of financial theory. The question that is of interest about financial market is the price of products that are exchanged on these markets. There exists several types of financial market and we will focus in the sequel on options market. An option is contingent claim, it means that its payoff depends on the value of the underlying asset, this could be a barrel of Brent Crude Oil, a stock of a tech company. The idea is to use them to hedge a position by having the right to choose at what price you will buy or sell the underlying. For this reason, they are extremely used by practitioners because it is a way to eliminate the risk associated to a trading strategy. We shall give formal definitions of an option.

Definition 1.1. *A European option is a financial contract that gives the owner the right but not the obligation to trade the underlying asset at a specified price K at a future date T.*

This is the most simple case of option since the right is at a future date T which is fixed, however variants exist: an option can allow the owner of the option to use this right at any moment, this is the case of the American option.

Definition 1.2. *An American option is a financial contract that gives the owner the right but not the obligation to trade the underlying asset at a specified price K at any period since the holding of the option until the date T.*

We should now talk about the payoff of these options. By "trade" in the previous definitions we mean either buy or sell the underlying. When the option allows the owner to buy the underlying asset, we talk about a **call** option, if it is a right to sell the underlying asset, we talk about a **put** option. This applies for all types of options we will mention here. As we have said, the payoff of the option depends on the value of the underlying, and since it gives the right to the owner to trade a specified price K, it seems natural to define the payoff of the option when it is exercised by

$$Y_t = \begin{cases} \max(S_t - K, 0) & \text{if it is a Call option} \\ \max(K - S_t, 0) & \text{if it is a Put option} \end{cases}$$

Indeed, the owner of a Call option expects the price of the underlying to increase and want to hedge his position by ensuring him a maximum price K he is willing to pay, if his expectation turns out to be wrong, then he just has to not use his right. A symmetric reasoning applies to the owner of a Put option and we derived the payoff formula given above. A such interesting way to hedge his position leads to the following question, at what price the owner of an option should sell it ? Now we introduce the payoff of a certain kind of option we will mention in the sequel which is the Asian option that has the particularity to have a payoff at time t that depends not only on the value of the underlying at time t but also on the value of the underlying preceding the time t which is

$$Y_t = \begin{cases} \max\left(\frac{1}{t+1}\sum_{i=0}^{t} S_i - K, 0\right) & \text{if it is a Call option} \\ \max\left(K - \frac{1}{t+1}\sum_{i=0}^{t} S_i, 0\right) & \text{if it is a Put option} \end{cases} \tag{1}$$

Now, we need to define what is a portfolio and thus introduce his link with a financial market. A portfolio is a combination of different assets that we detain for a given amount of time, each asset being hold in different proportions. In a given market, the number of assets is finite and their price at a future period is not known with certainty, this motivates to think of the price of an asset to be a random variable and one asset is very special : the no risky asset that we will explain later.
Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where $\omega \in \Omega$ represents the different pathways possible concerning the evolution of the price of an asset and $\mathcal{F}$ represents the knowledge we have about the potential issue, since Ω is discrete, $\mathcal{F}$ could be $\mathcal{P}(\Omega)$ and there is a probability measure such that for all $\omega \in \Omega$ we have $\mathbb{P}(\omega) > 0$. The way an investor want to manage his portfolio can be defined as follows

Definition 1.3. *The stochastic process*

$$\theta_t = (\theta_t^0, \theta_t^1, ..., \theta_t^N), \quad t \in \mathbb{N} \tag{2}$$

describes the proportion of each of the N assets holds by an investor. This process is called a strategy. Moreover, if we define $(\mathcal{F}_t)_{t\in\mathbb{N}}$ to be a filtration on $(\Omega, \mathcal{F})$ we have that θ_t is $\mathcal{F}_{t-1}$-measurable.

The fact that θ is a predictable process is not surprising. If at day 1 we decide to follow a given strategy, the next day we could change this strategy for sure but it depends on the price's assets evolution such that at day 2, our portfolio is composed given the price and the information that were avalaible at day 1 and so on. In other words, the strategy one makes at day t depends on the information available at day $t-1$.

Remark 1.4. *If we suppose $t \in \{0, 1\}$ and $\mathcal{F}_0 = (\emptyset, \Omega)$, that is a one period model, the strategy is fully deterministic.*

Definition 1.5. *The stochastic process*

$$S_t = (S_t^0, S_t^1, ..., S_t^N), \quad t \in \mathbb{N} \tag{3}$$

is the price process of the N assets that composed the strategy. We have that $(S_t)_t$ is $\mathcal{F}_t-$measurable.

Now we should talk about the risk free asset and assumptions we will assume on its price representation in the sequel. While investing in financial assets, it is useful to have a benchmark concerning the profitability of these assets and to compare this investment with an investment that is no risky, it is what we call in economics the opportunity cost. The risk free asset is this benchmark, it corresponds to the amount of money one put in a bank account. In the price process, we denote S_t^0 to be the risk free asset and we assume that its initial value is $S_0^0 = 1$ which does not hurt reality since at time 0 we cannot yet receive the risk free rate for the holding of this asset and we assume $S_t^0 = (1+r)^t$ where r is a fixed rate. We denote the value of the portfolio

$$V_t(\theta) = \langle \theta_t, S_t \rangle = \sum_{i=0}^{N} \theta_t^i \cdot S_t^i \tag{4}$$

Through the time, the value of a money varies, when we want to take a financial decision it relies certainly on a future period but to make the decision optimal and in order to make comparison available between different financial opportunities with different times horizon, it seems interesting to look at the discounted value generated by those projects, it works also for a portfolio. We denote the discounted value of the portfolio

$$\tilde{V}_t(\theta) = \langle \theta_t, \tilde{S}_t \rangle \tag{5}$$

where $\tilde{S}_t = \frac{S_t}{S_t^0}$. From that, we see the discounted value of the portfolio is likely to be less than its initial value, it makes sense with the economic theory that argues individuals have time preference concerning money, they want to first spend money and then save money for latter, which imposes to r to be positive.

When an investor holds a portfolio, from a period to another he has the possibility to reorganize his portfolio by changing the strategy, it means he has the possibility to invest more money in its portfolio or to retire money by selling some assets, however in finance, for an investor, it is likely to be that his portfolio will be fully reinvested during a long time since it is not a short term investment, we traduce this idea by this condition

$$\forall t \in \mathbb{N}, \quad \langle \theta_t, S_t \rangle = \langle \theta_{t+1}, S_t \rangle \tag{6}$$

which we call a self-financed strategy.

Example 1.6. *Now we will give an example that shows how the self financing strategy is important and also how an agent can hedge against the risk of selling an option. Consider a market with two assets, the risk free asset and a risky asset and 2 periods. The price of the risk free asset is $S_t^0 = (1+r)^t$ and the initial price of the risky asset is $S_0^1 = 40$. We are interested in a way to hedge the seller of a European call option with maturity $T = 2$ and $K = 40$. We put $r = 0$ for simplicity, this means that $S_t^0 = 1$. For $t \in \{0, 1\}$, $S_{t+1}^1 = S_t^1 \cdot \frac{3}{2}$ or $S_{t+1}^1 = S_t^1 \cdot \frac{1}{2}$. This mean at each period there are two possible outcomes, we associate ω_1 to the increase of $\frac{3}{2}$ and ω_2 to the decrease of $\frac{1}{2}$ of the stock price.*

$$\begin{cases} V_2(\theta)(\omega_1\omega_1) = \theta_1^0 + \theta_1^1 S_2^1(\omega_1\omega_1) = \theta_1^0 + \theta_1^1 90 = 50 \\ V_2(\theta)(\omega_1\omega_2) = \theta_1^0 + \theta_1^1 S_2^1(\omega_1\omega_2) = \theta_1^0 + \theta_1^1 30 = 0 \end{cases} \iff \begin{cases} \theta_1^0 = -25 \\ \theta_1^1 = \frac{5}{6} \end{cases}$$

$$\begin{cases} V_2(\theta)(\omega_2\omega_1) = \theta_1^0 + \theta_1^1 S_2^1(\omega_2\omega_1) = \theta_1^0 + \theta_1^1 30 = 0 \\ V_2(\theta)(\omega_2\omega_2) = \theta_1^0 + \theta_1^1 S_2^1(\omega_2\omega_2) = \theta_1^0 + \theta_1^1 10 = 0 \end{cases} \iff \begin{cases} \theta_1^0 = 0 \\ \theta_1^1 = 0 \end{cases}$$

This means $\theta_1 = (-25, \frac{5}{6})$ and

$$\begin{cases} V_1(\theta)(\omega_1) = -25 + \frac{5}{6} S_1^1(\omega_1) = -25 + \frac{5}{6} \cdot 60 = 25 \\ V_1(\theta)(\omega_2) = -25 + \frac{5}{6} S_1^1(\omega_2) = 0 + 0 \cdot 20 = 0 \end{cases}$$

Here, we started at $t = 2$ and we have conditioned on the first move of the risky asset's price, we clearly see that when the value of the underlying is under the strike, there is no need to hold a portfolio. Now we will use the self financing property of the strategy

$$\begin{cases} V_1(\theta)(\omega_1) = \theta_0^0 + \theta_0^1 60 = 25 \\ V_1(\theta)(\omega_2) = \theta_0^0 + \theta_0^1 20 = 0 \end{cases} \iff \begin{cases} \theta_0^0 = -\frac{25}{2} \\ \theta_0^1 = \frac{5}{8} \end{cases}$$

where $X_2(\omega_1\omega_1) = X_2(\omega_1)|X_1(\omega_1)$.
We see that the self financing strategy allows us to get a system to solve in order to find the strategy to hold at time $t = 0$. Thus, $\theta_0 = (-\frac{25}{2}, \frac{5}{8})$ and the value at time $t = 0$ of a portfolio with same payoff that the European call option is $\frac{25}{2}$. At $t = 1$ if the price of the stock A goes down to 20, then the seller will not have to pay the buy of the Call, and the owning of $\frac{5}{8}$ of the stock A will be exactly the sum necessary to payback the loan made at time $t = 0$. This

is a perfect hedging.

Now we are equipped to introduce a fundamental concept that we will use in the sequel. In economic theory, we want to study what we call to be the equilibrium of a market, it means that at this state, nobody has incentive to change his behavior or to disturb this state. In finance, we do not want to see people make money without taking risk since it cannot be an equilibrium, it leads us to define precisely what is an arbitrage opportunity by using the notion of strategy.

Definition 1.7. *A strategy is said to be admissible if it is self-financed and if we have* $V_t(\theta) \geq 0$ *for all* $t \in [0, T]$.

Definition 1.8. *An arbitrage strategy is an admissible strategy with initial value* $V_0(\theta) = 0$ *and* $\mathbb{P}(\{V_T(\theta) > 0\}) > 0$.

The idea behind an arbitrage strategy is to be able to make a gain without any resource. In financial theory, the idea of using an arbitrage opportunity has been studied and we can get the intuition easily : if there is an arbitrage opportunity, people will use it and therefore this arbitrage opportunity will be eliminate. This leads to the fundamental notion of arbitrage-free markets.

Definition 1.9. *A market is said to be arbitrage-free if it does not exist any arbitrage strategy.*

This notion is fundamental in the sense that in the sequel we will use a lot what we call an arbitrage reasoning when pricing option. The first consequence is that in an arbitrage free market, two different financial assets with same payoff will have the same price otherwise it would be easy by mean of short selling and invest in the risk free asset to construct an arbitrage strategy.

Example 1.10. *If we take the example 1.6 and assume that the market is arbitrage free, then a price for the option is the price of the portfolio that gives the same payoff, that is* $\frac{25}{2}$.

Before introducing a characterization of an arbitrage-free market, we recall the definition of a martingale. The idea behind a martingale is to have a process which represents a fair game, that is if we take a sequential game, the best expectation one can make about the reward of the next round is exactly the reward of the current round. In finance, it means there is no bullish or bearish movement.

Definition 1.11. *Consider a filtered probability space* $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$*. A martingale is a stochastic process* $X = (X_t)_{t\in\mathbb{N}}$ *such that*

- $\mathbb{E}[|X_t|] < \infty, \ \forall\, t \in \mathbb{N}$
- *The process* X *is* $\mathcal{F}_t$*-adapted*
- $\mathbb{E}[X_{t+1}|\mathcal{F}_t] = X_t, \ \forall\, t \in \mathbb{N}$

By the intuition we gave about martingale, it seems reasonable to think of an arbitrage-free market as a market in which nothing can be extract from the prices in order to make a gain. This motivates the following theorem known as the first fundamental theorem of asset pricing.

Theorem 1.12. *A market is said to be arbitrage-free if and only if there exists a probability measure* π *under which the discounted asset prices are martingales.*

Remark 1.13. *A probability measure* π *is also called a risk neutral measure probability.*

For a proof of this result, see [5]. However, we can give an intuition of this result using the reverse implication. Indeed, if there exists a probability π under which prices are martingale and there is an arbitrage strategy, say θ, it means that the value of the portfolio is a martingale transform, it implies that

$$\mathbb{E}_\pi[V_T(\theta)|\mathcal{F}_0] = V_0(\theta) = 0 \tag{7}$$

which implies that $V_T(\theta) = 0$ since θ is an admissible strategy. We see how martingales annihilate a possible arbitrage strategy.
As we have seen before, payoffs of Call and Put option depend on the price of the underlying which is a random variables, this means that Y_t the payoff is also a random variable. From now, we will denote by H this random variable. The standard way to price an option lies on the replication of H : we want to be able to hold a portfolio that will be exactly equal to H at its final date in order to simulate his payoff. More formally, we have

Definition 1.14. *H can be replicate if there exists an admissible strategy θ such that $V_T(\theta) = H$.*

This leads to the notion of complete market and the second fundamental theorem of asset pricing, the proof of the latter can be found in [5].

Definition 1.15. *A market is said to be complete if any asset with payoff H can be replicate.*

Theorem 1.16. *An arbitrage-free market is said to be complete if and only if there exists a unique probability π such that the discounted asset prices are martingales.*

The notion of completeness is fundamental in order to duplicate any options in the market. The intuition behind such assumption is that there is no transaction cost and it is easy to sell or buy an asset, thus it is hard to consider this as a realistic assumption but it allows us to solve the computational problem of pricing and hedging an option so we use it. It leads us to express the price of the portfolio as an expectation under a unique suitable probability measure which is the fundamental pricing formula under arbitrage free and complete market.

Consider H to be the payoff of a European Call option. The payoff is random and $\mathcal{F}_T$-measurable and there exists a strategy θ such that $V_T(\theta) = H$, we assume the risk free rate r to be constant and we are in an arbitrage free and complete market. Then, under the unique risk neutral probability measure π we have

$$\frac{V_t(\theta)}{(1+r)^t} = \mathbb{E}_\pi[\frac{V_T(\theta)}{(1+r)^T}|\mathcal{F}_t] \implies V_t(\theta) = \frac{1}{(1+r)^{T-t}}\mathbb{E}_\pi[H|\mathbb{F}_t]$$

since $V_t(\theta)$ is a martingale transform (linear combination of a martingale and a predictable process which is θ here). Thus, if the market is arbitrage-free and complete, we can duplicate in this manner any payoff.

Now that we have introduce the martingale and their characterization of the price process in an arbitrage-free and complete market, we have a powerful tool to price options. A model introduced in 1979 named after his authors Cox, Ross and Rubinstein is an interesting and practical application of this characterization.

2 Cox-Ross-Rubinstein model

The Cox-Ross-Rubinstein model appeared after the famous Black-Scholes formula introduced in 1973 to price option. The biggest advantage of this approach which is in discrete time is the computational efficiency.

Consider a T periods model with two assets, one is risked the other is not. The initial risky asset's price S_0^1 is known and at each period t the price has two pathways for the next period : $S_t u$ or $S_t d$ with $0 < d < u$. The free risk asset initial value is $S_0^0 = 1$ and for all $t \in [0, T]$ we have $S_t^0 = (1+r)^t$ where r is the risk free rate which is constant over time. The sample space is $\Omega = \{u, d\}^T$, clearly it is all pathways possible by the risky asset's price process. This framework is the one imposed by a binary tree, we consider the filtration generated by $(S_t)_t$ which imposes $\mathcal{F}_0 = \{\emptyset, \Omega\}$ since S_0 is constant. We assume $\mathbb{P}(\{\omega\}) > 0$ for each $\omega \in \Omega$. We denote by W_t the return of the asset from the period $t-1$ to the period t.

We start by showing that the market is arbitrage free if and only if $1 + r \in]d, u[$. Consider that $1 + r \leq d$, at time $t = 0$ we can borrow S_0^1 of the risk free asset in order to buy one unit of the risky asset. Then at time T, we have that the final portfolio value is at least equal to

$$V_T(\theta) = -S_0^1(1+r)^T + S_0^1 d^T \geq 0 \tag{8}$$

since $1+r \leq d$ and we know that $\mathbb{P}(\{V_T(\theta) > 0\}) > 0$, this shows us an arbitrage opportunity. Now, if $1+r \geq u$, clearly one can short sell one unit of the risky asset at time $t = 0$, from this exchange we will gain S_0^1, then if we put this amount in a bank account, the final portfolio value is at least equal to

$$V_T(\theta) = S_0^1(1+r)^T - S_0^1 u^T \geq 0 \tag{9}$$

since $1 + r \geq u$ and $\mathbb{P}(\{V_T(\theta) > 0\}) > 0$ which shows us an arbitrage opportunity. We conclude that $1 + r \in]d, u[$.

Now, we will give the general principle behind this model. This is a generalization of the example 1.6, the idea is that at time t, we have the price S_t^1, S_t^0 and a strategy $\theta_t = (\theta_t^0, \theta_t^1)$. We consider the time t when the strategy has not been changed yet which is important because it implies that $\theta_{t-1} = \theta_t$. Thus, if we denote $V_t(\theta)$ the value of the portfolio at time t we have

$$\langle \theta_{t-1}, S_t \rangle = \langle \theta_t, S_t \rangle = V_t(\theta) \tag{10}$$

To find the strategy at the period $t-1$, we consider the two paths possible by the underlying asset price which are $S_t^1 u$ and $S_t^1 d$ in order to solve this system

$$\begin{aligned} \theta_{t-1}^0 S_t^0 + \theta_{t-1}^1 S_{t-1}^1 u &= V_t(u) \\ \theta_{t-1}^0 S_t^0 + \theta_{t-1}^1 S_{t-1}^1 d &= V_t(d) \end{aligned} \tag{11}$$

We find

$$\begin{aligned} \theta_{t-1}^1 &= \frac{V_t(u) - V_t(d)}{S_{t-1}^1(u-d)} \\ \theta_{t-1}^0 &= \frac{V_t(d)u - V_t(u)d}{S_t^0(u-d)} \end{aligned} \tag{12}$$

By repeating this argument for each t, we see that every risky asset where the price S_t follows the binomial tree can be duplicate, hence in the CRR framework the market is complete.

Now, we would like to see what does the risk neutral probability π look like in this model. From theorem 1.12 we know that it is the unique probability measure π under which the discounted asset prices are martingales. This means we need to have

$$\mathbb{E}_\pi[\tilde{S}_t^1|\mathcal{F}_{t-1}] = \tilde{S}_{t-1}^1 \tag{13}$$

where $\tilde{S}_t^1 = \frac{S_t^1}{S_t^0}$. First, we remark that at each period, the prices of the risky asset can goes up (u) or down (d). Consider it goes up with probability p and down with probability $1-p$. Then we have

$$\mathbb{E}_\pi[\tilde{S}_t^1|\mathcal{F}_{t-1}] = p\frac{S_{t-1}^1 u}{(1+r)^t} + (1-p)\frac{S_{t-1}^1 d}{(1+r)^t} = \frac{S_{t-1}^1}{(1+r)^{t-1}} \tag{14}$$

This leads to

$$p = \frac{1+r-d}{u-d} \quad and \quad 1-p = \frac{u-(1+r)}{u-d} \tag{15}$$

and we are sure to have $p + (1-p) = 1$ which defines a probability measure and in fact $\pi = (p, 1-p)$ is the unique risk neutral probability measure for the binomial model. In order to be well defined, r need to be between d and u which is one of the assumption we have made before. If we consider $\theta_t = (\theta_t^0, \theta_t^1)$ the solution of the system (11), we can express for all $t \in [1, T]$, V_{t-1} the value of the portfolio at time $t-1$ as a function of $V_t(u)$ and $V_t(d)$

$$\begin{aligned}
V_{t-1} = \theta_{t-1}^0 S_{t-1}^0 + \theta_{t-1}^1 S_{t-1}^1 &= \left(\frac{V_t(d)u - V_t(u)d}{S_t^0(u-d)}\right) S_{t-1}^0 + \left(\frac{V_t(u) - V_t(d)}{S_{t-1}^1(u-d)}\right) S_{t-1}^1 \\
&= \frac{V_t(d)u - V_t(u)d}{(1+r)(u-d)} + \frac{V_t(u) - V_t(d)}{u-d} \\
&= \frac{V_t(d)u - V_t(u)d}{(1+r)(u-d)} + \frac{(V_t(u) - V_t(d))(1+r)}{(1+r)(u-d)} \\
&= \frac{V_t(u)(1+r-d)}{(1+r)(u-d)} + \frac{V_t(d)(u-(1+r)}{(1+r)(u-d)} \\
&= \frac{1}{1+r}[pV_t(u) + qV_t(d)]
\end{aligned} \tag{16}$$

Using the probability we have defined before. Thus we have the well-known risk neutral pricing formula

$$\forall\, t \in [0, T-1], \quad V_t = \frac{1}{1+r}[pV_{t+1}(u) + qV_{t+1}(d)] \tag{17}$$

where $q = 1-p$. Considering that V_t represents a Bernoulli process since it has two possible values at each node $V_t(u)$ or $V_t(d)$ we can see that

$$pV_{t+1}(u) + qV_{t+1}(d) = \mathbb{E}_\pi[V_{t+1}|\mathcal{F}_t] \tag{18}$$

and we derive

$$\forall\, t \in [0, T-1], \quad \tilde{V}_t = \mathbb{E}_\pi[\tilde{V}_{t+1}|\mathcal{F}_t] \implies \tilde{V}_0 = V_0 = \mathbb{E}_\pi[\tilde{V}_T|\mathcal{F}_0] \tag{19}$$

where the implication results from the fact that V_t is a martingale under π which can be done by induction and changing the sigma-field. Since the market is complete, we have

$$\mathbb{E}_\pi[\tilde{V}_T|\mathcal{F}_0] = \frac{1}{(1+r)^T}\mathbb{E}_\pi[f(S_T^1)|\mathcal{F}_0] = \frac{1}{(1+r)^T}\mathbb{E}_\pi[f(S_T^1)] \tag{20}$$

where $f(S_T^1)$ is a convex function since we are considering only European Call and Put option. The price process following a binomial structure, we get by using the Transfer theorem

$$V_0 = \frac{1}{(1+r)^T}\mathbb{E}_\pi[f(S_T^1)] = \frac{1}{(1+r)^T}\sum_{k=0}^{T}\binom{T}{k}p^k(1-p)^{T-k}f(S_0^1 u^k d^{T-k}) \tag{21}$$

Now, if we have a Call and Put option on the same underlying with same characteristics, it would be interesting to see how the pricing of both options is linked since the way we have just seen to price an option does not depend on the fact we are considering a Call or a Put option.

Now, we will establish the Put-Call parity for European options using the absence of arbitrage opportunity. We suppose the risk free interest rate r to be constant over time. Now, if the market is arbitrage-free, we know that two assets with same payoff have the same price. Knowing that, suppose we sell a European Call option and buy a European Put option with same characteristics that is T as maturity date and a strike K on the same underlying asset. For sure at time T we will have $K - S_T$ as payoff. Indeed, if the final value of the underlying S_T is greater than K we will have to pay the difference to the owner of the European Call option that is $K - S_T$ but in the other hand, we bought a Put option with same characteristics, this means that if $S_T < K$ we will exercise the Put option and the payoff will be $K - S_T$. Now we want to find a strategy that will provide the same payoff at time T. In order to get K, a fixed amount, at time T, we can just decided to put $K(1+r)^{-(T-t)}$ in a bank account which represents a cash outflow and in order to get $-S_T$ we just have to short sell the underlying asset at time t. Then, at time T we recover $K - S_T$. We conclude that this strategy allows us to get the same payoff that the one that consists to sell a European Call option and buy a European Put option, so we conclude that

$$C_t - P_t = S_t - K(1+r)^{-(T-t)} \tag{22}$$

Now we will take the same framework of the example 1.6 and show how the computation is made easier with martingale theory.

Example 2.1. *The framework is the same as example 1.6. We want to compute the value of a European call option with maturity* $T = 2$, $K = 40$ *and* $0 < d = \frac{1}{2} < u = \frac{3}{2}$. *Since this binary representation corresponds to a binomial model, we get (where* $\mathcal{F}_0$ *is the trivial sigma algebra)*

$$C_0 = \mathbb{E}_\pi[(S_2 - K)_+ | \mathcal{F}_0] = \mathbb{E}_\pi[(S_2 - K)_+] = \sum_{k=0}^{2} \binom{2}{k} p^k (1-p)^{2-k} (S_0 u^k d^{2-k} - K)_+$$

taking $p = \frac{1-d}{u-d} = \frac{1}{2}$ *we get*

$$C_0 = \mathbb{P}_\pi[S_2 = 10] \cdot 0 + \mathbb{P}_\pi[S_2 = 15] \cdot 0 + \mathbb{P}_\pi[S_2 = 90] \cdot 50 = \frac{25}{2}$$

which is the value of the portfolio we found in the example 1.6 by solving the system.

3 American Options

Now we will show how to price American call and put options. They have the same characteristic than European option except that it can be exercised until the maturity T ant not at the maturity T. For this reason, the price of an American option should be greater than the price of a European option of the same characteristic. First, we introduce the notion of stopping time. Sometimes we have to model behaviors that can stop at a precise moment and this, definitively. It is the case of an option for example, once we have exercised the right to buy or sell the underlying asset, the option no longer has any reason to be. Also, when we exercised an option, we do this using all the information we have at the moment, stopping times have this characteristic.

Definition 3.1. *Consider a filtered probability space* $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$. *A random variable* τ: $\Omega \to \mathbb{N} \cup \{+\infty\}$ *is a stopping time if*

$$\forall\, t \in \mathbb{N}, \quad \{\tau = t\} \in \mathcal{F}_t \tag{23}$$

In what follows, we suppose an arbitrage-free and complete market, and the payoff of the American option is a random variable $\mathcal{F}_t$-measurable where $(\mathcal{F}_t)_{t\in[0,T]}$ is the filtration generated by the price process of the underlying asset.

When considering an American option, we cannot use directly the martingale characterization of an arbitrage-free and complete market in order to price the option since it can be exercised at any period until the maturity. However, we know that at the maturity T, if the American option is not exercised before, the price of the option is its payoff since it is the last period : $Z_T = X_T$ where Z_t denotes the price of the American option and X_t the payoff of the American option at date t. Now if we take the period $T-1$, the seller of the option needs to face two possibilities : either the buyer of the option exercises it, or he decides to exercise it at the next period, the prices of the option needs to be such that the seller can face the two possibilities, to do so he needs to propose a price at least equal to the value of a strategy θ that duplicates the American option such that $V_T(\theta) = X_T$. Considering an arbitrage-free and complete market, we know this strategy θ exists. Then, by the martingale characterization of an arbitrage-free and complete market, we have

$$Z_{T-1} = \max\left(X_{T-1}, S^0_{T-1}\mathbb{E}_\pi\left[\frac{X_T}{S^0_T}|\mathcal{F}_{T-1}\right]\right) \tag{24}$$

but we know $X_T = Z_T$. Then by backward induction we have

$$\begin{cases} Z_T = X_T \\ Z_t = \max\left(X_t, S^0_t\mathbb{E}_\pi\left[\frac{Z_{t+1}}{S^0_{t+1}}|\mathcal{F}_t\right]\right) \quad \forall\, t \leq T-1 \end{cases} \tag{25}$$

which is called the Snell envelope of the process $(X_t)_{0\leq t\leq T}$. With the Snell Envelope, the seller of an American option is sure to propose at any period a price that hedge his position. Concerning the buyer of the American option, he will have to decide at what time he will exercise the option and this is for this reason we have motivated the notion of stopping time. To answer this question, we consider $\mathcal{T}$ the space of all stopping times with values in $[0, T]$ and the following problem

$$S = \sup_{\tau\in\mathcal{T}} \mathbb{E}_\pi[X_\tau] = \mathbb{E}_\pi[X_{\tau^*}] \tag{26}$$

where we want to find τ^*. It turns out that the buyer wants to exercise the American option when, conditionally on the information he has, he expects that an execution in the next periods will be less rewarding than the current period. This means that at this moment, we have

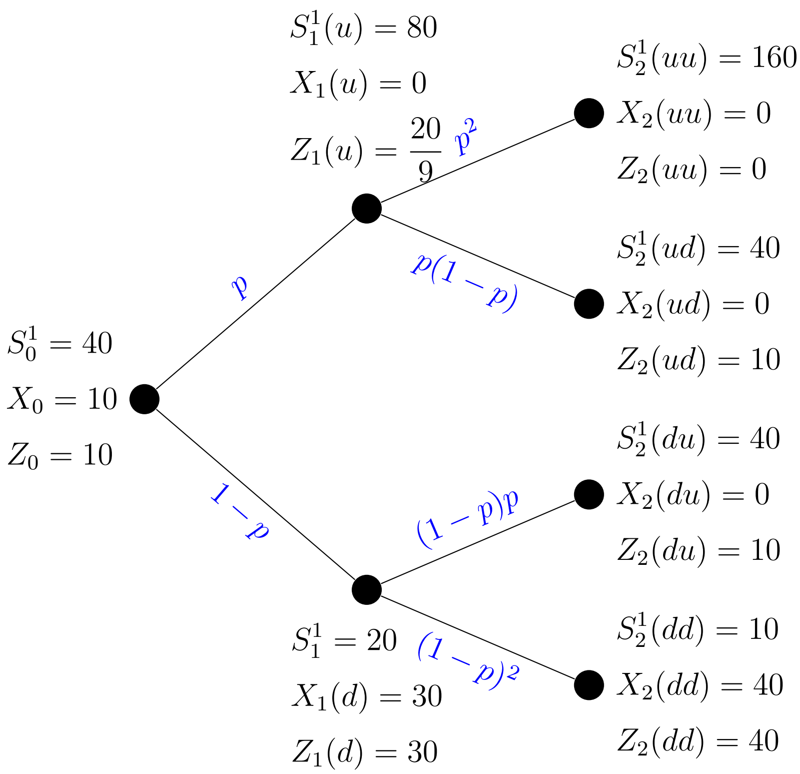
$$X_t \geq S^0_t\mathbb{E}_\pi\left[\frac{Z_{t+1}}{S^0_{t+1}}|\mathcal{F}_t\right] \tag{27}$$

Thus, we see that the following stopping time is of interest for the buyer

$$\tau^* = \min\{t \geq 0 : Z_t = X_t\} \tag{28}$$

This stopping time is also known to be an optimal stopping time satisfying equation (28) (see [5]). Moreover, it can be shown, using Doob's Optional sampling theorem, that the value of S in (28) is given by the initial value of the Snell envelope.

Example 3.2. *We consider the framework provided by the Binomial model, we want to price an American Put option. The parameters are* $T = 2, S_0^1 = 40, K = 50, u = 2, d = \frac{1}{2}, r = \frac{1}{2}, p = \frac{2}{3}, q = \frac{1}{3}$. *Using the Snell envelope, with the same notation used in the section.* $(Z_t)_{t\in[0,T]}$ *is the price proposed by the seller and* (X_t) *is the payoff of the Put option at time* t*, then the Binomial tree is*



so the initial price of American Put option is 10, which is consistent since the strike is 50, *if it would be less than* 10 *then there will be an arbitrage opportunity. Since* $Z_0 = X_0$, *the buyer has interest to exercise the Put at time* $t = 0$ *according to what we have said before.*

We end this section with a result, very intuitive, concerning the prices of European option and American options. If we take a European option and an American option with the same characteristics, we clearly see that the American option provides more flexibility for the owner of the option so we expect its price to be greater than the European option. Indeed, if we denote C_t^E and C_t^A the price of a European Call option and an American Call option respectively with same characteristics that is a maturity date T and a strike K, and furthermore the risk free rate r is constant over time, then we have by the Snell Envelope that $Z_T \geq X_T = C_T^E = C_T^A$, so this is verified for the maturity date. Now we suppose this is true for $t = T-1$, that is

$$Z_{T-1} \geq C_{T-1}^E = S_{T-1}^0 \mathbb{E}_\pi[\frac{X_T}{S_T^0}|\mathcal{F}_{T-1}] = \frac{1}{1+r}\mathbb{E}_\pi[X_T|\mathcal{F}_{T-1}]$$

Then we have again by the Snell Envelope

$$Z_{T-2} \geq S_{T-2}^0 \mathbb{E}_\pi\left[\frac{Z_{T-1}}{S_{T-1}^0}|\mathcal{F}_{T-2}\right] = \frac{1}{1+r}\mathbb{E}_\pi[Z_{T-1}|\mathcal{F}_{T-2}]$$
$$Z_{T-2} \geq \frac{1}{1+r}\mathbb{E}_\pi\left[\frac{1}{1+r}\mathbb{E}_\pi[X_T|\mathcal{F}_{T-1}]|\mathcal{F}_{T-2}\right]$$

using the induction hypothesis, and we conclude by changing the sigma field that

$$Z_{T-2} \geq \frac{1}{(1+r)^2}\mathbb{E}_\pi[X_T|\mathcal{F}_{T-2}] = C_{T-2}^E \tag{29}$$

Thus, the hypothesis is verified for $T-2$ and we conclude this is true for all $t \in [0, T]$.

4 Asian Option

Another kind of option that are interesting are the Asian options. First we need to motivate what we called exotic options, category in which belongs the Asian options. The payoff of standard options as European and American options depends only on the price of the underlying asset at the time it is exercised. This gives a powerful right to the buyer of the option and it reflects on the price of these options. To overcome this price that some agents are not ready to pay but who are ready to reduce the maneuvers offered by the option, exotic options have been created. First, we notice that an Asian option works as a European option except the payoff. This means that if we consider an Asian option with payoff H which can be replicated, we have the existence of a strategy θ such that

$$V_T(\theta) = H \quad and \quad V_0(\theta) = \mathbb{E}_\pi\left[\frac{H}{S_T^0}|\mathcal{F}_0\right] \tag{30}$$

where S_T^0 is the discounted factor. The problem is on the computation of an explicit formula for the pricing of these options since we need to take in account the previous prices of the asset, it means that at the maturity date we can have different payoffs even if the price S_T is the same. In what follows, we will place ourselves in the Cox-Ross-Rubinstein model and instead of considering

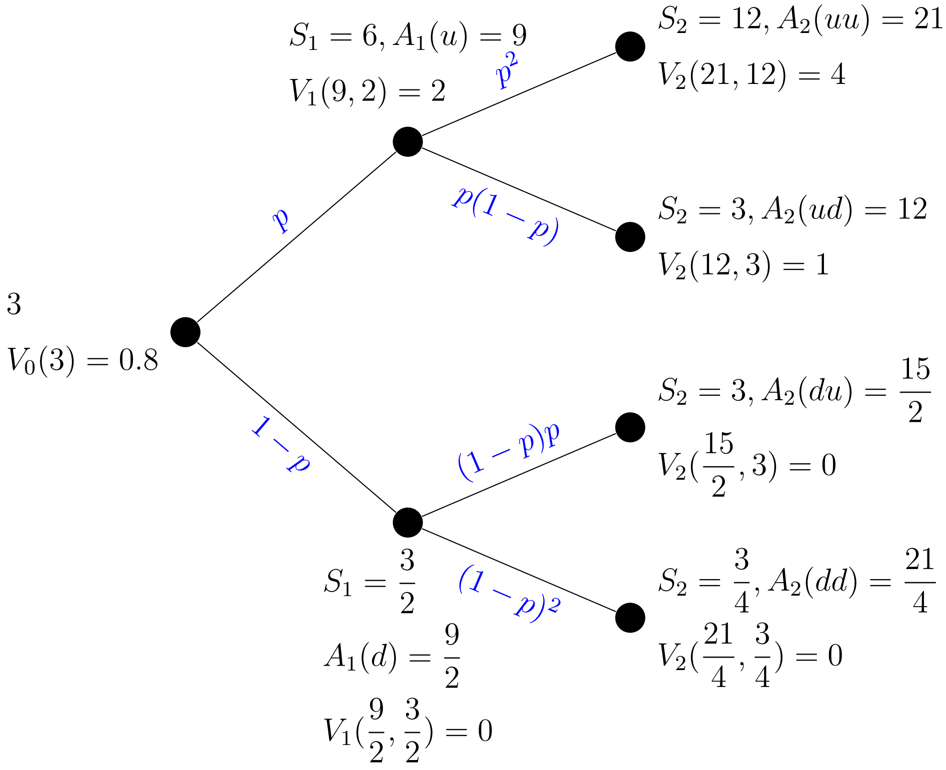
$$V_t = \frac{1}{1+r}[pV_{t+1}(u) + qV_{t+1}(d)]$$

we will write V_t as a function of all preceding values of the risky asset price

$$\forall\, t \in [0, T-1], \quad V_t(A_t, S_t) = \frac{1}{1+r}[pV_{t+1}(A_t + S_t u, S_t u) + qV_{t+1}(A_t + S_t d, S_t d)]$$

where $A_t = \frac{\sum_{k=0}^{t} S_k^1}{t+1}$. Then, by backward induction we are able to find the value of the Asian option as we did with a European option. The following example illustrates the mechanism to compute the value of the Asian option.

Example 4.1. *We will use the framework provided by the binomial model. Consider an Asian Call option on an underlying asset with initial value* $S_0 = 3$*, then at each period* t *the price goes up by* $S_{t-1}u$ *or goes down by* $S_{t-1}d$ *with* $u = 2$ *and* $d = \frac{1}{2}$*.The risk free rate is* $r = \frac{1}{4}$ *so* $p = q = \frac{1}{2}$*.*



5 Incomplete market

In this section we are interested only in European call and put options. As we have said before the complete market assumption is hard to justify from an economics point of view. The goal of this section is to remove this assumption, keeping an arbitrage-free market and see what happens concerning the pricing of options. First, if the market is incomplete, it means that there exists options that cannot be replicated, our interest is on these options. We start by giving an example to show how we fail to find the initial value of a portfolio that replicates the option.

Example 5.1. *Consider the following one period model : a market with a risky asset with inital value* $S_0^1 = 80$ *and a free risk asset with initial value* S_0^0*. The sample space is given by* $\Omega = (\omega_1, \omega_2, \omega_3)$ *such that* $S_1^1(\omega_1) = 120$*,* $S_1^1(\omega_2) = 80$ *and* $S_1^1(\omega_3) = 40$*. We consider a European Call option with maturity* $T = 1$ *and* $K = 80$*. In order to price the option we would like to solve*

$$\begin{cases} \theta_0^0 S_1^0 + \theta_0^1 120 = 40 \\ \theta_0^0 S_1^0 + \theta_0^1 80 = 0 \\ \theta_0^0 S_1^0 + \theta_0^1 40 = 0 \end{cases} \iff \begin{pmatrix} S_1^0 & 120 \\ S_1^0 & 80 \\ S_1^0 & 40 \end{pmatrix} \cdot \begin{pmatrix} \theta_0^0 \\ \theta_0^1 \end{pmatrix} = \begin{pmatrix} 40 \\ 0 \\ 0 \end{pmatrix}$$

Clearly, we cannot use what we have done so far since this system has no solution.

More generally, if we define a matrix $M \in \mathcal{M}_{n,n}(\mathbb{R})$ with column vectors that correspond to the price of each asset and row vectors that correspond to the different outcomes (or state of nature), what we have solved before can be written $M \cdot \theta = X$ where $\theta \in \mathbb{R}^n$ is the strategy and $X \in \mathbb{R}^n$ is the payoff we want duplicate. Thus, from linear algebra, we know this system has a solution (which means the market is complete) if the rank of the matrix M is the same than the rank of the augmented matrix formed from this system (Rouché-Fontené theorem). In the example above, the rank of M was 2 when the rank of the augmented matrix was 3. The condition stated above need to be respected for each node of the multi period model in order to be complete. This shows that in a such model, the market is not always complete. We need to mention the fact that if in the Cox-Ross-Rubinstein model the market is complete, it is not always the case since the probability measure under which the discounted price process is a martingale is not necessarily unique as we will see below.

5.1 One period Trinomial model

We consider a market similar to the one imposed by the binomial model except for the evolution of the prices of the risky asset which is the following

$$S_1^1 = \begin{cases} S_0^1 u & \textit{with probability } p_u \\ S_0^1 c & \textit{with probability } p_c \\ S_0^1 d & \textit{with probability } p_d \end{cases} \tag{31}$$

where $0 < d < c = 1 < u$. The new sample space is $\Omega = \{u, c, d\}$ and we consider the filtration generated by the price process that is $\mathcal{F}_1 = \sigma(S_1^1)$ and $\mathcal{F}_0 = \{\emptyset, \Omega\}$. First, we know from the example above that such a model is not always complete. However, we will try to compute a strategy that allows us to "complete the market" by adding a third asset in the model. This third asset could be for instance a European call option on the underlying risky asset with a different strike, this means that we assume we know the path process of this third asset. Then, we have $S_0 = (S_0^0, S_0^1, S_0^2)$ where

$$\forall k \in \{1, 2\}, \quad S_1^k = \begin{cases} S_0^k \cdot u_k & \textit{with probability } p_u \\ S_0^k \cdot c_k & \textit{with probability } p_c \\ S_0^k \cdot d_k & \textit{with probability } p_d \end{cases} \tag{32}$$

with $0 < d_k < c_k < u_k$. Then, the following system of linear equations

$$\begin{cases} \theta_0^0 S_1^0 + \theta_0^1 S_0^1 u_1 + \theta_0^2 S_0^2 u_2 = V_1(u) \\ \theta_0^0 S_1^0 + \theta_0^1 S_0^1 c_1 + \theta_0^2 S_0^2 c_2 = V_1(c) \\ \theta_0^0 S_1^0 + \theta_0^1 S_0^1 d_1 + \theta_0^2 S_0^2 d_2 = V_1(d) \end{cases} \tag{33}$$

has the solution

$$\begin{aligned} \theta_0^0 &= \frac{d_1(V_1^c u_2 - V_1^u c_2) + d_2(V_1^u c_1 - V_1^c u_1) + V_1^d(c_2 u_1 - c_1 u_2)}{(1+r)(d_2(c_1 - u_1) + c_2 u_1 - c_1 u_2 + d_1(u_2 - c_2))} \\ \theta_0^1 &= \frac{V_1^u c_2 - V_1^c u_2 + V_1^d(u_2 - c_2) + d_2(V_1^c - V_1^u)}{S_0^1(d_2(c_1 - u_1) + c_2 u_1 - c_1 u_2 + d_1(u_2 - m_2))} \\ \theta_0^2 &= \frac{V_1^u c_1 - V_1^c u_1 + V_1^d(u_1 - m_1) + d_1(V_1^c - V_1^u)}{S_0^2(d_2(u_1 - c_1) + d_1(c_2 - u_2) + c_1 u_2 - c_2 u_1)} \end{aligned} \tag{34}$$

where we have directly use the fact that the portfolio is self financed in order to express the initial strategy. Thus, we conclude that in this model extension, the market is complete. Moreover, we are able to propose a unique probability measure under which the discounted price process of the risky assets is a martingale. Indeed, we have

$$\begin{aligned}\mathbb{E}_\pi[\tilde{S}_1^1] &= \frac{1}{1+r}(p_u S_0^1 u_1 + p_c S_0^1 c_1 + p_d S_0^1 d_1) = S_0^1 \\ \mathbb{E}_\pi[\tilde{S}_1^2] &= \frac{1}{1+r}(p_u S_0^2 u_2 + p_c S_0^2 c_2 + p_d S_0^2 d_2) = S_0^2\end{aligned} \tag{35}$$

which imposes

$$\begin{cases} p_u u_1 + p_c c_1 + p_d d_1 = 1+r \\ p_u u_2 + p_c c_2 + p_d d_2 = 1+r \\ p_u + p_c + p_d = 1 \end{cases} \tag{36}$$

Under the risk free arbitrage hypothesis, that is $1+r \in]\min(d_1,d_2), \min(u_1,u_2)[$, we have that the risk neutral probability measure $\pi = (p_u, p_c, p_d)$ is uniquely determined by

$$\begin{aligned}p_u &= \frac{d_1(1+r-c_2) + (1+r)(c_2-d_2) - c_1(1+r-d_2)}{d_1(u_2-c_2) + u_1(c_2-d_2) - c_1(u_2-d_2)} \\ p_c &= \frac{d_1(u_2-1-r) + u_1(1+r-d_2) - (1+r)(u_2-d_2)}{d_1(u_2-c_2) + u_1(c_2-d_2) - c_1(u_2-d_2)} \\ p_d &= \frac{(1+r-c_1)(u_2-c_2) - (1+r-c_2)(u_1-c_1)}{d_1(u_2-c_2) + u_1(c_2-d_2) - c_1(u_2-d_2)}\end{aligned} \tag{37}$$

Thus, our model is now complete and risk free of arbitrage. This means we can also use the second fundamental theorem of asset pricing. We will illustrate this method with an example.

Example 5.2. *We want to price a European call option in a market with a risk free asset and a risky asset. Now, we assume the existence of a second risky asset which is a European call option on the same underlying asset, that is the first risky asset, than the one we want to price but with a different strike. This second risky asset is already priced according to a Trinomial process. The price of the underlying asset is denoted by* S_t^1 *and the price of the second risky asset is denoted by* S_t^2*. We take* $u_1 = 2, u_2 = 3, c_1 = 1, c_2 = \frac{3}{4}, d_1 = \frac{1}{2}, d_2 = \frac{1}{3}, r = 0, S_0^1 = 80$ *and* $S_0^2 = 30$*. This means that* S_1^2 *can take the values* 90, $\frac{90}{4}$ *or* 10 *and* S_1^1 *can take the values* 160, 80 *or* 40*. To find a strategy that gives us a perfect hedging, we solve*

$$\begin{cases} \theta_0^0 + 160\theta_0^1 + 90\theta_0^2 = 80 \\ \theta_0^0 + 80\theta_0^1 + \frac{90}{4}\theta_0^2 = 0 \\ \theta_0^0 + 40\theta_0^1 + 10\theta_0^2 = 0 \end{cases}$$

which has the solution $(\theta_0^0, \theta_0^1, \theta_0^2) = (\frac{80}{17}, -\frac{10}{17}, \frac{32}{17})$*. Thus, the seller of the option can get a perfect hedging by proposing the price* $\frac{80}{17} - \frac{10}{17} \cdot 80 + \frac{32}{17} \cdot 30 = \frac{240}{17}$*. Since we have completed the* *market, we can also use the risk neutral pricing formula corresponding to the model where* $\pi = (p_u = \frac{3}{17}, p_c = \frac{8}{17}, p_d = \frac{6}{17})$ *is the unique risk neutral probability measure in order to get*

$$V_0 = \frac{1}{1+r}[p_u(S_0^1 u_1 - K)_+ + p_c(S_0^1 c_1 - K)_+ + p_d(S_0^1 d_1 - K)_+] = \left[\frac{3}{17} \cdot 80\right] = \frac{240}{17}$$

Although the model is very easy to follow, we need to mention the fact that the assumptions we made are very strong and cannot be justified. Indeed, it seems not reasonable to suppose the existence of a contingent claim already priced according to a Trinomial tree. For this reason, we will go further in our study of incomplete markets by considering a new way for the seller of the option to hedge his position.

Definition 5.3. *Any self financing strategy* θ *satisfying* $V_T(\theta) \geq H$ *is called a super hedging strategy for* H*.*

By introducing this notion, the principle will be clear : if we are not able to provide a perfect hedging, we will try to provide an hedging that makes possible for the seller of the option, at a minimum, to meet its commitments. Now, we will consider the set of all portfolios initial value that are attainable and for which there exists at least one strategy such that the final value of the portfolio under this strategy dominates the payoff H of the contingent claim, that is

$$D = \{V_0 : \exists \theta \in \Theta \text{ such that } V_T(\theta) \geq H\} \tag{38}$$

where Θ is the set of all admissible strategy $(\theta_t)_{t\in[0,T]}$.In order to determine the price which is the most reasonable for the contingent claim, we need to remark that the seller will chose

$$D_+ = \inf D \tag{39}$$

as the price of the contingent claim. If it is not the case, the arbitrage free hypothesis is not respected. Indeed, consider a strategy θ such that $V_0(\theta) > D_+$. Then, the seller can propose the price $V_0(\theta)$ to the buyer in order to buy D_+ and invest $V_0(\theta) - V_+$ in the free risk asset. Then at time T, the seller will have to face H, which is possible using D_+ and he will get $(V_0(\theta) - D_+)(1+r)^T$ which is strictly positive. Thus, there is no price that can be set up to be greater than D_+ since we are in an arbitrage free market and we conclude that D_+ is an upper bound for the price of the contingent claim that has a payoff H. However, this price is not the only one in the sense that we are not considering a perfect hedging, this implies that buyers are not necessarily willing to pay this price.

The first observation we can make is that the strategy θ of buying the underlying asset is always a super hedging strategy. Consider a one period market with a risky asset and a risk free asset, we are interesting in the hedging strategy the seller of a Call option on the risky asset with strike K should follow. Denote S_t^1 the price of the risky asset at time t. We know that

$$\forall \omega_i \in \Omega,\ i \in \{1,2,3\}, \quad S_1^1(\omega_i) \geq 0 \; and \; S_1^1(\omega_i) \geq S_1^1(\omega_i) - K \tag{40}$$

since K is positive, this implies that $S_1^1 \geq (S_1^1 - K)_+$ so the value of the risky asset is greater than the payoff of the Call option. Similarly, if we consider a Put option on the risky asset we have

$$\forall \omega_i \in \Omega,\ i \in \{1,2,3\}, \quad K \geq 0 \; and \; K \geq K - S_1^1(\omega_i) \tag{41}$$

which implies that $K \geq (K - S_1^1)_+$. This shows that the strategy of holding the risky asset and the strategy of holding $\frac{K}{1+r}$ are respectively super hedging strategy for the sell of a Call and Put options. It also assures us that the set D is not empty, and that taking the infinimum is consistent with our intuition but they are not the most interesting since the amount required is too important for the buyer of the option.

Now, we will establish a procedure to find the smallest super hedging strategy in a one period Trinomial model. We denote $f(S_1^1)$ the payoff of the contingent claim, we assume that $1+r \in]d, u[$ in order to have an arbitrage free market. By the first fundamental theorem of asset pricing, we know that

$$\mathbb{E}_{\Pi}[\tilde{S}_1^1 | \mathcal{F}_0] = \frac{S_0^1}{1+r}(p_u u + p_c c + p_d d) = S_0^1 \tag{42}$$

and this gives us two conditions

$$\begin{cases} p_u u + p_c c + p_d d = 1 + r \\ p_u + p_c + p_d = 1 \end{cases} \tag{43}$$

from which we get

$$\Pi = \left\{ \pi = \left(\frac{p_c(d-c) + (1+r) - d}{u-d}, p_c, \frac{p_c(c-u) + u - (1+r)}{u-d} \right) : 0 < p_c < \frac{u-(1+r)}{u-c} \right\}$$

where we have written p_u and p_d as functions of p_c. This result shows the incompleteness of the Trinomial model by the fact that we do not have a unique risk neutral probability measure but rather a set. We deduce that the price of a self financing portfolio will depend on the value of p_c. Indeed, if we denote $V_1(\theta)$ the final value of the portfolio, we remark that

$$V_1(\theta) = \begin{cases} \theta_1^0 S_1^0 + \theta_1^1 S_0^1 u & with\ probability\ p_u \\ \theta_1^0 S_1^0 + \theta_1^1 S_0^1 & with\ probability\ p_c \\ \theta_1^0 S_1^0 + \theta_1^1 S_0^1 d & with\ probability\ p_d \end{cases} \tag{44}$$

and

$$\begin{aligned} V_0(\theta) &= \theta_0^0 S_0^0 + \theta_0^1 S_0^1 \\ &= \theta_1^0 S_0^0 + \theta_1^1 S_0^1 && by\ self\ financing\ property \\ &= \frac{1}{1+r}[\theta_1^0 S_0^0(1+r) + \theta_1^1 S_0^1(1+r)] \\ &= \frac{1}{1+r}[\theta_1^0 S_1^0(p_u+p_c+p_d) + \theta_1^1 S_0^1(p_u u + p_c c + p_d d)] && using\ (45) \\ &= \frac{1}{1+r}[p_u V_1^u(\theta) + p_c V_1^c(\theta) + p_d V_1^d(\theta)] \end{aligned} \tag{45}$$

which is useful since in the Trinomial model, by fixing the value of the parameter, we know the payoff function $f(S_1)$. Then, we try to find the strategy θ satisfying

$$\begin{aligned} \theta_0^0 S_1^0 + \theta_0^1 S_0^1 u &\geq f(S_0^1 u) \\ \theta_0^0 S_1^0 + \theta_0^1 S_0^1 &\geq f(S_0^1) \\ \theta_0^0 S_1^0 + \theta_0^1 S_0^1 d &\geq f(S_0^1 d) \end{aligned} \tag{46}$$

If we have the existence of two states of nature in which $f(S_1) = 0$ we can easily find the strategy θ. This way, we can get the minimal price of a strategy that super replicates the payoff of the option.

Moreover, if we consider the initial value of a portfolio replicating H the payoff of a contingent claim in an incomplete market and arbitrage free market, we know this price is not unique. In addition, we have seen that D_+ is an upper bound concerning the price of contingent claim otherwise an arbitrage opportunity could be set up. Since we assume to be in an arbitrage free market, it seems reasonable to have that the highest initial value of the contingent claim cannot exceed D_+. Since this initial value is a function of the risk neutral probability measure, this means that

$$\sup_{\pi\in\Pi} \mathbb{E}_\pi \left[\frac{H}{(1+r)^T} | \mathcal{F}_0 \right] \leq D_+ \tag{47}$$

where Π is the set of all risk neutral probability measure. But the reverse inequality is also true, for more details see [7]. Thus, in an incomplete and arbitrage free market we have

$$\sup_{\pi\in\Pi}\mathbb{E}_\pi\left[\frac{H}{(1+r)^T}|\mathcal{F}_0\right]=D_+ \tag{48}$$

This result is important because it links the price of a self-funded portfolio to the minimal super hedging in an incomplete and arbitrage free market, the following example illustrates these notions.

Example 5.4. *Consider a Call option and the following parameters* $u=2, d=\frac{1}{2}$, $c=1$, $r=\frac{1}{4}, S_0^1=30$ *and* $k=35$. *We know that* $f(S_0^1u)=25, f(S_0^1)=0$ *and* $f(S_0^1d)=0$. *Moreover, we have* $C_0=\frac{4}{5}[25p_u]=20p_u$ *and by (45) we have*

$$\begin{cases} 2p_u+p_c+\frac{1}{2}p_d=\frac{5}{4} \\ p_u+p_c+p_d=1 \end{cases} \implies \begin{cases} p_d=\frac{1}{2}-\frac{2}{3}p_c \\ p_u=1-p_c-p_d \end{cases} \tag{49}$$

which gives $p_u=\frac{1}{2}-\frac{1}{3}p_c$ *so we have that* $C_0=20(\frac{1}{2}-\frac{1}{3}p_c)$ *with* $0<p_c<\frac{3}{4}$. *We see that if* $p_c\in\left[\frac{1}{2},\frac{3}{4}\right[$, *then* $p_u\in\left]\frac{1}{4},\frac{1}{3}\right]$ *and* $C_0\in\left]\frac{20}{4},\frac{20}{3}\right]$ *whereas if* $p_c\in\left]0,\frac{1}{2}\right]$ *then* $p_u\in\left[\frac{1}{3},\frac{1}{2}\right[$ *and* $C_0\in\left[\frac{20}{3},\frac{20}{2}\right[$. *This shows that when* p_c *increases,* p_u *decreases and* C_0 *decreases which makes sense since if the probability of increasing for the price of the risky asset decreases, this means that it is more likely that the payoff will be* 0 *so the price of the Call option decreases, and conversely, if* p_c *decreases this means* p_u *increases, so it is more likely that the price of the risky asset at the next period will increase and the payoff will be strictly greater than* 0 *so the price of the Call option is higher.*

On the other hand, if we use (48) *we get*

$$\begin{aligned} &\frac{5}{4}\theta_0^0+60\theta_0^1\geq 25 \\ &\frac{5}{4}\theta_0^0+30\theta_0^1\geq 0 \\ &\frac{5}{4}\theta_0^0+15\theta_0^1\geq 0 \end{aligned} \tag{50}$$

however $\frac{5}{4}\theta_0^0+30\theta_0^1>\frac{5}{4}\theta_0^0+15\theta_0^1$ *if we consider that the use of the risky asset in the strategy is mandatory, which seems reasonable since we are facing several states of nature, so we can assume that* $\frac{5}{4}\theta_0^0+15\theta_0^1=0$ *and take* $\frac{5}{4}\theta_0^0+60\theta_0^1=25$ *in order to solve this system and get* *the minimal strategy that allows a super hedging :*

$$\begin{cases} \frac{5}{4}\theta_0^0 + 15\theta_0^1 = 0 \\ \frac{5}{4}\theta_0^0 + 60\theta_0^1 = 25 \end{cases} \implies \begin{cases} \theta_0^0 = -12\theta_0^1 = -\frac{60}{9} \\ \theta_0^1 = \frac{5}{9} \end{cases}$$

and the initial value of a such strategy is $V_0(\theta) = -\frac{60}{9} + \frac{150}{9} = \frac{90}{9} = 10$. *First, we see that the initial value of the minimal strategy that allows to dominate the payoff of the Call option is less than the value of the underlying asset. Moreover, by solving this system we make sure that* $\frac{5}{4}\theta_0^0 + 30\theta_0^1 \geq 0$. *The second interesting point is that*

$$\sup_{p_c \in \left(0, \frac{3}{4}\right)} C_0 = V_0(\theta) = 10 \tag{51}$$

6 Conclusion

In this master thesis, we saw that the pricing of option is made a lot easier by the introduction of a framework that allows to consider the price process as a martingale with an application due to Cox-Ross-Rubinstein in the case of a market without transaction costs in order to price several types of options. However, this last assumption is hard to justify in practice since any interesting market in finance is incomplete. To remove this restriction, we need either to make stronger assumption on the market in order to have a completed market or to consider imperfect hedging, the last one being more interesting because it does not rely on the existence of a second risky asset already priced. We have seen in the case of the seller how to get a super hedging strategy to finish with an important result about the minimal initial value of a super hedging strategy. However, a super hedging strategy, even if it is the cheapest, could have a price too high for the buyer, so it would be interesting to focus on criteria taking into account the initial value that an agent is able to bring.

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