State-Space Solutions and Properties II

Chong-Jin Ong

Department of Mechanical Engineering,
National University of Singapore

Outline

LTV System and Solution

- We now consider the LTV system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t)$$
$$y(t) = C(t)x(t) + D(t)u(t)$$

- Assumption is needed to ensure existence and uniqueness of solution - all entries of $A(t)$ are continuous functions of t.
- Like the LTI system, we begin with the scalar case:

$$\dot{x}(t) = a(t)x(t)$$

Then the solution of this equation with initial value $x(0)$ is

$$x(t) = e^{\int_0^t a(\tau)d\tau} x(0)$$

- It is easy to verify that

$$\frac{d}{dt} e^{\int_0^t a(\tau)d\tau} x(0) = a(t) e^{\int_0^t a(\tau)d\tau} x(0) = a(t)x(t)$$

LTV System and Solution

- In the case of a matrix equation, we have

$$x(t) = e^{\int_0^t A(\tau)d\tau} x(0) \tag{1}$$

- Using a series expansion of the matrix exponential, we have

$$e^{\int_0^t A(\tau)d\tau} = I + \int_0^t A(\tau)d\tau + \frac{1}{2}(\int_0^t A(\tau)d\tau)(\int_0^t A(\tau)d\tau) + \cdots$$

To verify that this is correct, we need

$$\frac{d}{dt} e^{\int_0^t A(\tau)d\tau} = A(t) + \frac{1}{2}A(t)(\int_0^t A(\tau)d\tau) + \frac{1}{2}(\int_0^t A(\tau)d\tau)A(t) + \cdots$$

However, unlike LTI system where $Ae^{At} = e^{At}A$,

$$A(t)(\int_0^t A(\tau)d\tau) \neq (\int_0^t A(\tau)d\tau)A(t)$$

Hence,

$$\frac{d}{dt} e^{\int_0^t A(\tau)d\tau} \neq A(t)e^{\int_0^t A(\tau)d\tau}$$

- Thus (1) is not a solution of $\dot{x}(t) = A(t)x(t)$. A different approach is needed.

Fundamental matrix

- Consider (1) and n linearly independent initial state $x_i(t_0)$ and n-unique solutions, $x_i(t)$, each for one of the initial states respectively.
- Arrange the solution as
$$X = [x_1 \; x_2 \; \cdots x_n]$$
Because each x_i satisfies (1),
$$\dot{X}(t) = A(t)X(t)$$
The matrix is called a *Fundamental Matrix* of (1).
- Because $x_i(t_0)$ is arbitrary, *Fundamental Matrix* is not unique.
- Fundamental matrices are non-singular for all t.
- We now define the state-transition matrix $\varphi(t, t_0)$ from (1) as
$$\varphi(t, t_0) := X(t)X^{-1}(t_0)$$
- It is easy to verify that $\varphi(t, t_0)$ is the unique solution of
$$\frac{\partial}{\partial t}\varphi(t, t_0) = A(t)\varphi(t, t_0), \quad \varphi(t_0, t_0) = I$$

Properties of State-Transition Matrix

- $\varphi(t,t) = I$.
- $\varphi^{-1}(t,t_0) = (X(t)X^{-1}(t_0))^{-1} = X(t_0)X^{-1}(t) = \varphi(t_0,t)$.
- $\varphi(t,t_0) = X(t)X^{-1}(t_1)X(t_1)X^{-1}(t_0) = \varphi(t,t_1)\varphi(t_1,t_0)$ for every t, t_0 and t_1.

Example: Consider the system

$$\dot{x}(t) = \begin{pmatrix} 0 & 0 \\ t & 0 \end{pmatrix} x(t)$$

or

$$\dot{x}_1(t) = 0, \quad \dot{x}_2(t) = t x_1(t)$$

Let $t_0 = 0$, then

$$\begin{aligned} x_1(t) &= x_1(0) \\ x_2(t) &= \int_0^t \tau x_1(0) d\tau + x_2(0) = 0.5t^2 x_1(0) + x_2(0). \end{aligned}$$

Fundamental matrix

- Thus, we have

$$x(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow x(t) = \begin{pmatrix} 1 \\ 0.5t^2 \end{pmatrix}; \quad x(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \Rightarrow x(t) = \begin{pmatrix} 1 \\ 0.5t^2 + 2 \end{pmatrix}$$

- Since the two initial states are l.i., the fundamental matrix is

$$X(t) = \begin{pmatrix} 1 & 1 \\ 0.5t^2 & 0.5t^2 + 2 \end{pmatrix} \text{ with } X^{-1}(t) = \begin{pmatrix} 0.25t_0^2 + 1 & -0.5 \\ -0.25t_0^2 & 0.5 \end{pmatrix}$$

and the state transition matrix is

$$\varphi(t, t_0) = X(t)X^{-1}(t_0) = \begin{pmatrix} 1 & 1 \\ 0.5t^2 & 0.5t^2 + 2 \end{pmatrix} \begin{pmatrix} 0.25t_0^2 + 1 & -0.5 \\ -0.25t_0^2 & 0.5 \end{pmatrix} =$$

$$\begin{pmatrix} 1 & 0 \\ 0.5(t^2 - t_0^2) & 1 \end{pmatrix}$$

Full Solution

- With the definition of $\varphi(t, t_0)$, the full solution can now be stated as

$$x(t) = \varphi(t, t_0)x(t_0) + \int_{t_0}^{t} \varphi(t, \tau)B(\tau)u(\tau)d\tau$$

To verify that this is indeed the solution, we have

$$\begin{aligned}
\frac{dx(t)}{dt} &= \frac{\partial}{\partial t}\varphi(t, t_0)x(t_0) + \frac{\partial}{\partial t}\int_{t_0}^{t} \varphi(t, \tau)B(\tau)u(\tau)d\tau \\
&= A(t)\varphi(t, t_0)x(t_0) + \int_{t_0}^{t} (\frac{\partial}{\partial t}\varphi(t, \tau)B(\tau)u(\tau))d\tau + \varphi(t, t)B(t)u(t) \\
&= A(t)\varphi(t, t_0)x(t_0) + \int_{t_0}^{t} (A(t)\varphi(t, \tau)B(\tau)u(\tau))d\tau + B(t)u(t) \\
&= A(t)[\varphi(t, t_0)x(t_0) + \int_{t_0}^{t} \varphi(t, \tau)B(\tau)u(\tau)d\tau] + B(t)u(t) \\
&= A(t)x(t) + B(t)u(t)
\end{aligned}$$