Tutorial 8 Quadratic Optimal Control

8.1 Consider a first-order system

$$\dot{x}(t) = ax(t) + u(t), \qquad x(0) = x_0 ,$$

$$y(t) = x(t) ,$$

with

$$J = \int_0^{\infty} \left[y^2(t) + ru^2(t) \right] dt .$$

find the optimal control.

Solution:
Let us assume a feedback control of the form

$$u(t) = -kx(t) ,$$

and try to find the optimum *k* by direct minimisation of the cost functional. We note first that

$$x(t) = x(0)\exp(a-k)t ,$$

so that after some algebra we can find that

$$J(k) = \begin{cases} \dfrac{-\left(1+rk^2\right)(a-k)^{-1}}{2} x^2(0) & , a-k<0 \\ \infty & , a-k>0 . \end{cases}$$

If $a < 0$, i.e., the original plant is stable, then we can choose $k = 0$ and have a finite cost $-(2a)^{-1}$. If $a > 0$ (unstable plant), then we must choose $k > a$ for a finite cost; i.e., we must move the unstable pole at $s = a$ at least into the left half plane (LHP), $s = a - k < 0$. If we just move ε into the LHP, however, $x(t) = x(0) \exp(a-k)t$, and the cost $J(k = \varepsilon + a)$ may be quite high. There will be an optimum value of *k* that will minimize the cost. To find this, we differentiate $J(k)$ with respect to *k* and set the result equal to zero to get

$$\bar{k}^2 r - \bar{k} 2ar - 1 = 0 ,$$

so that

$$\bar{k} = a \pm \sqrt{a^2 + r^{-1}} \ .$$

As $r \to \infty$ (very expensive control), we see that

$$\bar{k} \to 0 \quad or \quad 2a \, .$$

The value 0 will give an infinite cost if $a > 0$, and so in this case we must choose $\bar{k} = 2a$, so that the closed-loop pole is at $s = a - k = -a$, the mirror image of the unstable pole at $s = a$. On the other hand, if $a < 0$ (stable system), then the optimal choice is $\bar{k} = 0$, and the closed-loop pole is left at the location of the open-loop pole.

If control is cheap ($r \to 0$), then $\bar{k} = \pm\sqrt{1/r}$, but only the + sign will give a finite cost (we need $\bar{k} > a$). The cost can be made as small as we wish by choosing $\bar{k}$ large enough.

8.2 Consider the plant with transfer function $1 / s\,(s + 2)$. Its controllable form realization is

$$\dot{x} = \begin{bmatrix} -2 & 0 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \, ,$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} x \, .$$

Find the feedback gain to minimize the performance index

$$J = \int_0^\infty \left[y^2(t) + \frac{1}{9} u^2(t) \right] dt \, .$$

Solution:
This gives us $Q = c^T c$ and $R = 1/9$.
There are two methods to find the positive definite solution of the Riccati equation.
Method One: Let

$$P = \begin{bmatrix} p_{11} & p_{21} \\ p_{21} & p_{22} \end{bmatrix}.$$

It is a symmetric matrix. Thus, we have

$$\begin{bmatrix} p_{11} & p_{21} \\ p_{21} & p_{22} \end{bmatrix} \begin{bmatrix} -2 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} -2 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p_{11} & p_{21} \\ p_{21} & p_{22} \end{bmatrix}$$
$$-9 \begin{bmatrix} p_{11} & p_{21} \\ p_{21} & p_{22} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} p_{11} & p_{21} \\ p_{21} & p_{22} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Equating the corresponding entries yields

$$4p_{11} - 2p_{21} + 9p_{11}^2 = 0\,, \tag{2.1}$$

$$2p_{21} - p_{22} + 9p_{11}p_{21} = 0\,, \tag{2.2}$$

$$9p_{21}^2 - 1 = 0\,. \tag{2.3}$$

From (2.3), we have $p_{21} = \pm 1/3$. If $p_{21} = -1/3$, then the resulting P will not be positive definite. Thus we choose $p_{21} = 1/3$. The substitution of $p_{21} = 1/3$ into (2.1) yields

$$9p_{11}^2 + 4p_{11} - \frac{2}{3} = 0\,,$$

whose solutions are 0.129 and -0.68. If $p_{11} = -0.168$, then the resulting P will not be positive definite. Thus we choose $p_{11} = 0.129$. From (2.2), we can solve p_{22} as 1.05. Therefore, we have

$$P = \begin{bmatrix} 0.129 & 0.333 \\ 0.333 & 1.05 \end{bmatrix},$$

which can be easily verified as positive definite. Thus the feedback gain is given by

$$K = R^{-1}b^T P = 9[1 \quad 0]\begin{bmatrix} 0.129 & 0.333 \\ 0.333 & 1.05 \end{bmatrix} = [1.2 \quad 3], \tag{2.4}$$

and thus

$$\begin{aligned} \dot{x} = (A - bK)x &= \left(\begin{bmatrix} -2 & 0 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix} [1.2 \quad 3] \right) x \\ &= \begin{bmatrix} -3.2 & -3 \\ 1 & 0 \end{bmatrix} x. \end{aligned} \tag{2.5}$$

The characteristic polynomial of the matrix in (2.5) is

$$\det \begin{bmatrix} s + 3.2 & 3 \\ -1 & s \end{bmatrix} = s^2 + 3.2s + 3\,.$$

The quadratic optimal regulator problem using state-variable equations is closely related to the quadratic optimal transfer function. The conditions of controllability and observability are essential here.

Method Two:
One may use the following eigenvalue-eigenvector based algorithm.

Step 1: Form the $2n \times 2n$ matrix:

$$\Gamma = \begin{pmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{pmatrix},$$
$$= \begin{pmatrix} -2 & 0 & -9 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

and find its *4* eigenvalues:
-1.5811 + 0.7071i, -1.5811 - 0.7071i, 1.5811 + 0.7071i, 1.5811-0.7071.
Two are stable and two are unstable.

***Step 2*:** The eigenvectors corresponding to the two stable eigenvalues can be obtained as:

$$\begin{pmatrix} 0.8297 \\ \text{-0.4373 - 0.1956i} \\ \text{-0.0386 - 0.0652i} \\ \text{-0.1844 - 0.2061i} \end{pmatrix} \textit{and} \begin{pmatrix} 0.8297 \\ \text{-0.4373 + 0.1956i} \\ \text{-0.0386 + 0.0652i} \\ \text{-0.1844 + 0.2061i} \end{pmatrix}$$

Take out the first two rows to form v_1 and v_2, and the last two rows to form μ_1 and μ_2

$$v_1 = \begin{pmatrix} 0.8297 \\ \text{-0.4373 - 0.1956i} \end{pmatrix}$$

$$v_2 = \begin{pmatrix} 0.8297 \\ \text{-0.4373 +0.1956i} \end{pmatrix}$$

$$\mu_1 = \begin{pmatrix} \text{-0.0386 - 0.0652i} \\ \text{-0.1844 - 0.2061i} \end{pmatrix}$$

$$\mu_2 = \begin{pmatrix} \text{-0.0386 + 0.0652i} \\ \text{-0.1844 +0.2061i} \end{pmatrix}$$

Then, P is given by

$$P = [\mu_1, \dots, \mu_n][\nu_1, \dots, \nu_n]^{-1} = [\mu_1 \quad \mu_2][\nu_1 \quad \nu_2]^{-1}$$
$$= \begin{pmatrix} 0.1291 & 0.3333 \\ 0.3333 & 1.0541 \end{pmatrix}.$$

which is the same as before. From this example, you can see that the eigenvalues and eigenvectors might have complex values. But the final result P is always real valued positive definite matrix.

8.3 If

$$J = \frac{1}{2}\int_0^{\infty} \left(\|x\|_Q^2 + 2x^T W u + \|u\|_R^2 \right) dt \,,$$

is used as the criterion function, then find the optimal control u for the system

$$\dot{x} = Ax + Bu \,, \quad x(0) = x_0 \,.$$

Solution:

The given criterion function is transformed as following:

$$J = \int_0^T \left(x^T Q x + 2x^T W u + u^T R u \right) dt$$
$$= \int_0^T \left\{ x^T \left(Q - W R^{-1} W^T \right) x + \left(u + R^{-1} W^T x \right)^T R \left(u + R^{-1} W^T x \right) \right\} dt \,,$$

where

$$Q - W R^{-1} W^T \geq 0 \,,$$

is needed.

We define u_1 as

$$u_1 = u + R^{-1} W^T x \,.$$

Then, state variable equation is

$$\dot{x} = Ax + Bu$$
$$= Ax + B\left(u_1 - R^{-1} W^T x \right)$$
$$= \left(A - B R^{-1} W^T \right) x + B u_1$$
$$= A' x + B u_1 \,,$$

and the criterion function J is represented as

$$J = \int_0^T \left(x^T Q' x + u_1^T R u_1 \right) dt \ .$$

where

$$A' = A - BR^{-1} W^T ,$$
$$Q' = Q - WR^{-1} W^T .$$

New Riccati equation is

$$A'^T P + PA' + Q' - PBR^{-1} B^T P = 0 ,$$

that is equal to

$$A^T P + PA + Q - \left(PB + W \right) R^{-1} \left(W^T + B^T P \right) = 0 .$$

Therefore optimal control u is

$$u = -R^{-1} \left(W^T + B^T P \right) x .$$

8.4. Let the LQR problem have

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad R = r .$$

how does the optimal control vary with r ?

Solution:
It follows from the lecture notes that

$$u(t) = -\left[-2 + \sqrt{4 + \frac{1}{r}} \quad -3 + \sqrt{5 + 2\sqrt{4 + \frac{1}{r}}} \right] x(t) ,$$

and the poles are given by

$$\frac{1}{2} \left(-\sqrt{5 + 2\sqrt{4 + \frac{1}{r}}} \pm \sqrt{5 - 2\sqrt{4 + \frac{1}{r}}} \right) .$$

Therefore the location of the poles as r varies is shown in Figure 8.7.1.

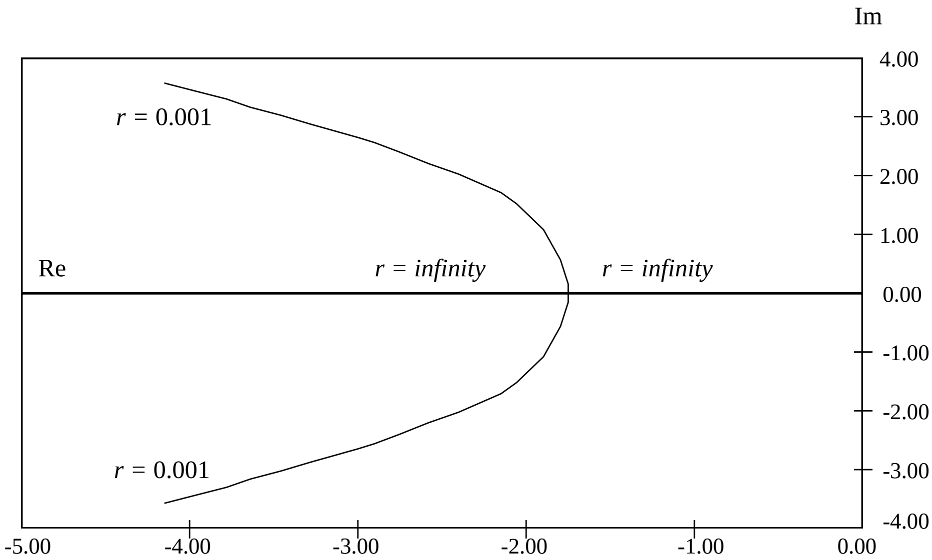

Figure 8.7.1.

It can be seen that the larger r, the higher price for the input. So the poles are closer to the origin, which means that the states converge to zero slower. If more penalty is weighed on the states, then r is smaller and the poles are further away to the left, which will lead to faster convergence of the state. But meanwhile, the cost of the input will be higher. There is no free lunch in this world. Optimal control is just to find out the best trade-off between the cost and speed.