Tutorial 12 Collection of Problems

12.1 Consider the following plant:

$$\dot{x} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u,$$

and

$$y = \begin{bmatrix} 1 & 1 \end{bmatrix} x .$$

(a) Find the state feedback $u = -Kx + v$, which assigns the closed-loop poles at $-1 \pm j$.

(b) Sketch the resultant feedback system.

(c) Obtain the closed-loop transfer function of the feedback system.

(d) Determine the output steady-state error in response to a unit step .

Solution:

(a) The characteristic polynomial with $-1 \pm j$ as its roots is

$$\{s - (-1 + j)\}\{s - (-1 - j)\} = s^2 + 2s + 2 .$$

The open-loop characteristic polynomial is

$$\det(sI - A) = \det \begin{bmatrix} s-1 & 0 \\ 0 & s+1 \end{bmatrix} = s^2 - 1 .$$

Let
$u = -Kx = -k_1 x_1 - k_2 x_2$

$$A-BK=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}-\begin{bmatrix}1\\ 1\end{bmatrix}\begin{bmatrix}k_1 & k_2\end{bmatrix}=\begin{bmatrix}1-k_1 & -k_2\\ -k_1 & -1-k_2\end{bmatrix}$$

$$\det[sI-(A-BK)]=\det\begin{bmatrix}s-1+k_1 & k_2\\ k_1 & s+1+k_2\end{bmatrix}$$

$$=(s-1+k_1)(s+1+k_2)-k_1k_2$$

$$=s^2+(k_1+k_2)+k_1-k_2-1$$

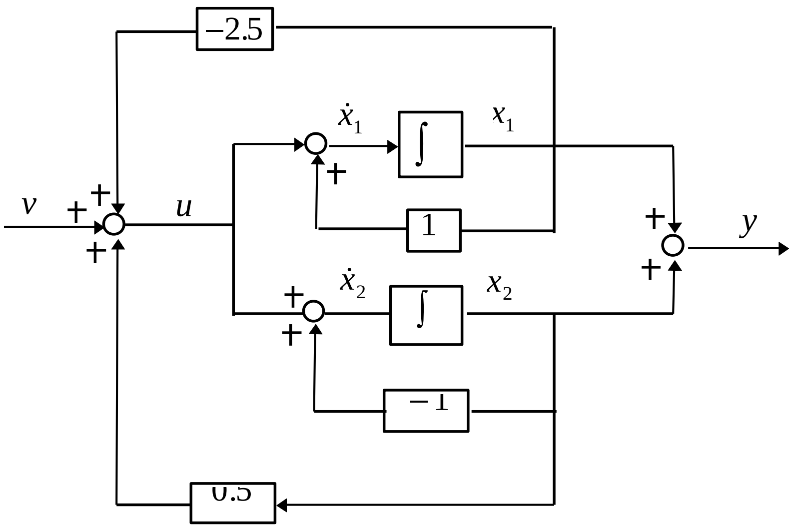
Compared it to the desired one, we have

$$k_1+k_2=2$$
$$k_1-k_2-1=2$$

So we obtain

$$k_1=2.5$$
$$k_2=-0.5$$

(b)



(c) The feedback system is described by

$$\dot{x}=Ax+Bu$$
$$u=-Kx+v$$

$$y = Cx$$

$$\therefore \qquad \dot{x} = (A - BK)x + Bv$$

$$y = Cx$$

$$\therefore \qquad G_{yv} = C\left(sI - (A - BK)\right)^{-1} B$$

$$A - BK = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 2.5 & -0.5 \end{bmatrix}$$

$$= \begin{bmatrix} -1.5 & 0.5 \\ -2.5 & -0.5 \end{bmatrix}$$

$$\left[sI - (A - BK)\right]^{-1} = \begin{bmatrix} s + 1.5 & -0.5 \\ 2.5 & s + 0.5 \end{bmatrix}^{-1} = \frac{1}{s^2 + 2s + 2} \begin{bmatrix} s + 0.5 & 0.5 \\ -2.5 & s + 1.5 \end{bmatrix}$$

$$G_{yv}(s) = C\left[sI - (A - BK)\right]^{-1} B = \frac{1}{s^2 + 2s + 2} \begin{bmatrix} s - 2 & s + 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \frac{2s}{s^2 + 2s + 2}$$

(d)

$$y(s) = G_{yv}(s)V(s)$$

Now

$$V(s) = \tfrac{1}{s}$$

$$\therefore \qquad y(s) = \frac{2s}{s(s^2 + 2s + 2)}$$

$$y(\infty) = \lim_{s\to 0} sy(s) = \lim_{s\to 0} \frac{2s}{s^2 + 2s + 2} = 0$$

$$e(\infty) = v(\infty) - y(\infty) = 1$$

Another way is to evaluate the steady state gain of $G_{yv}(s)$: $G_{yv}(0) = 0$
Therefore the steady state value of the output is zero.

12.2 Consider the double-integral plant:

$$\dot{x}_1(t) = x_2(t), \qquad x_1(0) = 1,$$
$$\dot{x}_2(t) = u(t), \qquad x_2(0) = 2.$$

The following cost function is of interest:

$$J = \frac{1}{2}\int_0^\infty \left[2x_1^2(t) + 4x_2^2(t) + 2x_1(t)x_2(t) + \frac{1}{2}u^2(t)\right]dt .$$

It is desired to solve the optimal control problem.

(a) Find the matrices A and B in the plant, and Q and R in the cost function.

(b) Obtain the positive-definite solution of the Riccati equation for this optimal control problem.

(c) Determine the optimal control law.

(d) Calculate the poles of the optimal feedback system.

(e) Find the optimal value of the cost function.

Solution:

(a) The plant can be rewritten as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} u$$

and

$$J = \frac{1}{2}\int_0^\infty \left\{ \begin{bmatrix} x_1 & x_2 \end{bmatrix} \underbrace{\begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}}_{Q} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + u \underbrace{\left[\tfrac{1}{2}\right]}_{R} u \right\} dt$$

(b) The corresponding Riccati equation is

$$\begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} - \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} 2 \begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix} = 0$$

$\Rightarrow$

$$2p_{12}^2 - 2 = 0$$

$$-p_{11} + 2p_{12}p_{22} - 1 = 0$$

$$2p_{22}^2 - 2p_{12} - 4 = 0$$

$\therefore$

$$P = \begin{bmatrix} 2\sqrt{3}-1 & 1 \\ 1 & \sqrt{3} \end{bmatrix}$$

(c) The optimal control is

$$u^* = -R^{-1}B^T Px = -2x_1 - 2\sqrt{3}x_2$$

(d) The closed-loop system is given by

$$\dot{x} = (A - BK)x = \begin{bmatrix} 0 & 1 \\ -2 & -2\sqrt{3} \end{bmatrix} x,$$

whose poles are at $-\sqrt{3} \pm 1$.

(e) The optimal value of the cost is

$$J^* = \frac{1}{2}x_0^T P x_0 = \frac{1}{2}\begin{bmatrix} 1 & 2 \end{bmatrix}\begin{bmatrix} 2\sqrt{3}-1 & 1 \\ 1 & \sqrt{3} \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \frac{1}{2}\left[6\sqrt{3}+3\right] = 6.7$$

12.3 (a) find a state feedback which decouples the plant:

$$\dot{x} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} u,$$

and

$$y = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} x.$$

(b) Derive the closed-loop transfer function of the decoupled system in (a).

(c) Design a controller in unity output feedback configuration, which decouples and stabilizes the plant:

$$G(s) = \begin{bmatrix} \frac{1}{(s+1)(s+2)} & \frac{1}{s+1} \\ \frac{-1}{s} & \frac{1}{s} \end{bmatrix}.$$

Solution:

(a) Construct

$$c_1^T B = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

$$\Rightarrow \sigma_1 = 1$$

$$c_1^T A = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 \end{bmatrix}$$

and

$$c_2^T B = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \end{bmatrix}$$

$$\Rightarrow \sigma_2 = 1$$

$$c_2^T A = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix}.$$

Then, we have

$$B^* := \begin{bmatrix} c_1^T A^{\sigma_1 - 1} B \\ c_2^T A^{\sigma_2 - 1} B \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

$$C^* := \begin{bmatrix} c_1^T A^{\sigma_1} \\ c_2^T A^{\sigma_2} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}.$$

Since B^* is nonsingular, the plant can be decoupled by the state feedback:

$$u = -Kx + Fr$$

where

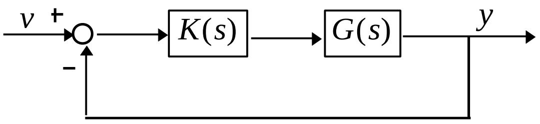
$$F = B^{*-1} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

$$K = B^{*-1} C^* = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \end{bmatrix}$$

(b) The decoupled system is given by

$$G_{yv}(s) = diag\left\{ s^{-\sigma_1},\ s^{-\sigma_2} \right\} = \begin{bmatrix} \frac{1}{s} & 0 \\ 0 & \frac{1}{s} \end{bmatrix}.$$

(c)



We note that

$$G(s) = \underbrace{\begin{bmatrix} (s+1)(s+2) & 0 \\ 0 & s \end{bmatrix}^{-1}}_{D^{-1}} \underbrace{\begin{bmatrix} 1 & (s+2) \\ -1 & 1 \end{bmatrix}}_{N}$$

and

$$\det N = 1 + (s+2) = s+3 \quad \text{stable.}$$

Thus, we let $K(s)$ be

$$K(s) = K_d(s) K_s(s)$$

Where

$$K_d(s) = \mathrm{adj(N(s))} = \begin{bmatrix} 1 & -(s+2) \\ 1 & 1 \end{bmatrix}$$

Then

$$G(s)\,\mathrm{K(s)} = \mathrm{G(s)} K_d(s) K_s(s) = \begin{bmatrix} \dfrac{s+3}{(s+1)(s+2)} & 0 \\ 0 & \dfrac{s+3}{s} \end{bmatrix} \begin{bmatrix} k_1(s) & 0 \\ 0 & k_2(s) \end{bmatrix}$$

Observing above form and note that (s+3) is stable, one simple way to design the stabilizer is

$$K_s(s) = \begin{bmatrix} \dfrac{k_1}{s+3} & 0 \\ 0 & \dfrac{k_2}{s+3} \end{bmatrix}$$

Then

$$G(s)K(s) = \begin{bmatrix} \dfrac{k_1}{(s+1)(s+2)} & 0 \\ 0 & \dfrac{k_2}{s} \end{bmatrix}$$

$k_1 = 1$ stabilizes $\frac{1}{(s+1)(s+2)}$ while $k_2(s) = 1$ stabilizes $\frac{1}{s}$. Therefore, the required decoupling controller is

$$K(s) = K_d(s)K_s(s) = \begin{bmatrix} 1 & -(s+2) \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{s+3} & 0 \\ 0 & \frac{1}{s+3} \end{bmatrix} = \begin{bmatrix} \frac{1}{s+3} & \frac{-(s+2)}{s+3} \\ \frac{1}{s+3} & \frac{1}{s+3} \end{bmatrix}.$$

12.4 Consider the following plant:

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -1 & 4 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u ,$$

and

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x .$$

(a) Show that the plant has the poles at 2, 3, -1.

(b) If two unstable poles are to be moved to -2 and -3 while keeping the stable pole unchanged, find the corresponding polynomial $\phi_c(s)$.

(c) Find the state feedback which assigns $\phi_c(s)$ as the closed-loop characteristic polynomial.

(d) Obtain the closed-loop transfer function of the feedback system.

(e) Determine the output steady-state error in response to a unit step change in the set-point.

Solution:

(a) We calculate

$$\det(sI - A) = s^3 - 4s^2 + s + 6 = (s+1)(s-2)(s-3)$$

so $s = -1,\ 2,\ 3$, are the poles of the plant.

(b) The desired poles are thus $s=-1,\ -2,\ -3$, so that the closed-loop characteristic polynomial $\phi_c(s)$ is

$$\phi_c(s)=(s+1)(s+2)(s+3)$$

$$=s^3+6s^2+11s+6$$

(c) Since (A, b) is already in controllable canonical form, the state feedback

$$\boldsymbol{u}=-\boldsymbol{K}\boldsymbol{x}$$

can be determined such that

$$\begin{bmatrix}-6 & -1 & 4\end{bmatrix}-K=\begin{bmatrix}-6 & -11 & -6\end{bmatrix}$$

$$\Rightarrow$$

$$-K=\begin{bmatrix}0 & -10 & -10\end{bmatrix}$$

$$K=\begin{bmatrix}0 & 10 & 10\end{bmatrix}$$

(d) The closed-loop transfer matrix is

$$H(s)=C(sI-A+BK)^{-1}B$$

$$=\begin{bmatrix}1 & 0 & 0\end{bmatrix}\begin{bmatrix}s & -1 & 0\\ 0 & s & -1\\ 6 & 11 & s+6\end{bmatrix}^{-1}\begin{bmatrix}0\\0\\1\end{bmatrix}$$

$$=\frac{1}{s^3+6s^2+11s+6}$$

(e) Since the system is stable, we can apply the final value theorem

$$y(\infty)=\lim_{s\to 0}sy(s)=\lim_{s\to 0}s\cdot\frac{1}{s^3+6s^2+11s+6}\cdot\frac{1}{s}$$

$$=\frac{1}{6}$$

So that

$$e(\infty) = v(\infty) - y(\infty) = 1 - \frac{1}{6} = \frac{5}{6}.$$

12.5 (a) For the plant:

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u,$$

find the optimal control which minimizes the following cost function:

$$J = \frac{1}{2}\int_0^\infty \left[x_1^2(t) + u^2(t)\right] dt.$$

(b) Design an observer for the plant:

$$\dot{x} = \begin{bmatrix} -0.5 & 0 \\ 0.2 & -0.4 \end{bmatrix} x + \begin{bmatrix} 27 \\ 0 \end{bmatrix} u,$$

and

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} x.$$

Solution:

(a) From the index, we know

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad R = 1.$$

The ARE is

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} + \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

which leads to

$$-p_2^2 + 1 = 0 \tag{i}$$

$$p_1 - p_2 p_3 = 0 \tag{ii}$$

$$2p_2 - p_3^2 = 0 \tag{iii}$$

(i) gives

$$p_2 = \pm 1$$

(ii) yields

$$p_3^2 = 2p_2$$

So

$$p_3 = \begin{cases} \pm\sqrt{2} & \text{for} \quad p_2 = 1 \\ \pm j\sqrt{2} & \text{for} \quad p_2 = -1 \end{cases}$$

The latter values for p_2 and p_3 should be discarded since they are complex but we want real solution. For $p_2 = 1$ and $p_3 = \sqrt{2}$, (ii) gives $p_1 = \sqrt{2}$. So the required solution to ARE is

$$P = \begin{bmatrix} \sqrt{2} & 1 \\ 1 & \sqrt{2} \end{bmatrix}$$

which is positive definite. The optimal control is given by

$$u^*(t) = -\begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 1 \\ 1 & \sqrt{2} \end{bmatrix} x(t)$$

$$= -\begin{bmatrix} 1 & \sqrt{2} \end{bmatrix} x(t)$$

The closed-loop state matrix is

$$A - BK^* = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & \sqrt{2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -\sqrt{2} \end{bmatrix}$$

which has eigenvalues at $-\sqrt{2}/2 \pm j\sqrt{2}/2$, and is thus stable, indeed, as predicted by the theory.

(b) Design a full-order observer:

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x})$$

$$A - LC = \begin{bmatrix} -0.5 & 0 \\ 0.2 & -0.4 \end{bmatrix} - \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} = \begin{bmatrix} -0.5 & -l_1 \\ 0.2 & -0.4 - l_2 \end{bmatrix}$$

If $l_1 = 0, l_2 = 0.6$, then

$$A - LC = \begin{bmatrix} -0.5 & 0 \\ 0.2 & -1 \end{bmatrix}$$

which has two eigenvalues at –0.5 and –1, so it is stable.

Therefore, $L = \begin{bmatrix} 0 \\ 0.6 \end{bmatrix}$ will do.

12.6 (a) Find a state feedback which decouples the plant:

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & -1 \end{bmatrix} u ,$$

and

$$y = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} x .$$

(b) Derive the closed-loop transfer function of the decoupled system in (a).

(c) Design a controller in unity output feedback configuration, which decouples and stabilizes the plant:

$$G(s) = \begin{bmatrix} \dfrac{2}{s+1} & \dfrac{1}{s+2} \\ \dfrac{1}{s(s+1)} & \dfrac{1}{s(s+2)} \end{bmatrix} .$$

Solution:

(a) The given plant has

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} .$$

we find that

$$C_1^T B = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

$$C_1^T AB = \begin{bmatrix} 1 & 1 \end{bmatrix} \quad \Rightarrow \quad \sigma_1 = 2$$

$$C_1^T A^2 = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix}$$

and

$$C_2^T B = \begin{bmatrix} 1 & 0 \end{bmatrix}, \qquad \Rightarrow \qquad \sigma_2 = 1$$

$$C_2^T A = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}.$$

Therefore, we have

$$B^* = \begin{bmatrix} C_1^T A^{\sigma_1 - 1} B \\ C_2 A^{\sigma_2 - 1} B \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

$$C^* = \begin{bmatrix} C_1 A^{\sigma_1} \\ C_2 A^{\sigma_2} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Since B^* is nonsingular, the plant can be decoupled by the state feedback:

$$u = -Kx + Fv$$

where

$$F = (B^*)^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix},$$

$$K = (B^*)^{-1} C^* = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

(b) The decoupled system has the transfer matrix:

$$G_{yv} = diag\left\{ s^{-\sigma_1},\ s^{-\sigma_2} \right\} = \begin{bmatrix} \frac{1}{s^2} & 0 \\ 0 & \frac{1}{s} \end{bmatrix}$$

(c)

$$G(s)=\begin{bmatrix}\frac{1}{(s+1)(s+2)} & 0\\ 0 & \frac{1}{s(s+1)(s+2)}\end{bmatrix}\begin{bmatrix}2(s+2) & s+1\\ s+2 & s+1\end{bmatrix}$$

and

$$\det N=(s+2)(\mathrm{s}+1) \qquad \text{stable.}$$

Thus, we let $K(s)$ be

$$K(s)=K_d(s)K_s(s)$$

Where

$$K_d(s)=\mathrm{adj(N(s))}=\begin{bmatrix}s+1 & -(s+1)\\ -(s+2) & 2(s+2)\end{bmatrix}$$

Then

$$G(s)\,\mathrm{K(s)}=\mathrm{G(s)}K_d(s)K_s(s)=\begin{bmatrix}1 & 0\\ 0 & \frac{1}{s}\end{bmatrix}\begin{bmatrix}k_1(s) & 0\\ 0 & k_2(s)\end{bmatrix}$$

Observing above form and note that (s+1) is stable, one simple way to design the stabilizer is

$$K_s(s)=\begin{bmatrix}\frac{1}{s+1} & 0\\ 0 & \frac{1}{s+1}\end{bmatrix}$$

Then

$$G(s)K(s)=\begin{bmatrix}\frac{1}{(s+1)} & 0\\ 0 & \frac{1}{s(s+1)}\end{bmatrix}$$

It can be easily verified that the closed loop is stable.
Therefore, the required decoupling controller is

$$K(s)=K_d(s)K_s(s)=\begin{bmatrix}(s+1) & -(s+1)\\ -(s+2) & 2(s+2)\end{bmatrix}\begin{bmatrix}\frac{1}{s+1} & 0\\ 0 & \frac{1}{s+1}\end{bmatrix}=\begin{bmatrix}1 & -1\\ \frac{-(s+2)}{s+1} & \frac{2(s+2)}{s+1}\end{bmatrix},$$

which is proper.

12.7 For the LQR problem with

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad R = r,$$

How does the optimal control law and closed-loop poles vary with r?

Solution:

For this problem, the optimal control is given by

$$u(t) = -\left[-2 + \sqrt{4 + \tfrac{1}{r}} \quad -3 + \sqrt{5 + 2\sqrt{4 + \tfrac{1}{r}}} \right] x(t)$$

It follows that the closed-loop system

$$\dot{x} = (A - BK)x$$

has the poles:

$$\frac{1}{2}\left(-\sqrt{5 + 2\sqrt{4 + \tfrac{1}{r}}} \pm \sqrt{5 - 2\sqrt{4 + \tfrac{1}{r}}} \right)$$

12.8 Consider the following system:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = A \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + B \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}, \quad \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = C \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

which it is assumed can be decoupled and for which $c_1 B \neq 0$ and $c_2 B \neq 0$, where c_1 and c_2 are the rows of the matrix C.

(a) Find $u = -Kx + Fr,$ such that the closed-loop system is decoupled and

(b) Determine the transfer function matrix of the decoupled closed-loop system.

Solution: We should have

$$B^* = \begin{pmatrix} c_1^T A^{\sigma_1 - 1} B \\ c_2^T A^{\sigma_2 - 1} B \\ \vdots \\ c_m^T A^{\sigma_m - 1} B \end{pmatrix}, \quad C^* = \begin{bmatrix} c_1^T A^{\sigma_1} \\ c_2^T A^{\sigma_2} \\ \vdots \\ c_m^T A^{\sigma_m} \end{bmatrix},$$

Since $c_1 B \neq 0$ and $c_2 B \neq 0$, it follows that $\sigma_1 = \sigma_2 = 1$. Thus,

$$B^* = \begin{bmatrix} c_1 B \\ \cdots\cdots \\ c_2 B \end{bmatrix} = CB \text{ and } C^* = \begin{bmatrix} c_1 A \\ \cdots\cdots \\ c_2 A \end{bmatrix} = CA.$$

Consequently,

$$F = B^{*-1} = (CB)^{-1} \text{ and } K = B^{*-1}C^{*} = (CB)^{-1} CA.$$

The transfer function of the closed-loop system is

$$H(s) = C(sI - A + BK)^{-1} BF.$$

If we expand $H(s)$ in negative power series of s, we obtain

$$H(s) = C\left[\frac{1}{s} + \frac{A - BK}{s^2} + \frac{(A-BK)^2}{s^3} + \cdots\right]BF$$

Using the foregoing expressions for F and K, the matrix $C(A - BK)$ takes on the form

$$C(A - BK) = C\left[A - B(CB)^{-1} CA\right] = CA - CB(CB)^{-1} CA = 0$$

Consequently $C(A - BK)^{i} = 0$, for $i \geq 1$ and therefore the transfer function of the closed-loop system reduces to

$$H(s) = \frac{CBF}{s} = \frac{CB(CB)^{-1}}{s} = \frac{I}{s}, \text{ where } I \text{ the unit matrix}$$

We can observe that the closed-loop system has been decoupled into two subsystems, each of which is a simple integrator.

12.9 (a) For the system

$$G(s) = \frac{1}{(s+1)(s+2)}$$

Design a stable control system with zero steady-state errors in response to step disturbance and step set-point change.

(b) Design a reduced-order observer for the system:

$$\frac{dx_1}{dt} = x_2 + u_1,$$
$$\frac{dx_2}{dt} = -4x_1 - 4x_2 + u_2,$$
$$y = x_1.$$

Solution:

(a) The plant is given as

$$G(s)=\frac{1}{(s+1)(s+2)}$$

We adopt unity feedback system, use polynomial approach to design servo controller.
Since both inputs are of step type, we have $\phi(s)=s$, and the controller takes the form of

$$D(s)=\frac{\tilde{D}(s)}{\phi(s)}=\frac{1}{s}\tilde{D}(s)$$

Try PI type controller

$$D(s)=\frac{k(s+\alpha)}{s}$$

Choose $\alpha=2$ to cancel the plant's fast pole at $s=-2$. Then, the open-loop is

$$G(s)D(s)=\frac{k}{s(s+1)}$$

The characteristic polynomial of the closed-loop is

$$s(s+1)+k=0$$
$$s^2+s+k=0$$

Take $k=1$ giving a stable system with a reasonable damping of $\xi=0.5$, and natural frequency of $\omega_n=1$.
Thus, the designed controller is

$$D(s)=\frac{s+2}{s}$$

(b) The plant

$$\dot{x}_1=x_2+u_1,$$
$$\dot{x}_2=-4x_1-4x_2+u_2,$$
$$y=x_1.$$

It can be expressed as

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}0 & 1\\ -4 & -4\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}u_1\\ u_2\end{bmatrix},$$

$$y=\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}.$$

Define $T=\begin{bmatrix}t_1 & t_2\end{bmatrix}$ and

$$\begin{bmatrix} y(t) \\ \xi(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ t_1 & t_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

The form of the observer is

$$\dot{\xi} = d\xi + eu + gy .$$

Choose $d = -3$. With $e = TB = T$, the observer now is

$$\dot{\xi} = -3\xi + t_1 u_1 + t_2 u_2 + gy$$

g and T can be solved from $TA - dT = gC$ or

$$\begin{bmatrix} t_1 & t_2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -4 & -4 \end{bmatrix} + 3\begin{bmatrix} t_1 & t_2 \end{bmatrix} = g\begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 3t_1 - 4t_2 & t_1 - t_2 \end{bmatrix} = \begin{bmatrix} g & 0 \end{bmatrix}.$$

Try $g = 1$, solve for t_1 and t_2 and check the rank of $\begin{bmatrix} 1 & 0 \\ t_1 & t_2 \end{bmatrix}$. This yields $t_1 = -1$ and $t_2 = -1$, and the matrix in question will have full rank.

This results in the final observer design as

$$\dot{\xi} = -3\xi - u_1 - u_2 + y .$$

12.10. Consider the broom-balancing system shown in the figure (a) below with

$$x(t) = \begin{bmatrix} z(t) & \dot{z}(t) & \theta(t) & \dot{\theta}(t) \end{bmatrix},$$

$$\dot{x} = Ax + bu = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 11 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} u,$$

$$y = z = c^T x = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} x.$$

completely observable from *z*. Use an observer:

$$\dot{\hat{x}} = (A - Lc^T)\hat{x} + bu + Ly,$$

with

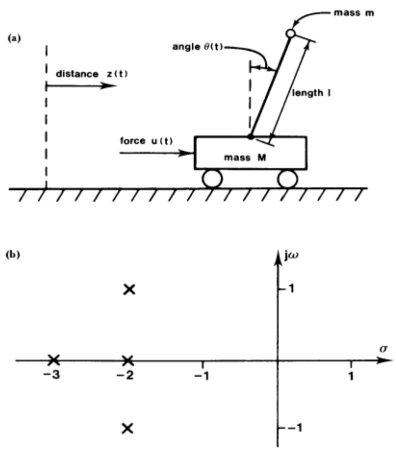
$$L = \begin{bmatrix} k_1 & k_2 & k_3 & k_4 \end{bmatrix}^{\mathrm{T}}$$

such that the poles of the observer are shown in figure (b).

Solution: One writes observer state equation in an element wise way,

$$\dot{\hat{x}} = \begin{bmatrix} -k_1 & 1 & 0 & 0 \\ -k_2 & 0 & -1 & 0 \\ -k_3 & 0 & 0 & 1 \\ -k_4 & 0 & 11 & 0 \end{bmatrix} \hat{x} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} u + \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} y.$$

must be stable



(a) Broom -balancing system and (b) desired observer pole configuration

Observer's pole configuration is given in figure (b), giving

$$\begin{aligned} \varphi_d(s) &= (s+2)(s+3)(s+2+j)(s+2-j) \\ &= s^4 + 9s^3 + 31s^2 + 49s + 30 \end{aligned}$$

We want

$$\begin{aligned} \varphi_d(s) &= \det(sI - A + Lc) \\ &= s^4 + k_1 s^3 + (k_2 - 11)s^2 \\ &\quad -(11k_1 + k_3)s - (11k_2 + k_4) \end{aligned}$$

,

which is true if we choose the coefficients as

$$k_1 = 9$$
$$k_2 = 42$$

$$k_3 = -49 + 11k_1 = -148$$
$$k_4 = -30 + 11k_2 = -464$$

Then, the observer can estimate x(t) asymptotically.

12.11.

For a plant with the transfer function,

$$G(s) = \frac{2s + a}{(s+1)(s+4)},$$

we want to design a stable servo control system such that the closed-loop system can achieve the damping ratio, $\xi = 0.5$, settling time, $\tau_s = 4$, asymptotic tracking of the set-point, $r(t) = e^{-t}$, and asymptotic rejection of the disturbance, $d(t) = e^t$.

(i) Find the servo-mechanism, *1/Q(s).*
(ii) Determine the solvability condition of this servo control problem in terms of the plant parameter, *a*.
(iii) Design a controller for this servo control problem if *a=2.*

Solution:

(i)

$$R(s) = \frac{1}{s+1}, \ \mathrm{D(s)} = \frac{1}{s-1}$$
$$\Rightarrow Q(s) = s - 1$$
$$\therefore \frac{1}{Q(s)} = \frac{1}{s-1}$$

(ii) The servo problem is solvable iff G(s) has no zero coinciding with the root(s) of Q(s), s=1, It is the case iff $a \neq -2$

(iii) There are many ways to design the controller, and therefore the answer is not unique.
Method 1:
For a=2, the generalized plant is,

$$\frac{G}{Q} = \frac{2}{(s+4)(s-1)}$$

Take the stabilizer as,

$$K_2(s) = as + b$$

Then the loop is,

$$\frac{GK_2}{Q} = \frac{2(as+b)}{(s-1)(s+4)}$$

The characteristic equation is,

$$1+\frac{GK_2}{Q}=0$$
$$(s-1)(s+4)+2(as+b)=0$$
$$s^2+(3+2a)s+2b-4=0$$

Note,

$$\xi=0.5,\ \tau_s=\frac{4}{\xi\omega_n}\Rightarrow\omega_n=\frac{4}{\xi\tau_s}=2$$
$$\therefore s^2+2\xi\omega_n s+\omega_n^2=s^2+2s+4=0$$

Comparing the two equations above, we get,

$$a=-\frac{1}{2},\ \mathrm{b}=4$$

Therefore the overall controller is,

$$K(s)=\frac{K_2}{Q}=\frac{-0.5s+4}{s-1}$$

Method 2:
As the generalized plant is,

$$\frac{G}{Q}=\frac{2}{(s+4)(s-1)}$$

It can be treated as a strictly proper plant transfer function with n=2. A compensator with the order of 1 is designed,

$$K_2(s)=\frac{b_1s+b_0}{s+a_0}$$

So the loop is,

$$\frac{G}{Q}K_2=\frac{2(b_1s+b_0)}{(s+4)(s-1)(s+a_0)}$$

The characteristic equation is,

$$s^3+(a_0+3)s^2+(3a_0+2b_1-4)s+(2b_0-4a_0)=0$$

It is of third order. We place the extra pole far away by making $\lambda=5$,

$$(s+\lambda\omega_n)(s^2+2\xi\omega_n s+\omega_n^2)=(s+10)(s^2+2s+4)=0$$
$$s^3+12s^2+24s+40=0$$

Comparing the two equations above gives,

$$K_2(s)=\frac{0.5s+38}{s+9}$$

Thus, the overall controller is,

$$K(s)=\frac{K_2}{Q}=\frac{0.5s+38}{(s+9)(s-1)}$$

Method 3:
Here the servo-mechanism is chosen as,

$$\frac{1}{Q(s)} = \frac{1}{s^2 - 1}$$

The generalized plant becomes,

$$\frac{G}{Q} = \frac{2}{(s+4)(s^2-1)}$$

To cancel the stable pole, the stabilizer is taken as,

$$K_2(s) = (as+b)(s+4)$$

The loop becomes,

$$\frac{G}{Q}K_2 = \frac{2(as+b)}{(s^2-1)}$$

The characteristic equation is second-order now,

$$s^2 + 2as + 2b - 1 = 0$$

To get the desired characteristic equation like what is in Method 1, we have,

$$K_2(s) = (s+4)(s+2.5)$$

The overall controller is,

$$K(s) = \frac{K_2}{Q} = \frac{(s+4)(s+2.5)}{s^2-1}$$