# Tutorial 11 State Estimation

**11.1** Design an observer for instrument servo:

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & -\alpha \end{bmatrix} x + \begin{bmatrix} 0 \\ \beta \end{bmatrix} u\,,$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x\,.$$

**Solution:**

Consider an observor in the form:

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + l\left[y - c^T\hat{x}\right],$$

where $\hat{x}(t)$ denotes the estimate of $x(t)$, $l = [l_1, \quad l_2]^{\mathrm{T}}$ and

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -\alpha \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ \beta \end{bmatrix}, \qquad c^T = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

Let the estimation error be denoted by $\tilde{x}(t)$ so that

$$\tilde{x} = x - \hat{x}\,.$$

It then readily follows that

$$\dot{\tilde{x}} = (A - lc^T)\tilde{x}\,.$$

As *(A, $c^T$ )* is observable it is possible to find $l$ to arbitrary place the pole of *(A - $l\,c^T$ ).*

The characteristic polynomial of ($A$ - $l\,c^T$ ) is computed as

$$\det\left(sI - A + lc^T\right) = \det\begin{bmatrix} s + l_1 & -1 \\ l_2 & s + \alpha \end{bmatrix} = (s + l_1)(s + \alpha) + l_2 = s^2 + (l_1 + \alpha)s + l_1\alpha + l_2\,.$$

If the desired observer characteristic polynomial is $s^2 + \gamma_1 s + \gamma_2$, then by equating the coeffiences, we have

$$l_1 + \alpha = r_1\,,$$

$$l_1\alpha + l_2 = r_2\,.$$

Thus, the observer gain is

$$l = \begin{bmatrix} \gamma_1 - \alpha \\ \gamma_2 - (\gamma_1 - \alpha)\alpha \end{bmatrix}.$$

**11.2** The second-order system

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u ,$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} x .$$

is known as a *harmonic oscillator* since it has a pair of undamped oscillatory poles on the imaginary axis (at $s = \pm j$), investigate stabilization of the system and its implementation.

**Solution**:

If we attempt to stabilize this system with output feedback $u=ky$, the closed-loop system matrix becomes

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} k \begin{bmatrix} 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1+k \\ -1 & 0 \end{bmatrix},$$

with characteristic polynomial $p(s) = s^2 + k + 1$, which is not stable for any value of the gain $k$. The reader will find it instructive to sketch the root-locus diagram for both positive and negative values of $k$.

If the other state were available for feedback, we could use a state feedback law

$$u = \begin{bmatrix} k_1 & k_2 \end{bmatrix} x ,$$

resulting in the closed-loop system matrix

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix} = \begin{bmatrix} k_1 & 1+k_2 \\ -1 & 0 \end{bmatrix},$$

with characteristic polynomial $p(s) = s^2 - k_1 s + k_2 + 1$, and the closed-loop poles could be placed anywhere. To be more specific, the gain matrix $\begin{bmatrix} k_1 & k_2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \end{bmatrix}$ would result in stable closed-loop poles at s=-0.5±j1.32.

This control law can be realized by estimating the state with an observer. Here, we design a reduced-order observer.

Only $x_1(t)$ needs to be reconstructed since $x_2(t) = y(t)$.

Note $T = [t_1 \quad t_2]$. We have

$$\begin{bmatrix} y(t) \\ \xi(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ t_1 & t_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix},$$

where the matrix $\begin{bmatrix} 0 & 1 \\ t_1 & t_2 \end{bmatrix}$ must have rank 2 to ensure that the inverse exists. This is the case iff $t_1 \neq 0$.

The form of the observer is

$$\dot{\xi} = d\xi + eu + gy$$

where $\xi, d, e$ and $g$ are scalars. Choose $d = -3$. With e=TB, the observer now looks like

$$\begin{aligned} \dot{\xi} &= -3\xi + TBu + gy \\ &= -3\xi + \begin{bmatrix} t_1 & t_2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} u + gy \\ &= -3\xi + t_1 u + gy \end{aligned}$$

$g$ and $t_1$ can be found from $dT - TA + gC = 0$, or

$$\begin{array}{ccccc} TA & - & dT & = & gC \\ \begin{bmatrix} t_1 & t_2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} & + & 3\begin{bmatrix} t_1 & t_2 \end{bmatrix} & = & g\begin{bmatrix} 0 & 1 \end{bmatrix} \\ \begin{bmatrix} -t_2 & t_1 \end{bmatrix} & + & \begin{bmatrix} 3t_1 & 3t_2 \end{bmatrix} & = & \begin{bmatrix} 0 & g \end{bmatrix} \end{array}$$

or $g = 3t_1 + t_2$ and $t_1 + 3t_2 = 0$. Try $g = 1$, solve for $t_1$ and $t_2$ and check the rank of $\begin{bmatrix} 0 & 1 \\ t_1 & t_2 \end{bmatrix}$. This yields $t_1 = \frac{1}{10}$, $t_2 = \frac{3}{10}$ and the matrix in question will have full rank.

This results in the final observer design as

$$\dot{\xi} = -3\xi + \frac{1}{10}u + y.$$

$$\begin{bmatrix} \hat{x}_1(t) \\ \hat{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{1}{10} & \frac{3}{10} \end{bmatrix}^{-1} \begin{bmatrix} y(t) \\ \xi(t) \end{bmatrix} = \begin{bmatrix} -3 & 10 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y(t) \\ \xi(t) \end{bmatrix}$$

or

$$\dot{\xi} = -3\xi + \frac{1}{10}u + y$$
$$\hat{x}_1(t) = -3y(t) + 10\xi(t)\,.$$
$$\hat{x}_2(t) = y(t)$$

Combining above equations, where $k_1 = -1,\ \ k_2 = 1$, we have

$$u = -\hat{x}_1 + y = 4y(t) - 10\xi(t)$$
$$\dot{\xi} = -4\xi + 1.4y$$

The closed-loop system is thus

$$\begin{bmatrix} \dot{x} \\ \dot{\xi} \end{bmatrix} = \begin{bmatrix} 0 & 5 & -10 \\ -1 & 0 & 0 \\ 0 & 1.4 & -4 \end{bmatrix} \begin{bmatrix} x \\ \xi \end{bmatrix},$$

$$y = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ \xi \end{bmatrix},$$

and the reader should verify that its poles are at $s = -3$ and $s = -0.5 \pm j1.32$, as expected.