Solution to Tutorial 1

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Q1

(a) and (b) are straight forward.
(c) Not possible because $det(A) = 0$.

Q2

$$e^{At} = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k = I + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots \tag{1}$$

$$\begin{aligned}\frac{de^{At}}{dt} &= A + A^2t + \frac{1}{2!}A^3t^2 + \cdots \\ &= A(I + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots) = Ae^{At} \\ &= (I + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots)A = e^{At}A.\end{aligned} \tag{2}$$

Taking the Laplace Transform of (1), we have

$$\mathcal{L}(e^{At}) = \mathcal{L}(\sum_{k=0}^{\infty} \frac{t^k}{k!} A^k) = \sum_{k=0}^{\infty} s^{-(k+1)} A^k = s^{-1} \sum_{k=0}^{\infty} (s^{-1}A)^k$$

From (2), we have

$$\begin{aligned}\mathcal{L}(\frac{de^{At}}{dt}) &= s\mathcal{L}(e^{At}) - e^{A\cdot 0} = A\mathcal{L}(e^{At}) \\ &\implies (sI - A)\mathcal{L}(e^{At}) = e^0 = I \\ &\implies \mathcal{L}(e^{At}) = (sI - A)^{-1}\end{aligned}$$

Q3

Let the output of $u_1(t)$ and $u_2(t)$ be

$$y_1(t) = \frac{d(tu_1(t))}{dt} = u_1(t) + t\dot{u}_1(t)$$
$$y_2(t) = \frac{d(tu_2(t))}{dt} = u_2(t) + t\dot{u}_2(t)$$

respectively. Then, for input $u_c(t) = \alpha_1 u_1(t) + \alpha_2 u_2(t)$, the output is

$$\begin{aligned} y_c(t) &= \frac{d(\alpha_1 t u_1(t) + \alpha_2 t u_2(t))}{dt} = \alpha_1(u_1(t) + t\dot{u}_1(t)) + \alpha_2(u_2(t) + t\dot{u}_2(t)) \\ &= \alpha_1 y_1(t) + \alpha_2 y_2(t) \end{aligned}$$

for any α_1, α_2. Hence the system is linear

Q4

Eigenvalues of $\begin{bmatrix} -4 & 1 & 0 \\ 0 & -3 & 1 \\ 0 & 0 & -2 \end{bmatrix}$ are $-4, \ -3$ and -2.

The corresponding eigenvectors are $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1/3 \\ 2/3 \\ 2/3 \end{pmatrix}$.

Q5

$\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$ and $det(sI - A) = s^2 - 5s - 2$. Then $A^2 - 5A - 2I = 0$.

Q6

$$A = \begin{bmatrix} 4 & 2 & 1 & 3 \\ 6 & 3 & 1 & 4 \\ 2 & 1 & 0 & 1 \end{bmatrix} \implies rank(A) = 2.$$

The linearly independent vectors are $\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$. Hence, basis vectors of

$Range(A) = \left\{ \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$.

To find basis vector of Null space of A. Let $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$, then

$$\begin{pmatrix} 4 & 2 & 1 & 3 \\ 6 & 3 & 1 & 4 \\ 2 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = 0$$

Using Gaussian Elimination or noting that the above equation can be written as

$$\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} (2x_1 + x_2 + x_4) + \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} (x_3 + x_4) = 0,$$

we can get

$$\left.\begin{matrix} 2x_1 + x_2 + x_4 = 0 \\ x_3 + x_4 = 0 \end{matrix}\right\} \implies \begin{matrix} x_2 = -x_4 - 2x_1 \\ x_3 = -x_4 \end{matrix}$$

Let $x_4 = 1, x_1 = 0$ then $n_1 = \begin{pmatrix} 0 \\ -1 \\ -1 \\ 1 \end{pmatrix}$ and $x_4 = 0, x_1 = 1$ then $n_1 = \begin{pmatrix} 1 \\ -2 \\ 0 \\ 0 \end{pmatrix}$ are the two basis vectors of the nullspace.

Q7

If λ is an eigenvalue of A, then

$$|A - \lambda I| = 0 \implies |\lambda I| \cdot |\frac{1}{\lambda} A - I| = 0$$

$$\implies |\lambda I| \cdot |\frac{1}{\lambda} A - A^{-1} \cdot A| = |\lambda I| \cdot |\frac{1}{\lambda} I - A^{-1}| \cdot |A| = 0$$

Since $|\lambda I| \neq 0$ and $|A| \neq 0$, therefore,

$$|\frac{1}{\lambda} I - A^{-1}| = 0$$

which implies that $\frac{1}{\lambda}$ is an eigenvalue of A^{-1}.

Q8 (Difficult)

Prove by contradiction. Suppose the n eigenvectors are linearly dependent, we want to show that this leads to a contradiction. Let the n eigenvectors be $x_1, x_2, \cdots, x_n$, then there exists scalars $\alpha_1, \alpha_2, \cdots, \alpha_n$, not all zeros, such that

$$\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n = 0 \tag{3}$$

Let the first non-zero α be α_1 (if not, the vectors can be rearranged so that it is so). Pre-multiply by $(A - \lambda_2 I) \cdots (A - \lambda_n I_n)$ to (3), we get

$$(A - \lambda_2 I) \cdots (A - \lambda_n I_n) \alpha_1 x_1 + (A - \lambda_2 I) \cdots (A - \lambda_n I_n) \alpha_2 x_2 + \cdots = 0 \tag{4}$$

Since

$$(A - \lambda_j I) \alpha_j x_j = \alpha_j (A x_j - \lambda_j x_j) = 0$$

and

$$(A - \lambda_i I) \alpha_j x_j = \alpha_j (A x_j - \lambda_i x_j) = \alpha_j x_j (\lambda_j - \lambda_i)$$

This implies that (4) must have all terms being zero except the left most. Hence,

$$\alpha_1(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)\cdots(\lambda_1 - \lambda_n) = 0$$

which implies that $\alpha_1 = 0$ since $\lambda_1 \neq \lambda_2 \neq \lambda_3 \cdots \neq \lambda_n$, resulting in a contradiction.

Q9 (Difficult)
Let $A \in R^{n\times n}, B \in R^{r\times n}$. For (a) & (b)
($\Rightarrow$) Let λ be a non-zero eigenvalue of AB, then

$$\begin{aligned} ABx &= \lambda x \\ \Rightarrow BABx &= \lambda Bx \\ \Rightarrow BA\cdot\xi &= \lambda\xi, \quad \text{where } \xi = Bx \end{aligned}$$

This implies the λ is a non-zero e-value of BA.
($\Leftarrow$) Let μ be a non-zero eigenvalue of BA, i.e.,

$$\begin{aligned} BAy &= \mu y \\ \Rightarrow ABAy &= \mu Ay \\ \Rightarrow AB\cdot\eta &= \mu\eta, \end{aligned}$$

which implies that μ is a non-zero eigenvalue of AB.
For (c), the result follows from (a) and (b).

Q10 (Difficult)

Let (λ_i, x_i) be an eigen-pair for A. This means that

$$Ax_i = \lambda_i x_i \tag{5}$$

Multiply the above on the left by x_i^H (the conjugate transpose of x_i), we have

$$x_i^H Ax_i = \lambda_i x_i^H x_i \tag{6}$$

Also, taking the conjugate transpose of (5) yields

$$(Ax_i)^H = x_i^H A^H = \lambda_i^H x_i^H.$$

Suppose the above is multiplied on the right by x_i, it becomes

$$x_i^H A^H x_i = \lambda_i^H x_i^H x_i$$

which, when subtracted from (6) and noting that $A^H = A$ and $x_i^H x_i \neq 0$, shows that

$$\lambda_i - \lambda_i^H = 0.$$

Hence, λ_i is real.