Tutorial 10 Decoupling Control

10.1 Consider the 2-input/2-output system described by

$$\dot{x} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 2 & 3 \\ -3 & -3 \end{bmatrix} u,$$

$$y = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} x.$$

Find the state-feedback decoupler.

Solution:
It is readily checked that

$$\begin{aligned} &c_1^T A^0 B = C_1^T B = \begin{pmatrix} 1 & 0 \end{pmatrix}, \quad \text{or} \quad \sigma_1 = 1; \\ &c_2^T B = \begin{pmatrix} 0 & 0 \end{pmatrix}, \\ &c_2^T AB = \begin{pmatrix} 4 & 3 \end{pmatrix}, \quad \text{or} \quad \sigma_2 = 2, \end{aligned}$$

so that

$$B^* = \begin{bmatrix} 1 & 0 \\ 4 & 3 \end{bmatrix}.$$

Since B^* is nonsingular, the system can be decoupled by state feedback $u = -Kx + Fv$. It is easily checked that

$$B^{*-1} = \frac{1}{3}\begin{bmatrix} 3 & 0 \\ -4 & 1 \end{bmatrix} = F,$$

Use the constructive procedure of Theorem 9.1, one finds that

$$K = B^{*-1}C^* = \begin{bmatrix} -1 & 0 & 0 \\ 1.67 & 1.33 & 3 \end{bmatrix}.$$

The resulting closed loop transfer function is

$$H(s)=\begin{bmatrix}\frac{1}{s} & 0\\ 0 & \frac{1}{s^2}\end{bmatrix}.$$

Now that the system is decoupled further stabilization is easily done on the single loop basis.

10.2 The linearized equations governing the equatorial motion of a satellite in a circular orbit is

$$\dot{x}=Ax+Bu=\begin{bmatrix}0 & 1 & 0 & 0\\ 3\omega^2 & 0 & 0 & 2\omega\\ 0 & 0 & 0 & 1\\ 0 & -2\omega & 0 & 0\end{bmatrix}x+\begin{bmatrix}0 & 0\\ 1 & 0\\ 0 & 0\\ 0 & 1\end{bmatrix}u,$$

$$y=Cx=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\end{bmatrix}x.$$

where the outputs and inputs are radial and tangential

$$y=\begin{bmatrix}r\\ \theta\end{bmatrix},\qquad u=\begin{bmatrix}u_r\\ u_\theta\end{bmatrix}.$$

These are coupled by the orbital dynamics, as indicated by the transfer function matrix

$$G(s)=\begin{bmatrix}\dfrac{1}{s^2+\omega^2} & \dfrac{2\omega}{s\left(s^2+\omega^2\right)}\\[2ex] \dfrac{-2\omega}{s\left(s^2+\omega^2\right)} & \dfrac{s^2-3\omega^2}{s^2\left(s^2+\omega^2\right)}\end{bmatrix}.$$

Find a state-feedback decoupler.

Solution:

To determine whether this system can be decoupled, we compute

$$c_1^T B=\begin{bmatrix}0 & 0\end{bmatrix},$$

$$c_1^T AB=\begin{bmatrix}1 & 0\end{bmatrix},\ \text{or}\ \sigma_1=2.$$

$$\begin{aligned} c_2^T B \quad &= \begin{bmatrix} 0 & 0 \end{bmatrix}, \\ c_2^T AB &= \begin{bmatrix} 0 & 1 \end{bmatrix}, \text{ or } \sigma_2 = 2. \end{aligned}$$

We then have

$$B^* = \begin{bmatrix} c_1^T AB \\ c_2^T AB \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I,$$

is nonsingular. Thus decoupling can be achieved with

$$\begin{aligned} u(t) &= -Kx(t) + Fw(t) \\ &= -B^{*-1}C^* x(t) + B^{*-1} w(t) \\ &= -\begin{bmatrix} 3\omega^2 & 0 & 0 & 2\omega \\ 0 & -2\omega & 0 & 0 \end{bmatrix} x(t) + w(t) \end{aligned},$$

which results in the close-loop system

$$\begin{aligned} \dot{x} &= (A - BK)x + BFw = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} w, \\ y &= Cx = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} x. \end{aligned} \tag{1}$$

The transfer function matrix may easily be shown to be

$$H(s) = \begin{bmatrix} \frac{1}{s^2} & 0 \\ 0 & \frac{1}{s^2} \end{bmatrix}.$$

We design pole placement state feedback control for this decoupled process as follows

$$w = K_f x + \overline{w} = \begin{bmatrix} k_1 & k_2 & 0 & 0 \\ 0 & 0 & k_3 & k_4 \end{bmatrix} x + \begin{bmatrix} \overline{w}_r \\ \overline{w}_\theta \end{bmatrix},$$

resulting in the closed-loop system

$$\dot{x}=\left(A-BK+BK_f\right)x+B\overline{w}=\begin{bmatrix}0&1&0&0\\k_1&k_2&0&0\\0&0&0&1\\0&0&k_3&k_4\end{bmatrix}x+\begin{bmatrix}0&0\\1&0\\0&0\\0&1\end{bmatrix}\overline{w},$$

$$y=Cx=\begin{bmatrix}1&0&0&0\\0&0&1&0\end{bmatrix}x\,.$$

If we choose the feedback gains $k_1=k_3=-1$ and $k_2=k_4=-2$, each of the two independent subsystems will be stable with a double pole at $s=-1$. The transfer function matrix may easily be shown to be

$$H(s)=\begin{bmatrix}\dfrac{1}{(s+1)^2}&0\\0&\dfrac{1}{(s+1)^2}\end{bmatrix}.$$

This closed-loop system is not only stable (perturbations from the nominal trajectory will always decay to zero), but adjustments to r and θ be made independently via the external inputs $\overline{w}_r$ and $\overline{w}_\theta$.

10.3 Design the controller in a unity output feedback configuration such that it decouples and stabilizes the plant

$$G(s)=\begin{bmatrix}\dfrac{1}{s+1}&\dfrac{1}{s+1}\\\dfrac{1}{s+2}&\dfrac{2}{s+2}\end{bmatrix}.$$

Solution:

Write

$$G(s)=\begin{bmatrix}\dfrac{1}{s+1}&0\\0&\dfrac{1}{s+2}\end{bmatrix}N_r=\begin{bmatrix}\dfrac{1}{s+1}&0\\0&\dfrac{1}{s+2}\end{bmatrix}\begin{bmatrix}1&1\\1&2\end{bmatrix}$$

Choose the decoupler

$$K_d = N_r^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

So that the decoupled plant, GK_d, is

$$GK_d = \begin{bmatrix} \frac{1}{s+1} & 0 \\ 0 & \frac{1}{s+2} \end{bmatrix}.$$

It can be easily verified that the closed loop is stable in the unity feedback configuration. So a simple gain matrix can decouple and stabilize the system at the same time.