Solution to Tutorial 3

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Q1

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \cdots & & \ddots & \vdots \\ -\alpha_n & & \cdots & & -\alpha_1 \end{pmatrix} \quad B = b = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

$$Ab = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \cdots & & \ddots & \vdots \\ -\alpha_n & & \cdots & & -\alpha_1 \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ -\alpha_1 \end{pmatrix} := \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ e_1 \end{pmatrix}$$

$$A^2b = A \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ e_1 \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ -a_2 \end{pmatrix} + e_1 \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ -\alpha_1 \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ e_1 \\ -\alpha_2 - \alpha_1 e_1 \end{pmatrix} := \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ e_1 \\ e_2 \end{pmatrix}$$

Similarly,

$$A^3b = A \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ 0 \\ -\alpha_3 \end{pmatrix} + e_1 \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ -\alpha_2 \end{pmatrix} + e_2 \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ 1 \\ -\alpha_1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ \vdots \\ 1 \\ e_1 \\ e_2 \\ -\alpha_3 - \alpha_2 e_1 - \alpha_1 e_2 \end{pmatrix} := \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ e_1 \\ e_2 \\ e_3 \end{pmatrix}$$

From this, $e_1 = \alpha_1,\ e_2 = -\alpha_2 - \alpha_1 e_1,\ e_3 = -\alpha_3 - \alpha_2 e_1 - \alpha_1 e_2$

$$\Rightarrow e_k = -\sum_{i=0}^{k-1} \alpha_{i+1} e_{k-1-i}, \quad e_0 = 1$$

Since $det(U) \neq 0 \Rightarrow$ System in controllable canonical form is always controllable.

Q2
A SISO system expressed in Controllable Canonical Form has a structure as

$$\dot{x} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & & 1 \\ -\alpha_n & & & \cdots & -\alpha_1 \end{pmatrix} x + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} u,$$

$$y = [\beta_n \quad \cdots \quad \beta_1]x + du$$

Hence,

$$G(s) = c(sI - A)^{-1}b + d, = \frac{1}{det(sI - A)} \cdot c \cdot adj(sI - A)b + d$$

Since

$$(sI - A) = \begin{pmatrix} s & -1 & 0 & \cdots & 0 \\ 0 & s & -1 & \cdots & 0 \\ \vdots & & \ddots & & -1 \\ \alpha_n & \alpha_{n-1} & & \cdots & s + \alpha_1 \end{pmatrix}$$

$$\Rightarrow \quad det(sI - A) = s \begin{vmatrix} s & -1 & 0 & \cdots & 0 \\ 0 & s & -1 & \cdots & 0 \\ \vdots & & \ddots & & -1 \\ \alpha_{n-1} & \alpha_{n-2} & & \cdots & s + \alpha_1 \end{vmatrix} + (-1)^{n-1}\alpha_n \begin{vmatrix} -1 & 0 & 0 & \cdots & 0 \\ s & -1 & 0 & \cdots & 0 \\ \vdots & & \ddots & & 0 \\ 0 & 0 & \cdots & s & -1 \end{vmatrix}$$

$$= s \left[s \begin{vmatrix} s & -1 & 0 & \cdots & 0 \\ 0 & s & -1 & \cdots & 0 \\ \vdots & & \ddots & & -1 \\ \alpha_{n-2} & & \cdots & & s + \alpha_1 \end{vmatrix} + (-1)^{n-2}\alpha_{n-1} \begin{vmatrix} -1 & 0 & 0 & \cdots & 0 \\ s & -1 & 0 & \cdots & 0 \\ \vdots & & \ddots & & 0 \\ 0 & 0 & \cdots & s & -1 \end{vmatrix} \right] + \alpha_n (-1)^{n-1} (-1)^{n-1}$$

$$\vdots$$

$$= s[s[s\cdots s[\left|\begin{pmatrix} s & -1 \\ \alpha_2 & s+\alpha_1 \end{pmatrix}\right| + \alpha_3] + \alpha_4]\cdots + \alpha_{n-1}] + \alpha_n$$

Hence,

$$cadj(sI-A)b = [\beta_n \ \cdots \ \beta_1] \begin{bmatrix} \times & \times & \cdots & \\ \vdots & \vdots & \vdots & \\ k_1 & k_2 & \cdots & k_n \end{bmatrix}^T \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

where

$$k_1 = \begin{pmatrix} -1 & 0 & \cdots & & 0 \\ s & -1 & 0 & \cdots & 0 \\ 0 & s & -1 & \cdots & \\ \vdots & & \ddots & & \\ 0 & & \cdots & s & -1 \end{pmatrix} = (-1)^{n-1}(-1)^{n-1} = 1$$

$$k_2 = \begin{pmatrix} s & 0 & \cdots & & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & s & -1 & 0 & \cdots \\ \vdots & & \ddots & & \\ 0 & & \cdots & s & -1 \end{pmatrix} = s(-1)^{n-2}(-1)^{n-2} = s$$

$$\vdots$$

$$k_n = s^{n-1}$$

So,

$$c\,adj(sI-A)b = [\beta_n \ \cdots \ \beta_1] \begin{bmatrix} \times & \times & \cdots & 1 \\ \times & \times & \times & s \\ \vdots & \vdots & \vdots & \vdots \\ \times & \times & \cdots & s^{n-1} \end{bmatrix} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} = \beta_n + \beta_{n-1}s + \cdots + \beta_1 s^{n-1}.$$

Hence,

$$G(s) = \frac{\beta_1 s^{n-1} + \beta_2 s^{n-2} + \cdots + \beta_n}{s^n + \alpha_1 s^{n-1} + \cdots + \alpha_n}.$$

Q3

$$A = \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix}$$

$$|\lambda I - A| = \begin{vmatrix} \lambda & 1 \\ -1 & \lambda+2 \end{vmatrix} = \lambda(\lambda+2)+1 = \lambda^2+2\lambda+1 = (\lambda+1)^2 = 0$$

$$\Rightarrow \quad \lambda = -1, -1.$$

$\Rightarrow$ System is asymptotically stable.

Q4

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -3 \end{bmatrix}$$

Let $Q = I$ and $P = \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{12} & P_{22} & P_{23} \\ P_{13} & P_{23} & P_{33} \end{bmatrix}$. Then, with

$$A^T P + PA = -I$$

we have

$$\begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & -3 \\ 0 & 1 & -3 \end{bmatrix} \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{12} & P_{22} & P_{23} \\ P_{13} & P_{23} & P_{33} \end{bmatrix} + \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{12} & P_{22} & P_{23} \\ P_{13} & P_{23} & P_{33} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -3 \end{bmatrix} = - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

or

$$\begin{bmatrix} -P_{13} & -P_{23} & -P_{33} \\ P_{11}-3P_{13} & P_{12}-3P_{23} & P_{13}-3P_{33} \\ P_{12}-3P_{13} & P_{22}-3P_{13} & P_{23}-3P_{33} \end{bmatrix} + \begin{bmatrix} -P_{13} & P_{11}-3P_{13} & P_{12}-3P_{13} \\ -P_{23} & P_{12}-3P_{23} & P_{22}-3P_{23} \\ -P_{33} & P_{13}-3P_{33} & P_{23}-3P_{33} \end{bmatrix} = -I$$

This implies that there are 6 equations and there are 6 unknowns. Solving, we have

$$\begin{aligned} P_{11} &= 2.312 \\ P_{12} &= 1.937 \\ P_{13} &= 0.5 \\ P_{22} &= 3.25 \\ P_{23} &= 0.8125 \\ P_{33} &= 0.4375 \end{aligned}$$

To show that P is positive definite, note that

$$|2.3125| > 0, \quad \begin{vmatrix} 2.3125 & 1.9375 \\ 1.9375 & 3.25 \end{vmatrix} > 0, \quad \begin{vmatrix} 2.3125 & 1.9375 & 0.5 \\ 1.9375 & 3.25 & 0.8125 \\ 0.5 & 0.8125 & 0.4375 \end{vmatrix} > 0$$

Hence, system is asymptotically stable.

Q5

(i)

$$\dot{x} = \begin{pmatrix} -1 & -2 & -2 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix} x + \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} u$$

$$y = [1 \quad 1 \quad 0]x$$

System is controllable & observable $\Rightarrow$ realization is minimal. Since all eigenvalues have negative real part $\Rightarrow$ asymptotically stable.
Since asymptotic stability $\Rightarrow$ BIBO stability, system is BIBO stable.

(ii)

$$\dot{x} = \begin{pmatrix} 0 & 2 \\ 1 & -1 \end{pmatrix} x + \begin{pmatrix} -1 \\ 1 \end{pmatrix} u$$

$$y = [0 \quad 1]x$$

System is observable but not controllable $\Rightarrow$ realization is not minimal. Eigenvalues are 1 and -2 $\Rightarrow$ system is not asymptotically stable.

$$G(s) = c(sI - A)^{-1}b = [0 \quad 1] \begin{bmatrix} s & -2 \\ -1 & s+1 \end{bmatrix}^{-1} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \frac{1}{s+2}$$

$\Rightarrow$ System is BIBO stable.

Q6

$$\begin{bmatrix} A & B \\ 0 & D \end{bmatrix} \cdot \begin{bmatrix} A^{-1} & -A^{-1}BD^{-1} \\ 0 & D^{-1} \end{bmatrix} = \begin{bmatrix} AA^{-1} & -AA^{-1}BD^{-1} + BD^{-1} \\ 0 & DD^{-1} \end{bmatrix} = I$$

$$\begin{bmatrix} A & 0 \\ C & D \end{bmatrix} \cdot \begin{bmatrix} A^{-1} & 0 \\ -D^{-1}CA^{-1} & D^{-1} \end{bmatrix} = \begin{bmatrix} AA^{-1} & 0 \\ CA^{-1} - DD^{-1}CA^{-1} & DD^{-1} \end{bmatrix} = I$$

Q7

Since

$$A[B \quad AB \quad A^2B \cdots A^{n-1}B] = [AB \quad A^2B \quad A^3B \cdots A^nB]$$

This implies that $rank[B \quad AB \cdots A^{n-1}B] = rank[AB \quad A^2B \quad A^3B \cdots A^nB]$ if and only if A is full rank.