

Linear System (ME5401/EE5101)

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Outline

- 1 Preliminaries
- 2 Problem with Classical Control
- 3 Mathematical Description of Systems
- 4 Defintion of Linear Systems
- 5 Input-Output Description of Systems
- 6 State Space Description
- 7 Examples

Preliminaries

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References:

- 1 Linear System Theory and Design by C.T. Chen, 3rd Edition, Published by Oxford University Press.
- 2 Linear Systems by T. Kailath. Published by Prentice Hall 1980.
- 3 Modern Control Theory by W. Brogen, 3rd Edition, Prentice-Hall, 1991.
- 4 Fundamental of State Space Linear Systems, by John Bay, First Edition, McGraw Hill, 1999.
- 5 Modern Control Engineering by K. Ogata, 2nd Edition, Prentice Hall, 1990.
- 6 Control System Design by B. Friedland. McGraw Hill 1987.

Preliminaries

Outline of course:

- 1 Introduction to Linear Control System
- 2 Review of Linear Algebra
- 3 State space solutions and properties
- 4 Controllability and Observability
- 5 Canonical Forms and Realization
- 6 Stability
- 7 Pole Placement
- 8 Quadratic Optimal Control
- 9 Decoupling Control
- 10 State Estimation
- 11 Servo Control

Additional Requirement: Project (30%)

Introduction

Review of Classical Control Systems

Definition:

- A Control System is a system in which some physical quantities are controlled by regulating certain energy inputs.
- A system is a group of physical components assembled to perform a specific function. The components may be electrical, mechanical, thermal, biomedical, chemical, pneumatic or any combination of the above.
- A physical quantity may be temperature, pressure, electrical voltage, mechanical position, velocity, pH value, liquid level etc.
- Applications of Control System include
Space vehicle, missile guidance, autopilot, robotic systems, Power system, assembly line, Chemical processes, Quality Control etc.

Introduction

Features of Classical and Modern Control Theories

Classical:

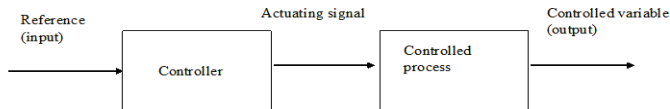
- Uses extensively the Laplace Operator in the description of a system.
- Based on Input/Output relationship called Transfer Function. All initial conditions are assumed zero in T.F. representation.
- First developed for single-input-single-output system, extension to multi-input-multi-output systems.
- Only for systems described by linear differential equations with constant coefficients.
- Uses heavily concepts like frequency response, root-locus, phase and gain margin.
- Formulation is not well-suited for Digital computer simulation.

Modern Control Theory:

- Uses time domain description known as state space representation.
- Naturally adapted for MIMO system and easily extended to linear time-varying, nonlinear and distributed systems.
- Incorporates initial conditions into its descriptions.
- Well suited for computer simulation.
- Provides a complete description of the system.

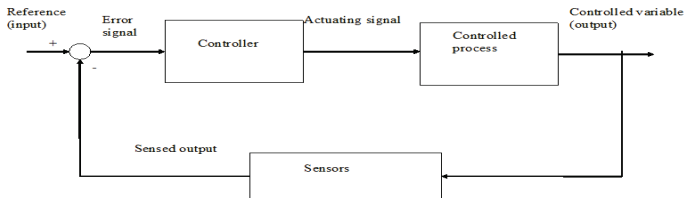
Review of Classical Control Systems

Open-loop control system



- Output has no effect on control action. Examples: Traffic light, microwave oven, dryers, toasters.

Closed-loop control system



- Actuating signal and hence output, depends on the input and the current state of the output. Examples: Servo Controlled motor, missile, robots etc.

Review of Classical Control Systems

Laplace Transformation

- Given a function $f(t)$ where $t \geq 0$.
- Let $s = \sigma + j\omega$ be a complex variable where σ, ω are both variables.
- The Laplace transform of $f(t)$ denoted by $F(s)$ is

$$F(s) = \mathcal{L}[f(t)] = \int_0^{\infty} f(t)e^{-st}dt$$

- Existence; $F(s)$ exists if the integral converges.

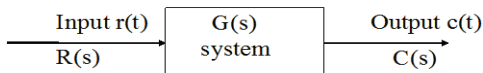
Properties of Laplace Transform

- 1 $\mathcal{L}[Af(t)] = A\mathcal{L}[f(t)]$ where A is a constant.
- 2 $\mathcal{L}[A_1f_1(t) + A_2f_2(t)] = A_1\mathcal{L}[f_1(t)] + A_2\mathcal{L}[f_2(t)]$

Review of Classical Control Systems

Transfer Function:

Consider the system given by



where

$$a_0 \frac{d^n}{dt^n} c(t) + a_1 \frac{d^{n-1}}{dt^{n-1}} c(t) + \cdots + a_n c(t) = b_0 \frac{d^m}{dt^m} r(t) + b_1 \frac{d^{m-1}}{dt^{m-1}} r(t) + \cdots + b_m r(t)$$

with $n \geq m$.

- The transfer function of a linear, time-invariant, differential equation is defined as

$$G(s) = \frac{\mathcal{L}[\text{output}]}{\mathcal{L}[\text{input}]} = \frac{C(s)}{R(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_{m-1} s + b_m}{a_0 s^n + a_1 s^{n-1} + \cdots + a_n}$$

- In deriving a transfer function, all initial conditions are assumed zero.

Review of Classical Control Systems

Other Definitions:

- Proper Transfer Function: The transfer function $G(s)$ is called proper if $n \geq m$ (i.e., $G(s)|_{s=\infty} = \text{constant}$).
- Strictly proper transfer function: The transfer function $G(s)$ is called strictly proper if $n > m$ (i.e., $G(s)|_{s=\infty} = 0$).
- Order of a system: The highest power of “ s ” in the denominator of $G(s)$ is called the order of the system.
- Poles of $G(s)$: The roots of the denominator polynomial of $G(s)$.
- Zeros of $G(s)$: The roots of the numerator polynomial of $G(s)$.
- When is $G(s)$ stable?

Example: Suppose

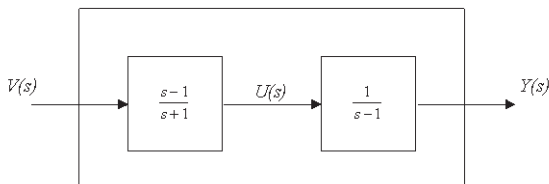
$$G(s) = \frac{s^2 + 2s + 2}{s(s^2 + 3s + 2)}$$

By definition, $n = 3, m = 2 \Rightarrow G(s)$ is strictly proper.

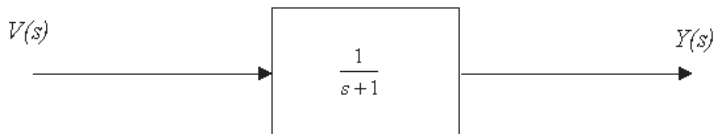
Order of system is 3, Poles of system are $s = 0, -2, -1$ and Zeros of system are $s = -1 \pm j1$.

Problem with Classical Control

Consider the following system



where the plant $H_f(s) = \frac{1}{s-1}$ is unstable and the compensator is chosen to be $H_c = \frac{s-1}{s+1}$. Overall transfer function is given by $\frac{Y(s)}{V(s)} = H_f(s)H_c(s) = \frac{1}{s+1}$.



Will this work in reality?

Problem with Classical Control

If we let $x_2(t) = y(t)$, then from Figure, we have

$$\dot{x}_2(t) = x_2(t) + u(t) \quad (1)$$

Similary, we have $\frac{U(s)}{V(s)} = \frac{s-1}{s+1}$ from the figure, or, $U(s) = V(s) - \frac{2}{s+1}V(s)$. If we let $x_1(s) = -\frac{2}{s+1}V(s)$, we have $u(s) = v(s) + x_1(s)$, then

$$\dot{x}_1(t) + x_1(t) = -2v(t). \quad (2)$$

If we include the initial conditions of (1) and (2), it can be shown that

$$x_1(t) = x_1(0)e^{-t} - 2 \int_0^t e^{-\tau} v(t-\tau) d\tau$$

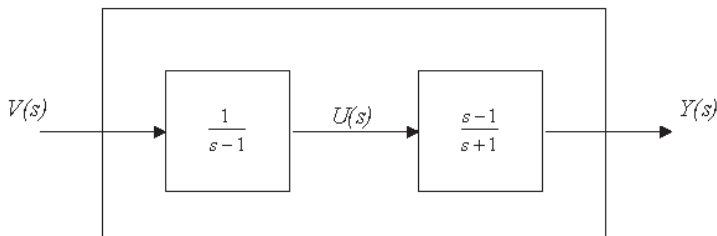
and

$$y(t) = x_2(t) = e^t x_2(0) + \frac{1}{2}(e^t - e^{-t})x_1(0) + \int_0^t e^{-\tau} v(t-\tau) d\tau$$

- Output will grow without bounds because of the e^t term (unless initial conds. can be guaranteed to be zero or $x_1(0) = -2x_2(0)$).

Problem with Classical Control

- Further insight into the problem can be obtained when the two blocks are interchanged. Consider the case



- Overall system $\frac{Y(s)}{V(s)} = \frac{1}{s+1}$ is the same BIBO stable system as before.

Let $x_1(s) = U(s)$ and $x_2(s) = -\frac{2}{s+1}x_1(s)$. Then, $y(t) = x_1(t) + x_2(t)$.

Problem with Classical Control

By incorporating the initial conditions, it can be shown that

$$x_1(t) = x_1(0)e^t + \int_0^t e^\tau v(t - \tau) d\tau$$

$$x_2(t) = x_2(0)e^{-t} + (e^{-t} - e^t)x_1(0) + \int_0^t e^{-\tau} v(t - \tau) d\tau - \int_0^t e^\tau v(t - \tau) d\tau$$

and

$$y(t) = (x_1(0) + x_2(0))e^{-t} + \int_0^t e^{-\tau} v(t - \tau) d\tau$$

- Now the system is stable as far as $y(t)$ goes, even if init cond. are non-zero.
- However, system is internally unstable because $x_1(t)$ and $x_2(t)$ have terms that grow as e^t ; after awhile, the state will saturate or burn out.

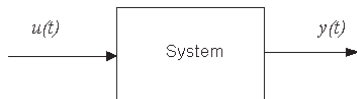
Conclusions: The internal behavior of a system is more complicated than indicated by the external behavior!

Mathematical Description of Systems

Class of systems:

The class of systems studied in this course has

- Some input terminals and some output terminals.
- If an excitation or input is applied, a unique response or output can be measured at the output terminals.



- A system with only one input terminal and one output terminal is termed a single-input-single-output (SISO) system.
- A system with two or more input terminals/or two or more output terminals is called a multi-variable system or multi-input-multi-output (MIMO) system.

Continuous time system:

- A system is called a continuous-time system if it accepts continuous-time signals as its input and generates continuous-time signals at its output.
- If the input has one terminal, $u(t)$ denotes a scalar quantity, otherwise, $u(t) = [u_1(t) \ u_2(t) \ \cdots \ u_p(t)]^T$ denotes a $p \times 1$ vector.

Mathematical Description of Systems

Causality:

- A system is called memoryless if its output $y(t_0)$ depends only on the input applied at time t_0 ; it is independent of the input applied before or after t_0 .
- Most system have memory. By this, we meant that $y(t_0)$ depends on $u(t)$ for $t < t_0, t = t_0$ and $t > t_0$.
- A system is called causal or non-anticipatory system if its current output depends on past and current inputs but not on future input. If a system is non-causal then its current output will depend on future input. No physical system has such capability. (This corresponds to the $n < m$ condition in the transfer function representation.)
- Current output of a causal system is affected by its past inputs. How far back in time will the past input affect the current output?
- In general, the time should go back to $-\infty$. However, tracking $u(t), -\infty \leq t \leq t_0$ is not convenient.
- The concept of state deals with this problem.

Mathematical Description of Systems

States and associated definitions:

- The state $x(t_0)$ of a system is the information at t_0 that, together with input $u(t)$ for $t \geq t_0$ determines uniquely the output $y(t)$ for all $t \geq t_0$.
- State Variables: The state variables of a dynamical system are the smallest set of variables that determine the state of the system.
- State Vector: The n state variables can be considered as elements or components of the n -dimensional vector $x(t)$, called the state vector.
- State Space: State space is defined as the n -dimensional space in which the components of the state vector represent its' coordinate axes.
- State Trajectory: The path produced in the state space by the state vector as it changes with the passage of time.

Lumpedness:

- A system is said to be lumped if its number of state variables is finite or the state vector is finite.
- A system is said to be distributed if its state has infinitely many state variables.

Linear Systems

Based on the above, define the triplet

$$\{x(t_0), u(t) \text{ for } t \geq t_0, y(t) \text{ for } t \geq t_0\}$$

as the output $y(t)$ for $t \geq t_0$ using $u(t)$ for $t \geq t_0$ with state $x(t_0)$ at time t_0 .

Definition: A system is called a linear system if for every t_0 and any two triplets $\{x_1(t_0), u_1(t) \text{ for } t \geq t_0, y_1(t) \text{ for } t \geq t_0\}$, $\{x_2(t_0), u_2(t) \text{ for } t \geq t_0, y_2(t) \text{ for } t \geq t_0\}$, we have for any real α the following two triplets:

$$\{x_1(t_0) + x_2(t_0), u_1(t) + u_2(t) \text{ for } t \geq t_0, y_1(t) + y_2(t) \text{ for } t \geq t_0\} \text{(additivity)}$$

$$\text{and } \{\alpha x(t_0), \alpha u(t) \text{ for } t \geq t_0, \alpha y(t) \text{ for } t \geq t_0\} \text{(homogeneity)}$$

The above two properties can be combined into one as

$$\{\alpha_1 x_1(t_0) + \alpha_2 x_2(t_0), \alpha_1 u_1(t) + \alpha_2 u_2(t) \text{ for } t \geq t_0, \alpha_1 y_1(t) + \alpha_2 y_2(t) \text{ for } t \geq t_0\}$$

for any real constants α_1 and α_2 .

- The above property is known as superposition.
- A nonlinear system is one where the superposition property does not hold.

Linear Systems

- If $u(t) = 0$ for all $t \geq t_0$, $y(t)$ is excited by $x(t_0)$ only and is known as zero-input response, $y_{zi}(t)$.
- The corresponding triplet is $\{x(t_0), 0, y_{zi}(t) \text{ for } t \geq t_0\}$.
- If $x(t_0) = 0$, $y(t)$ is excited by $u(t)$ for $t \geq t_0$ only and is known as zero-state response, $y_{zs}(t)$.
- The corresponding triplet is $\{0, u(t) \text{ for } t \geq t_0, y_{zs}(t) \text{ for } t \geq t_0\}$.
- Using the additivity property,

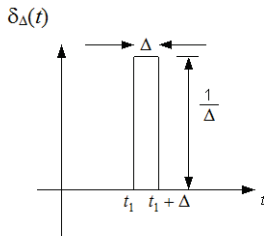
$$y(t) = y_{zi}(t) + y_{zs}(t)$$

- The above shows that the response of every linear system can be decomposed into the sum of two responses.
- These two responses can be studied separately and their sum yields the complete response.

Input-Output Description

We now develop the mathematical equation that describes the zero-state response of a SISO system and assume that all its initial states are zero.

Let $\delta_{\Delta}(t - t_1)$ be the pulse function as shown below:



where the pulse is of width Δ and height $1/\Delta$ located at time t_1 . Thus,

$$\delta_{\Delta}(t - t_1)\Delta = 1 \text{ for } t_1 \leq t < t_1 + \Delta$$

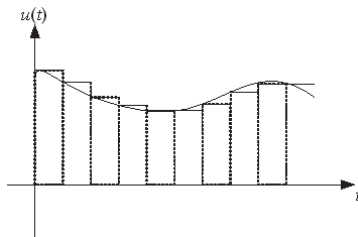
$$\delta_{\Delta}(t - t_1)\Delta = 0 \text{ otherwise.}$$

Then, any input function $u(t)$ can be approximated by

$$u(t) \approx \sum_i u(t_i)\delta_{\Delta}(t - t_i)\Delta \quad (3)$$

Input-Output Description

as shown below



Let $g_{\Delta}(t, t_i)$ be the output at time t excited by a pulse $u(t) = \delta_{\Delta}(t - t_i)$ applied at time t_i . Then, by linearity, we have

<u>Input</u>	<u>Output</u>
$\delta_{\Delta}(t - t_i)$	$g_{\Delta}(t, t_i)$
$\delta_{\Delta}(t - t_i)u(t_i)\Delta$	$g_{\Delta}(t, t_i)u(t_i)\Delta$ (homogeneity)
$\sum_i \delta_{\Delta}(t - t_i)u(t_i)\Delta$	$\sum_i g_{\Delta}(t, t_i)u(t_i)\Delta$ (additivity)

Thus, the output $y(t)$ excited by the input $u(t)$ can be approximated by

$$y(t) = \sum_i g_{\Delta}(t, t_i)u(t_i)\Delta \quad (4)$$

Input-Output Description

Let $\Delta \rightarrow 0$, and denote

$$\lim_{\Delta \rightarrow 0} \delta_{\Delta}(t - t_i) \text{ as } \delta(t - t_i)$$
$$\lim_{\Delta \rightarrow 0} g_{\Delta}(t, t_i) \text{ as } g(t, t_i)$$

- In (3), summation becomes integration and approximation becomes equality.
- Similarly, t_i becomes a continuum τ , Δ becomes $d\tau$ and (4) becomes

$$y(t) = \int_{-\infty}^{\infty} g(t, \tau) u(\tau) d\tau \quad (5)$$

- $g(t, \tau)$ refers to the output observed at time t due to impulse applied at time τ and hence, is known as the impulse response.
- By causality,

$$g(t, \tau) = 0 \text{ for all } t < \tau.$$

This implies that the upper limit of the integral of (5) becomes t .

Input-Output Description

A system is said to be relaxed at t_0 if its initial states at t_0 is 0. Hence, $y(t), t \geq t_0$ is excited exclusively by $u(t), t \geq t_0$ for a relaxed system at t_0 , or (5) becomes

$$y(t) = \int_{t_0}^t g(t, \tau) u(\tau) d\tau \quad (6)$$

- The condition of lumpedness is not used in the above. Hence, (6) is good for lumped or distributed system.

If the system has p inputs and q outputs, then (6) can be extended to

$$y(t) = \int_{t_0}^t G(t, \tau) u(\tau) d\tau$$

where

$$G(t, \tau) = \begin{pmatrix} g_{11}(t, \tau) & g_{12}(t, \tau) & \dots & g_{1p}(t, \tau) \\ g_{21}(t, \tau) & g_{22}(t, \tau) & \dots & g_{2p}(t, \tau) \\ \vdots & \vdots & & \vdots \\ g_{q1}(t, \tau) & g_{q2}(t, \tau) & \dots & g_{qp}(t, \tau) \end{pmatrix}$$

- $G(t, \tau)$ is known as the impulse response matrix of the system.

Input-Output Description

- Recall that for any function $f(t)$, $f(t + T)$ is its T -time shifted function.
- Recall also that $g(t, \tau + T)$ is the impulse response of system when impulse is applied at time $\tau + T$.
- Now, a system is time-invariant if it satisfies

$$g(t, \tau) = g(t + T, \tau + T) \text{ for any } T$$

- In particular, if $T = -\tau$, then

$$g(t + T, \tau + T) = g(t - \tau, 0) = g(t - \tau)$$

- Equation (6) becomes

$$y(t) = \int_{t_0}^t g(t - \tau) u(\tau) d\tau \quad (7)$$

- Note that $g(\cdot)$ now becomes a function of a single variable.
- (7) is used extensively in classical control analysis.
- Extension of (7) to MIMO case is direct resulting in impulse matrix $G(t - \tau)$.

State Space Representation

Every linear lumped system can be described by a set of equations of the form

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) \quad (8)$$

$$y(t) = C(t)x(t) + D(t)u(t) \quad (9)$$

where

$$x \in \mathbb{R}^n, u \in \mathbb{R}^p, y \in \mathbb{R}^q$$

$$A(t) \in \mathbb{R}^{n \times n}, B(t) \in \mathbb{R}^{n \times p}, C(t) \in \mathbb{R}^{q \times n}, D(t) \in \mathbb{R}^{q \times p}$$

- Equation (8) is a set of N first-order differential equations with p inputs.
- Equation (9) consists of q algebraic equations relating $x(t)$ and $u(t)$ to the output $y(t)$.
- Collectively, (8) and (9) are known as the state space equations.
- When $A(t), B(t), C(t), D(t)$ are constants, the SS equations describe an LTI system:

$$\dot{x}(t) = Ax(t) + Bu(t) \quad (10)$$

$$y(t) = Cx(t) + Du(t) \quad (11)$$

Linearization

Most physical systems are nonlinear and time varying described by nonlinear differential equation of the general form

$$\dot{x} = f(x, u) = \begin{pmatrix} f_1(x, u) \\ \vdots \\ f_n(x, u) \end{pmatrix} \quad (12)$$

$$y = h(x, u) \quad (13)$$

where f and h are nonlinear functions. The behavior of system described by such equations can be very complicated.

- Some nonlinear systems can be approximated by linear equations under certain conditions.
- Suppose for some input function $u_0(t)$ and some initial states, $x_0(t)$ is the solution of (12). Then,

$$\dot{x}_0(t) = f(x_0(t), u_0(t))$$

- Consider a perturbed system arising from $u(t) = u_0(t) + \delta u(t)$ and the solution of (12) is also perturbed in the form of $x_0(t) + \delta x(t)$. Linearizing via Taylor's series expansion about $(x_0(t), u_0(t))$ yields

Linearization

$$\begin{aligned}\dot{x}_0 + \delta\dot{x} &= f(x_0(t) + \delta x(t), u_0(t) + \delta u(t)) \\ &= f(x_0(t), u_0(t)) + \frac{\partial f}{\partial x}|_{x_0(t), u_0(t)} \delta x + \frac{\partial f}{\partial u}|_{x_0(t), u_0(t)} \delta u + \dots\end{aligned}$$

- If we neglect higher-order terms, we have

$$\delta\dot{x} = \frac{\partial f}{\partial x}|_{x_0(t), u_0(t)} \delta x + \frac{\partial f}{\partial u}|_{x_0(t), u_0(t)} \delta u$$

which is of the form $\delta\dot{x} = A(t)\delta x + B(t)\delta u$ with

$$A(t) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \dots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \dots & & \frac{\partial f_2}{\partial x_n} \\ \vdots & & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \dots & \dots & \frac{\partial f_n}{\partial x_n} \end{pmatrix} \Big|_{x_0(t), u_0(t)}, B(t) = \begin{pmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} & \dots & \frac{\partial f_1}{\partial u_p} \\ \frac{\partial f_2}{\partial u_1} & \dots & & \frac{\partial f_2}{\partial u_p} \\ \vdots & & \ddots & \vdots \\ \frac{\partial f_n}{\partial u_1} & \dots & \dots & \frac{\partial f_n}{\partial u_p} \end{pmatrix} \Big|_{x_0(t), u_0(t)}$$

The nonlinear output equation

$$y_0(t) = h(x_0(t), u_0(t))$$

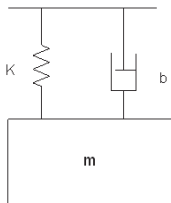
can be similarly linearized to give the approximate output

$$\delta y(t) = C(t)\delta x + D(t)\delta u$$

with $\delta y(t) = y(t) - y_0(t)$ and

$$C(t) = \begin{pmatrix} \frac{\partial h_1}{\partial x_1} & \frac{\partial h_1}{\partial x_2} & \cdots & \frac{\partial h_1}{\partial x_n} \\ \frac{\partial h_2}{\partial x_1} & \cdots & & \frac{\partial h_2}{\partial x_n} \\ \vdots & & \ddots & \vdots \\ \frac{\partial h_q}{\partial x_1} & \cdots & & \frac{\partial h_q}{\partial x_n} \end{pmatrix} \Big|_{x_0(t), u_0(t)}, D(t) = \begin{pmatrix} \frac{\partial h_1}{\partial u_1} & \frac{\partial h_1}{\partial u_2} & \cdots & \frac{\partial h_1}{\partial u_p} \\ \frac{\partial h_2}{\partial u_1} & \cdots & & \frac{\partial h_2}{\partial u_p} \\ \vdots & & \ddots & \vdots \\ \frac{\partial h_q}{\partial u_1} & \cdots & & \frac{\partial h_q}{\partial u_p} \end{pmatrix} \Big|_{x_0(t), u_0(t)}$$

Example 1: Mass-Spring-Damper System



From Newton's second law, we have

$$m \frac{d^2 z(t)}{dt^2} = f(t) - b \frac{dz(t)}{dt} - k z(t)$$

Choose $x_1 = z$ and $x_2 = \dot{z}$. Then,

$$\dot{x}_1 = \dot{z} = x_2$$

$$\dot{x}_2 = \ddot{z} = m^{-1}(f(t) - b x_2 - k x_1)$$

or, in matrix form

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{m} \end{pmatrix} f$$

If we are interested in the position of mass, then

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

If we are interested in the force acting on the spring, then

$$y(t) = kx_1 = \begin{bmatrix} k & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + 0 \cdot f$$

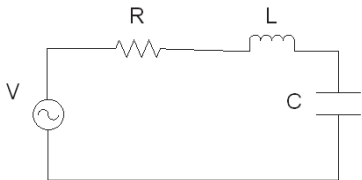
If we are interested in both the force acting on the dashpot and the force on the spring, then

$$y(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} 0 & b \\ k & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + 0 \cdot f$$

or if we want to know about the restoring forces acting on the mass, then

$$y(t) = \begin{bmatrix} k & b \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + 0 \cdot f$$

Example 2: RLC Electrical Circuit



Let $x_1(t)$ be the voltage across the capacitor, $x_2(t)$ be the current in the circuit. By Kirchoff's law, we have

$$\begin{aligned}v(t) &= R \cdot x_2(t) + L \cdot \frac{dx_2(t)}{dt} + x_1(t) \\ \dot{x}_1 &= \frac{1}{C} x_2(t)\end{aligned}$$

Rearranging, we have

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{C} \\ -\frac{1}{L} & -\frac{R}{L} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{L} \end{pmatrix} v$$

Suppose we are interested in the voltage across the inductor, then we have

$$y(t) = L \frac{dx_2(t)}{dt} = L \dot{x}_2 = L \left(-\frac{1}{L} x_1 - \frac{R}{L} x_2 \right) + L \frac{1}{L} v = -x_1 - R x_2 + v$$

$$\text{or, } y(t) = [-1 \quad -R] \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + v$$

If we denote the voltage across the resistor as x_3 , then $x_3 = Rx_2$. The state equation can also be written in terms of x_1 and x_3 as

$$\begin{aligned}\dot{x}_1(t) &= \frac{1}{c}x_2 = \frac{1}{cR}x_3(t) \\ \dot{x}_2(t) &= \frac{1}{R}\dot{x}_3(t) = -\frac{1}{L}x_1 - \frac{R}{L}x_2 + \frac{1}{L}v \\ \text{or, } \dot{x}_3(t) &= -\frac{R}{L}x_1 - \frac{R^2}{L} \cdot \frac{1}{R}x_3 + \frac{R}{L}v\end{aligned}$$

which in matrix form is

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_3(t) \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{cR} \\ -\frac{R}{L} & -\frac{R}{L} \end{pmatrix} \begin{pmatrix} x_1 \\ x_3 \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{R}{L} \end{pmatrix} v$$

Suppose we are still interested in the voltage across the inductor, then

$$\begin{aligned}y(t) &= L \frac{dx_2(t)}{dt} = \frac{L}{R} \dot{x}_3 = \frac{L}{R} \left(-\frac{R}{L}x_1 - \frac{R}{L}x_3 \right) + v \\ &= -x_1 - x_3 + v\end{aligned}$$

$$y(t) = [-1 \quad -1] \begin{pmatrix} x_1 \\ x_3 \end{pmatrix} + v$$