Stability

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Outline

Introduction

- Stability of a system is an important concept in the study of linear dynamical system.
- Every working system must be stable, no unstable can be used in practice.
- The concept of Stability can take different forms depending on its definitions.

Introduction

We begin with the standard input-output stability

- Recall that for a LTI s.s. system,

$$y(t) = Ce^{At}x_0 + \int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau$$

- Input-output stability considers just the input-output representation and this means $x_0 = 0$.
- In the case of a SISO system, the output expression can also be represented as

$$y(t) = \int_0^t g(t-\tau)u(\tau)d\tau = \int_0^t g(\tau)u(t-\tau)d\tau$$

- In Laplace domain, it becomes $Y(s) = G(s)U(s)$ and hence $y(t) = \mathcal{L}^{-1}[G(s)]$, the impulse response of the system when $u(t)$ is an impulse.

BIBO Stability

- Definition: An input $u(t)$ is said to be bounded if $u(t)$ does not grow to positive or negative infinity or, equivalently, there exists a constant m such that
$$|u(t)| \leq m < \infty \text{ for all } t \geq 0.$$
- A system is Bounded-Input-Bound-Output (BIBO) stable if for any bounded input, the output is bounded, i.e.,
$$\text{For all } |u(t)| \leq k_1 < \infty \forall t \geq 0, |y(t)| \leq k_2 < \infty$$
- Theorem 5.1: A SISO system is BIBO stable if and only if
$$\int_0^\infty |g(t)|dt \leq k < \infty \tag{1}$$
for some constant k.
- (1) is also known as absolute integrable.

BIBO Stability

Proof: ($\Rightarrow$) Suppose $\int_0^\infty |g(t)|dt \leq k < \infty$, then

$$|y(t)| = |\int_0^t g(\tau)u(t-\tau)d\tau| \leq \int_0^t |g(\tau)||u(t-\tau)|d\tau$$
$$\leq \int_0^\infty |g(\tau)||u(t-\tau)|d\tau \leq k_1 \int_0^\infty |g(\tau)|d\tau \leq k_1 k$$

Thus, output is bounded for all $t \geq 0$.
($\Leftarrow$) Suppose $\int_0^\infty |g(t)|dt = \infty$, we show that a bounded input exists for which the output goes unbounded. Choose the bounded input according to

$$u(t-\tau) = \begin{cases} 1, & \text{if } g(\tau) > 0; \\ 0, & \text{if } g(\tau) = 0; \\ -1, & \text{if } g(\tau) < 0. \end{cases}$$

Then $|y(t)| = |\int_0^t g(\tau)u(t-\tau)d\tau| = \int_0^t |g(\tau)|d\tau$ and $\int_0^\infty |g(\tau)|d\tau = \infty$. This means output is unbounded under a bounded input $\Rightarrow$ system is not BIBO stable.

BIBO Stability

- Theorem 5.2: The SISO system with proper rational transfer function $G(s)$ is BIBO stable if and only if all the poles of $G(s)$ are in the open left-half s-plane, or equivalently, all poles of $G(s)$ have negative real parts.
- Proof: If $G(s)$ is a proper transfer function, it can be expressed as

$$G(s) = \gamma + \sum_{i,j} \frac{\beta_{ij}}{(s-\lambda_i)^{k_j}}$$

  This means that the impulse response $g(t)$ is a sum of finite number of terms $t^{k_j-1}e^{\lambda_i t}$ and possibly the δ function (corresponding to the inverse Laplace of a constant). Since $t^{k_j-1}e^{\lambda_i t}$ is absolutely integrable if and only if λ_i has negative real part. Hence, the system is BIBO stable if and only if all poles of $G(s)$ have negative real parts.
- The above is easily extended to MIMO system.
- The MIMO transfer function $G(s)$ of dimension $p \times m$ is BIBO stable if and only if all poles of every entry of $G(s)$ have negative real parts.

Internal Stability

- Internal Stability deals with zero-input response while BIBO stability deals with zero-state response.
- Hence, the easiest is to consider the stability of

$$\dot{x} = Ax, \quad x(0) = x_0 \tag{2}$$

To start, consider the concept an equilibrium point

- The equilibrium point x_e is defined as those states that satisfy

$$Ax_e = 0 \tag{3}$$

If A is a non-singular matrix, the only equilibrium point is $x_e = 0$. Otherwise, there is an infinite number of equilibrium points.

- (Shifting of origin) It is convenient to shift the origin of the state-space representation of the system to the equilibrium. This is done by letting

$$\hat{x}(t) = x(t) - x_e \Rightarrow \dot{\hat{x}} = \dot{x}(t) - \dot{x_e} = \dot{x}(t) \tag{4}$$

$$\Rightarrow \dot{\hat{x}} = Ax(t) = A(\hat{x}(t) + x_e) = A\hat{x}(t) \tag{5}$$

Hence, Internal Stability is done with the equilibrium point as the origin.

Internal Stability

- Definition: The origin (or the equilibrium point) of the system

$$\dot{x} = Ax, \quad x(0) = x_0$$

  is stable in the sense of Lyapunov (i.s.L.) if, for every real number $\epsilon > 0$, there exists a $\delta(\epsilon) > 0$, such that if $\|x(0)\| \leq \delta$ then $\|x(t)\| \leq \epsilon$ for all $t \geq 0$.

- This has the equivalent notion of boundedness in the response.

Figure: Stability in the sense of Lyapunov

Internal Stability

- Definition: The origin (or the equilibrium point) of the system

$$\dot{x} = Ax, \quad x(0) = x_0$$

is asymptotically stable in the sense of Lyapunov (i.s.L.) if
(i) the origin is stable in the sense of Lyapunov and (ii) every initial state $x(0)$ results in

$$\|x(t)\| \to 0 \text{ as } t \to \infty$$

- Asymptotic stability ensures that every motion will eventually approach the origin.

Figure: Asymptotic Stability in the sense of Lyapunov

Internal Stability

- Definition: (Instability in the sense of Lyapunov) The origin is said to be unstable if for some real number $\epsilon > 0$ and any real number $\delta > 0$ no matter how small, there is always an initial state $x(0)$ inside $S(\delta) = \{x | \|x\| \leq \delta\}$ such that the trajectory $x(t)$ starting at $x(0)$ will leave $S(\epsilon)$ at some time $t < \infty$.

Figure: Instability in the sense of Lyapunov

Lyapunov Theorem

- The concept of Lyapunov Stability is general - can be extended to general nonlinear system.
- In fact, it is the most common tool for ensuring stability for general nonlinear system.
- We now introduce the Lyapunov Direct Method for general nonlinear autonomous system of the form

$$\dot{x} = f(x)$$

- It uses a scalar function $V(x)$, commonly known as the Lyapunov Function.
- Can be seen as a generalized energy function.

Lyapunov Theorem

- Properties of a candidate $V(x)$ are
  1. $V(x)$ is continuous with respect to x and has continuous $\frac{dV}{dx}$ in a domain $D \subset \mathbb{R}^n$.
  2. $V(0) = 0$.
  3. $V(x) > 0$ in D except at $x = 0$.
- **Theorem 5.3** : If $V(x)$ satisfy the three properties above and that $\frac{dV}{dt} \leq 0$ in D. Then, the origin is stable in the sense of Lyapunov. In addition, if $\frac{dV}{dt} < 0$ in D except at $x = 0$, then the origin of $\dot{x} = f(x)$ is asymptotically stable in the sense of Lyapunov.
- Proof: Not done here.
- For the case of LTI system, more specific and stronger results are known.

Lyapunov Theorem for LTI System

- **Theorem 5.4 :** Given $\dot{x} = Ax$. The system is said to be stable i.s.L. if and only if all eigenvalues of A have zero or negative real parts and those with zero real parts has no Jordan block of order 2 or higher. The system is Asymptotically Stable i.s.L. if and only if all the eigenvalues of A have negative real parts.

- Proof: Since stability is independent of coordinate representation, consider A in Jordan form. In this form, the solution is $x(t) = e^{\Lambda t}x(0)$. To show that A is stable i.s.L., we need only to show that $e^{\Lambda t}$ is bounded.
  (This follows because if $\|e^{\Lambda t}\| \leq m$ for all t, then $\|x(t)\| < \|e^{\Lambda t}\|\|x(0)\| < \epsilon$ if $\|x(0)\| < \delta := \frac{\epsilon}{m}$).
  Consider 3 cases: (i) If all eigenvalues of A have negative real parts, then $e^{\Lambda t}$ consists of sum of terms like $t^{k-1}e^{\lambda_i}$ for $k = 1, 2, \cdots$. These terms goes toward zero as t approaches infinity. Hence, the origin is asymptotically stable i.s.L. (ii) If there is one λ_i having $Re(\lambda_i) > 0$, the corresponding term will grow without bound for some $x(0)$. (iii) If $\lambda_i = 0 + j\omega_i$, the $e^{\lambda_i t}$ term will have terms that are either a constant (when $\omega_i = 0$) or proportional to $t^{k-1}sin(\omega_i t)$ or $t^{k-1}cos(\omega_i t)$. These terms are not bounded unless $k = 1$. The $k = 1$ case corresponds to A having no Jordan block of order 2 or higher.

Lyapunov Theorem for LTI System

- Theorem 5.5 Given

$$\dot{x} = Ax + Bu \tag{6}$$
$$y = Cx + Du \tag{7}$$

  Then asymptotic stability i.s.L. implies BIBO stability

- Proof: Since asymptotic stability i.s.L. is for

$$\dot{x} = Ax$$

  and from Theorem 5.4, this implies that all eigenvalues of A have negative real parts. On the other hand, BIBO stability is defined for the transfer matrix of (6)

$$y(s) = G(s)u(s) = [C(sI - A)^{-1}B + D]u(s)$$

  and the poles of $G(s)$ are eigenvalues of A. Hence, all poles of $G(s)$ have negative real parts and this establishes that (6) is BIBO stable.

Lyapunov Theorem for LTI System

Example: Given

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$

$$y = \begin{pmatrix} -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

(1) Is the system asymptotically stable?
Answer: No, since eigenvalues of A are 1 and -2.
(2) Is the system BIBO stable?
Answer: Yes, since the T.F. is

$$y(s) = C(sI - A)^{-1}B + D = \begin{pmatrix} -1 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{s-1} & 0 \\ 0 & \frac{1}{s+2} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{s+2}.$$

Lyapunov Equation

- **Theorem 5.6 :** The linear system

$$\dot{x}(t) = Ax, \quad x(0) = x_0$$

is asymptotic stability i.s.L. if and only if for any $Q \in \mathbb{R}^{n \times n}$ which is positive definite (and symmetric), there exists a $P \in \mathbb{R}^{n \times n}$ which is symmetric and positive definite satisfying the equation

$$A^T P + PA = -Q, \tag{8}$$

known as the Lyapunov equation.

Lyapunov Equation

- Proof: ($\Rightarrow$) Suppose the exist $P = P^T \succ 0$ and $Q = Q^T \succ 0$ satisfying (8). We now show that A is asymptotically stable. Let

$$V(x) = x^T P x \Rightarrow \dot{V}(x) = \frac{dV(x)}{dt} = \dot{x}^T P x + x^T P \dot{x}$$
$$= x^T A^T P x + x^T P A x = x^T (A^T P + P A) x = -x^T Q x$$

Note that since $Q \succ 0$, we have $\dot{V}(x) = 0$ if and only if $x = 0$ and $\dot{V}(x) < 0$ for all $x \neq 0$, or

$$\dot{V}(x) = -x^T Q x \leq -\lambda_{min}(Q) x^T x \leq -\frac{\lambda_{min}(Q)}{\lambda_{max}(P)} x^T P x = -\alpha V(x)$$

where $\alpha = \frac{\lambda_{min}(Q)}{\lambda_{max}(P)} \succ 0$. (since $P, Q \succ 0$)
$\Rightarrow V(t) \leq e^{-\alpha t} V(0)$, or $V(t) \to 0$ exponentially.

Lyapunov Equation

- Proof: ($\Leftarrow$) Suppose the system is asymptotically stable. We want to show that there exists a $P \succ 0$ that satisfy (8). Let

  $$P = \int_0^\infty e^{A^T t} Q e^{At} dt \tag{9}$$

  Since all eigenvalues of A have negative real parts, (9) exists and is finite. Using this choice of P,

  $$\begin{aligned} A^T P + PA &= \int_0^\infty A^T e^{A^T t} Q e^{At} dt + \int_0^\infty e^{A^T t} Q e^{At} A dt \\ &= \int_0^\infty \frac{d}{dt}(e^{A^T t} Q e^{At}) dt = e^{A^T t} Q e^{At}|_0^\infty = -Q \end{aligned}$$

  Thus, P is a solution of equation (8).

Lyapunov Equation

- To show that $P \succ 0$, consider

$$x^T P x = \int_0^\infty x^T e^{A^T t} Q e^{At} x dt = \int_0^\infty x^T e^{A^T t} D^T D e^{At} x dt = \int_0^\infty \|De^{At} x\|^2 dt$$

  where the property $Q = D^T D$ is used.

- Since D and e^{At} are both nonsingular matrices, $\|De^{At}x\|^2$ is zero if and only if $x = 0$ and is greater than zero for all $x \neq 0$. Hence $P \succ 0$.
- It can also be shown that the choice of P of (9) is unique (see Kailath pg. 179).

Lyapunov Equation

Example:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Applying the A matrix into the Lyapunov Equation, we have

$$\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} + \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

Collecting the individual elements of the above,

$$\begin{aligned} & -2p_{12} = -1, \; or \; p_{12} = 0.5 \\ & p_{11} - p_{12} - p_{22} = 0, \quad \text{and} \quad 2p_{12} - 2p_{22} = -1 \\ & \Rightarrow p_{11} = 1.5 \quad \text{and} \quad p_{22} = 1 \end{aligned}$$

Hence, $p = \begin{pmatrix} 1.5 & 0.5 \\ 0.5 & 1 \end{pmatrix}$. Checking for $P > 0$:

$$p_{11} = 1.5 > 0, \quad \begin{vmatrix} 1.5 & 0.5 \\ 0.5 & 1 \end{vmatrix} = \frac{5}{4} > 0.$$

Solving the Lyapunov Equation

- The above example shows that $A^T P + PA = -Q$ can be solved as a linear equation

$$Mx = b \tag{10}$$

  where x corresponds the elements of P.
- Since P is symmetric, there are $\frac{n(n-1)}{2}$ number of variables.
- Solving using (10) is expensive ($O(n^6)$), especially when n is large.
- In practice, other more efficient methods are used.