State-Space Solutions and Properties

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Outline

Introduction

- We begin with solution of LTI system.
- Focuses on the solutions of state-space models.
- Input-Output Description model is not amiable to numerical computation of its solution.
- The system consider throughout the first part of this lecture is

$$\dot{x} = Ax + Bu \tag{1a}$$

$$y = Cx + Du \tag{1b}$$

- We want to obtain responses $x(t), y(t)$ for $t \geq t_0$ to initial state $x(t_0)$ and input $u(t), t \geq t_0$

Homogeneous Solution

Consider the simple scalar case of

$$\dot{x} = ax \tag{2}$$

where $a, x \in \mathbb{R}$. Let $x(t_0) = x_0$. From elementary calculus,

$$x(t) = e^{a(t-t_0)}x_0$$

It is easy to verify that

$$\dot{x} = ae^{a(t-t_0)}x_0 = ax$$
$$x(t_0) = e^0 x_0 = x_0$$

Consider the matrix differential equation

$$\dot{x} = Ax, \quad x(t_0) = x_0 \tag{3}$$

Let the solution be

$$x(t) = \varphi(t, t_0)x_0 \tag{4}$$

where $\varphi(t, t_0) \in \mathbb{R}^{n \times n}$ is called the *state transition matrix*.

Solution of $\dot{x} = Ax$

For $\varphi(t, t_0)$ to be the solution, we must have

$$\dot{x}(t) = \dot{\varphi}(t, t_0)x_0 = Ax = A\varphi(t, t_0)x_0$$

$$x(t_0) = \varphi(t_0, t_0)x_0 = x_0$$

This requires that

$$\dot{\varphi}(t, t_0) = A\varphi(t, t_0) \tag{5a}$$

$$\varphi(t_0, t_0) = I \tag{5b}$$

for all t_0 and all $t \geq t_0$. The state transition matrix can be expressed as

$$\varphi(t, t_0) = e^{A(t-t_0)} \tag{6}$$

where $e^{A(t-t_0)}$ is called the matrix exponential and is defined as

$$e^{A(t-t_0)} = I + A(t - t_0) + \frac{A^2(t - t_0)^2}{2!} + \frac{A^3(t - t_0)^3}{3!} + \cdots \tag{7}$$

To verify that $e^{A(t-t_0)}$ is indeed the solution:

Solution of $\dot{x} = Ax$.

$$\dot{\varphi}(t,t_0) = \frac{d}{dt}e^{A(t-t_0)} = A + A^2(t-t_0) + \frac{A^3(t-t_0)^2}{2!} + \frac{A^4(t-t_0)^3}{3!} + \cdots$$

$$= A[I + A(t-t_0) + \frac{A^2(t-t_0)^2}{2!} + \frac{A^3(t-t_0)^3}{3!} + \cdots] = Ae^{A(t-t_0)} = A\varphi(t,t_0)$$

and

$$\varphi(t_0,t_0) = e^{A(t_0-t_0)} = I$$

Thus the solution of (4) can be written as

$$x(t) = e^{A(t-t_0)}x_0. \tag{8}$$

- The e^{At} is featured strongly in the solution of LTI system. In view of this, properties of it are first reviewed.

Properties of e^{At}

Note that

$$e^{At} = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!}$$

1. $\frac{d}{dt} e^{At} = Ae^{At} = e^{At}A$ (see Tutorial 1, P2)
2. $e^{At} = \mathcal{L}^{-1}[(sI - A)^{-1}]$.

   From Tutorial 1, P2, we have

$$(sI - A)^{-1} = \frac{I}{s} + \frac{A}{s^2} + \frac{A^2}{s^3} + \frac{A^3}{s^4} + \cdots$$

$$\mathcal{L}^{-1}[(sI - A)^{-1}] = I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots$$

   Thus,

$$\mathcal{L}^{-1}[(sI - A)^{-1}] = e^{At}.$$

Properties of e^{At}

3 $e^{At_1}e^{At_2} = e^{A(t_1+t_2)}$.

$$e^{At_1}e^{At_2} = (I + At_1 + \frac{A^2t_1^2}{2!} + \frac{A^3t_1^3}{3!} + \cdots) \times (I + At_2 + \frac{A^2t_2^2}{2!} + \frac{A^3t_2^3}{3!} + \cdots)$$

The coefficients of A^k in the expansion is

$$\sum_{i=0}^{\infty} \frac{t_1^i t_2^{k-i}}{i!(k-i)!} = \frac{t_2^k}{k!} + \frac{t_1 t_2^{k-1}}{(k-1)!} + \frac{t_1^2 t_2^{k-2}}{2!(k-2)!} + \frac{t_1^3 t_2^{k-3}}{3!(k-3)!} + \cdots$$

Thus,

$$e^{At_1}e^{At_2} = \sum_{k=0}^{\infty} A^k \left(\sum_{i=0}^{\infty} \frac{t_1^i t_2^{k-i}}{i!(k-i)!} \right) = \sum_{k=0}^{\infty} \frac{A^k (t_1+t_2)^k}{k!} = e^{A(t_1+t_2)}$$

Properties of e^{At}

4 Matrix exponential is non-singular at all t.

$$\begin{aligned} e^{A(t-t)} &= e^{At}e^{-At} \\ e^{A(0)} &= e^{At}e^{-At} \\ I &= e^{At}e^{-At} \end{aligned}$$

5 $[e^{At}]^T = e^{A^T t}$.

6 $e^{At}e^{Bt} = e^{(A+B)t}$ if and only if $AB = BA$.

Existence and Uniqueness

We now answer two questions:

1. Does a solution always exist for system (3)?
2. If so, is the solution unique?

The answer to [1] lies with the evaluation of (7). Existence is ensured when the series converges.
For what value of A and t will the infinite series converge?

$$
\begin{aligned}
e^{At} &= I + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots \\
\Rightarrow \|e^{At}\| &= \|I + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots\| \\
&\leq \|I| \; + \|A\||t| + \frac{1}{2!}\|A^2\||t^2| + \cdots\| \leq (1 + \|A\||t| + \frac{1}{2!}\|A\|^2|t|^2 + \cdots) \\
&\leq 1 + \alpha|t| + \frac{1}{2!}\alpha^2|t|^2 + \cdots = e^{\alpha|t|}
\end{aligned}
$$

where $\|A\| \leq \alpha$ is used. Hence, convergence of e^{At} is governed by convergence of $e^{\alpha|t|}$ which is known to converge for all α and all t from the standard ratio test for series.

Existence and Uniqueness

As for the answer to [2], let $x_1(t)$ and $x_2(t)$ be two separate solutions of (3) and define

$$z(t) = x_1(t) - x_2(t)$$

it follows that

$$\begin{aligned} \dot{z}(t) &= \dot{x}_1 - \dot{x}_2 = Ax_1(t) - Ax_2(t) \\ &= A(x_1(t) - x_2(t)) = Az(t) \end{aligned}$$

and

$$z(0) = x_1(0) - x_2(0) = 0.$$

Hence, the z system follows

$$\dot{z}(t) = Az(t), \quad z(0) = 0$$

which implies $z(t) = e^{At}z(0) = 0$ for all $t \geq 0$ and hence $x_1(t) = x_2(t)$.

Methods of Computing e^{At}

1 Series Evaluation:

$$e^{At} = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!}$$

Example 1: $A = \begin{pmatrix} 0 & 1 \\ 0 & -2 \end{pmatrix}$

$$\begin{aligned}
e^{At} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & -2 \end{pmatrix} t + \frac{1}{2!} \begin{pmatrix} 0 & 1 \\ 0 & -2 \end{pmatrix}^2 t^2 + \cdots \\
&= \begin{pmatrix} 1 & \frac{1}{2} - \frac{1}{2}[1 - 2t + \frac{(2t)^2}{2!} - \frac{(2t)^3}{3!} + \cdots] \\ 0 & [1 - 2t + \frac{(2t)^2}{2!} - \frac{(2t)^3}{3!} + \cdots] \end{pmatrix} \\
&= \begin{pmatrix} 1 & \frac{1}{2}(1 - e^{-2t}) \\ 0 & e^{-2t} \end{pmatrix}
\end{aligned}$$

Methods of Computing e^{At}

Example 2: $A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

$$\begin{aligned} e^{At} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} t + \frac{1}{2!} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}^2 t^2 + \cdots \\ &= \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix} \end{aligned}$$

2 Using Inverse Laplace Transform: From property 2 of e^{At}, we have

$$e^{At} = \mathcal{L}^{-1}[(sI - A)^{-1}]$$

Example 3:

$A = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}$, $sI - A = \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} = \begin{pmatrix} s & -1 \\ 2 & s+3 \end{pmatrix}$

$$\mathcal{L}^{-1}(sI - A)^{-1} = \mathcal{L}^{-1} \begin{pmatrix} \frac{s+3}{(s+1)(s+2)} & \frac{1}{(s+1)(s+2)} \\ \frac{-2}{(s+1)(s+2)} & \frac{s}{(s+1)(s+2)} \end{pmatrix} = \begin{pmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{pmatrix}$$

Methods of Computing e^{At}

3 Using Cayley-Hamilton Theorem:

- Consider $f(\lambda) = e^{\lambda t}$.
- First compute the eigenvalues of A.
- Find a polynomial $h(\lambda)$ of degree $(n-1)$ that equals $e^{\lambda t}$ on the spectrum of A.
- Then $e^{At} = h(A)$.

Example 4: $A = \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}$ with eigenvalues 5 and 2.

Choose $h(\lambda) = \alpha_0(t) + \lambda\alpha_1(t)$. Then, conditions of $h(\lambda) = f(\lambda)$ require that

$$e^{5t} = \alpha_0(t) + 5\alpha_1(t)$$
$$e^{2t} = \alpha_0(t) + 2\alpha_1(t)$$

Solving the above, we get

$$\alpha_0(t) = \frac{e^{2t} - 2e^{5t}}{3}; \quad \alpha_1(t) = \frac{e^{5t} - 2e^{2t}}{3}$$

Therefore, $e^{At} = \alpha_0(t)I + \alpha_1(t)A$.

Methods of Computing e^{At}

Example 5: $A = \begin{pmatrix} -2 & 3 \\ -\frac{1}{3} & -4 \end{pmatrix}$ with eigenvalues -3 and -3.
Again, let

$$h(\lambda) = \alpha_0(t) + \lambda\alpha_1(t)$$

When $\lambda = -3$, we have

$$e^{-3t} = \alpha_0(t) - 3\alpha_1(t)$$

The second equation is obtained by differentiating $h(\lambda)$ w.r.t. λ and evaluated at $\lambda = -3$, yielding

$$\frac{df(\lambda)}{d\lambda}|_{\lambda=-3} = \frac{h(\lambda)}{d\lambda}|_{\lambda=-3}$$

$$\Rightarrow te^{-3t} = \alpha_1(t)$$

Solving, we get $\alpha_0(t) = e^{-3t} + 3te^{-3t}$ and

$$e^{At} = \alpha_0(t)I + \alpha_1(t)A$$

Methods of Computing e^{At}

4 Using Method of Diagonalization:

From Chapter 2, we have

$$f(A) = Qf(\bar{A})Q^{-1}$$

where A and $\bar{A}$ are similar. Hence, choose $Q = [q_1\; q_2 \cdots q_n]$ where q_i is the eigenvector of A corresponding to eigenvalue λ_i, then,

$$e^{At} = Qe^{\bar{A}t}Q^{-1}$$

In addition, if all the eigenvalues are distinct, we have

$$\bar{A} = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}, \text{ and } e^{\bar{A}t} = \begin{pmatrix} e^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 t} & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & e^{\lambda_n t} \end{pmatrix}$$

Using the above,

$$e^{At} = Qe^{\bar{A}t}Q^{-1}$$

If the eigenvalues are not all distinct, the above expression can be obtained in Jordan Form.

Methods of Computing e^{At}

Example 6: $A = \begin{pmatrix} 0 & 1 \\ 0 & -2 \end{pmatrix}$ with eigenvalues 0 and -2.
Since eigenvalues are distinct, the matrix A can be diagonalized. The eigenvector matrix is

$$T = [v_1 \; v_2] = \begin{pmatrix} 1 & 1 \\ 0 & -2 \end{pmatrix}$$

Therefore,

$$e^{At} = Te^{\bar{A}t}T^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} e^{0t} & 0 \\ 0 & e^{-2t} \end{pmatrix} \begin{pmatrix} 1 & 0.5 \\ 0 & -0.5 \end{pmatrix} = \begin{pmatrix} 1 & 0.5(1 - e^{-2t}) \\ 0 & e^{-2t} \end{pmatrix}.$$

Full Solution

We now derive the solution of the full S.S. system of

$$\dot{x} = Ax + Bu \tag{9a}$$

$$y = Cx + Du \tag{9b}$$

with $x(0) = x_0$.

- There are several ways to derive the solution and we show one of them.
- Known as "variation of constants".
- Assume a particular solution of the form

  $$x_p(t) = e^{At} \cdot C(t)$$

  where $C(t)$ is a function of time. Differentiating, we have

  $$\dot{x}_p(t) = e^{At} \cdot \dot{C}(t) + Ae^{At}C(t)$$

  Substituting into the state equation, we have, by comparing terms,

  $$e^{At}\dot{C}(t) = Bu(t)$$

  $$\dot{C}(t) = e^{-At}Bu(t)$$

  $$C(t) = \int_{\alpha}^{t} e^{-A\tau}Bu(\tau)d\tau$$

  The lower limit α is undefined at this moment.

Full Solution

Hence,

$$x_p(t) = e^{At} \cdot C(t) = e^{At} \cdot \int_{\alpha}^{t} e^{-A\tau} Bu(\tau) d\tau = \int_{\alpha}^{t} e^{A(t-\tau)} Bu(\tau) d\tau$$

The complete solution is given by the complementary solution and the particular solution, yielding

$$x(t) = e^{A(t-t_0)} x(t_0) + \int_{\alpha}^{t} e^{A(t-\tau)} Bu(\tau) d\tau$$

We now determine the value of α. Note that $x(t_0) = x(0)$,

$$x(0) = e^{A(t_0-t_0)} x(0) + \int_{\alpha}^{t_0} e^{A(t_0-\tau)} Bu(\tau) d\tau$$

This implies that

$$x(t) = e^{A(t-t_0)} x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)} Bu(\tau) d\tau$$

and the output expression is

$$y(t) = Cx(t) + Du(t)$$

$$y(t) = Ce^{A(t-t_0)} x(t_0) + C \int_{t_0}^{t} e^{A(t-\tau)} Bu(\tau) d\tau + Du(t)$$

Response to Impulse

If $u(t)$ is an unit-impulse, i.e., $\int_0^\infty \delta(t)dt = 1$ and

$$u(t) = \delta(t) = \begin{cases} \infty, & \text{if } t = 0 \\ 0, & t \neq 0. \end{cases}$$

$$\int_0^t f(t-\tau)\delta(\tau)d\tau = f(t)$$

Then the zero-state response is

$$\begin{aligned} y(t) &= \int_{t_0}^t Ce^{A(t-\tau)}B\delta(\tau)d\tau + D\delta(t) \\ &= Ce^{At}B + D\delta(t) \end{aligned}$$

Example

Consider the example

$$\dot{x}(t) = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$

where the input $u(t)$ is a unit-step function. To do so, note that

$$e^{At} = \mathcal{L}^{-1}[(sI - A)^{-1}] = \begin{pmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{pmatrix}$$

Then the solution is

$$\begin{aligned} x(t) = & \begin{pmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{pmatrix} x(0) + \\ & \int_0^t \begin{pmatrix} 2e^{-(t-\tau)} - e^{-2(t-\tau)} & e^{-(t-\tau)} - e^{-2(t-\tau)} \\ -2e^{-(t-\tau)} + 2e^{-2(t-\tau)} & -e^{-(t-\tau)} + 2e^{-2(t-\tau)} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} 1 d\tau \\ = & \begin{pmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{pmatrix} x(0) + \begin{pmatrix} \frac{1}{2} - e^{-t} + \frac{1}{2} e^{-2t}) \\ e^{-t} - e^{-2t} \end{pmatrix} \end{aligned}$$

Non-uniqueness of s.s representation

In Chapter 2, we show that matrices and vectors can be represented under different basis.

- Suppose the system

$$\dot{x} = Ax + Bu \tag{10a}$$
$$y = Cx + Du \tag{10b}$$

  is given under the standard basis. How will it be represented under a different basis.

- Let $Q = [q_1 \; q_2 \cdots q_n]$ be the basis used. Then, vector x under Q basis has a representation $x = Q\bar{x}$. Hence

$$\dot{\bar{x}} = Q^{-1}\dot{x} = Q^{-1}Ax + Q^{-1}Bu = Q^{-1}AQ\bar{x} + Q^{-1}Bu$$

$$y = CQ\bar{x} + Du$$

- Hence, system of (10) is expressed as

$$\dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}u \tag{11a}$$
$$y = \bar{C}\bar{x} + \bar{D}u \tag{11b}$$

  where $\bar{A} = Q^{-1}AQ, \bar{B} = Q^{-1}B, \bar{C} = CQ$ and $\bar{D} = D$.

Invariance of Eigenvalues

As (10) and (11) are different representations of the same system, it is not surprising that they have the same characteristic equation or the same set of eigenvalues. To see this,

$$\begin{aligned} det(\lambda I - \bar{A}) &= det(\lambda Q^{-1}Q - Q^{-1}AQ) \\ &= det(Q^{-1}(\lambda I - A)Q) = detQ^{-1}det(\lambda I - A)detQ = det(\lambda I - A) \end{aligned}$$

Example: $A = \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}, \bar{A} = \begin{pmatrix} 1 & \frac{4}{5} \\ 0 & 2 \end{pmatrix}$ are two representation of the same system having the same eigenvalues.

T.F. Matrix and S.S. Equation

- Consider (10).
- Take Laplace Transform on both sides, we get

$$\begin{aligned} sX(s) - x_0 &= AX(s) + BU(s) \\ (sI - A)X(s) &= x_0 + BU(s) \\ X(s) &= (sI - A)^{-1}BU(s) \\ \text{or, } Y(s) &= [C(sI - A)^{-1}B + D]U(s) \end{aligned}$$

  where we have assumed $x_0 = 0$.
- Comparing this with $Y(s) = G(s)U(s)$ we have

$$G(s) = C(sI - A)^{-1}B + D$$

  known as the Transfer Function matrix of the system.
- In a SISO system

$$G(s) = C(sI - A)^{-1}B + D = \frac{1}{det(sI - A)}(Cadj(sI - A)B + det(sI - A)D)$$

T.F. Matrix and S.S. Equation

- In the case of a SISO system, $Cadj(sI-A)B+det(sI-A)D$ is also a polynomial is s, or
$$G(s)=\frac{N(s)}{D(s)}$$
- From classical control, poles of system are roots of $D(s)$.
- But the values of s such that $D(s)=det(sI-A)=0$ are eigenvalues of A.
- Hence,
Poles of T.F. $G(s)$ are the eigenvalues of A.

Example:

$$\dot{x}=\begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}+\begin{pmatrix} 0 \\ \frac{1}{m} \end{pmatrix}f$$
$$y=\begin{pmatrix} 1 & 0 \end{pmatrix}x$$

Example

Using $G(s) = C(sI - A)^{-1}B + D$, we get

$$sI - A = \begin{pmatrix} s & -1 \\ \frac{k}{m} & s + \frac{b}{m} \end{pmatrix}$$

$$C(sI - A)^{-1}B = \frac{1}{s(s + \frac{b}{m}) + \frac{k}{m}} \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} s + \frac{b}{m} & 1 \\ -\frac{k}{m} & s \end{pmatrix} \begin{pmatrix} 0 \\ \frac{1}{m} \end{pmatrix}$$

$$G(s) = \frac{1}{s(s + \frac{b}{m}) + \frac{k}{m}} \begin{pmatrix} s + \frac{b}{m} & 1 \end{pmatrix} \begin{pmatrix} 0 \\ \frac{1}{m} \end{pmatrix} = \frac{1}{ms^2 + bs + k}$$

Modes of System

- Consider $\dot{x} = Ax, x(0) = x_0$ with A having distinct eigenvalues.
- By choosing $Q = [q_1\ q_2 \cdots q_n]$ where q_i are the eigenvectors of A and

$$Q^{-1} = \begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{pmatrix}$$

where p_i is the i^{th} row vector of Q^{-1}.
- We have

$$x(t) = e^{At}x_0 = Qe^{\Lambda t}Q^{-1}x_0$$

where $e^{\Lambda t} = diag\{e^{\lambda_1 t}, e^{\lambda_1 t}, \cdots, e^{\lambda_n t}\}$ is diagonal. Hence, the solution can be rewritten as

$$x(t) = \sum_{i=1}^{n} e^{\lambda_i t} q_i p_i x_0 = \sum_{i=1}^{n} e^{\lambda_i t} q_i \mu_i$$

where $\mu_i = p_i x_0, i = 1, \cdots, n$.
- The above shows that the solution is the sum of n modes.

Example

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ -4 & -5 \end{pmatrix} x$$

The eigenvalues are -1 and -4 with evectors $[1 \; -1]^T$ and $[-1 \; 4]^T$. Therefore,

$$Q = \begin{pmatrix} 1 & -1 \\ -1 & 4 \end{pmatrix} \text{ and } Q^{-1} = \frac{1}{3} \begin{pmatrix} 4 & 1 \\ 1 & 1 \end{pmatrix}$$

Therefore, $x(t) = \mu_1 e^{\lambda_1 t} q_1 + \mu_2 e^{\lambda_2 t} q_2$ where $\mu_1 = p_1 x_0$ and $\mu_2 = p_2 x_0$. Thus,

$$x_1(t) = \frac{1}{3}[(4x_{10} + x_{20})e^{-t} - (x_{10} + x_{20})e^{-4t}]$$
$$x_2(t) = \frac{1}{3}[(-4x_{10} - x_{20})e^{-t} + (4x_{10} + 4x_{20})e^{-4t}]$$