Tutorial 9 Servo Control

9.1 Consider the second order system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} w_2 \\ w_3 \end{bmatrix},$$

where w_2 and w_3 are unknown constant disturbance signals. It is desired that the state x_1 follows a reference input w_1 = constant. Design a control with state and dynamic feedback such that all the roots of the characteristic equation of the closed-loop system are at -3, and $\lim x_1(t) = w_1$. Sketch $x_1(t)$ for $t \geq 0$ for $w_1 = w_2 = w_3 = 1$ and with zero initial conditions.

Solution:

Let $u = k_1 x_1 + k_2 x_2 + g\int (w_1 - x_1)dt$, and with $\dot{z} = e = w_1 - x_1$ define for convenience $\dot{x}_3 = g\dot{z}$. Then the state equation of the closed-loop system can be written as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ k_1 & k_2 - 1 & 1 \\ -g & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ g & 0 & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}.$$

The characteristic equation is

$$s^3 + (3 - k_2)s^2 + (2 - 2k_2 - k_1)s + g = 0,$$

and the desired characteristic equation is

$$s^3 + 9s^2 + 27s + 27 = 0.$$

Thus, $k_1 = -13, k_2 = -6, g = 27$. The output transform is

$$X_1(s) = \frac{gW_1(s) + sW_2(s) + s(s - k_2 + 1)W_3(s)}{s^3 + 9s^2 + 27s + 27},$$

and for $W_1(s) = W_2(s) = W_3(s) = 1/s$, it is easy to see that

$$\lim_{t \to \infty} x_1(t) = \lim_{s \to 0} sX_1(s) = 1 = w_1 .$$

9.2 A broom-balancing system is given by

$$\dot{x} = Ax + Bu + d = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 11 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} u + \begin{bmatrix} 0 \\ 4 \\ 0 \\ 6 \end{bmatrix} w,$$

$$\underline{z} = c'\, x = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} x.$$

and requires that it be maintained equal to the reference input (or set point) z_r. Note that we have *not* designated the angle Θ as an output since any control law which stabilizes the system must maintain the pendulum in balance regardless of z_r.

Solution:
The steady-state errors in z can be removed using the integral

$$q(t) = \int_0^t [z(\tau) - z_r(\tau)]d\tau \ ,$$

that is, by augmenting the system as :

$$\begin{bmatrix} \dot{x} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 11 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ q \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ 4w \\ 0 \\ 6w \\ -z_r \end{bmatrix},$$

$$z = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ q \end{bmatrix}.$$

It can be easily verified that the original system is controllable. So this composite system is controllable if and only if the matrix

$$\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 11 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix},$$

has rank 5 (i.e., is nonsingular). Since the determinant of this matrix is -10, it is nonsingular and the system must be controllable. Therefore, we know that it can be stabilized using a state feedback control law of the form

$$u(t) = k' \begin{bmatrix} x(t) \\ q(t) \end{bmatrix} = \begin{bmatrix} k_1 & k_2 & k_3 & k_4 & k_5 \end{bmatrix} \begin{bmatrix} x(t) \\ q(t) \end{bmatrix}.$$

Let the required closed-loop characteristic polynomial is

$$\phi_d(s) = s^5 + 7s^4 + 20s^3 + 30s^2 + 24s + 8 .$$

Substitution of control into plant yields the closed-loop system

$$\begin{bmatrix} \dot{x} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ k_1 & k_2 & k_3 - 1 & k_4 & k_5 \\ 0 & 0 & 0 & 1 & 0 \\ -k_1 & -k_2 & 11 - k_3 & -k_4 & -k_5 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ q \end{bmatrix} + \begin{bmatrix} d \\ -z_r \end{bmatrix}, \tag{47}$$

$$z = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ q \end{bmatrix},$$

with characteristic polynomial

$$\begin{aligned} \phi_d(s) &= \det(sI - A_1) \\ &= s^5 + (k_4 - k_2)s^4 + (k_3 - k_1 - 11)s^3 + (10k_2 - k_5)s^2 + 10k_1 s + 10k_5 . \end{aligned} \tag{48}$$

Matching ϕ_c to ϕ_d, we find that the required feedback gains are

$$k_5 = 0.8,$$

$$k_1 = 2.4,$$

$$k_2 = \frac{30 + k_5}{10} = 3.08,$$

$$k_3 = 20 + k_1 + 11 = 33.4,$$

$$k_4 = 7 + k_2 = 10.08.$$

and the closed-loop system (47) becomes

$$\begin{bmatrix} \dot{x} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 2.4 & 3.08 & 32.4 & 10.08 & 0.8 \\ 0 & 0 & 0 & 1 & 0 \\ -2.4 & -3.08 & -22.4 & -10.08 & -0.8 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ q \end{bmatrix} + \begin{bmatrix} d \\ -z_r \end{bmatrix} := A_1 \begin{bmatrix} x \\ q \end{bmatrix} + \begin{bmatrix} d \\ -z_r \end{bmatrix}, \quad (49)$$

$$z = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ q \end{bmatrix} := c_1' \begin{bmatrix} x \\ q \end{bmatrix}.$$

Now let us verify that the steady-state behavior of (49) is satisfactory. If the disturbance d and reference input z_r are both constant (i.e., step functions), then it can be shown by setting the derivatives to be zero in equation (49), that

$$\lim_{t \to \infty} z(t) = -c' A_1^{-1} \begin{bmatrix} d \\ -z_r \end{bmatrix}$$

$$= -\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0.1 & 0 & 0.1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -3.85 & -2.8 & -12.6 & -4.05 & -3 \end{bmatrix} \begin{bmatrix} d \\ -z_r \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} d \\ -z_r \end{bmatrix} = z_r .$$

Thus $z(t)$ approaches the steady-state value z_r, as required.

To check on Θ (which is x_3), observe that

$$Q = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ q \end{bmatrix} = c'_2 \begin{bmatrix} x \\ q \end{bmatrix},$$

so that

$$\lim_{t \to \infty} \Theta(t) = -c_2' A_1^{-1} \begin{bmatrix} d \\ -z_r \end{bmatrix} = -\begin{bmatrix} 0 & 0.1 & 0 & 0.1 & 0 \end{bmatrix} \begin{bmatrix} d \\ -z_r \end{bmatrix}$$

$$= 0.1(d_1 + d_4) = -w .$$

9.3 For the system

$$G(s) = \frac{1}{(s+1)(s+2)}$$

design a stable control system with zero steady-state errors in response to step disturbance and step set-point change.

Solution:
Let us use polynomial approach to design servo controller. Since both inputs are of step type, we have $Q(s) = s$, and the controller takes the form of

$$D(s) = \frac{\tilde{D}(s)}{Q(s)} = \frac{1}{s}\tilde{D}(s)$$

Try PI type controller

$$D(s) = \frac{k(s+\alpha)}{s}$$

choose $\alpha = 2$ to cancel the plant's fast pole at s=-2. Then, the open-loop is

$$G(s)D(s) = \frac{K}{s(s+1)}$$

The characteristic polynomial of the closed-loop is

$$s(s+1) + k = 0$$

Take k=1 giving a stable system with a reasonable damping of $\xi = 0.5$, and natural frequency of $w_n = 1$. Thus, the designed controller is

$$D(s) = \frac{s+2}{s}$$

9.4 For the system

$$G(s) = \frac{1}{s(s+1)}$$

design a stable control system with zero steady-state errors in response to step disturbance and ramp set-point change.

Solution:
For step disturbance, zero steady state error needs $\frac{1}{s}$ in the controller; for ramp set-point, the open-loop should have a factor of $\frac{1}{s^2}$. But G(s) already has $\frac{1}{s}$. Thus the controller only needs an integrator $\frac{1}{s}$. Let the controller be

$$K(s) = \frac{k_1(s)}{s}$$

Then the open-loop is

$$GK = \frac{1}{s^2(s+1)} k_1(s)$$

Take

$$k_1(s) = (as+b),$$

then

$$GK = \frac{as+b}{s^2(s+1)}$$

The characteristic equation is

$$p(s) = s^3 + s^2 + as + b$$

We can easily design a and b to stabilize the system. Let the three poles be p_1, p_2, p_3, and the characteristic polynomial is

$$(s-p_1)(s-p_2)(s-p_3)$$

As long as the three poles satisfy the condition that

$$p_1 + p_2 + p_3 = -1$$

The resulting characteristic polynomial will be in the form of

$$p(s) = s^3 + s^2 + (p_1p_2 + p_1p_3 + p_2p_3)s - p_1p_2p_3$$

Then we can easily choose a and b as

$$a = p_1p_2 + p_1p_3 + p_2p_3$$
$$b = -p_1p_2p_3$$

Since the controller is only first order controller, we do not have the complete freedom to place the poles to anywhere we like. If we want to have that freedom, we need to increase the order of the controller. For instance take the second order controller in the form of

$$K(s) = \frac{b_2 s^2 + b_1 s + b_0}{s(s+a_0)}$$

then we can place the poles of the closed loop to any positions.