Tutorial 7 Pole Placement

7.1 Consider a plant with transfer function

$$G(s) = \frac{10}{s(s+1)} .$$

Describe the system in state space representation. Choose suitable closed-loop poles and calculate the corresponding feedback gain.

Solution:
Consider a plant with transfer function

$$G(s) = \frac{10}{s(s+1)} .$$

The plant could be an armature-controlled d.c. motor driving an antenna and can be represented by the block diagram enclosed by the dotted line in Figure T7.1. If we assign the output of each block as a state variable as shown, then we can readily obtain

$$\dot{x}_1 = x_2 ,$$

and

$$\dot{x}_2 = -x_2 + 10u .$$

Thus, the plant can be described by the following state-variable equation

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 10 \end{bmatrix} u ,$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x .$$

Now we introduce feedback from x_1 and x_2 as shown in Figure T2.1 with real constant gains k_1 and k_2. This is called *state feedback*. With the feedback, the transfer function from *r* to *y* becomes,

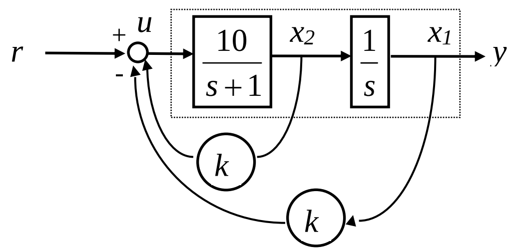

Figure T7.1. Block diagrams

$$G_0(s) = \frac{\dfrac{10}{s(s+1)}}{1+\dfrac{10k_2}{s+1}+\dfrac{10k_1}{s(s+1)}} = \frac{10}{s(s+1)+10k_2 s+10k_1}$$
$$= \frac{10}{s^2+(1+10k_2)s+10k_1}\,.$$

We see that by choosing k_1 and k_2, the poles of $G_0(s)$ can be arbitrarily assigned provided complex-conjugate poles are assigned in pairs. For example, if we assign the poles of $G_0(s)$ as $-2 \pm j2$, then k_1 and k_2 can be determined from

$$(s+2+j2)(s+2-j2) = s^2+4s+8$$
$$= s^2+(1+10k_2)s+10k_1\,,$$

or

$$(1+10k_2)=4\,, \qquad 10k_1=8\,,$$

as k_1 = 0.8 and k_2 = 0.3. This shows that by state feedback, the poles of the resulting system can be arbitrarily assigned.

7.2 Consider the system

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 10 \end{bmatrix} u\,,$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x\,,$$

with transfer function

$$G(s) = \frac{10}{s^2+s}\,.$$

Find the feedback gain k^T in $u = r - k^T x$ such that the resulting equation has $-2 \pm j2$ as its eigenvalues.

Solution:
We compute the characteristic polynomial of A

$$\det(sI-A) = \det\begin{bmatrix} s & -1 \\ 0 & s+1 \end{bmatrix} = s(s+1) = s^2+s\,,$$

and the desired characteristic polynomial

$$\phi_d(s) = (s+2+j2)(s+2-j2) = s^2 + 4s + 8,$$

$$\varphi_d(A) = A^2 + 4A + 8I_2$$

$$= \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} + 4\begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} + 8\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 3 \\ 0 & 5 \end{bmatrix}$$

$$\mathsf{C} = [b \quad Ab] = \begin{bmatrix} 0 & 10 \\ 10 & -10 \end{bmatrix},$$

$$[0 \quad 1]\mathsf{C}^{-1} = [0.1 \quad 0].$$

Hence, Ackermann's formula yields

$$k^T = \begin{bmatrix} 0.1 & 0 \end{bmatrix}\begin{bmatrix} 8 & 3 \\ 0 & 0.5 \end{bmatrix} = \begin{bmatrix} 0.8 & 0.3 \end{bmatrix}.$$

One short cut to solve this problem:

Notice that the system is in controllable canonical form if we let v=10u, and treat v as the input. Then we have the solution immediately that

$$v = \bar{k}^T x$$

where

$$\bar{k} = [\gamma_0 - \alpha_0 \quad \gamma_1 - \alpha_1] = [8-0 \quad 4-1] = [8 \quad 3],$$

So the solution for the original input u is
$u = 0.1v = 0.1\bar{k}^T x$.

Thus we have

$$k = 0.1\bar{k}^T = \begin{bmatrix} 0.8 & 0.3 \end{bmatrix}.$$

To verify the result, we compute

$$sI - A + bk = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 0 \\ 10 \end{bmatrix}\begin{bmatrix} 0.8 & 0.3 \end{bmatrix} = \begin{bmatrix} s & -1 \\ 8 & s+4 \end{bmatrix}.$$

Thus, the transfer function of the resulting system is

$$G_0(s) = [1 \quad 0]\begin{bmatrix} s & -1 \\ 8 & s+4 \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ 10 \end{bmatrix}$$
$$= [1 \quad 0]\frac{1}{s^2+4s+8}\begin{bmatrix} s+4 & 1 \\ -8 & s \end{bmatrix}\begin{bmatrix} 0 \\ 10 \end{bmatrix}$$
$$= \frac{10}{s^2+4s+8},$$

as expected. We see that the state feedback shifts the poles of $G(s)$ from 0 and -1 to $-2 \pm j2$. However, it has no effect on the numerator of $G(s)$.

7.3 Find the feedback gain k^T that assigns the closed-loop poles $-1 \pm j$ to the system:

$$(A,b) = \left(\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}\right).$$

Solution:
The desired characteristic equation with $-1 \pm j$ as solution is

$$\phi_d(s) = \{s-(-1+j)\}\{s-(-1-j)\} = s^2+2s+2,$$

$$\mathsf{C} = [b \quad Ab] = \begin{bmatrix} -1 & -1 \\ 1 & 3 \end{bmatrix},$$

Hence, Ackermann's formula yields

$$k^T = [0,\ 1]\mathsf{C}^{-1}\phi_d(A) = [-1.5 \quad 2.5]$$

7.4 Find the feedback gain k^T that assigns the closed-loop poles -1, $-1 \pm j$ to the following system:

$$(A,b) = \left(\begin{bmatrix} 0 & 1 & 2 \\ 1 & 3 & 4 \\ 0 & 5 & 6 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right).$$

Solution:
The characteristic equation with -1, $-1 \pm j$ as solution is

$$\phi_d(s) = (s+1)\{s-(-1+j)\}\{s-(-1-j)\} = s^3+3s^2+4s+2.$$

$$\mathsf{C} = \begin{bmatrix} b & Ab & A^2b \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 5 \end{bmatrix},$$

Hence, Ackermann's formula yields

$$k^T = [0,0,\ 1]\mathsf{C}^{-1}\phi_d(A) = \begin{bmatrix} 12 & 115 & 144 \end{bmatrix}.$$

7.5 The model of a rigid satellite in a frictionless environment is given by

$$T(t) = J\frac{d^2\theta}{dt^2},$$

where $T(t)$ is the applied torque caused by firing the thrustors, $\theta(t)$ is the altitude angle, and J=10 is the satellite's moment of inertia.

(i) Write the state equation of the system with the state variables chosen as angular position and velocity.
(ii) Design a control system by pole placement such that the closed-loop system has a 2% settling time $t_s = 4s$, and a damping ratio ς = 0.707.
(iii) Suppose there is no signal that appears in the rate path. What is the nature of the system response in this failure mode? Hint: no signal from the rate path implies that $k\dot{\theta} = 0,$, or let k=0 for analysis.

Solution:

(i) Let $x_1 = \theta,\ x_2 = \dot{\theta}$, and u=T, then

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ \frac{1}{10} \end{bmatrix} u,$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x.$$

(ii) Since $t_s = 4/\varsigma\omega_n$ and ς = 0.707, we have $\omega_n = \sqrt{2}$ and the desired characteristic polynomial becomes $\phi_f(s) = s^2 + 2\varsigma\omega_n s + \omega_n^2 = s^2 + 2s + 2$. Now using Ackermann's formula we have

$$\begin{aligned} k &= \begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} B & AB \end{bmatrix}^{-1}\phi_f(A) \\ &= \begin{bmatrix} 20 & 20 \end{bmatrix}. \end{aligned}$$

then we get the control law

$$u = -kx + r.$$

(iii) In the failure mode, there is no signal from $x_2 = \dot{\theta}$ so that

$u = -k^T x + r = -20x_1 + r$. The closed-loop system becomes

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ \frac{1}{10} \end{bmatrix} r .$$

7.6 Consider the system

$$\dot{x} = \begin{bmatrix} 0 & -1 \\ -2 & -1 \end{bmatrix} x + \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} u ,$$

$$y = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} x .$$

Investigate the pole placement problem for eigenvalues at -1 and -2.

Solution:
System is controllable and observable, that is irreducible (minimal), but unstable since the eigenvalues of A are 1 and -2. We need to position the closed-loop poles at λ_1=-1, λ_2=-2. Then by using the control law

$$u = r - Kx = r - \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} x ,$$

we have

$$\phi_f(s) = \det[sI - (A - BK)] = \det \begin{bmatrix} s + k_{21} & k_{22} + 1 \\ 2 + 2k_{11} + k_{21} & s + (1 + 2k_{12} + k_{22}) \end{bmatrix} ,$$
$$= s^2 + (k_{21} + k_{22} + 2k_{12} + 1)s + 2k_{21}k_{12} - 2k_{22} - 2k_{11}k_{22} - 2k_{11} - 2$$

and

$$\phi_d(s) = (s+1)(s+2) = s^2 + 3s + 2 .$$

If we expand the determinant to obtain $\phi_f(s)$ and match its coefficients with the ones in $\phi_d(s)$, we will obtain a set of nonlinear equations.

$$k_{21}+k_{22}+2k_{12}+1=3$$
$$2k_{21}k_{12}-2k_{22}-2k_{11}k_{22}-2k_{11}-2=2$$

Since there are only two equations with four design parameters, there is no unique solution for *K*. One possible solution is $k_{11}=-2,\ k_{12}=1,\ k_{21}=0$, and $k_{22}=0$, which is dyadic, i.e. *k* is rank deficient. A possible full rank solution can also be given by $k_{11}=-1.5,\ k_{12}=0.5,\ k_{21}=0$, and $k_{22}=1$.

7.7 Find the control law $u=-Kx$ for the system

$$\dot{x}=\begin{bmatrix}0&0&0&1\\1&0&0&-2\\-22&-11&-4&0\\-23&-6&0&-6\end{bmatrix}x+\begin{bmatrix}0&0\\0&0\\0&1\\1&3\end{bmatrix}u,$$

$$y=\begin{bmatrix}0&0&0&1\\0&0&1&0\end{bmatrix}x.$$

so that the poles of the closed-loop system are at -1, $-1\pm j$, -2.

Solution:
Full rank *Method.*
Get the controllability matrix:

$$\mathrm{W}_c=(B\quad AB\quad A^2B\quad A^3B)$$
$$=\begin{bmatrix}0&0&1&3&-6&-18&25&75\\0&0&-2&-6&13&39&-56&-168\\0&1&0&-4&0&16&-11&-97\\1&3&-6&-18&25&75&-90&-270\end{bmatrix}$$

Select 4 independent vectors from left to right, and construct

$$\mathrm{C}=(b_1\quad Ab_1\quad A^2b_1\quad b_2)=\begin{bmatrix}0&1&-6&0\\0&-2&13&0\\0&0&0&1\\1&-6&25&3\end{bmatrix}.$$

Then

$$\mathrm{C}^{-1} = \begin{bmatrix} 28 & 11 & -3 & 1 \\ 13 & 6 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{array}{l} \\ \\ \leftarrow q_3^T, \\ \leftarrow q_4^T \end{array}$$

$$T = \begin{bmatrix} q_3^T \\ q_3^T A \\ q_3^T A^2 \\ q_4^T \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

so that

$$\bar{A} = TAT^{-1} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -6 & -11 & -6 & 0 \\ -11 & 0 & 0 & -4 \end{bmatrix} \quad \bar{B} = TB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 3 \\ 0 & 1 \end{bmatrix},$$

The controllable canonical form of the system is given by

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -6 & -11 & -6 & 0 \\ -11 & 0 & 0 & -4 \end{bmatrix} \bar{x} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 3 \\ 0 & 1 \end{bmatrix} u,$$

where

$$\bar{x} = Tx = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \mathrm{x},$$

$$x = T^{-1}\bar{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \bar{x}.$$

Design the feedback gain matrix

$$\bar{K} = \begin{bmatrix} \bar{k}_{11} & \bar{k}_{12} & \bar{k}_{13} & \bar{k}_{14} \\ \bar{k}_{21} & \bar{k}_{22} & \bar{k}_{23} & \bar{k}_{24} \end{bmatrix}$$

Compute

$$\bar{B}\bar{K} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \bar{k}_{11} & \bar{k}_{12} & \bar{k}_{13} & \bar{k}_{14} \\ \bar{k}_{21} & \bar{k}_{22} & \bar{k}_{23} & \bar{k}_{24} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \bar{k}_{11}+3\bar{k}_{21} & \bar{k}_{12}+3\bar{k}_{22} & \bar{k}_{13}+3\bar{k}_{23} & \bar{k}_{14}+3\bar{k}_{24} \\ \bar{k}_{21} & \bar{k}_{22} & \bar{k}_{23} & \bar{k}_{24} \end{bmatrix}$$

Form the closed loop matrix:

$$\bar{A} - \bar{B}\bar{K} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -6-(\bar{k}_{11}+3\bar{k}_{21}) & -11-(\bar{k}_{12}+3\bar{k}_{22}) & -6-(\bar{k}_{13}+3\bar{k}_{23}) & -(\bar{k}_{14}+3\bar{k}_{24}) \\ -11-\bar{k}_{21} & -\bar{k}_{22} & -\bar{k}_{23} & -4-\bar{k}_{24} \end{bmatrix}$$

Since the desirable characteristic equation is

$$\begin{aligned} \det(sI - \bar{A} + \bar{B}\bar{K}) &= (s+1)(s+1-j)(s+1+j)(s+2) \\ &= (s^2+3s+2)(s^2+2s+2) \\ &= s^4+5s^3+10s^2+10s+4 \,. \end{aligned}$$

Choose A_d as

$$A_d = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -4 & -10 & -10 & -5 \end{bmatrix}.$$

which meets requirements. Compare it with $\bar{A} - \bar{B}\bar{K}$, we have

$$\bar{K} = \begin{bmatrix} 15 & -41 & -36 & -4 \\ -7 & 10 & 10 & 1 \end{bmatrix}$$

therefore,

$$K = \overline{K}T = \begin{bmatrix} 15 & -41 & -36 & -4 \\ -7 & 10 & 10 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -11 & 15 & -4 & -36 \\ -4 & -7 & 1 & 10 \end{bmatrix}.$$