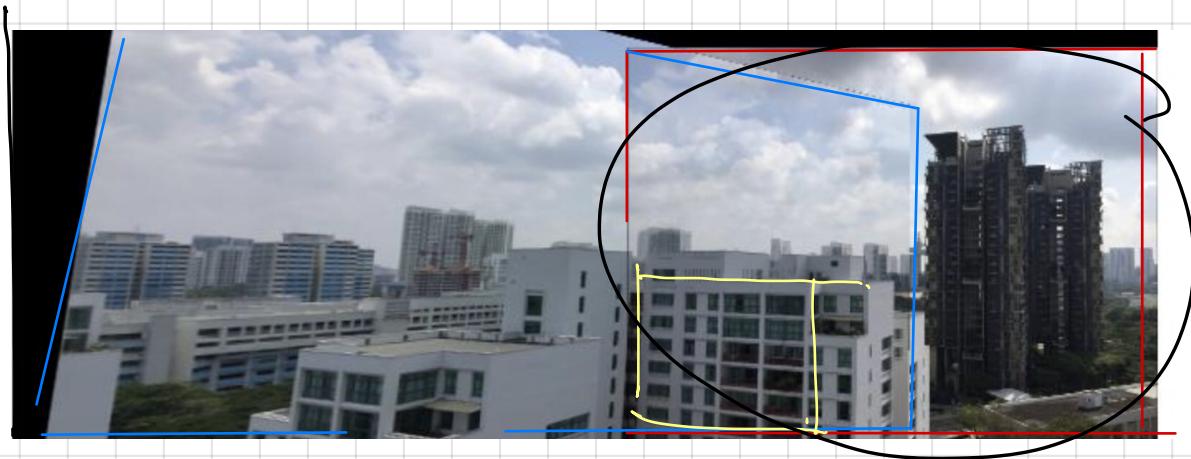


# LECTURE 4: IMAGE STITCHING

#1



Algorithm to stitch 2 images:

1. Extract the keypoints from the two images.
2. Based on the keypoints (i.e. the descriptors), find all the matches.
3. From the matches, compute the best homography matrix using RANSAC.
4. Stitch the two images, by transforming one image to the other using the computed homography matrix.

## [r] Keypoint Matching

Goal: To determine the corresponding keypoints from two images that match best.

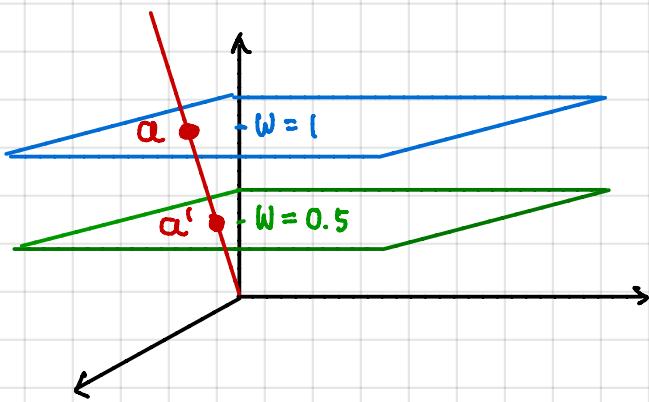
$$\text{Algorithm: } \text{Best}_i = \min_j \sqrt{\sum_k \| \bar{v}_i[k] - \bar{v}_j[k] \|^2}$$

where:  
 $i$  = the index of a keypoint on image 1.  
 $j$  = the index of a keypoint on image 2.  
 $\bar{v}_i$  = the descriptor of keypoint  $i$   
 $k$  = the index of the descriptor

Real number  
space

## [2] Homogeneous Coordinates

In  $\mathbb{R}^2$  (or a 2D space), the homogeneous coordinates are expressed as:  $\begin{bmatrix} x \\ y \\ w \end{bmatrix}$



$$a = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} ; a' = \begin{bmatrix} x' \\ y' \\ 0.5 \end{bmatrix}$$

In the inhomogeneous coord.,  $a$  &  $a'$  are exactly the same point :

$$a = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x' \\ y' \\ 0.5 \end{bmatrix} = \begin{bmatrix} x'/0.5 \\ y'/0.5 \\ 1 \end{bmatrix}$$

Q: Why use homogeneous coordinates ?

Theoretical reason:

One of the main benefits is the ability to represent a point at infinity :  $\begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$

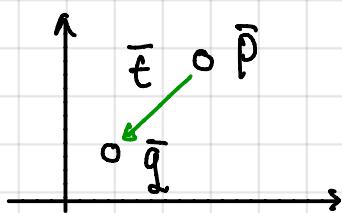
Practical reason:

Homogeneous coordinates allow us to express the translation operation in a matrix form .

Homogeneous coord. :

Inhomogeneous  
coordinates

Translation in a matrix form is impossible:



$$\bar{p} - \bar{t} = \bar{q}$$

$$\begin{bmatrix} p_x \\ p_y \end{bmatrix} - \begin{bmatrix} t_x \\ t_y \end{bmatrix} = \begin{bmatrix} q_x \\ q_y \end{bmatrix}$$

How to do this in a matrix form:  $\bar{q} = T \bar{p}$ , without using the homogeneous coord.?

$$\begin{aligned} \bar{q} &= T \bar{p} \\ \begin{bmatrix} q_x \\ q_y \\ 1 \end{bmatrix} &= \begin{bmatrix} 1 & 0 & -tx \\ 0 & 1 & -ty \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ 1 \end{bmatrix} \end{aligned}$$

### [3] Hierarchy of Transformations in 2D ( $\mathbb{R}^2$ )

#3

1. Euclidean group : Rotation & Translation:

$$\begin{bmatrix} r_{11} & r_{12} & t_x \\ r_{21} & r_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

Invariant properties: length & area (will be the same when we apply these translations)

2. Similarity group: Scaling:

$$\begin{bmatrix} s r_{11} & s r_{12} & t_x \\ s r_{21} & s r_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

Invariant properties: ratio of lengths ; angle

3. Affine group: Shearing:

$$\begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$$

Invariant properties: parallelism, ratio of lengths on midpoints, etc.

4. Projective group:

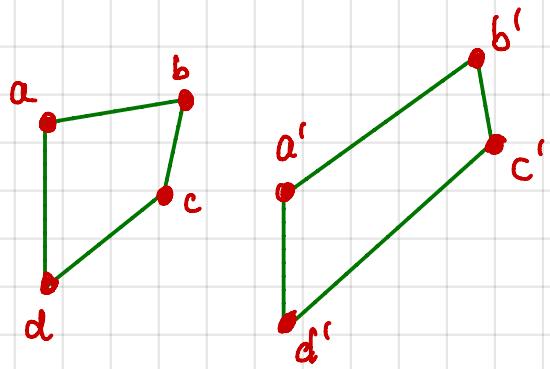
$$\begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}$$

Invariant properties: concurrency, collinearity ,etc

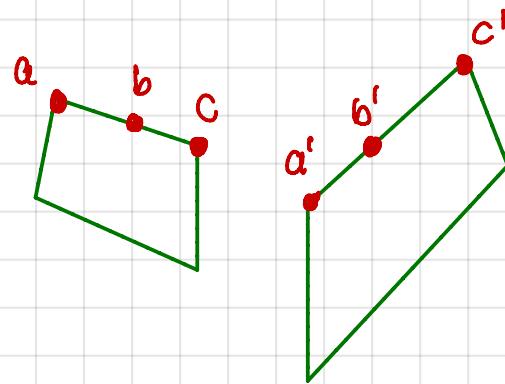
!  
intersecting lines  
still intersect.

!  
points on a line still points  
on a line

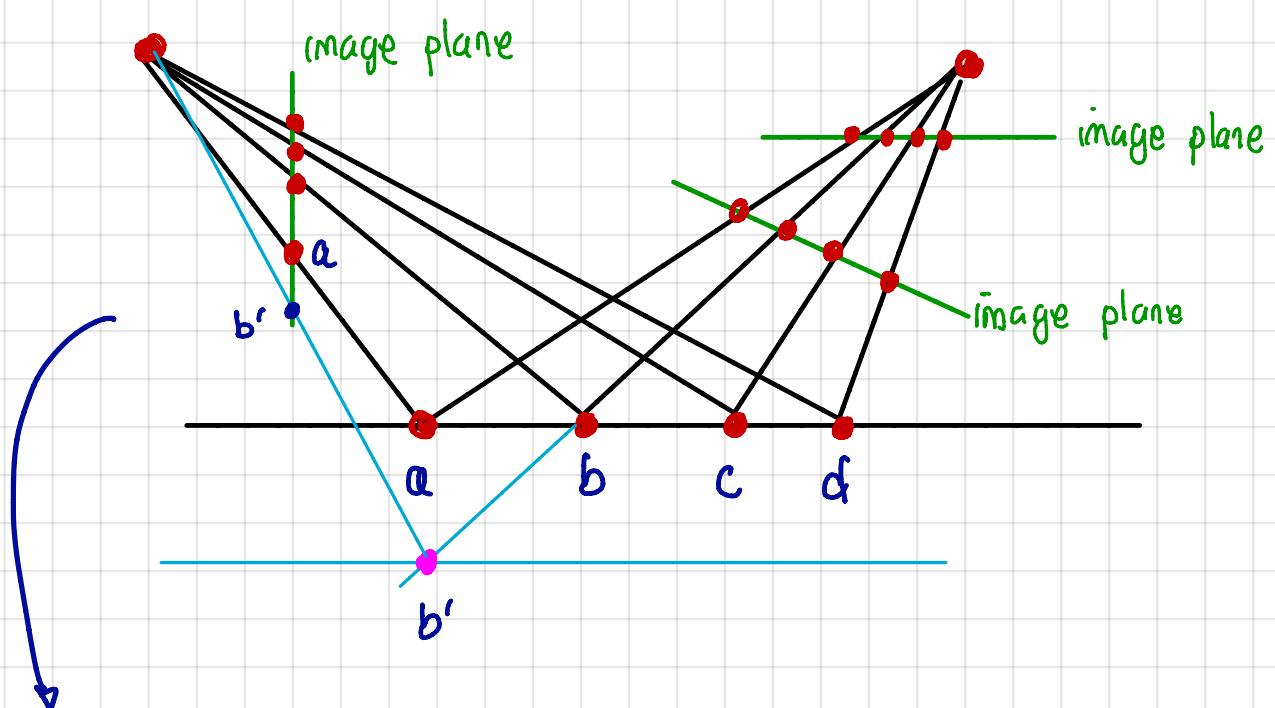
Concurrency preservation:



Collinearity preservation:



ID:

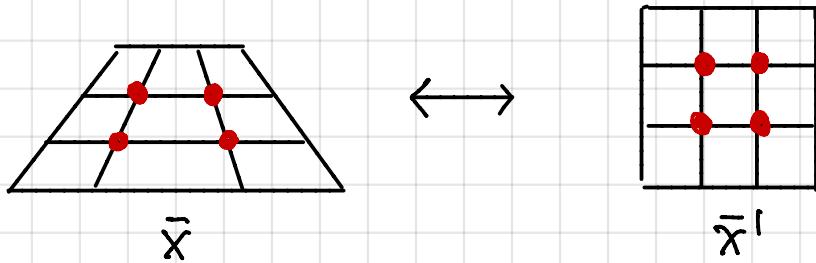


For  $b'$ , collinearity is violated!

Hence, collinearity can be preserved if the actual points lie on a planar surface.

## [4] Homography

In short, homography is a matrix that transforms:



Any two images of the same planar surface in space are related by a homography (assuming a pinhole camera model):

$$\bar{x}' = H \bar{x}$$

How to compute  $H$ , given pairs of  $\bar{x}$  and  $\bar{x}'$ ?

$$\bar{x}' = H \bar{x}$$

3x1      3x3      3x1

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

To convert matrix  $H$  to a vector:

$$( \Rightarrow ) \quad x' = \frac{h_{11}x + h_{12}y + h_{13}}{h_{31}x + h_{32}y + h_{33}} \quad ; \quad y' = \frac{h_{21}x + h_{22}y + h_{23}}{h_{31}x + h_{32}y + h_{33}}$$

$$( \Rightarrow ) \quad h_{11}x + h_{12}y + h_{13} - h_{31}x x' - h_{32}y x' - h_{33}x x' = 0$$

$$h_{21}x + h_{22}y + h_{23} - h_{31}x y' - h_{32}y y' - h_{33}x y' = 0$$

$$( \Rightarrow ) \quad \begin{bmatrix} x & y & 1 & 0 & 0 & 0 & -x'x & -x'y & -x' \\ 0 & 0 & 0 & x & y & 1 & -y'x & -y'y & -y' \end{bmatrix} \begin{bmatrix} h_{11} \\ h_{12} \\ \vdots \\ h_{33} \end{bmatrix} = 0$$

$$\begin{bmatrix} x & y & 1 & 0 & 0 & 0 & -x^T x & -x^T y & -x^T \end{bmatrix} \begin{bmatrix} h_{11} \\ h_{12} \\ \vdots \\ h_{33} \end{bmatrix} = 0 \rightarrow \text{Homogeneous linear system}$$

$2 \times 9$   
 $9 \times 1$

Assuming we have more than 4 pairs of matched keypoints:

$$A \bar{h} = 0 ; n \geq 4$$

(n x 2) x g g x 1  
(Overdetermined system)

To avoid a trivial solution, where  $\bar{h} = 0$ , we need to put a constraint on  $\bar{h}$ , such as  $|\bar{h}| = 1$ :

$A \bar{h} = 0$   
 s.t.  $|\bar{h}| = 1$

or:

$\min |A \bar{h}|$   
 s.t.  $|\bar{h}| = 1$



Solution for getting  $\bar{h}$  given  $A$ : use SVD

$$A = UDV^T \rightarrow \bar{h} \text{ is the last row of } V^T$$

(see page #13 for more detailed discussion).

Notes:

1. Considering :  $\|\bar{h}\| = 1$ , the g unknowns becomes 8, since we can obtain  $h_{33}$  given  $h_{11} \dots h_{32}$ .
2. If the number of unknowns in  $\bar{h}$  is 8, then we only need 4 pairs to recover  $\bar{h}$ . And, the system becomes a closed form (instead of overdetermined)



A closed-form system is solvable directly, even without SVD.



Q: How to do this?

$$A: \begin{bmatrix} x & y & 1 & 0 & 0 & 0 & -x'x & -x'y & -x' \\ 0 & 0 & 0 & x & y & 1 & -y'x & -y'y & -y' \end{bmatrix} \begin{bmatrix} h_{11}/h_{33} \\ h_{12}/h_{33} \\ \vdots \\ h_{32}/h_{33} \\ 1 \end{bmatrix} = 0$$

$2 \times 9$

$g \times 1$

if  $\tilde{h}_{ij} = h_{ij}/h_{33}$

then:

$$\begin{aligned} x \tilde{h}_{11} + y \tilde{h}_{12} + \tilde{h}_{13} - x'x \tilde{h}_{31} - x'y \tilde{h}_{32} - x' &= 0 \\ x \tilde{h}_{21} + y \tilde{h}_{22} + \tilde{h}_{23} - y'x \tilde{h}_{31} - y'y \tilde{h}_{32} - y' &= 0 \end{aligned}$$

$$\Leftrightarrow \begin{bmatrix} x & y & 1 & 0 & 0 & 0 & -x'x & -x'y & -x' \\ 0 & 0 & 0 & x & y & 1 & -y'x & -y'y & -y' \end{bmatrix} \begin{bmatrix} \tilde{h}_{11} \\ \tilde{h}_{12} \\ \vdots \\ \tilde{h}_{33} \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

$2 \times 8$

$2 \times 1$

if we have 4 pairs:

$A \tilde{h} = \bar{b}$
$8 \times 8 \quad 8 \times 1 \quad 8 \times 1$

$$\rightarrow \tilde{h} = A^{-1} \bar{b}$$

3.

$$\tilde{h} = A^{-1} \bar{b}$$

$8 \times 1 \quad 8 \times 8 \quad 8 \times 1$

$$\bar{b} = \begin{bmatrix} x_1' \\ y_1' \\ x_2' \\ y_2' \\ x_3' \\ y_3' \\ x_4' \\ y_4' \end{bmatrix}$$

$$\tilde{h}^2 = \begin{bmatrix} h_{11}/h_{33} \\ h_{21}/h_{33} \\ \vdots \\ h_{32}/h_{33} \end{bmatrix}$$

Q: How can we recover  $\bar{h}$  from  $\tilde{h}$ ?

A:  $|\tilde{h}| = 1 \rightarrow \sum_{i,j} \tilde{h}_{ij}^2 = 1$

$$\Leftrightarrow h_{11}^2 + h_{12}^2 + \dots + h_{33}^2 = 1$$

$$\Leftrightarrow \frac{h_{11}^2}{h_{33}^2} + \frac{h_{12}^2}{h_{33}^2} + \dots + 1 = \frac{1}{h_{33}^2}$$

$$\Leftrightarrow \underbrace{\tilde{h}_{11}^2 + \tilde{h}_{12}^2 + \dots + \tilde{h}_{32}^2}_{\text{known!}} + 1 = \frac{1}{h_{33}^2}$$

Unknown!

$$\Leftrightarrow \tilde{h}_{33}^2 = \frac{1}{\sum_{i,j} \tilde{h}_{ij}^2 + 1}$$

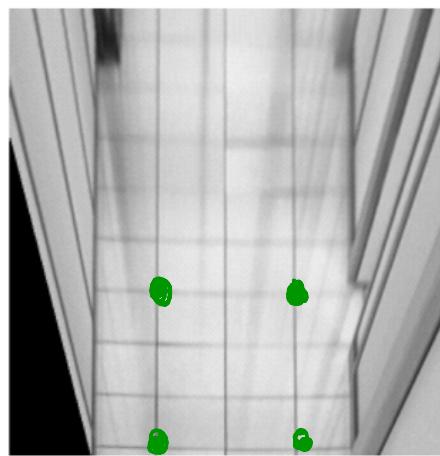
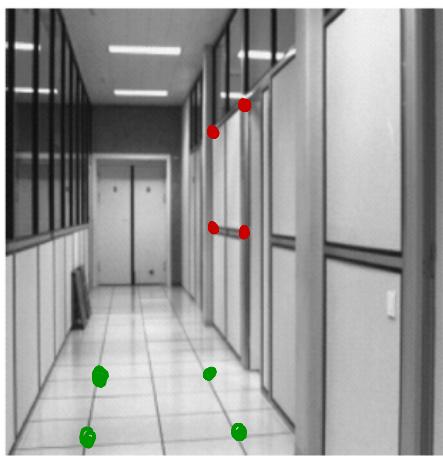
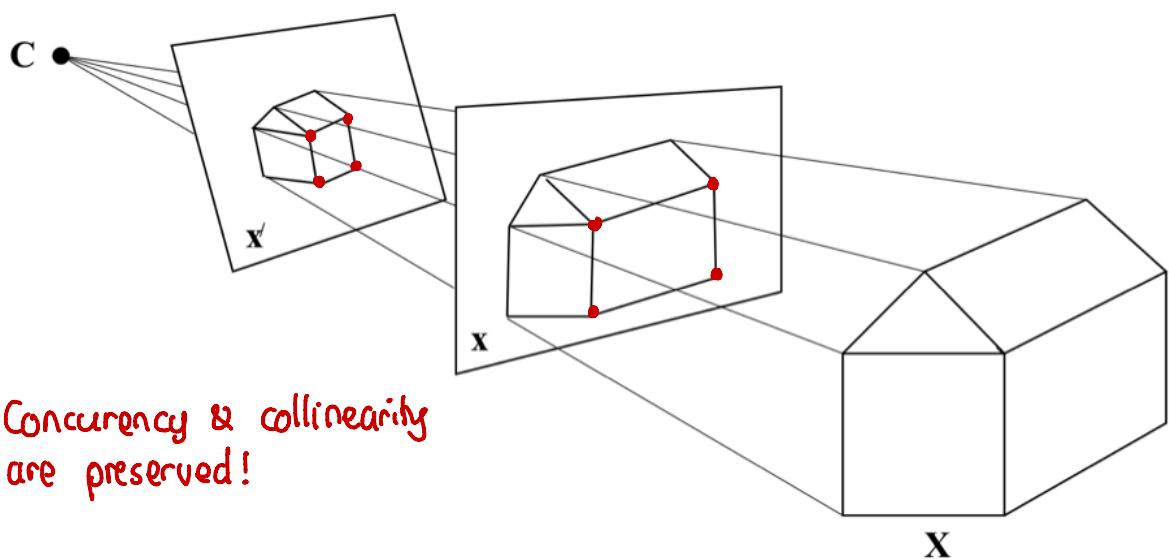
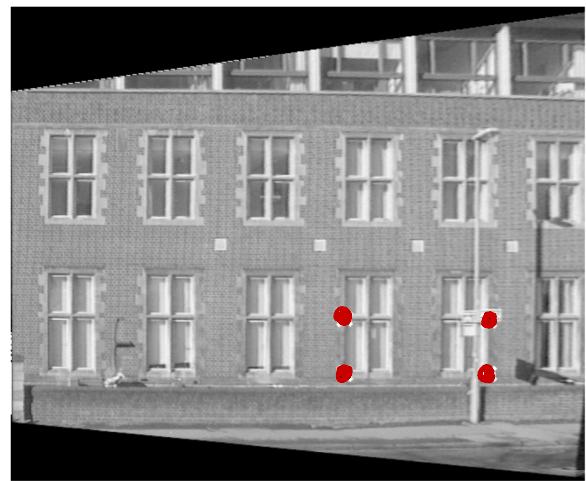
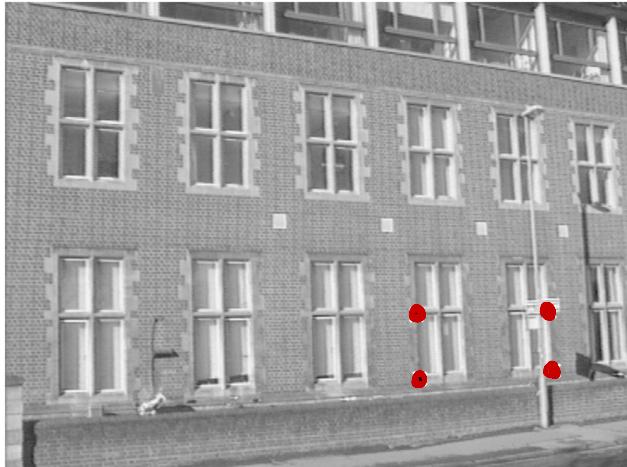
Hence:

$$h_{ij} = \tilde{h}_{ij} \tilde{h}_{33}$$



Note that, an over-determined system (where  $n > 4$ ) is more robust than a closed-form system, where noise can affect the values in  $A$ .

Example:



## [5] RANSAC : Random Sample Consensus

#10

Algorithm (General) :

1. Select a minimum subset of points randomly
2. Fit the function (e.g. a line) using the points
3. Count the inliers (i.e. the number of other points outside the subset that agree with this solution)
4. Repeat the above process for T iterations
5. Choose the function's parameters that have the most inliers.

Algorithm (RANSAC + Homography) :

but not that large

1. Select n pairs of keypoints randomly ;  $n \geq 5$
2. Compute the homography matrix from the n pairs
3. Compute inliers, where the distance,  $d(x_{in}^i, Hx_{in}^i) < \epsilon$
4. Repeat the above process for T iterations & keep the largest set of inliers
5. Recompute H on all of the inliers



Either using SVD or Least Squares .

least squares: MLE :

$$H^* = \underset{\{H\}}{\operatorname{argmin}} \sum_n (x'_n - H x_n)^2$$

↓

8 unknown parameters

[solvable using gradient descent → Levenberg-Maquardt]

$$H^* = \underset{\{H\}}{\operatorname{argmin}} E(H)$$

$$\text{where: } E(H) = \sum_n \left( \begin{bmatrix} x'_n \\ y'_n \end{bmatrix} - H \begin{bmatrix} x_n \\ y_n \end{bmatrix} \right)^2$$

$$= \sum_n \begin{pmatrix} x'_n - h_{11} x_n - h_{12} y_n - h_{13} \\ y'_n - h_{21} x_n - h_{22} y_n - h_{23} \\ 1 - h_{31} x_n - h_{32} y_n - h_{33} \end{pmatrix}^2$$

Minimization:

$$\frac{\partial E(H)}{\partial h_{11}} = \sum_n \frac{\partial}{\partial h_{11}} (x'_n - h_{11} x_n - h_{12} y_n - h_{13})^2$$

$$= -2 \sum_n (x'_n - h_{11} x_n - h_{12} y_n - h_{13}) x_n = 0$$

$h_{11}$  depends on unknowns:  $h_{12}$  &  $h_{13}$  → Hence it can't be solved directly.

Gradient Descent:

$$h_{11}^{\text{new}} = h_{11}^{\text{old}} - \eta \frac{\partial E(H)}{\partial h_{11}} \quad \boxed{\frac{\partial E(H)}{\partial h_{11}}} \quad |_{h_{11}=h_{11}^{\text{old}}}$$

Q: How to get the initialization of this?

A: From the result of RANSAC.

## [6] Stitching Images

### [•] Stitching 2 images :

1. Using the homography  $H_{21}$  :  $\bar{x}_1 = H \bar{x}_2$ , transform image 2 to image 1.
2. Since the transformation changes the size of image 1, prepare an empty array that is large enough to handle image 1 + the transformed image 2.
3. This array becomes your canvas.

### [•] Adding image 3 to the canvas

1. Compute the homography between the canvas & image 3,  $H_{C3}$ , which is defined as:

$$x_C = H_{C3} x_3$$

2. Enlarge the canvas so that it can handle image 3.
3. Transform image 3 to the canvas.

## [7] Homogeneous Systems & SVD

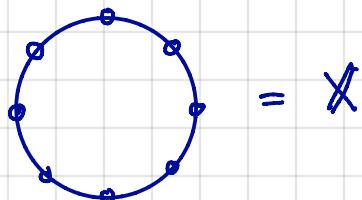
$$A = U D V^T$$

MxN      MxM      MxN      NxN

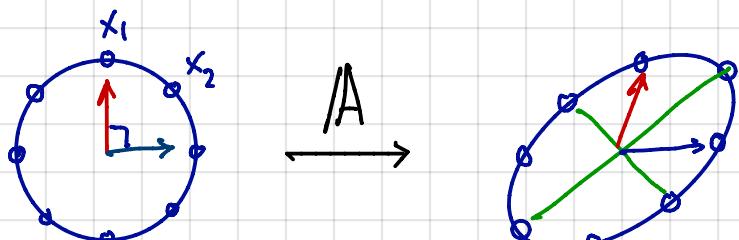
orthogonal matrix  
 (rotation)      diagonal matrix      orthogonal matrix  
 (rotation)

## [•] SVD illustration :

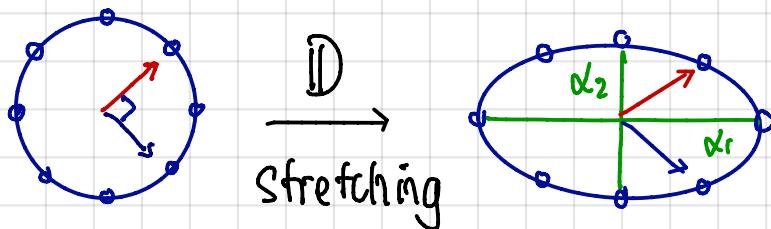
Assume we have a collection of points in matrix,  $\mathbf{X}$ .



If we do :  $A \times$  means  $\times$  is transformed by matrix  $A$



$$\text{rotation} = V^T \downarrow \quad \uparrow \quad \text{U} = \text{rotation}$$



# [●] SVD & Homogeneous Systems

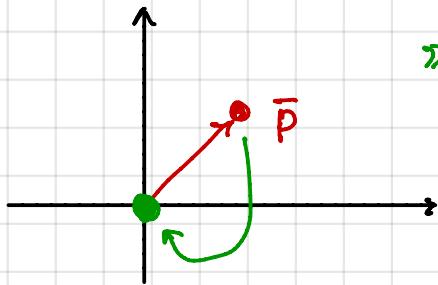
#14

We want to solve:  $A\bar{p} = 0$

$\bar{p} \in \mathbb{R}^2$ :

where  $A$  is known &  $|\bar{p}| = 1$

» What is  $\bar{p}$  so that when we transform  $\bar{p}$  using  $A$ , the outcome is zero (= the origin).



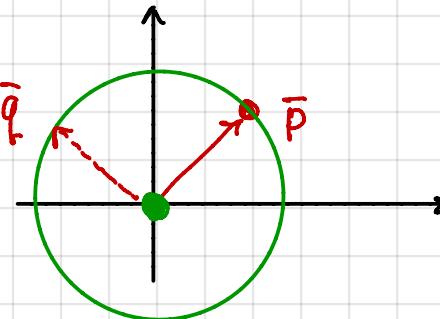
Using SVD:

$$A\bar{p} = UDV^T\bar{p} = 0$$

$$UDV^T\bar{p} = 0$$

let:  $\bar{q} = V^T\bar{p}$   
↳ a rotation matrix,

which doesn't make  $\bar{q} = 0$ .



$$\underbrace{UD}_{b}\bar{q} = 0 \rightarrow \text{Assuming } \mathbb{R}^2:$$

This is another rotation  
that doesn't make  
the transformation to zero.

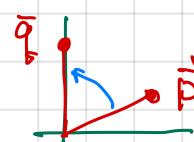
$$D\bar{q} = \begin{bmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{bmatrix} \begin{bmatrix} q_x \\ q_y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\alpha_1 > \alpha_2 \quad \downarrow$$

we want to find  $q_x$  &  $q_y$   
so that the equation holds.

Due to the constraint:  $\|\bar{p}\| = 1$   
 it also implies:  $\|\bar{q}\| = 1$ .  
 Hence, we can define  $\bar{q} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ , which assuming  $\alpha_2 = 0$ :

$$\begin{bmatrix} \alpha_1 & 0 \\ 0 & \alpha_2=0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$



We choose  $\bar{q} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

since  $\alpha_1 > \alpha_2$

#15

In some cases,  $\alpha_2$  is indeed zero; however, in most cases  $\alpha_2 > 0$ .  
 Although, usually:  $\alpha_1 > \alpha_2$ .

If  $\alpha_2 > 0$ , then it's impossible to make  $D\bar{q} = 0$ . For this reason:

$$\begin{aligned} \min |A\bar{p}| \\ \text{s.t. } |\bar{p}| = 1 \end{aligned}$$

and,  $\bar{q} = \begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix}$  will enable us to get the minimum, but not zero.

Implying  $\bar{p}$  is the last column of  $V$ .

$$\text{If } \bar{q} = \begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix} \rightarrow \bar{q} = V^T \bar{p}$$

$$\bar{p} = V \bar{q} = \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} V_{12} \\ V_{22} \end{bmatrix}$$

[•] Proof that  $\bar{p}$  = the last column of  $V$ , minimizing  $|A\bar{p}|$

$$\min |A\bar{p}| = \min |UDV^T \bar{p}| \rightarrow \text{Orthogonal matrix's property:}$$

$$= \min |DV^T \bar{p}| ; \bar{q} = V^T \bar{p} \quad |U\bar{z}| = |\bar{z}|$$

$$= \min |D\bar{q}| \quad \left. \begin{array}{l} \bar{q} = [0, 0, \dots, 1]^T \\ \text{s.t. } |\bar{q}| = 1 \end{array} \right\} \quad \text{corresponds to smallest element of } D$$

$\uparrow$   
 we want this kind  
 of  $\bar{q}$ .

Since:  $\bar{p} = V \bar{q}$ , thus:  $\bar{p}$  is simply the last column of  $V$ .

## • Example 1:

$$\text{1. } A\bar{p} = 0 \text{ and } A = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} \rightarrow Q: \text{what is } \bar{p} = ?$$

$$\text{SVD: } A = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ -1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix} \begin{bmatrix} \sqrt{10}/2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

$\mathbb{U}$        $D$        $V^T$

$$\text{Hence: } \bar{p} = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

$\alpha_2 = 0 \rightarrow$  what does it imply?

This value makes  $A\bar{p} = 0$ :

$$A\bar{p} = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The stretching makes the second axis disappear!

## Example 2:

$$\min |A\bar{p}| \text{ for } A = \begin{bmatrix} 2 & 3 \\ 4 & 2 \\ 1 & 2 \end{bmatrix}$$

$$\text{SVD: } A = \begin{bmatrix} 2 & 3 \\ 4 & 2 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 \times 3 & -0.6 & 0.5 & -0.6 \\ 3 \times 2 & -0.7 & -0.7 & 0.1 \\ & -0.3 & 0.5 & 0.8 \end{bmatrix} \begin{bmatrix} 3 \times 2 & 5.9 & 0 \\ & 0 & 1.7 \\ & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 \times 2 & -0.7 & -0.6 \\ & -0.6 & 0.7 \end{bmatrix}$$

$\mathbb{U}$        $D$        $V^T$

$$\text{Hence: } \bar{p} = \begin{bmatrix} -0.6 \\ 0.7 \end{bmatrix}$$

The smallest eigenvalue is

$$|A\bar{p}| = \left| \begin{bmatrix} 2 & 3 \\ 4 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -0.6 \\ 0.7 \end{bmatrix} \right| = 1.69$$

its corresponding eigenvector is  $\begin{bmatrix} -0.6 \\ 0.7 \end{bmatrix}$

↳ it's not zero, yet it's the minimum value

## • SVD: What is it?

$$A = \mathbb{U} D V^T \rightarrow A^T A = (\mathbb{U} D V^T)^T (\mathbb{U} D V^T)$$

$$= V D^T \underline{\mathbb{U}^T \mathbb{U}} D V^T = V D^T D V^T$$

$D^T D = D^2 =$  the eigenvalues of  $A^T A$

Orthogonal matrices:

$$\mathbb{U}^T \mathbb{U} = \mathbb{I}$$

$V$ 's columns are the eigenvectors of  $A^T A$

(using the same derivation  $(AA^T)$ ),  $\mathbb{U}$ 's columns are the eigenvectors of  $A A^T$