Controllability and Observability Part II

Chong-Jin Ong

Department of Mechanical Engineering,
National University of Singapore

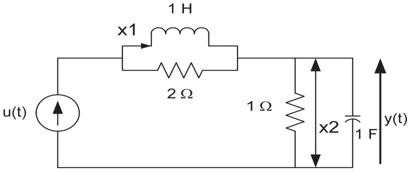
Outline

Observability

- Observability is a closely related concept to Controllability.
- Controllability deals with the ability to steer the states using the input.
- Observability studies the ability to estimate the states from the output.
- Loosely, observability refers to the ability to “see” the effects of all the states from the output.

Why is observability important?

- The design of many controllers requires the information of the states.
- However, some of the state variables are not accessible for direct measurement $\Rightarrow$ Estimation of unmeasurable states is needed.
- Estimation of states is possible if and only if the system observable.
- If $u(t) = 0$ for all $t \geq 0$ and $x_1(0) = a \neq 0$ for some unknown a, and $x_2(0) = 0$, $\Rightarrow y(t)$ is identically zero for all time.



Definition

- While a general definition of observability exists, the one used here is for LTI system.
- **Definition:** A linear time-invariant system is observable if **every** unknown initial state $x(0)$ can be determined from the knowledge of $u(t)$ and the observation of $y(t)$ over a **finite** time interval. Otherwise, the LTI system is said to be unobservable.
- For a system to be observable, **all** initial states $x(0)$ can be determined using a **finite** time interval. If only a subset of the initial states can be determined or that it takes infinite amount of time to determine all initial states, the system is not observable.

- The study of observability is concerned with the unforced system:

$$\dot{x} = Ax, \quad y = Cx$$

  as a result of the assumption of $u(t)$ being completely known.
- The time solution of the complete state-space system is

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau, \text{ and,}$$

$$y(t) = Ce^{At}x(0) + C\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau + Du(t)$$

- Under complete knowledge of $\{A, B, C, D\}$ and $u(t)$, the overall system can be restated as those above.
- We first provide one condition for observability.
- The others are based on a duality result.

Observability

Theorem 4.2 The n-dimensional LTI system or the pair (A, C) is observable if and only if the $n \times n$ matrix (the observability grammian)

$$M(0,t) := \int_0^t e^{A^T\tau} C^T C e^{A\tau} d\tau$$

is non-singular for any $t > 0$.
Proof: $(\Rightarrow)$ Suppose $M(0,t)$ is non-singular for all $t > 0$. We have

$$y(t) = Ce^{At}x(0)$$

$$\Rightarrow e^{A^T t} C^T y(t) = e^{A^T t} C^T C e^{At} x(0), \text{ or,}$$

$$\Rightarrow \int_0^t e^{A^T\tau} C^T y(\tau) d\tau = \int_0^t e^{A^T\tau} C^T C e^{A\tau} d\tau x(0) = M(0,t)x(0)$$

This implies that

$$x(0) = M^{-1}(0,t) \int_0^t e^{A^T\tau} C^T y(\tau)$$

Thus, system is observable.

Observability

Proof: ($\Leftarrow$) Suppose $M(0,t)$ is singular for some $t > 0$ but the system is observable. We want to show that this leads to a contradiction. $M(0,t)$ is singular means that there exists a non-zero vector α s.t.

$$\alpha^T M(0,t)\alpha = 0$$

$$\Rightarrow \int_0^t \alpha^T e^{A^T t} C^T C e^{At} \alpha dt = 0$$

$$\Rightarrow \int_0^t \|Ce^{At}\alpha\|^2 dt = 0$$

$$\Rightarrow Ce^{At}\alpha = 0 \text{ for all } t$$

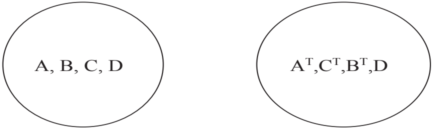
This means that the output

$$y(t) = Ce^{At}x(0) = 0$$

if $x(0) = \alpha$. Hence, $y(t)$ is identically zero and the system is not observable, which leads to a contradiction. $\square$

Duality

- (A, B) is controllable if and only if (A^T, B^T) is observable.
- (A, B) controllable $\Rightarrow W(0,t) = \int_0^t e^{A\tau} B B^T e^{A^T\tau} d\tau$ is non-singular.
- (A^T, B^T) observable $\Rightarrow M(0,t) = \int_0^t e^{A\tau} B B^T e^{A^T\tau} d\tau$ is non-singular.
- Using the same argument, we have
- (A, C) is observable if and only if (A^T, C^T) is controllable.



Original System Dual System

Conditions

The following statements are equivalent.

1. The pair (A, C) is observable.
2. The observability Grammian $M(0,t) := \int_0^t e^{A^T\tau} C^T C e^{A\tau} d\tau$ is non-singular for all $t > 0$.
3. The $mn \times n$ observability matrix

$$O = \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix}$$

   is full column rank.
4. The $(n+m) \times n$ matrix

$$\begin{pmatrix} A - \lambda I \\ C \end{pmatrix}$$

   has full column rank at every eigenvalue of A.

Proof: Either directly or by invoking duality result and then followed by the corresponding proofs given in the controllability results.

Observability under Similarity Transformation

Like the case of controllability, the observability matrix, under a new basis representation is

$$\bar{O} = \begin{pmatrix} \bar{C} \\ \bar{C}\bar{A} \\ \vdots \\ \bar{C}\bar{A}^{n-1} \end{pmatrix} = \begin{pmatrix} CT \\ CTT^{-1}AT \\ \vdots \\ CA^{n-1}T \end{pmatrix} = OT$$

Hence, observability is not affected by similarity transformation.

Observability Grammian

- The Observability Grammian

$$M(0,\ell) := \int_0^\ell e^{A^T\tau} C^T C e^{A\tau} d\tau$$

  has to be full rank for all $\ell > 0$.
- Like the Controllability case, the full rank condition of $M(0,\ell)$ can be shown to be independent of ℓ.
- WLOG, consider the case where $\ell \to \infty$ and

$$M := \lim_{\ell\to\infty} M(0,\ell) = \int_0^\infty e^{A^T\tau} C^T C e^{A\tau} d\tau$$

Computation of Observability Grammian

- The expression of $M = \int_0^\infty e^{A^T\tau} C^T C e^{A\tau} d\tau$ is hard to compute.
- Like the controllability case, there is an expression that allow easy computation of M. In particular, M satisfies the Lyapunov Equation

$$A^T M + MA = -C^T C$$

  such that the solution of the above yields M.
- This follows because (assuming A is asymptotically stable),

$$A^T M + MA = \int_0^\infty \frac{d}{dt}(e^{A^T t} C^T C e^{At})dt = -C^T C$$

Gilbert Decomposition

- Question: What can be done with an uncontrollable (unobservable) system?
- **Theorem 4.3:** Given a LTI system S. If the controllability matrix of S has rank $n_1 (n_1 < n)$, then there exists an equivalent transformation which transforms the system into $\bar{S}$:

$$\begin{pmatrix} \dot{\bar{x}}_c \\ \dot{\bar{x}}_{\bar{c}} \end{pmatrix} = \begin{pmatrix} \bar{A}_c & \bar{A}_{12} \\ 0 & \bar{A}_{\bar{c}} \end{pmatrix} \begin{pmatrix} \bar{x}_c \\ \bar{x}_{\bar{c}} \end{pmatrix} + \begin{pmatrix} \bar{B}_c \\ 0 \end{pmatrix} u$$

$$y = \begin{pmatrix} \bar{C}_c & \bar{C}_{\bar{c}} \end{pmatrix} \begin{pmatrix} \bar{x}_c \\ \bar{x}_{\bar{c}} \end{pmatrix} + \bar{D}u$$

and the n_1-dimensional subsystem $\bar{S}_c$ of $\bar{S}$

$$\dot{\bar{x}}_c = \bar{A}_c \bar{x}_c + \bar{B}_c u$$
$$y = \bar{C}_c \bar{x}_c + \bar{D}u$$

is controllable and has the same transfer function matrix as S.

Proof

Proof: If S is not controllable then

$$rank U = rank[B \ AB \cdots \ A^{n-1}B] = n_1 < n$$

Let $p_1, \cdots, p_{n_1}$ be any linearly independent vectors of U. Note that $Ap_i, i = 1, \cdots, n_1$ can be written as a l.c. of $p_1, \cdots, p_{n_1}$. Hence, choose P for $\bar{x} = Px$ as

$$P^{-1} = [p_1, \cdots, p_{n_1}, p_{n_1+1}, \cdots, p_n]$$

where the last $(n - n_1)$ columns are arbitrary so long as P^{-1} is non-singular. Using this, $\bar{S}$ becomes

$$\bar{A} = PAP^{-1}, \ \bar{B} = PB, \ \bar{C} = CP^{-1}, \bar{D} = D$$

For columns of B, note that B is in the range space of U and can be expressed as a l.c. of $p_1, \cdots, p_{n_1}$, or

$$B = P^{-1}\bar{B} = [p_1, \cdots, p_{n_1}, p_{n_1+1}, \cdots, p_n] \begin{pmatrix} \bar{B}_c \\ 0 \end{pmatrix}$$

Similarly,

$$AP^{-1} = A[p_1, \cdots, p_{n_1}, p_{n_1+1}, \cdots, p_n] = [p_1, \cdots, p_{n_1}, p_{n_1+1}, \cdots, p_n] \begin{pmatrix} \bar{A}_c & \bar{A}_{12} \\ 0 & \bar{A}_{\bar{c}} \end{pmatrix}$$

Hence, $\bar{S}$ has the said structure.

Proof

To see that $\bar{S}_c$ is controllable, note that

$$
\begin{aligned}
n_1 = rank U = rank \bar{U} &= rank[\bar{B} \ \bar{A}\bar{B} \cdots \ \bar{A}^{n-1}\bar{B}] \\
&= rank \begin{pmatrix} \bar{B}_c & \bar{A}_c\bar{B}_c & \cdots & \bar{A}_c^{n-1}\bar{B}_c \\ 0 & 0 & \cdots & 0 \end{pmatrix} = rank[\bar{B}_c \ \bar{A}_c\bar{B}_c \ \cdots \ \bar{A}_c^{n-1}\bar{B}_c]
\end{aligned}
$$

To see that T.F. of S and $\bar{S}_c$ are the same, note that T.F. of $\bar{S}$ is

$$
\begin{pmatrix} \bar{C}_c & \bar{C}_{\bar{c}} \end{pmatrix} \begin{pmatrix} sI - \bar{A}_c & -\bar{A}_{12} \\ 0 & sI - \bar{A}_{\bar{c}} \end{pmatrix}^{-1} \begin{pmatrix} \bar{B}_c \\ 0 \end{pmatrix} + \bar{D}
$$

$$
= \begin{pmatrix} \bar{C}_c & \bar{C}_{\bar{c}} \end{pmatrix} \begin{pmatrix} (sI - \bar{A}_c)^{-1} & (sI - \bar{A}_c)^{-1}\bar{A}_{12}(sI - \bar{A}_{\bar{c}})^{-1} \\ 0 & (sI - \bar{A}_{\bar{c}})^{-1} \end{pmatrix} \begin{pmatrix} \bar{B}_c \\ 0 \end{pmatrix} + \bar{D}
$$

$$
= \bar{C}_c(sI - \bar{A}_c)^{-1}\bar{B}_c + \bar{D} = \text{T. F. of } \bar{S}_c
$$

Note that the system of $\bar{S}$ can be considered as two subsystems of the form

$$\begin{aligned}\dot{\bar{x}}_c &= \bar{A}_c \bar{x}_c + \bar{A}_{12} \bar{x}_{\bar{c}} + \bar{B}_c u \\ \dot{\bar{x}}_{\bar{c}} &= \bar{A}_{\bar{c}} \bar{x}_{\bar{c}}\end{aligned}$$

where the dynamics of $\bar{x}_c$ is affected by $\bar{x}_{\bar{c}}$ and u, while the dynamics of $\bar{x}_{\bar{c}}$ is unaffected by $\bar{x}_c$ and u. Hence,

$$\dot{\bar{x}}_{\bar{c}} = \bar{A}_{\bar{c}} \bar{x}_{\bar{c}}$$

is an uncontrollable subsystem.

- **Definition:** If $\bar{A}_{\bar{c}}$ is stable, the system S is known as stablizable.
- On the other hand, S is not stablizable if $\bar{A}_{\bar{c}}$ is unstable.

Example

$$\dot{x} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} x + \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} u$$

$$y = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} x$$

$$rankU = rank \begin{pmatrix} 0 & 1 & 1 & 1 & 2 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 2 & 1 \end{pmatrix} = 2 < 3$$

Choose $P^{-1} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ where the last column is arbitrary so long as P is non-singular. Then

$$\bar{A} = PAP^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\bar{B} = PB = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \bar{C} = CP^{-1} = \begin{pmatrix} 1 & 2 & 1 \end{pmatrix}$$

An equivalent result exists for an unobservable system (expected in view of duality),

- **Theorem 4.4:** Given a LTI system S. If the observability matrix has rank $n_2 (n_2 < n)$, then there exists an equivalent transformation which transforms S into $\bar{S}$:

$$\begin{pmatrix} \dot{\bar{x}}_o \\ \dot{\bar{x}}_{\bar{o}} \end{pmatrix} = \begin{pmatrix} \bar{A}_o & 0 \\ \bar{A}_{12} & \bar{A}_{\bar{o}} \end{pmatrix} \begin{pmatrix} \bar{x}_o \\ \bar{x}_{\bar{o}} \end{pmatrix} + \begin{pmatrix} \bar{B}_o \\ \bar{B}_{\bar{o}} \end{pmatrix} u$$

$$y = \begin{pmatrix} \bar{C}_o & 0 \end{pmatrix} \begin{pmatrix} \bar{x}_o \\ \bar{x}_{\bar{o}} \end{pmatrix} + \bar{D}u$$

  and the n_2-dimensional subsystem $\bar{S}_o$ of $\bar{S}$

$$\dot{\bar{x}}_o = \bar{A}_o \bar{x}_o + \bar{B}_o u$$
$$y = \bar{C}_o \bar{x}_o + \bar{D}u$$

  is observable and has the same transfer function matrix as S.

- **Definition:** If $\bar{A}_{\bar{o}}$ is stable, the system S is known as detectable.
- On the other hand, S is not detectable if $\bar{A}_{\bar{o}}$ is unstable.

Proof

Proof: Since $rankO = n_2 < n$. Choose $(n - n_2)$ l.i. vectors in the null space of the observability matrix. Let $q_{n_2+1}, \cdots, q_n$ be such vectors and choose non-singular Q as

$$Q = [q_1 \cdots q_{n_2} \cdots q_n]$$

Consider the state transformation $\bar{x} = Q^{-1}x$, we have

$$\bar{A} = Q^{-1}AQ, \ \bar{B} = Q^{-1}B, \ \bar{C} = CQ, \bar{D} = D$$

Again, the structure of $\bar{C}$ is

$$CQ = [\bar{C}_o \ 0]$$

Notice also that $Aq_{n_2+1}, \cdots, Aq_n$ are in the null space of O. Hence,

$$A[q_1 \cdots q_{n_2} \cdots q_n] = Q \begin{pmatrix} \bar{A}_o & 0 \\ \bar{A}_{12} & \bar{A}_{\bar{o}} \end{pmatrix}.$$

(Exercise) Note that $(\bar{A}_o, \bar{C}_o)$ is observable and T.F. of S = T.F. of $\bar{S}_o$.

Example

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}, \; B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \; C = \begin{pmatrix} -1 & 1 \end{pmatrix}$$

Note that $O = \begin{pmatrix} C \\ CA \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}$. Choose basis for null space as $[1\ 1]^T$. Then

$$Q = \begin{pmatrix} q_1 & q_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } Q^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

Using this Q, the transformed system becomes

$$\begin{pmatrix} \dot{\bar{x}}_o \\ \dot{\bar{x}}_{\bar{o}} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} \bar{x}_o \\ \bar{x}_{\bar{o}} \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u, \quad y = [-1 \quad 0]\bar{x}$$

Gilbert-Kalman Canonical Decomposition

Theorem 4.5 Any LTI system can be converted into the following form using an appropriate similarity transformation.

$$\begin{pmatrix} \dot{\bar{x}}_{c\bar{o}} \\ \dot{\bar{x}}_{co} \\ \dot{\bar{x}}_{\bar{c}\bar{o}} \\ \dot{\bar{x}}_{\bar{c}o} \end{pmatrix} = \begin{pmatrix} \bar{A}_{c\bar{o}} & \bar{A}_{12} & \bar{A}_{13} & \bar{A}_{14} \\ 0 & \bar{A}_{co} & 0 & \bar{A}_{24} \\ 0 & 0 & \bar{A}_{\bar{c}\bar{o}} & \bar{A}_{34} \\ 0 & 0 & 0 & \bar{A}_{\bar{c}o} \end{pmatrix} \begin{pmatrix} \bar{x}_{c\bar{o}} \\ \bar{x}_{co} \\ \bar{x}_{\bar{c}\bar{o}} \\ \bar{x}_{\bar{c}o} \end{pmatrix} + \begin{pmatrix} \bar{B}_{c\bar{o}} \\ \bar{B}_{co} \\ 0 \\ 0 \end{pmatrix} u$$

$$y = \begin{pmatrix} 0 & \bar{C}_{co} & 0 & \bar{C}_{\bar{c}o} \end{pmatrix} + \bar{D}u$$

The transfer matrix of the system is given by

$$G(S) = \bar{C}_{co}(sI - \bar{A}_{co})^{-1}\bar{B}_{co} + \bar{D}$$

Proof: Combining proofs of Theorems 4.3 and 4.4.

Minimality, Controllability and Observability

- From the Gilbert-Kalman decomposition, it is clear that if a LTI system is either uncontrollable or unobservable, there exists a dynamical system of lesser dimension that has the same transfer function matrix as the original dynamical system.
- As a result, the following definition is now given.
- Definition: A LTI dynamical system, S, is said to be minimal when there is no other LTI system of lesser dimension that has the same transfer function matrix as S.