Canonical Forms

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Outline

1. SISO system: Controllable Canonical Form
2. SISO system: Observable Canonical Form

Controllable Canonical Form

- Consider the system

$$\dot{x} = Ax + Bu \tag{1a}$$
$$y = Cx \tag{1b}$$

where $x \in \mathbb{R}^n, u \in \mathbb{R}, y \in \mathbb{R}$.

- The controllability matrix is given by

$$U = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} \in \mathbb{R}^{n \times n}$$

Theorem 6.1: If $rank(U) = n$, then the system (1) can be transformed into a controllable form given by

$$\dot{\bar{x}} = \begin{bmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 1 \\ -a_0 & -a_1 & \cdots & -a_{n-2} & -a_{n-1} \end{bmatrix} \bar{x} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u \tag{2a}$$

$$y = \begin{bmatrix} b_0 & b_1 & \cdots & b_{n-2} & b_{n-1} \end{bmatrix} \bar{x} \tag{2b}$$

Controllable Canonical Form

- In (3), the a_i and b_i are the coefficients of the denominator and numerator of polynomials of the transfer function representation of system (1) of

$$\frac{Y(s)}{U(s)} = \frac{b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \cdots + a_1 s + a_0}$$

Proof: Since $rank(U) = n$, U^{-1} exists. Let q^T be the n^{th} row of U^{-1}. Then, from $U^{-1}U = I$, we have

$$q^T A^i B = 0 \text{ for all } i = 0, 1, \cdots, n-2$$

$$q^T A^{n-1} B = 1$$

Let the matrix T of transformation $\bar{x} = Tx$ be

$$T = \begin{bmatrix} q^T \\ q^T A \\ \vdots \\ q^T A^{n-1} \end{bmatrix}$$

Then, $\bar{B} = TB = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$

Controllable Canonical Form

and from

$$\bar{A}T = TA$$

$$\bar{A}\begin{bmatrix} q^T \\ q^T A \\ \vdots \\ q^T A^{n-1} \end{bmatrix} = \begin{bmatrix} q^T \\ q^T A \\ \vdots \\ q^T A^{n-1} \end{bmatrix} A$$

The first row of the above equation is

$$\bar{A}_{1\cdot} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \end{bmatrix}$$

Similarly, the remaining rows of $\bar{A}$ can be inferred. For the last row of $\bar{A}$, we have

$$\bar{A}_{n\cdot}T = q^T A^n = q^T \left[-\sum_{i=0}^{n-1} a_i A^i \right]$$

and so

$$\bar{A}_{n\cdot} = \begin{bmatrix} -a_0 & -a_1 & \cdots & -a_{n-2} & -a_{n-1} \end{bmatrix}$$

Controllable Canonical Form

Remarks

- (i) The transfer function of the Controllable Canonical Form is

$$\frac{Y(s)}{U(s)} = \frac{b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \cdots + a_1 s + a_0}$$

- (ii) The CCF is always controllable. Se Tutorial 3.
- (iii) The matrix T is non-singular because $|det(TU)| = 1$ (verify this!)

Controllable Canonical Form

Example:

$$\dot{x} = \begin{pmatrix} -2 & -3 \\ -4 & -5 \end{pmatrix} x + \begin{pmatrix} 1 \\ 2 \end{pmatrix} u$$
$$y = \begin{pmatrix} 1 & 1 \end{pmatrix} x$$

Note that $\frac{y(s)}{u(s)} = \frac{3s-1}{s^2+7s-2}$. The controllability matrix is

$$U = [B \quad AB] = \begin{pmatrix} 1 & -8 \\ 2 & -14 \end{pmatrix} \rightarrow U^{-1} = \begin{pmatrix} -7 & 4 \\ -1 & 0.5 \end{pmatrix}$$

The last row of U^{-1} is $q^T = [-1 \quad 0.5]$. Thus, $T = \begin{pmatrix} q^T \\ q^T A \end{pmatrix} = \begin{pmatrix} -1 & 0.5 \\ 0 & 0.5 \end{pmatrix}$ and

$$\dot{\bar{x}} = TAT^{-1}\bar{x} + TBu = \begin{pmatrix} 0 & 1 \\ 2 & -7 \end{pmatrix} \bar{x} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$
$$y = CT^{-1}\bar{x} = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 0 & 2 \end{pmatrix} \bar{x} = \begin{pmatrix} -1 & 3 \end{pmatrix} \bar{x}$$

Observable Canonical Form

Consider the system of (1) and the corresponding observability matrix is

$$O = \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix} \in \mathbb{R}^{n\times n}$$

Theorem 6.2: If $rank(O) = n$, then the system (1) can be transformed into a observability form given by

$$\dot{\bar{x}} = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 1 & 0 & -a_{n-2} \\ 0 & 0 & \cdots & 1 & -a_{n-1} \end{bmatrix} \bar{x} + \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_{n-2} \\ b_{n-1} \end{bmatrix} u \tag{3a}$$

$$y = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \bar{x} \tag{3b}$$

where a_i and b_i are the coefficients as given in Theorem 6.1.

Observable Canonical Form

Proof: Since $rank(O) = n$, O^{-1} exists. Let the last column of O^{-1} be w. From $OO^{-1} = I$, we have

$$CA^i w = 0 \text{ for all } i = 0, 1, \cdots, n-2$$
$$CA^{n-1} w = 1$$

Then we form the transformation matrix Q where $x = Q\bar{x}$ be

$$Q = \begin{bmatrix} w & Aw & \cdots & A^{n-1}w \end{bmatrix}$$

Then, under the new coordinate system, we have

$$\bar{A} = Q^{-1}AQ,\ \bar{B} = Q^{-1}B,\ \bar{C} = CQ,\ \bar{D} = D$$

It is easy to see from $\bar{C} = CQ$ that

$$C\begin{bmatrix} w & Aw & \cdots & A^{n-1}w \end{bmatrix} = \begin{bmatrix} Cw & CAw & \cdots & CA^{n-1}w \end{bmatrix} = \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix}$$

and

$$Q\bar{A} = AQ$$
$$\begin{bmatrix} w & Aw & \cdots & A^{n-1}w \end{bmatrix}\bar{A} = \begin{bmatrix} Aw & A^2w & \cdots & A^n w \end{bmatrix}$$

Observable Canonical Form

Looking at the first column of the above equation, we can see that

$$\bar{A}_{\cdot 1} = TB = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

The same can be seen for the other columns. For the last column, the use of Caley-Hamilton Principle is needed. As a result, we have the following system observable canonical representation.

$$\dot{\bar{x}} = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 1 & 0 & -a_{n-2} \\ 0 & 0 & \cdots & 1 & -a_{n-1} \end{bmatrix} \bar{x} + \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_{n-2} \\ b_{n-1} \end{bmatrix} u \tag{4}$$

$$y = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \bar{x} \tag{5}$$

Observable Canonical Form

Remarks

- (i) The transfer function of the Observable Canonical Form is

$$\frac{Y(s)}{U(s)} = \frac{b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \cdots + a_1 s + a_0}$$

- (ii) The OCF is always observable - its observability matrix is always full rank (check it yourself).
- (iii) The matrix Q is non-singular because $|det(QO)| = 1$ (verify this!)

Observable Canonical Form

Example:

$$\dot{x} = \begin{pmatrix} -2 & -3 \\ -4 & -5 \end{pmatrix} x + \begin{pmatrix} 1 \\ 2 \end{pmatrix} u$$

The observability matrix is

$$O = \begin{pmatrix} C \\ CA \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -6 & -8 \end{pmatrix} \rightarrow O^{-1} = \begin{pmatrix} 4 & 0.5 \\ -3 & -0.5 \end{pmatrix}$$

with $w = \begin{pmatrix} 0.5 \\ -0.5 \end{pmatrix}$. The Q matrix is

$$Q = \begin{pmatrix} 0.5 & 0.5 \\ -0.5 & 0.5 \end{pmatrix} \text{ and } Q^{-1} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

Hence, we have

$$\dot{\bar{x}} = Q^{-1}AQ\bar{x} + Q^{-1}Bu$$

$$= \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -2 & -3 \\ -4 & -5 \end{pmatrix} \begin{pmatrix} 0.5 & 0.5 \\ -0.5 & 0.5 \end{pmatrix} \bar{x} + \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} u$$

$$= \begin{pmatrix} 0 & 2 \\ 1 & -7 \end{pmatrix} \bar{x} + \begin{pmatrix} -1 \\ 3 \end{pmatrix} u$$

$$y = CQx = [1 \quad 1] \begin{pmatrix} 0.5 & 0.5 \\ -0.5 & 0.5 \end{pmatrix} \bar{x} = [0 \quad 1]\bar{x}$$