Solution to Tutorial 2

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Q1

Assuming that $I \neq 0$, we have

$$I\ddot{\theta} + b\dot{\theta} + k\theta = H\omega \ cos\theta$$

Let $x_1 = \theta, x_2 = \dot{\theta}$ then

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\frac{k}{I}x_1 - \frac{b}{I}x_2 + \frac{H}{I}\omega \ cos x_1
\end{aligned}$$

Note that this system is nonlinear. To linearize the system, consider

$$x = x_0 + \delta x \text{ and } \omega = \omega_0 + \delta\omega$$

where x_0 and ω_0 are the state and the input function about which the linearization is done. In this case,

$$\begin{aligned}
\dot{\delta x} &= \frac{\partial f}{\partial x}|_{(x_0,\omega_0)}\delta x + \frac{\partial f}{\partial u}|_{(x_0,\omega_0)}\delta u \\
\dot{\delta x}_1 &= 0 \cdot \delta x_1 + 1 \cdot \delta x_2 \\
\dot{\delta x}_2 &= -\frac{k}{I}\delta x_1 - \frac{b}{I}\delta x_2 - \frac{H}{I}\omega \ sin x_1|_{(0,0)}\delta x_1 + \frac{H}{I}cos x_1|_{(0,0)}\delta\omega \\
\begin{pmatrix} \dot{\delta x}_1 \\ \dot{\delta x}_2 \end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ -k/I & -b/I \end{pmatrix}\begin{pmatrix} \delta x_1 \\ \delta x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ H/I \end{pmatrix}\delta\omega.
\end{aligned}$$

Q2

$$A = \begin{pmatrix} 0 & 0 & -2 \\ 0 & 1 & 0 \\ 1 & 0 & 3 \end{pmatrix}$$

$$|\lambda I - A| = \begin{vmatrix} \lambda & 0 & 2 \\ 0 & \lambda - 1 & 0 \\ -1 & 0 & \lambda - 3 \end{vmatrix} = \lambda(\lambda-1)(\lambda-5) + 2(\lambda-1)$$

$$= (\lambda-1)(\lambda-1)(\lambda-2) = 0.$$

Hence, the eigenvalues are $\lambda = 1, \lambda = 1$ and $\lambda = 2$.

Let

$$e^{\lambda t} = \alpha_0(t) + \lambda\alpha_1(t) + \lambda^2\alpha_2(t) \tag{1}$$

$$\text{For } \lambda = 1, \Rightarrow e^t = \alpha_0(t) + \alpha_1(t) + \alpha_2(t) \tag{2}$$

$$\text{For } \lambda = 2, \Rightarrow e^{2t} = \alpha_0(t) + 2\alpha_1(t) + 4\alpha_2(t) \tag{3}$$

Differentiating (1) w.r.t λ, we have

$$te^{\lambda t} = \alpha_1(t) + 2\lambda\alpha_2(t)$$

$$\text{which, when } \lambda = 1 \Rightarrow te^t = \alpha_1(t) + 2\alpha_2(t) \tag{4}$$

From (2),(3) and (4), we have

$$\begin{aligned} \alpha_0(t) &= -2te^t + e^{2t} \\ \alpha_1(t) &= 2e^t + 3te^t - 2e^{2t} \\ \alpha_2(t) &= -e^t - te^t + e^{2t} \end{aligned}$$

Hence,

$$e^{At} = \alpha_0(t)I + \alpha_1(t)A + \alpha_2(t)A^2$$

Q3

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} x = Ax.$$

The eigenvalues of A are computed from

$$|\lambda I - A| = \begin{vmatrix} \lambda & -1 \\ 2 & \lambda + 3 \end{vmatrix} = (\lambda+2)(\lambda+1) = 0$$

$$\Rightarrow \quad \lambda = -2, \lambda = -1.$$

For $\lambda = -2$, eigenvector is $v_1 = [-1 \;\; 2]^T$.

For $\lambda = -1$, eigenvector is $v_2 = [-1 \;\; 1]^T$.

Hence,

$$T = \begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix} \text{ and } T^{-1} = \begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix}.$$

Therefore,

$$\begin{aligned} x(t) = Te^{\Lambda t}T^{-1}x_0 &= \begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} e^{-2t} & 0 \\ 0 & e^{-t} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix} \\ &= (x_{10} + x_{20}) \begin{pmatrix} -1 \\ 2 \end{pmatrix} e^{-2t} + (-2x_{10} - x_{20}) \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-t} \\ &= \begin{pmatrix} -1 \\ 2 \end{pmatrix} 2e^{-2t} - \begin{pmatrix} -1 \\ 1 \end{pmatrix} 3e^{-t} \end{aligned}$$

Q4

Check by direct verification of controllability matrix U and observability matrix O.

Q5

Since

$$\begin{aligned} A \cdot A^{-1} &= I \\ \Rightarrow (A \cdot A^{-1})^T &= I \\ \Rightarrow (A^{-1})^T \cdot A^T &= I \\ \Rightarrow (A^{-1})^T &= (A^T)^{-1} \end{aligned}$$

Q6

$$A = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}$$

Consider the matrix $[\gamma I - A \quad b]$ for whatever b, we have, when $\gamma = \lambda$

$$[\gamma I - A \quad b] = \begin{pmatrix} 0 & 0 & 0 & b_1 \\ 0 & 0 & -1 & b_2 \\ 0 & 0 & 0 & b_3 \end{pmatrix}.$$

In this case, the maximum rank of $[\gamma I - A \quad b]$ is 2 for all possible choices of b. Hence, $[\gamma I - A \quad b]$ is not full row rank. Hence, system is not controllable.

Q7

$$\dot{x} = \begin{pmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{pmatrix} x + \begin{pmatrix} b_1 \\ \bar{b}_1 \end{pmatrix} u, \qquad y = [c_1 \quad \bar{c}_1]x$$

Let $x = Q\bar{x}$ where $Q = \begin{pmatrix} -\bar{\lambda}b_1 & b_1 \\ -\lambda\bar{b}_1 & \bar{b}_1 \end{pmatrix}$, then

$$\bar{A} = Q^{-1}AQ, \quad \bar{B} = Q^{-1}B, \quad \bar{C} = CQ$$

and

$$\begin{aligned} Q\bar{A} &= AQ \\ \begin{pmatrix} -\bar{\lambda}b_1 & b_1 \\ -\lambda\bar{b}_1 & \bar{b}_1 \end{pmatrix}\begin{pmatrix} \bar{a}_{11} \\ \bar{a}_{21} \end{pmatrix} &= \begin{pmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{pmatrix}\begin{pmatrix} -\bar{\lambda}b_1 \\ -\lambda\bar{b}_1 \end{pmatrix} = \begin{pmatrix} -\lambda\bar{\lambda}b_1 \\ -\lambda\bar{\lambda}\bar{b}_1 \end{pmatrix} \\ \Rightarrow \quad \begin{pmatrix} \bar{a}_{11} \\ \bar{a}_{21} \end{pmatrix} &= \begin{pmatrix} 0 \\ -\lambda\bar{\lambda} \end{pmatrix}. \end{aligned}$$

Similarly,

$$\begin{aligned} \begin{pmatrix} -\bar{\lambda}b_1 & b_1 \\ -\lambda\bar{b}_1 & \bar{b}_1 \end{pmatrix}\begin{pmatrix} \bar{a}_{12} \\ \bar{a}_{22} \end{pmatrix} &= \begin{pmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{pmatrix}\begin{pmatrix} b_1 \\ \bar{b}_1 \end{pmatrix} = \begin{pmatrix} \lambda b_1 \\ \bar{\lambda}\bar{b}_1 \end{pmatrix} \\ \Rightarrow \quad \begin{pmatrix} \bar{a}_{12} \\ \bar{a}_{22} \end{pmatrix} &= \begin{pmatrix} 1 \\ \lambda + \bar{\lambda} \end{pmatrix}. \end{aligned}$$

For $\bar{B} = Q^{-1}B$, we have

$$\begin{aligned} Q\bar{B} &= B \\ \begin{pmatrix} -\bar{\lambda}b_1 & b_1 \\ -\lambda\bar{b}_1 & \bar{b}_1 \end{pmatrix}\begin{pmatrix} \hat{b}_1 \\ \hat{b}_2 \end{pmatrix} &= \begin{pmatrix} b_1 \\ \bar{b}_1 \end{pmatrix} \end{aligned}$$

$$\Rightarrow \quad \hat{b}_1 = 0, \quad \hat{b}_2 = 1. \text{ Hence, } \bar{B} = [0 \quad 1]^T$$

For $\bar{C}$, we have

$$\begin{aligned} \bar{C} = CQ = [c_1 \quad \bar{c}_1]\begin{pmatrix} -\bar{\lambda}b_1 & b_1 \\ -\lambda\bar{b}_1 & \bar{b}_1 \end{pmatrix} &= [-(c_1\bar{\lambda}b_1 + \bar{c}_1\lambda\bar{b}_1) \quad c_1b_1 + \bar{c}_1\bar{b}_1] \\ &= [-2Re(\bar{\lambda}b_1c_1) \quad 2Re(b_1c_1)]. \end{aligned}$$