Equi-Integrability is a Special Case of Equi-Continuity

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1 Introductions

In real analysis we have learned the equi-integrability of a family of integrable functions:

Definition 1: Equi-integrability A family $\mathcal{F}$ of measurable functions on a measurable set E of finite measure is defined to be equi-integrable over E if for every $\epsilon > 0$, there exists a $\delta > 0$ such that for every $f \in \mathcal{F}$, the following statement holds:

$$\text{if } A \subseteq E \text{ is measurable and } m(A) < \delta, \text{ then } \int_A |f| < \epsilon.$$

The definition of equi-integrability is quite similar that of equi-continuity we have learned in mathematical analysis:

Definition 2: Equi-continuity Let X and Y be metric spaces with d_X and d_Y as their metrics, respectively. Let F be a family of functions from X to Y. F is defined to be equi-continuous at a point $x_0 \in X$ if for every $\epsilon > 0$, there exists a $\delta > 0$ such that for every $f \in \mathcal{F}$, the following statement holds:

$$\text{for all } x \in X, \text{ if } d_X(x_0, x) < \delta, \text{ then } d_Y(f(x_0), f(x)) < \epsilon.$$

The above observations lead to some questions: Are there any relationship between the above two definitions? And further, is equi-integrability a special case of equi-continuity (for integration in particular metric spaces)?

2 Notations and Preliminaries

To answer the questions, we should find a metric space on which integration can be properly defined.

Notation Let $\mathcal{M}$ be the collection of all measurable subsets of $\mathbb{R}$ with finite measure. We define a distance on $\mathcal{M}$ as following: For $A, B \in \mathcal{M}$, $d(A, B) := m(A \triangle B)$, where $A \triangle B = (A \setminus B) \cup (B \setminus A)$.

Remark: Notice that the integrals of a same function on two finite domains are the same if the domains only differ with a set of measure zero, and in the article we only need to

consider the integral of certain functions on its domain instead of the function itself on the domain. Therefore we regard $A, B \in \mathcal{M}$ as the same if $d(A,B) := m(A \triangle B) = 0$ (in this case, we denote their relation by $A \doteq B$).
Now we can see that $(\mathcal{M}, d)$ is indeed a metric space.
Lemma The distance $d(A,B) := m(A \triangle B)$ is a well-defined metric on $\mathcal{M}$.
Proof Straightly by the definition, for all $A, B \in \mathcal{M}$, d satisfies:

- $d(A,B) \geq 0$ and $d(A,B) = 0$ if and only if $A \doteq B$.
- $d(A,B) = d(B,A)$.

for the third axiom, for any $A, B, C \in \mathcal{M}$, by definition $A \triangle C = (A \setminus C) \cup (C \setminus A)$.
Notice that

$$\begin{cases} A \setminus C &= [A \setminus (B \cup C)] \ \cup \ [(A \cap B) \setminus C]; \\ C \setminus A &= [C \setminus (B \cup A)] \ \cup \ [(B \cap C) \setminus A]. \end{cases}$$

We also have

$$\begin{cases} [A \setminus (B \cup C)] &\subseteq \ A \setminus B; \\ [(A \cap B) \setminus C] &\subseteq \ B \setminus C; \\ [C \setminus (B \cup A)] &\subseteq \ C \setminus B; \\ [(B \cap C) \setminus A] &\subseteq \ B \setminus A. \end{cases}$$

Therefore, $A \triangle C = (A \setminus C) \cup (C \setminus A) \subseteq [A \setminus B] \cup [B \setminus C] \cup [C \setminus B] \cup [B \setminus A] = [A \triangle B] \cup [B \triangle C]$.
Then we get the third axiom:

- $d(A,C) \leq d(A,B) + d(B,C)$ □

Now we define the metric space to be considered in the following part.
Definition 3 Let E be a measurable set of finite measure (the same as in the definition 1). Denote by $(\mathcal{M}_E, d)$ the subspace of $(\mathcal{M}, d)$ that contains all measurable subsets of E. Note that the metric d is directly inherited from $(\mathcal{M}, d)$.
Now we consider the integration operation as a functional:
Definition 4 Let f be an integrable function defined on E, we define the functional $\Phi_f : \mathcal{M}_E \to \mathbb{R}_+ \cup \{0\}$ by $\Phi_f(S) := \int_S |f|$, for all $S \in \mathcal{M}_E$.
Check for Well-definedness: for $S_1, S_2 \subseteq E$, if $S_1 \doteq S_2$, i.e. $m(S_1 \triangle S_2) = 0$ then

$$\int_{S_1} |f| = \int_{S_1 \cap S_2} |f| + \int_{S_1 \setminus S_2} |f| = \int_{S_1 \cap S_2} |f| = \int_{S_1 \cap S_2} |f| + \int_{S_2 \setminus S_1} |f| = \int_{S_2} |f|$$

Therefore $\Phi_f(S_1) = \Phi_f(S_2)$, which means Φ_f is well-defined.
Let $\mathcal{F}$ be a collection of integrable function on E, let's consider the collection of functional: $\mathbf{\Phi}_{\mathcal{F}} = \{\Phi_f | f \in \mathcal{F}\}$. In the following part, we show that the equi-integrability of $\mathcal{F}$ on E is equivalent to the equi-continuity of $\mathbf{\Phi}_{\mathcal{F}}$ "at E" (note that in the space $\mathcal{M}_E$, we can treat a set as a "point").
Definition 5: Equi-continuity of collection of functional on $\mathcal{M}_E$ Similar to the definition of usual equi-continuity, we say $\mathbf{\Phi}_{\mathcal{F}}$ is equi-continuous "at set E" if for every $\epsilon > 0$, there exists a $\delta > 0$ such that for every $f \in \mathcal{F}$, the following statement holds:

$$\text{for all } S \in \mathcal{M}_E, \text{ if } d(S,E) < \delta, \text{ then } |\Phi_f(E) - \Phi_f(S)| < \epsilon.$$

3 Conclusion with Proof

Then we have the main result of this article:

Corollary Let E be a measurable set of finite measure, then

$$[\mathcal{F} \text{ is equi-integrable on } E] \Leftrightarrow [\mathbf{\Phi}_{\mathcal{F}} \text{ is equi-continuous ''at set } E\text{''}].$$

Remark Therefore, equi-integrability can be regarded as a special case of equi-continuity (of collections of functionals on sets).

Proof Assume that $\mathbf{\Phi}_{\mathcal{F}}$ is equi-continuous "at set E". Then under all conditions of definition 5, we have:

$$\begin{aligned} & |\Phi_f(E) - \Phi_f(S)| \ < \ \epsilon \\ \Leftrightarrow \quad & \left| \int_E |f| - \int_S |f| \right| \ < \ \epsilon \\ \Leftrightarrow \quad & \left| \int_E |f|(\chi_E - \chi_S) \right| \ < \ \epsilon \\ \Leftrightarrow \quad & \int_{E \setminus S} |f| \ < \ \epsilon, \end{aligned} \tag{1}$$

for any S satisfies the condition of in definition 5.

Notice that for any measurable $A \subseteq E$ such that $m(A) < \delta$, we can find $\tilde{S} = E \setminus A$, then $m(E \setminus \tilde{S}) = m(A) < \delta$. Since $S \subseteq E$ as well, we have $d(\tilde{S}, E) = m(\tilde{S} \triangle E) = m(E \setminus \tilde{S}) + m(\tilde{S} \setminus E) = m(E \setminus \tilde{S}) < \delta$, which means $\tilde{S}$ satisfies the condition of definition 5 as well. Therefore, by (1), $\int_A |f| = \int_{E \setminus \tilde{S}} |f| \ < \ \epsilon$. We have proved the $\Leftarrow$ direction of the corollary. We can easily verified the $\Rightarrow$ direction of corollary by reverse the process of (1). $\square$