A Correction of a Lemma Talked on Nov. 6th

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During the lecture on Nov. 6^{th}, Prof. Yao has mentioned and proved a generalized version of Poisson Limit Theorem, which states that:

Lemma 1 (tentative) Let $\{X_n\}_{n\geq 1}$ be a collection of independent Bernoulli r.v. with parameters $\{p_n\}_{n\geq 1}$, respectively. Suppose $\sum_{i=1}^{n} p_i \to \lambda$ as $n \to \infty$, where λ is a positive constant.
Then the distribution of $(X_1 + X_2 + ... + X_n)$ converge to $\mathcal{P}(\lambda)$.

However, the statement is somehow not complete. One can easily find some counterexamples: e.g. $\{p_1 = \lambda$, while $p_i = 0$ for all $i \geq 2\}$ or e.g. $\{p_i = \frac{1}{2^i}\lambda$ for all i $\}$...

The key point here is that we need some extra condition to make each p_i arbitrarily small when n is sufficiently large. We have discussed the lemma with Prof. Yao after class, and Prof. Yao modified the lemma with extra condition $\sum_{i=1}^{n} p_i^2 \to 0$ as $n \to \infty$. And the statement works very well under such condition. I will rewrite the lemma with proof below:

Lemma 1' (revised) Let $\{X_n\}_{n\geq 1}$ be a collection of independent Bernoulli r.v. with parameters $\{p_n\}_{n\geq 1}$, respectively. Suppose $\sum_{i=1}^{n} p_i \to \lambda$ as $n \to \infty$, where λ is a positive constant. Moreover, $\sum_{i=1}^{n} p_i^2 \to 0$ as $n \to \infty$.
Then the distribution of $(X_1 + X_2 + ... + X_n)$ converge to $\mathcal{P}(\lambda)$.

Proof Denote $S_n = X_1 + X_2 + ... + X_n$, $\lambda_n = p_1 + p_2 + ... + p_n$, then $\lambda_n \to \lambda$.
Let $Y_n \sim \mathcal{P}(\lambda_n)$ and $Y \sim \mathcal{P}(\lambda)$. Consider the MGF of $\{X_i\}_{i=1,2,...,n}$, S_n, Y_n and Y, denoted by $\{M_{X_i}\}_{i=1,2,...,n}$, M_{S_n}, M_{Y_n} and M_Y, resp. We have: $M_{X_i}(t) = e^t p_i + q_i$ for $i = 1, 2, ..., n$, and

$$M_Y(t) = \mathbb{E}[e^{tY}] = \sum_{y=0}^{\infty} e^{-\lambda} \frac{(e^t\lambda)^y}{y!} = e^{\lambda(e^t-1)}.$$

Therefore,

$$\ln M_Y(t) = \lambda(e^t - 1). \tag{1}$$

Similarly,

$$\ln M_{Y_n}(t) = \lambda_n(e^t - 1) = \sum_{i=1}^{n} p_i(e^t - 1), \tag{2}$$

and we can easily find that $|\ln M_{Y_n}(t) - \ln M_Y(t)| \to 0$ as $n \to 0$.
Since X_i's are independent, we have

$$
\begin{aligned}
M_{S_n}(t) = M_{X_1+\ldots+X_n}(t) &= \prod_{i=1}^{n} M_{X_i}(t); \\
&= \prod_{i=1}^{n} \left(e^t p_i + q_i\right); \\
&= \prod_{i=1}^{n} \left[p_i(e^t - 1) + 1\right].
\end{aligned}
$$

Therefore,

$$
\ln M_{S_n}(t) = \sum_{i=1}^{n} \ln\left[p_i(e^t - 1) + 1\right]. \tag{3}
$$

By (2) and (3), we have:

$$
\begin{aligned}
|\ln M_{S_n}(t) - \ln M_{Y_n}(t)| &= \left|\sum_{i=1}^{n} \ln\left[p_i(e^t - 1) + 1\right] - p_i(e^t - 1)\right| \\
&\le \sum_{i=1}^{n} \left|\ln\left[p_i(e^t - 1) + 1\right] - p_i(e^t - 1)\right| \qquad (4)
\end{aligned}
$$

Notice that, by the extra condition, for each i, $p_i \to 0$ as $n \to 0$.For any t fixed, consider the Taylor expansion of $\ln\left[p_i(e^t - 1) + 1\right]$ (as a function of p_i), we have

$$
\ln\left[p_i(e^t - 1) + 1\right] = p_i(e^t - 1) - \frac{1}{2}\left[p_i(e^t - 1)\right]^2 + o(p_i^2).
$$

Therefore,

$$
\text{RHS of (4)} = \sum_{i=1}^{n} \left|\frac{1}{2}\left[p_i(e^t - 1)\right]^2 + o(p_i^2)\right| = \frac{(e^t - 1)^2}{2} \sum_{i=1}^{n} \left[p_i^2 + o(p_i^2)\right],
$$

which converge to 0 as $n \to \infty$ by the extra condition.
Therefore,

$$
|\ln M_{S_n}(t) - \ln M_Y(t)| \le |\ln M_{S_n}(t) - \ln M_{Y_n}(t)| + |\ln M_{Y_n}(t) - \ln M_Y(t)|,
$$

whose RHS converge to 0 as $n \to \infty$, implying when n is sufficiently large, the pmf of S_n and Y can be arbitrarily close. $\square$

Remark I also found another theorem related to these Poisson Approximation:

Theorem (Le Cam's Inequality) Let $\{X_n\}_{n\ge1}$ be a collection of independent Bernoulli r.v. with parameters $\{p_n\}_{n\ge1}$, respectively. Denote $S_n = \sum_{i=1}^n X_i$, $\lambda_n = \sum_{i=1}^n p_i$. Then:

$$\sum_{k=0}^{\infty}\left|\mathbb{P}(S_n = k) - \frac{\lambda_n^k e^{-\lambda_n}}{k!}\right| < 2\sum_{i=1}^{n} p_i^2.$$

You can found the proof in the reference.

Notice that, the RHS of the inequality is highly consistent with the extra condition added in the above lemma, i.e. $\sum_{i=1}^n p_i^2 \to 0$ as $n \to \infty$. As we can see, if the RHS of the inequality can be arbitrarily small as n approaches ∞, then the pmf of S_n should be arbitrarily close to the pmf of Poisson r.v. with parameter $\lambda = \lim\limits_{n\to\infty} \lambda_n$.

Reference:

Hodges, S. L. and Le Cam, L. (1960). The Poison approximation to the binomial distribution, *Ann. Math. Statist.*, 31, 737-740.