ESM 5734 Home Work - 2

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1. $P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}\left[(x^2-1)^n\right]$, $n = 0, 1, 2, \ldots$

(a) $P_0(x) = \frac{1}{2^0 \times (0!)} \times 1 = 1$

(b) • $P_1(x) = \frac{1}{2 \cdot 1!} \frac{d}{dx}\left[x^2-1\right] = \frac{2x}{2} = x$

(c) • $P_2(x) = \frac{1}{4 \cdot 2!} \frac{d^2}{dx^2}\left[(x^2-1)^2\right] = \frac{1}{8} \frac{d^2}{dx^2}\left[x^4-2x^2+1\right]$

$= \frac{1}{8} \cdot \frac{d}{dx}\left(4x^3-4x\right)$

$= \frac{1}{8}\left(12x^2-4\right) = \frac{3x^2}{2} - \frac{1}{2}$

(d) • $P_3(x) = \frac{1}{8 \times 3!} \frac{d^3}{dx^3}\left[(x^2-1)^3\right] = \frac{1}{48} \frac{d^2}{dx^2}\left[3(x^2-1)^2 \cdot 2x\right]$

$= \frac{6}{48} \frac{d^2}{dx^2}\left(x^5-2x^3+x\right) = \frac{1}{8} \frac{d}{dx}\left[5x^4-6x^2+1\right]$

$= \frac{1}{8}\left[20x^3-12x\right]$

$P_3(x) = \frac{5x^3}{2} - \frac{3x}{2}$

• $P_4(x) = \frac{1}{16 \times 4!} \frac{d^4}{dx^4}\left[(x^2-1)^4\right]$

$= \frac{1}{16 \times 4!} \frac{d^3}{dx^3}\left[4(x^2-1)^3 \cdot 2x\right] = \frac{1}{2 \times 4!} \frac{d^2}{dx^2}\left[3(x^2-1)^2 \cdot (2x^2) + (x^2-1)^3\right]$

$$P_4(x) = \frac{1}{48}\frac{d^2}{dx^2}\left[6(x^6-2x^4+x^2) + (x^2-1)^3\right]$$

$$= \frac{1}{48}\frac{d}{dx}\left[36x^5 - 48x^3 + 12x + 3(x^2-1)^2(2x)\right]$$

$$= \frac{1}{48}\left[36\cdot5\cdot x^4 - 48\cdot3x^2 + 12 + 6\cdot5x^4 - 6\cdot6x^2 + 6\right]$$

$$= \frac{1}{48}\left[x^4[42\cdot5] - x^2[48\cdot3 + 6\cdot6] + 18\right]$$

$$= x^4\left(\frac{35}{8}\right) - x^2\left(3 + \frac{3}{4}\right) + \frac{3}{8}$$

$$P_4(x) = \frac{35x^4}{8} - \frac{15}{4}x^2 + \frac{3}{8}$$

$\rightarrow$ $x^4 = C_0P_0 + C_1P_1 + C_2P_2 + C_3P_3 + C_4P_4$

$$= C_0[1] + C_1[x] + C_2\left[\frac{3x^2-1}{2}\right] + C_3\left[\frac{5x^3-3x}{2}\right] + C_4\left[\frac{35x^4}{8} - \frac{15x^2}{4} + \frac{3}{8}\right]$$

$\rightarrow$ $C_0 - \frac{C_2}{2} + \frac{3C_4}{8} = 0 \longrightarrow C_0 - \frac{2}{7} + \frac{3}{8}\times\frac{8}{35} = 0 \rightarrow \boxed{C_0 = \frac{1}{5}}$

$C_1 - \frac{3C_3}{2} = 0 \quad \boxed{C_1 = 0}$

$\frac{3C_2}{2} - \frac{15}{4}C_4 = 0 \longrightarrow C_2 = \frac{2}{3}\times\frac{15}{4}\times\frac{8}{35} = \boxed{\frac{4}{7} = C_2}$

$\frac{5C_3}{2} = 0 \quad \boxed{C_3 = 0}$

$\frac{35}{8}C_4 = 1 \Rightarrow \boxed{C_4 = \frac{8}{35}}$

$$\therefore \boxed{x^4 = \frac{1}{5} + \frac{4}{7}P_2 + \frac{8}{35}P_4}$$

(b) $\int_{-1}^{1} P_2(x) P_3(x) \cdot dx$

$$= \int_{-1}^{1} \left(\frac{3x^2-1}{2}\right)\left(\frac{5x^3}{2} - \frac{3x}{2}\right) \cdot dx$$

$$= \frac{1}{4}\int_{-1}^{1} (15x^5 - 14x^3 + 3x) \cdot dx = \frac{1}{4}\left[\frac{15x^6}{6} - \frac{14x^4}{4} + \frac{3x^2}{2}\right]_{-1}^{1}$$

$$\boxed{\int_{-1}^{1} P_2(x) P_3(x) \cdot dx = 0}$$

- $\int_{-1}^{1} P_2^2(x) \cdot dx$

$$= \int_{-1}^{1} \left(\frac{3x^2-1}{2}\right)^2 \cdot dx = \int_{-1}^{1} \frac{9x^4 - 6x^2 + 1}{4} \cdot dx = \left[\frac{9x^5}{5} - \frac{6x^3}{3} + x\right]_{-1}^{1} \cdot \frac{1}{4}$$

$$= \frac{1}{4}\left[\frac{9}{5} \times 2 - (2 \times 2) + 2\right] = \frac{9}{10} - 1 + \frac{1}{2} = \frac{1}{2} - \frac{1}{10} = \frac{4}{10}$$

$$\boxed{\int_{-1}^{1} P_2^2(x) \cdot dx = \frac{2}{5}}$$

$u'(0)=3$ $\quad$ $u(1)=2$ $\qquad$ $-u''+u=2$, $0<x<1$

(2) $\quad$ 0 $\quad$ 1 $\qquad$ $u'(0)=3$, $u(1)=2$

(a) $-u''+u=2$

multiply both sides by a test function $\phi(x)$

$$-u''\phi(x)+u\phi(x)=2(\phi(x))$$

$$-\int_0^1 u''\phi(x)\,dx+\int_0^1 u\phi(x)\,dx=2\int_0^1\phi(x)\,dx$$

$$-\left[u'(x)\phi(x)\right]_0^1+\int_0^1 u'(x)\phi'(x)\,dx+\int_0^1 u(x)\phi(x)\,dx=2\int_0^1\phi(x)\,dx$$

$$\rightarrow \left[\int_0^1 u'(x)\phi'(x)+\int_0^1 u(x)\phi(x)\,dx=2\int_0^1\phi(x)\,dx+(u'(1)\phi(1))-(3\cdot\phi(0))\right] \text{—} (A)$$

(This is the weak formulation) of the given problem

(b) Consider some basis functions $\psi_1,\psi_2,\psi_3,\psi_4$ to define $u(x)$ & $\phi(x)$

where
$$\phi(x)=c_1\psi_1+c_2\psi_2+c_3\psi_3+c_4\psi_4+\dots$$
$$u(x)=d_1\psi_1+d_2\psi_2+d_3\psi_3+d_4\psi_4+\dots$$

$\therefore$ eq (A) becomes

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\int_0^1 c_i\psi_i'\,d_j\psi_j'\,dx+\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\int_0^1 c_i\psi_i\,d_j\psi_j\,dx=2\int_0^1\sum_{i=1}^{\infty}c_i\psi_i\,dx+u'(1)\cdot\sum_{i=1}^{\infty}c_i\psi_i(1)-3\sum_{i=1}^{\infty}(c_i\psi_i(0))$$

$\rightarrow$ By GALERKIN APPROXIMATION we can truncate it for 4 terms can can be written as

$$\sum_{i=1}^{4}\sum_{j=1}^{4}c_i\underbrace{\left[\int_0^1\psi_i'\psi_j'\,dx+\int_0^1\psi_i\psi_j\,dx\right]}_{K_{ij}}d_j=\sum_{i=1}^{4}c_i\underbrace{\left[\int_0^1 2\psi_i\,dx+u'(1)\psi_i(1)-3\psi_i(0)\right]}_{F_i}$$

$$\rightarrow \boxed{\sum_{i=1}^{4}\sum_{j=1}^{4}c_i K_{ij}d_j=\sum_{i=1}^{4}c_iF_i}$$

so we will get 4 equations for different values of c_i & one Boundary condition which is $u(1)=2$ and we can solve the variables by solving equations.

(c) Let $c_1=1,\ c_2, c_3, c_4=0$ $\quad \phi(x)=1,\ \phi'(x)=0$

Given:
$\psi_1(x)=1$
$\psi_2(x)=3x-1$
$\psi_3(x)=x^2$
$\psi_4(x)=x^3$

$$\int_0^1 (3d_2+2d_3x+3d_4x^2)(0)\cdot dx + \int_0^1 (d_1+d_2(3x-1)+d_3x^2+d_4x^3)(1)\cdot dx$$

$$= 2\int_0^1 1\cdot dx + u'(1)\cdot 1 - (3)(1)$$

$$\left[d_1x + \frac{3d_2x^2}{2} - d_2x + \frac{d_3x^3}{3} + \frac{d_4x^4}{4}\right]_0^1 = \left[2x\right]_0^1 + u'(1) - 3$$

$$\left[d_1 + \frac{d_2}{2} + \frac{d_3}{3} + \frac{d_4}{4} - u'(1) = -1\right] — ①$$

→ Let $c_1=0, c_2=1, c_3, c_4=0$ $\quad \phi(x)=3x-1,\ \phi'(x)=3$
$\phi(0)=-1,\ \phi(1)=2$

$$\int_0^1 (3d_2+2d_3x+3d_4x^2)(3)\cdot dx + \int_0^1 (d_1+d_2(3x-1)+d_3x^2+d_4x^3)(3x-1)\cdot dx$$

$$= 2\int_0^1 (3x-1)\cdot dx + u'(1)(2) + 3$$

$$\left[9d_2x + \frac{6d_3x^2}{2} + \frac{9d_4x^3}{3}\right]_0^1 + \left[d_1\left(\frac{3x^2}{2}-1\right) + d_2\left(\frac{9x^3}{3} - \frac{6x^2}{2} + x\right) + d_3\left(\frac{3x^4}{4} - \frac{x^3}{3}\right) + d_4\left(\frac{3x^5}{5} - \frac{x^4}{4}\right)\right.$$

$$= 2\left[\frac{3x^2}{2} - x\right]_0^1 + 2u'(1) + 3$$

$$\left[\frac{d_1}{2} + 10d_2 + \frac{41}{12}d_3 + \frac{67}{20}d_4 - 2u'(1) = 4\right] — ②$$

→ Let $c_1, c_2, c_4=0;\ c_3=1$ $\quad \phi(x)=x^2;\ \phi'(x)=2x$
$\phi(0)=0;\ \phi(1)=1$

$$\int_0^1 (3d_2+2d_3x+3d_4x^2)(2x)\cdot dx + \int_0^1 (d_1+d_2(3x-1)+d_3x^2+d_4x^3)(x^2)\cdot dx$$

$$= 2\int_0^1 x^2\cdot dx + u'(1)(1) - 3(0)$$

$$\left[\frac{d_1}{3} + \frac{41}{12}d_2 + \frac{23}{15}d_3 + \frac{5}{3}d_4 - u'(1) = \frac{2}{3}\right] — ③$$

→ Let $c_1, c_2, c_3 = 0$; $c_4 = 1$ $\phi(x) = x^3$; $\phi'(x) = 3x^2$

$\phi(0) = 0$; $\phi(1) = 1$

$$\int_0^1 (3d_2 + 2d_3 x + 3d_4 x^2)(3x^2)\cdot dx + \int_0^1 (d_1 + d_2(3x-1) + d_3 x^2 + d_4 x^3)(x^3)\cdot dx$$

$$= 2\int_0^1 x^3 \cdot dx + u'(1)\cdot[1] - u'(0)(0)$$

$$\left[\frac{d_1}{4} + \frac{67}{20} d_2 + \frac{5}{3} d_3 + \frac{68}{35} d_4 - u'(1) = \frac{1}{2}\right] - ④$$

→ ~~Essential~~ Boundary condition at $x = 1 \to u(1) = 2$

$$d_1 + d_2(3-1) + d_3(1) + d_4(1) = 2$$

$$\left[d_1 + 2d_2 + d_3 + d_4 + 0(u'(1)) = 2\right] - ⑤$$

→ Now we have 5 eqs and 5 variables, so we can solve by using Matrix Inversion method.

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & -1 \\ \frac{1}{2} & 10 & \frac{41}{12} & \frac{67}{20} & -2 \\ \frac{1}{3} & \frac{41}{12} & \frac{23}{15} & \frac{5}{3} & -1 \\ \frac{1}{4} & \frac{67}{20} & \frac{5}{3} & \frac{68}{35} & -1 \\ 1 & 2 & 1 & 1 & 0 \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ u'(1) \end{Bmatrix} = \begin{Bmatrix} 1 \\ 4 \\ \frac{2}{3} \\ \frac{1}{2} \\ 2 \end{Bmatrix}$$

→ $$\begin{Bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ u'(1) \end{Bmatrix} = \begin{Bmatrix} 0.712 \\ 0.997 \\ -1.077 \\ 0.372 \\ 1.9442 \end{Bmatrix}$$

$$\boxed{\begin{aligned} \therefore u &= \sum_1^4 d_j \psi_j(x) \\ &= 0.712 + 0.997(3x-1) \\ &\quad + 1.077(x^2) + 0.372x^3 \end{aligned}}$$

(d) → from analytical point we can solve $-u'' + u = 2$ by finding Complementary function and Particular Inegral.

Let $e^{mx} = u$)

for C.F

$-u'' + u = 0$

$-(m^2 + 1)e^{mx} = 0$

$\boxed{m = \pm 1}$

$\therefore C.F = c_1 e^{x} + c_2 e^{-x}$

$P.I = \dfrac{2 \cdot e^{(0 \cdot x)}}{-D^2 + 1}$

$= 2$ $(D = 0.)$

$\therefore u(x) = C.F + PI$

$= c_1 e^{x} + c_2 e^{-x} + 2$

→ Since $u(1) = 2$

$c_1 e + \frac{c_2}{e} = 0$

$(c_2 = -c_1 e^2)$

$u'(0) = 3$

$(c_1 e^{x} - c_2 e^{-x})_{x=0} = 3$

$c_1 - c_2 = 3 \rightarrow c_1[1 + e^2] = 3 \Rightarrow \left(c_1 = \frac{3}{1+e^2}\right)$

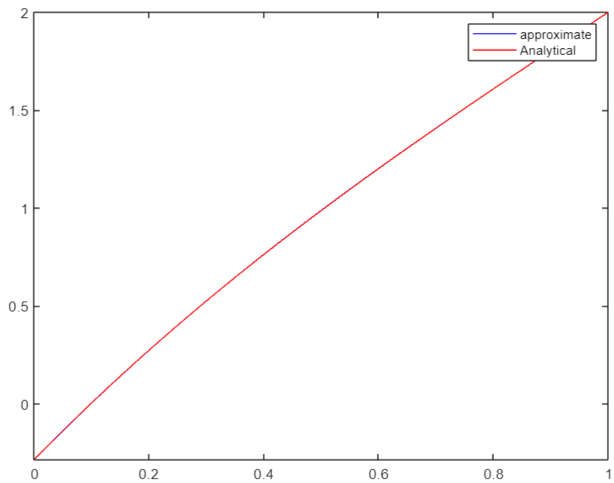
$\therefore c_2 = \frac{-3e^2}{1+e^2}$

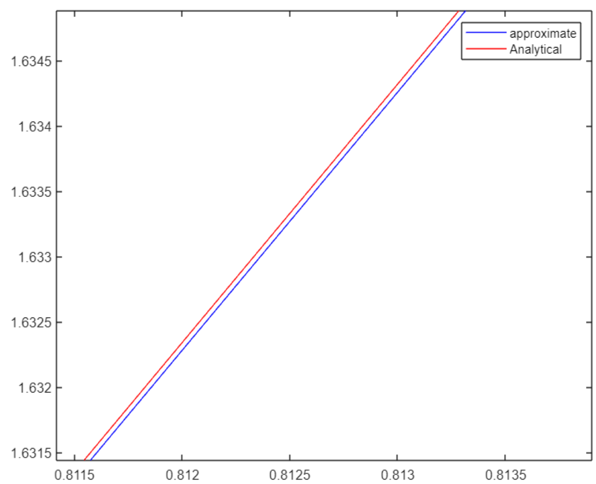
$$\therefore \boxed{u(x) = \frac{3}{(1+e^2)} \cdot e^{x} - \frac{3e^2}{(1+e^2)} \cdot e^{-x} + 2}$$

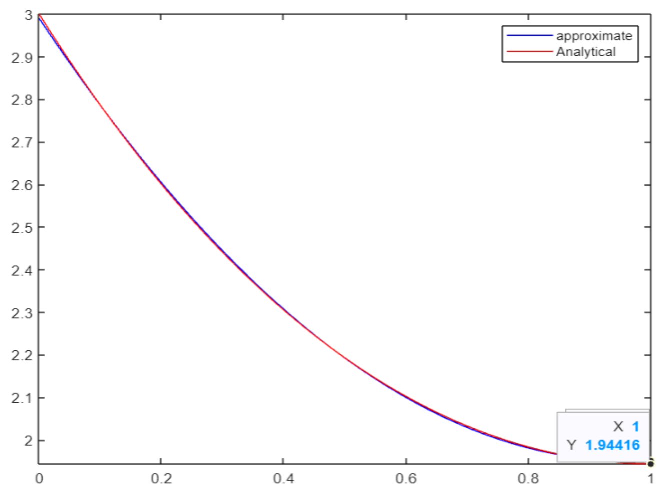
(d) Plotting Approximate and Analytical Solutions in MATLAB

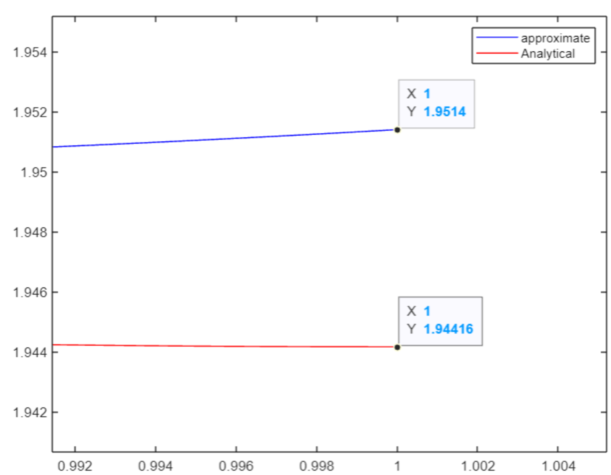
```
%plots of both approximate and Analytical Sollutions
figure(1)
fplot(@(x) 0.7119+0.9967*(3*x-1)-1.0772*(x)^2+0.3719*(x)^3,[0,1],'b')
hold on
fplot(@(x) (3/(1+exp(2)))*exp(x)-(3*exp(2)/(1+exp(2)))*exp(-x)+2,[0,1],'r')
legend('approximate','Analytical')
hold off
%Plots of derivaties of approximate and Analtical Solutions
figure(2)
fplot(@(x) (0.9967*3)-(2*1.0772*(x))+(3*0.3719*(x)^2),[0,1],'b')
hold on
fplot(@(x) (3/(1+exp(2)))*exp(x)+(3*exp(2)/(1+exp(2)))*exp(-x),[0,1],'r')
legend('approximate','Analytical')
hold off
```

Fig 1 and 2 are plots of Approximate and Analytical Solutions of u(x) in 0<x<1
FIg 3 and 4 are plots of Approximate and Analytical Solutions of du/dx (0<x<1)









(e) $u'(1) = \frac{3e^{(1)}}{(1+e^2)} - \left[\frac{-3e^2}{(1+e^2)}\right]e^{-1} = \frac{6e}{1+e^2} = 1.94416$ (from Analytical Solution)

→ From matrix inversion we found out d_1, d_2, d_3, d_4 & $u'(1)$ as well.

$u'(1)$ was found out to be 1.9442.

→ Also I have calculate the derivate of obtained approximate solution by derivating the equation $u(x) = d_1\psi_1 + d_2\psi_2 + d_3\psi_3 + d_4\psi_4$

which came out to be $u'(1) = 1.9514$

$\therefore$ error in approximate solution $= \frac{u'(1)_{approx} - u'(1)_{analytical}}{u'(1)_{analytical}} \times 100$

$= \frac{(1.9514 - 1.94416)}{1.94416} \times 100$

$= 0.3723\%$

(f) to reduce the error in $u'(1)$ we need to ~~shape~~ smoothen the curve even better so we need to increase the number of basis functions and also can choose better suitable basis functions.