ESM-5734 Homework-1

L.CHERITH
9065 48745
cherithl@vt.edu

1. Finite Element method is a technique to numerically find an approximate solution of a given Initial Boundary Value Problem.

2. $f(x) = 4x^2, \ \forall \ 0 \leq x \leq \frac{1}{2}$

$\quad = 2(1-x), \ \forall \ 1/2 \leq x \leq 1$

$f^N(x) = \sum_{1}^{N} a_n(\sin n\pi x)$

$$\rightarrow a_n = \frac{2}{L}\int_0^L f(x) \cdot \sin\left(\frac{n\pi x}{L}\right) \cdot dx$$

$$\therefore \ a_n = 2\int_0^1 f(x) \cdot \sin(n\pi x) \cdot dx$$

$$= 2\left[\int_0^{\frac{1}{2}} 4x^2 \cdot \sin(n\pi x) \cdot dx + \int_{\frac{1}{2}}^1 2(1-x)\sin(n\pi x) \cdot dx\right] \quad — (1)$$

- $\displaystyle\int_0^{\frac{1}{2}} 4x^2 \cdot \sin(n\pi x) \cdot dx$

$$= -\left[\frac{4x^2 \cos(n\pi x)}{n\pi}\right]_0^{\frac{1}{2}} - \left[\int_0^{\frac{1}{2}} -\frac{8x \cdot \cos(n\pi x)}{n\pi} \cdot dx\right]$$

$$= -\frac{\cos\left(\frac{n\pi}{2}\right)}{n\pi} + \left[\left[\frac{8x \sin(n\pi x)}{n^2\pi^2}\right]_0^{\frac{1}{2}} - \int_0^{\frac{1}{2}} \frac{8\sin(n\pi x)}{n^2\pi^2} \cdot dx\right]$$

$$= \frac{4\sin\left(\frac{n\pi}{2}\right)}{n^2\pi^2} - \frac{\cos\left(\frac{n\pi}{2}\right)}{n\pi} + \left[\frac{8\cos(n\pi x)}{n^3\pi^3}\right]_0^{\frac{1}{2}}$$

$$= \frac{8\cos\left(\frac{n\pi}{2}\right)}{n^3\pi^3} + \frac{4\sin\left(\frac{n\pi}{2}\right)}{n^2\pi^2} - \frac{\cos\left(\frac{n\pi}{2}\right)}{n\pi} - \frac{8}{n^3\pi^3} \quad — (A)$$

• $\int_{\frac{1}{2}}^{1} 2(1-x)\sin(n\pi x)\cdot dx$

$= \int_{\frac{1}{2}}^{1} 2\sin(n\pi x)\cdot dx - \int_{\frac{1}{2}}^{1} 2x\sin(n\pi x)\cdot dx$

$= \left[-\frac{2\cos(n\pi x)}{n\pi}\right]_{\frac{1}{2}}^{1} + \left[\frac{2x\cos(n\pi x)}{n\pi}\right]_{1/2}^{1} - \int_{\frac{1}{2}}^{1} \frac{2\cos(n\pi x)}{n\pi}\cdot dx$

$= \left[\frac{2\cos(\frac{n\pi}{2})}{n\pi} - \cancel{\frac{2\cos(n\pi)}{n\pi}}\right] + \left[\cancel{\frac{2\cos(n\pi)}{n\pi}} - \frac{\cos(\frac{n\pi}{2})}{n\pi}\right] - \left[\frac{2\sin(n\pi x)}{n^2\pi^2}\right]_{\frac{1}{2}}^{1}$

$= \frac{\cos(\frac{n\pi}{2})}{n\pi} + \frac{2\sin(\frac{n\pi}{2})}{n^2\pi^2} - \frac{2\sin(n\pi)}{n^2\pi^2}$ — (B)

Sub (A) & (B) in (1)

$a_n = 2\left[\frac{8\cos(\frac{n\pi}{2})}{n^3\pi^3} + \frac{4\sin(\frac{n\pi}{2})}{n^2\pi^2} - \cancel{\frac{\cos(\frac{n\pi}{2})}{n\pi}} - \frac{8}{n^3\pi^3} + \cancel{\frac{\cos(\frac{n\pi}{2})}{n\pi}} + \frac{2\sin(\frac{n\pi}{2})}{n^2\pi^2} - \cancel{\frac{2\sin(n\pi)}{n^2\pi^2}}^{\,0}\right]$

$$\boxed{a_n = \frac{12\sin\frac{n\pi}{2}}{n^2\pi^2} + \frac{16\cos(\frac{n\pi}{2})}{n^3\pi^3} - \frac{16}{n^3\pi^3}}$$

→ $a_1 = \frac{12}{\pi^2} - \frac{16}{\pi^3}$; $a_2 = -\frac{4}{\pi^3}$; $a_3 = -\frac{4}{3\pi^2} - \frac{16}{27\pi^3}$; $a_4 = 0$

$a_5 = \frac{12}{25\pi^2} - \frac{16}{125\pi^3}$; $a_6 = -\frac{4}{27\pi^3}$; $a_7 = -\frac{12}{49\pi^2} - \frac{16}{343\pi^3}$; $a_8 = 0$

$a_9 = \frac{4}{27\pi^2} - \frac{16}{729\pi^3}$; $a_{10} = -\frac{4}{125\pi^3}$

1st Trial

③

$n=1$

$$\frac{\int_0^{0.5}[4x^2-a_1\sin\pi x]^2\cdot dx+\int_{0.5}^{1}[2(1-x)-a_1\sin\pi x]^2\cdot dx}{\int_0^{0.5}(4x^2)^2 dx+\int_{0.5}^{1}[2(1-x)]^2\cdot dx}=\frac{0.0163+0.0054}{0.266667}$$

$= 0.08$ not less than 0.01.

2nd Trial

$n=2$

$$\frac{\int_0^{0.5}[4x^2-a_1\sin\pi x-a_2\sin 2\pi x]^2\cdot dx+\int_{0.5}^{1}[2(1-x)-a_1\sin\pi x-a_2\sin 2\pi x]^2\cdot dx}{\int_0^{0.5}(4x^2)^2\cdot dx+\int_{0.5}^{1}(2(1-x))^2\cdot dx}$$

$$=\frac{0.0065741+0.0068905}{0.26667}=0.050$$ not less than 0.01

3rd Trial

$n=3$

$$=\frac{0.00265952+0.00507048}{0.26667}=0.0289$$ not less than 0.01

4th trial

$n=5$

$$=\frac{0.000333+0.00025096}{0.26667}=0.002$$

$\therefore$ for $\boxed{n=5}$ the given normalised error is < 0.01

3. Let us try to write $\sin 3\pi x$ as a linear combination of $\sin \pi x$ and $\sin 2\pi x$

(a & b are const)

$$\sin 3\pi x = a \sin \pi x + b \sin 2\pi x$$

→ by series expansion $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$

$$(3\pi x) - \frac{(3\pi x)^3}{3!} + \frac{(3\pi x)^5}{5!} + \dots = a\left[\pi x - \frac{(\pi x)^3}{3!} + \frac{(\pi x)^5}{5!} + \dots\right] + b\left[2\pi x - \frac{(2\pi x)^3}{3!} + \frac{(2\pi x)^5}{5!} + \dots\right]$$

∵ LHS = RHS

all x^n coefficients ~~has~~ must be equal for all x

→ Coeff of x: $3\pi = a\pi + 2b\pi \rightarrow a + 2b = 3$ — ①

Coeff of x^3: $27\pi^3 = a\pi^3 + 8b\pi^3 \rightarrow a + 8b = 27$ — ②

Coeff of x^5: $243\pi^5 = a\pi^5 + 32b\pi^5 \rightarrow a + 32b = 243$ — ③

→ Solving ① & ②

$6b = 24$ $\boxed{b = 4}$ $\boxed{a = -5}$

→ If they are linearly dependant a, b values should also satisfy equation ③ substituting a, b in ③

$$-5 + 32(4) = 123 \neq 243$$

∴ the given three functions are linearly independant.

∴ we can't express $\sin(3\pi x)$ as linear combination of $\sin(\pi x)$ & $\sin(2\pi x)$

4. $\sin \pi x = a + bx + cx^2 + dx^3 \quad \forall\, x \in [0, 1]$

(a) $r(x) = a + bx + cx^2 + dx^3 - \sin \pi x$

- $r(0) = 0$

$a + b(0) + c(0) + d(0) - \sin\pi(0) = 0 \Rightarrow \boxed{a = 0}$

- $r(1/4) = 0$

$0 = \frac{b}{4} + \frac{c}{16} + \frac{d}{64} - \frac{1}{\sqrt{2}} \Rightarrow 16b + 4c + d = 32\sqrt{2}$ — ①

- $r(1/2) = 0$

$0 = \frac{b}{2} + \frac{c}{4} + \frac{d}{8} - 1 \Rightarrow 4b + 2c + d = 8$ — ②

- $r(3/4) = 0$

$0 = \frac{3b}{4} + \frac{9c}{16} + \frac{27d}{64} - \frac{1}{\sqrt{2}} \Rightarrow 48b + 36c + 27d = 32\sqrt{2}$ — ③

$$\begin{bmatrix} 16 & 4 & 1 \\ 4 & 2 & 1 \\ 48 & 36 & 27 \end{bmatrix} \begin{Bmatrix} b \\ c \\ d \end{Bmatrix} = \begin{Bmatrix} 32\sqrt{2} \\ 8 \\ 32\sqrt{2} \end{Bmatrix}$$

$AX = b$

$X = A^{-1} b$

$$A^{-1} = \begin{bmatrix} \frac{3}{16} & \frac{-3}{4} & \frac{1}{48} \\ -\frac{5}{8} & 4 & -\frac{1}{8} \\ \frac{1}{2} & -4 & \frac{1}{6} \end{bmatrix}$$

$$A^{-1} \cdot b = \begin{Bmatrix} \left(6\sqrt{2} - 6 + \frac{2\sqrt{2}}{3}\right) \\ \left(-20\sqrt{2} + 32 - 4\sqrt{2}\right) \\ \left(16\sqrt{2} - 32 + \frac{16\sqrt{2}}{3}\right) \end{Bmatrix}$$

$\therefore \begin{Bmatrix} b \\ c \\ d \end{Bmatrix} =$

$$\therefore \begin{Bmatrix} b \\ c \\ d \end{Bmatrix} = \begin{Bmatrix} \frac{20\sqrt{2}}{3} - 6 \\ 32 - 24\sqrt{2} \\ \frac{64\sqrt{2}}{3} - 32 \end{Bmatrix} = \begin{Bmatrix} 3.4281 \\ -1.9411 \\ -1.8301 \end{Bmatrix}$$

$\therefore a = 0,\ b = 3.4281,\ c = -1.9411,\ d = -1.8301$ //

(b) $\left[\int_0^1 r^2 \cdot dx\right] = I$ for I to be least value $\frac{\partial I}{\partial a}, \frac{\partial I}{\partial b}, \frac{\partial I}{\partial c}, \frac{\partial I}{\partial d} = 0$

- $$\frac{\partial I}{\partial a} = \int_0^1 2r \cdot \frac{\partial r}{\partial a} \cdot dx = 0$$

$$\Rightarrow \int_0^1 \left[2a + 2bx + 2cx^2 + 2dx^3 - 2\sin\pi x\right] \cdot [1] \cdot dx = 0$$

$$\Rightarrow 2a(1) + \frac{2b}{2}(1)^2 + \frac{2c}{3}(1)^3 + \frac{2d(1)^4}{4} + \frac{[2\cos\pi - 2\cos(0)]}{\pi} = 0$$

$$a + \frac{b}{2} + \frac{c}{3} + \frac{d}{4} = \frac{2}{\pi}$$

$$\left(12a + 6b + 4c + 3d = \frac{24}{\pi}\right) - ①$$

- $$\frac{\partial I}{\partial b} = \int_0^1 2r \cdot \frac{\partial r}{\partial b} \cdot dx = 0$$

$$\int_0^1 \left[ax + bx^2 + cx^3 + dx^4 - \sin\pi x \cdot x\right] \cdot dx = 0$$

$$\frac{a}{2} + \frac{b}{3} + \frac{c}{4} + \frac{d}{5} = \frac{1}{\pi}$$

$$\left[30a + 20b + 15c + 12d = \frac{60}{\pi}\right] - ②$$

- $\frac{\partial I}{\partial c} = \int_0^1 2r \cdot \frac{\partial r}{\partial c} \cdot dx = 0$

$$\Rightarrow \int_0^1 [ax^2 + bx^3 + cx^4 + dx^5 - x^2 \sin \pi x] \cdot dx = 0$$

$$\frac{a}{3} + \frac{b}{4} + \frac{c}{5} + \frac{d}{6} = \frac{1}{\pi} - \frac{4}{\pi^3} \quad — ③$$

- $\frac{\partial I}{\partial d} = \int_0^1 2r \cdot \frac{\partial r}{\partial d} \cdot dx = 0$

$$\Rightarrow \int_0^1 [ax^3 + bx^4 + cx^5 + dx^6 - x^3 \sin \pi x] \cdot dx = 0$$

$$\frac{a}{4} + \frac{b}{5} + \frac{c}{6} + \frac{d}{7} = \frac{1}{\pi} - \frac{6}{\pi^3} \quad — ④$$

→ from ①, ②, ③, ④ $\quad AX = b$

$$X = A^{-1} \cdot b$$

$$\begin{Bmatrix} a \\ b \\ c \\ d \end{Bmatrix} = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} \end{bmatrix}^{-1} \begin{Bmatrix} \frac{2}{\pi} \\ \frac{1}{\pi} \\ \frac{1}{\pi} - \frac{4}{\pi^3} \\ \frac{1}{\pi} - \frac{6}{\pi^3} \end{Bmatrix}$$

$$\begin{Bmatrix} a \\ b \\ c \\ d \end{Bmatrix} = \begin{Bmatrix} \frac{12\pi^2 - 120}{\pi^3} \\ \frac{-60\pi^2 + 720}{\pi^3} \\ \frac{60\pi^2 - 720}{\pi^3} \\ 0 \end{Bmatrix} \quad \therefore \begin{bmatrix} a = -0.0505 \\ b = 4.1225 \\ c = -4.1225 \\ d = 0 \end{bmatrix}$$

//