Dynamical systems with constraints

Computational Intelligence, Lecture 12

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Lagrange equations are a basic tool for modelling multi-body mechanical systems. General form is given as:

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{\mathbf{q}}} - \frac{\partial T}{\partial \mathbf{q}} = \tau \tag{1}$$

where T is kinetic energy of the system, $\mathbf{q}$ is the vector of generalized coordinates and τ is the vector of generalized torques.

For systems with *constraints*, for example $\mathbf{f}(\mathbf{q}) = 0$, Lagrange equations take a different form:

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{\mathbf{q}}} - \frac{\partial T}{\partial \mathbf{q}} = \tau + \frac{\partial \mathbf{f}}{\partial \mathbf{q}}^{\top} \lambda \tag{2}$$

Lagrange equations can be easily converted to a *manipulator equations* form:

$$\mathbf{H}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{g} = \mathbf{T}\mathbf{u} \tag{3}$$

where $\mathbf{H}$ is a generalized inertia matrix, $\mathbf{C}$ is a matrix of generalized inertial forces, $\mathbf{g}$ is a generalized gravity force, $\mathbf{T}$ is a control map (control matrix), and $\mathbf{u}$ is a vector of control inputs.

Manipulator equations for systems with constraints $\mathbf{f}(\mathbf{q}) = 0$ are very similar:

$$\mathbf{H}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{g} = \mathbf{T}\mathbf{u} + \mathbf{F}^{\top}\lambda \tag{4}$$

where $\mathbf{F} = \frac{\partial \mathbf{f}}{\partial \mathbf{q}}$ and λ are Lagrange multipliers, encoding reaction forces ta ht enforce constraints.

It is not sufficient to provide (4) to describe the behaviour of the system, since it depends both of the differential equation and the algebraic constraint.

If we want to study how generalized accelerations $\ddot{\mathbf{q}}$ behave, we can differentiate $\mathbf{f}(\mathbf{q}) = 0$ twice and add the result to the manipulator equation:

$$\begin{cases} \mathbf{H}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{g} = \mathbf{T}\mathbf{u} + \mathbf{F}^{\top}\lambda \\ \mathbf{F}\ddot{\mathbf{q}} + \dot{\mathbf{F}}\dot{\mathbf{q}} = 0 \end{cases} \tag{5}$$

Equation (5) can be solved with respect to variables $\ddot{\mathbf{q}}$ and λ.

Introducing change of variables $\mathbf{x} = \begin{bmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \end{bmatrix}$ we can linearize dynamics (3) to obtain form:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} + \mathbf{c} \tag{6}$$

where $\mathbf{A}$ is the state matrix, $\mathbf{B}$ is the control matrix and $\mathbf{c}$ is the affine term of the affine dynamics model.

For systems with constraints the same linearization takes form:

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} + \mathbf{S}\lambda + \mathbf{c} \\ \mathbf{G}\dot{\mathbf{x}} = 0 \end{cases} \tag{7}$$

where $\mathbf{S}$ is linearized constraint Jacobian and $\mathbf{G} = \begin{bmatrix} \mathbf{F} & \mathbf{0} \\ \dot{\mathbf{F}} & \mathbf{F} \end{bmatrix}$.

Implicit (minimal) representation of a constrained system
Part 1

We can observe that constraint $\mathbf{G}\dot{\mathbf{x}} = 0$ implies that all feasible state velocities $\dot{\mathbf{x}}$ lie in the null space of $\mathbf{G}$. This means that we can introduce a new lower dimensional variable $\mathbf{z}$ to describe $\mathbf{x}$ (assuming initial value of $\mathbf{x}$ lies in the column space of $\mathbf{N}$):

$$\mathbf{N}\mathbf{z} = \mathbf{x} \tag{8}$$

where $\mathbf{N} = \text{null}(\mathbf{G})$ - orthonormal basis in the null space of $\mathbf{G}$.

Implicit (minimal) representation of a constrained system

Part 2

Let us re-express dynamics (7) in terms of $\mathbf{z}$ by multiplying it by $\mathbf{N}^\top$ on the left:

$$\mathbf{N}^\top \dot{\mathbf{x}} = \mathbf{N}^\top \mathbf{A}\mathbf{x} + \mathbf{N}^\top \mathbf{B}\mathbf{u} + \mathbf{N}^\top \mathbf{S}\lambda + \mathbf{N}^\top \mathbf{c} \tag{9}$$

We can prove that $\mathbf{N}^\top \mathbf{S} = 0$ for all mechanical systems (for example, by observing that mechanical constrains do not do work) or check that our particular $\mathbf{S}$ lies in the row space of our $\mathbf{G}$.

Noting that $\dot{\mathbf{z}} = \mathbf{N}^\top \dot{\mathbf{x}}$ and $\mathbf{x} = \mathbf{N}\mathbf{z}$ we get:

$$\dot{\mathbf{z}} = \mathbf{N}^\top \mathbf{A}\mathbf{N}\mathbf{z} + \mathbf{N}^\top \mathbf{B}\mathbf{u} + \mathbf{N}^\top \mathbf{c} \tag{10}$$

Defining $\mathbf{A}_N = \mathbf{N}^\top \mathbf{A}\mathbf{N}$, $\mathbf{B}_N = \mathbf{N}^\top \mathbf{B}$ and $\mathbf{c}_N = \mathbf{N}^\top \mathbf{c}$ we get:

$$\dot{\mathbf{z}} = \mathbf{A}_N \mathbf{z} + \mathbf{B}_N \mathbf{u} + \mathbf{c}_N \tag{11}$$

Implicit (minimal) representation of a constrained system

Part 3

Since we achieved that our constrained dynamics is written in the standard LTI form:

$$\dot{\mathbf{z}} = \mathbf{A}_N \mathbf{z} + \mathbf{B}_N \mathbf{u} + \mathbf{c}_N, \tag{12}$$

we can use standard LTI control methods on it, for example finding optimal feedback gains via pole placement or LQR:

$$\mathbf{K}_N = \mathrm{lqr}(\mathbf{A}_N, \mathbf{B}_N, \mathbf{Q}, \mathbf{R}) \tag{13}$$

where $\mathbf{Q}$ and $\mathbf{R}$ are matrices defining cost function for the LQR problem.

For any LTI system, including the LTI form of a constrained system we saw previously, inverse dynamics can be solved precisely by a pseudo-inverse, as long as there exist a solution. The following condition verifies it:

$$(\mathbf{I} - \mathbf{B}\mathbf{B}^{+})(\dot{\mathbf{x}} - \mathbf{A}\mathbf{x} - \mathbf{c}) = 0, \tag{14}$$

The condition checks if vector $(\dot{\mathbf{x}} - \mathbf{A}\mathbf{x} - \mathbf{c})$ lies in the column space of $\mathbf{B}$. If it holds, precise solution to inverse kinematics can be found as:

$$\mathbf{u}_{ID} = \mathbf{B}^{+}(\dot{\mathbf{x}} - \mathbf{A}\mathbf{x} - \mathbf{c}). \tag{15}$$

For a constrained mechanical system we can solve inverse dynamics without the need for linearization. Consider the following dynamics:

$$\mathbf{H}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{g} = \mathbf{T}\mathbf{u} + \mathbf{F}^\top \lambda \tag{16}$$

For that we represent constraint Jacobian $\mathbf{F}^\top$ as its QR decomposition: $\mathbf{F}^\top = \mathbf{Q}\begin{bmatrix}\mathbf{R} \\ \mathbf{0}\end{bmatrix}$, where $\mathbf{Q}^\top\mathbf{Q} = \mathbf{Q}\mathbf{Q}^\top = \mathbf{I}$ and $\mathbf{R}$ is invertible.

$$\mathbf{H}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{g} = \mathbf{T}\mathbf{u} + \mathbf{Q}\begin{bmatrix}\mathbf{R} \\ \mathbf{0}\end{bmatrix}\lambda \tag{17}$$

Let us multiply the equation by $\mathbf{Q}^\top$:

$$\mathbf{Q}^\top(\mathbf{H}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{g}) = \mathbf{Q}^\top \mathbf{T}\mathbf{u} + \begin{bmatrix} \mathbf{R} \\ \mathbf{0} \end{bmatrix} \lambda \tag{18}$$

Introducing switching variables (to divide upper and lower part of the equations) $\mathbf{S}_1 = \begin{bmatrix} \mathbf{I} & \mathbf{0} \end{bmatrix}$ and $\mathbf{S}_2 = \begin{bmatrix} \mathbf{0} & \mathbf{I} \end{bmatrix}$ and multiplying equations by one and the other we get two systems:

$$\begin{cases} \mathbf{S}_1\mathbf{Q}^\top(\mathbf{H}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{g}) = \mathbf{S}_1\mathbf{Q}^\top \mathbf{T}\mathbf{u} + \mathbf{R}\lambda \\ \mathbf{S}_2\mathbf{Q}^\top(\mathbf{H}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{g}) = \mathbf{S}_2\mathbf{Q}^\top \mathbf{T}\mathbf{u} \end{cases} \tag{19}$$

The main advantage we achieved is that now we can calculate both $\mathbf{u}$ and λ

Resulting expression for $\mathbf{u}$ is:

$$\mathbf{u} = (\mathbf{S}_2\mathbf{Q}^\top\mathbf{T})^+\mathbf{S}_2\mathbf{Q}^\top(\mathbf{H}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{g}) \tag{20}$$

Expression for λ is:

$$\lambda = \mathbf{R}^{-1}\mathbf{S}_1\mathbf{Q}^\top(\mathbf{H}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{g} - \mathbf{T}\mathbf{u}) \tag{21}$$

We can notice a pseudo-inverse, implying that the no-residual solution does not have to exist.

We can easily write inverse dynamics as a QP:

$$\begin{aligned} \underset{\mathbf{u},\lambda}{\text{minimize}} \quad & ||\mathbf{u}||, \\ \text{subject to} \quad & \begin{cases} \mathbf{H}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{g} = \mathbf{T}\mathbf{u} + \mathbf{F}^\top \lambda \\ \mathbf{F}\ddot{\mathbf{q}} + \dot{\mathbf{F}}\dot{\mathbf{q}} = 0 \end{cases} \end{aligned} \tag{22}$$

If there are some constraints or limits on the control input (torque limits, for instance) or the reaction forces are restricted (by friction cones, for instance), those can be directly added.

Implement a model-predictive controller for an LTI system with implicit or explicit constraints.

Lecture slides are available via Moodle.

You can help improve these slides at:
github.com/SergeiSa/Computational-Intelligence-Slides-Fall-2020

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