

# Linear Programming

## Computational Intelligence, Lecture 9

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# Linear Programming

## General form

A linear program (LP) is an optimization problem of the form:

$$\begin{array}{ll} \underset{\mathbf{x}}{\text{minimize}} & \mathbf{f}^\top \mathbf{x}, \\ \text{subject to} & \begin{cases} \mathbf{Ax} \leq \mathbf{b}, \\ \mathbf{Cx} = \mathbf{d}. \end{cases} \end{array} \quad (1)$$

It is one of the older and widely used classes of convex optimization problems.

Note that the solution of such problem will always lie on the boundary of its domain.

# Linear Programming

## LP with no solution - examples

Here are some examples of LP which have no solutions:

$$\underset{\mathbf{x}}{\text{minimize}} \quad \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad (2)$$

This one is has no boundaries at all, hence no solution. Next one has boundaries, but they do not restrict motion along the descent direction for the cost function.

$$\begin{aligned} \underset{\mathbf{x}}{\text{minimize}} \quad & \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \\ \text{subject to} \quad & \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq 1 \end{aligned} \quad (3)$$

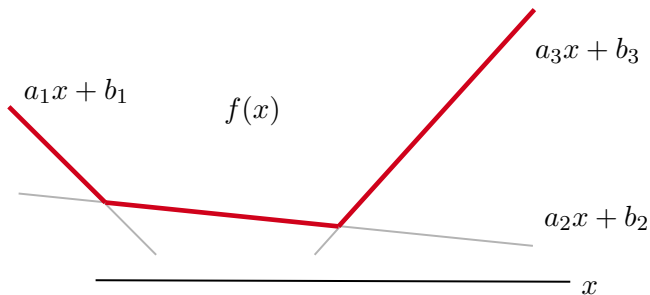
# Convex piece-wise linear functions

## Problem statement

Convex piece-wise linear functions have the form:

$$f(\mathbf{x}) = \max(\mathbf{a}_i^\top \mathbf{x} + b_i) \quad (4)$$

Figure below shows geometric interpretation of such function for a one-dimensional case.



# Convex piece-wise linear functions

## Solution as LP

We can formulate a minimization problem using convex piece-wise linear functions:

$$\underset{\mathbf{x}}{\text{minimize}} \quad \max(\mathbf{a}_i^\top \mathbf{x} + b_i) \quad (5)$$

Which can be equivalently transformed into the following LP:

$$\begin{aligned} &\underset{\mathbf{x}}{\text{minimize}} \quad t \\ &\text{subject to} \quad \mathbf{a}_i^\top \mathbf{x} + b_i \leq t \end{aligned} \quad (6)$$

We can observe that optimal (minimal)  $t$  will have to lie on one of the linear functions  $\mathbf{a}_i^\top \mathbf{x} + b_i$ , i.e. on the original piece-wise linear function  $f(\mathbf{x})$ . And optimal value on  $t$  corresponds to the smallest value of the original function  $f(\mathbf{x})$ .

# Sum of piece-wise linear functions

## Solution as LP

Sum of convex piece-wise linear functions have the form:

$$f(\mathbf{x}) + g(\mathbf{x}) = \max(\mathbf{a}_i^\top \mathbf{x} + b_i) + \max(\mathbf{c}_i^\top \mathbf{x} + d_i) \quad (7)$$

Their representation as LP is:

$$\begin{array}{ll} \underset{\mathbf{x}}{\text{minimize}} & t_1 + t_2 \\ \text{subject to} & \begin{cases} \mathbf{a}_i^\top \mathbf{x} + b_i \leq t_1 \\ \mathbf{c}_i^\top \mathbf{x} + d_i \leq t_2 \end{cases} \end{array} \quad (8)$$

# Convex piece-wise linear functions

## Code

```
0 func = @(t) t^2;
  derivative_func = @(t) 2*t;
2
  approx_points = [-1, -0.3, 0, 0.3, 1];
4 n = length(approx_points);
  a = zeros(n, 1);
6 b = zeros(n, 1);

8 for i = 1:n
    t = approx_points(i);
10    a(i) = derivative_func(t);
    b(i) = func(t) - a(i)*t ;
12 end

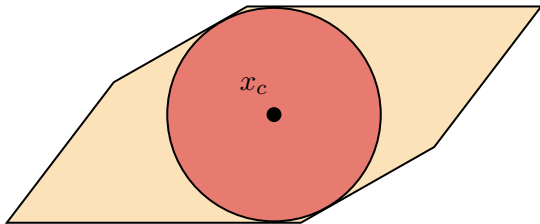
14 f = [1; 0];
  lin_A = [-ones(n, 1), a];
16 lin_b = -b;
  x = linprog(f, lin_A, lin_b, [], []);
```



# Chebyshev center of a polyhedron

## Problem statement

Chebyshev center of a polyhedron is the center of the largest ball inscribed in a polyhedron:



Equation describing this ball can be written as:

$$\mathcal{B} = \{\mathbf{x}_c + \mathbf{u} : \|\mathbf{u}\|_2 \leq r\} \quad (9)$$

where  $r$  is the radius of the ball and  $\mathbf{x}_c$  is its center.

# Chebyshev center of a polyhedron

## Solution as LP, part one

For the ball  $\mathcal{B}$  to be inscribed in a polygon  $\mathcal{P} = \{\mathbf{x} : \mathbf{Ax} \leq \mathbf{b}\}$ , the following should hold:

$$\sup\{\mathbf{a}_i^\top (\mathbf{x}_c + \mathbf{u}) : \|\mathbf{u}\|_2 \leq r\} \leq b_i \quad (10)$$

Note that the largest value of  $\mathbf{a}_i^\top \mathbf{u}$  under condition  $\|\mathbf{u}\|_2 \leq r$  is  $r\|\mathbf{a}_i^\top\|$ : it can indeed achieve this value if  $\mathbf{a}_i$  and  $\mathbf{u}$  are co-directional, but a larger one is not possible. Therefore:

$$\sup\{\mathbf{a}_i^\top (\mathbf{x}_c + \mathbf{u}) : \|\mathbf{u}\|_2 \leq r\} = \mathbf{a}_i^\top \mathbf{x}_c + r\|\mathbf{a}_i^\top\| \leq b_i \quad (11)$$

# Chebyshev center of a polyhedron

Solution as LP, part two

Finally, we can write down the solution of the problem as a linear optimization:

$$\begin{array}{ll}\underset{r, \mathbf{x}_c}{\text{maximize}} & r \\ \text{subject to} & \mathbf{a}_i^\top \mathbf{x}_c + r \|\mathbf{a}_i^\top\| \leq b_i\end{array}\tag{12}$$

# Chebyshev center of a polyhedron

## Code

Below we can see MATLAB code for solving the problem:

```
0 V = randn(10, 2);
2 k = convhull(V);
  P = V(k, :);
4
  [domain_A, domain_b] = vert2con(P);
6 norm_A = vecnorm(domain_A');

8 f = [-1; 0; 0];
  A = [reshape(norm_A, [], 1), domain_A];
10 b = domain_b;

12 x = linprog(f, A, b, [], []);

14 center = [x(2), x(3)];
   r = x(1);
```

# Homework

Implement linear approximation of a convex function and solve it as LP

Lecture slides are available via Moodle.

You can help improve these slides at:

[github.com/SergeiSa/Computational-Intelligence-Slides-Fall-2020](https://github.com/SergeiSa/Computational-Intelligence-Slides-Fall-2020)

Check Moodle for additional links, videos, textbook suggestions.