

# Three Whitehead Theorems and Three Puppe Sequences

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# Two goals

## The Whitehead Theorem

$$\begin{aligned} &\forall X : D(\infty\text{-Grpd}_0), \forall Y : D(\infty\text{-Grpd}_0), \\ &\forall f : X \rightarrow Y, \forall g : X \rightarrow Y, \\ &(\forall n : \text{Nat}, \pi_n f = \pi_n g) \rightarrow f = g \end{aligned}$$

## The Puppe Sequence

$$\begin{aligned} &\cdots \rightarrow \pi_1.\text{obj } E_0 \rightarrow \pi_1.\text{obj } B_0 \rightarrow \\ &\pi_0.\text{obj } ((1_{B_0}) \bullet ((\omega.\text{hom } (1_{B_0})).\text{hom } f)) \rightarrow \\ &\pi_0.\text{obj } (E_0) \rightarrow \pi_0.\text{obj } (B_0) \end{aligned}$$

- The homotopy fiber

# Definition of $\infty$ -category

```
def horn_filling_condition (X : SSet) (n i : Nat): Prop :=
  ∀ f :  $\Lambda[n, i] \rightarrow X$ , ∃ g :  $\Delta[n] \rightarrow X$ ,
  f = SSet.hornInclusion n i » g

/-- A simplicial set is called an  $\infty$ -category
if it has the extension property for all inner horn inclusions
 $\Lambda[n, i] \rightarrow \Delta[n]$ ,  $n \geq 2$ ,  $0 < i < n$ . -/
def InfCategory := {X : SSet //
  ∀ (n i : Nat),
   $n \geq 2 \wedge 0 < i \wedge i < n \rightarrow$  horn_filling_condition X n i}

#check (inferInstance : Category SSet) -- OK
#check (inferInstance : Category InfCategory) -- fail
```

# Automatic typeclass inference?

```
-- instance : Category InfCategory := inferInstance -- ?

-- instance : Category InfCategory := by -- ?
--   dsimp only [InfCategory]
--   infer_instance

instance : Category InfCategory where
  Hom X Y := NatTrans X.1 Y.1
  id X := NatTrans.id X.1
  comp  $\alpha$   $\beta$  := NatTrans.vcomp  $\alpha$   $\beta$ 

#check (inferInstance : Category SSet) -- OK
#check (inferInstance : Category InfCategory) -- OK
```

## Related Work

- Spectral Sequences in Homotopy Type Theory
  - ▶ Floris van Doorn et al.
  - ▶ <https://github.com/cmu-phil/Spectral>
  - ▶ Lean 2
- Myers, David Jaz, Hisham Sati, and Urs Schreiber. “Topological Quantum Gates in Homotopy Type Theory.” *arXiv preprint arXiv:2303.02382* (2023).
  - ▶ simulating and verifying topological quantum gates
  - ▶ Professor Schreiber suggested we formalize the construction of their “Gauss-Manin connections.”