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The Whitehead Theorem and Two Variations

$\forall$(C:D(∞-Cat)),$\forall$(D:D(∞-Cat)),$\forall$(F:D(∞-Cat).Hom C D),$\forall$(G:D(∞-Cat).Hom C D),($\forall$(n:Nat),($\vec{\Pi}_n$ F = $\vec{\Pi}_n$ G)) $\to$ F = G

$\forall$(X:D(∞-Grpd)),$\forall$(Y:D(∞-Grpd)),$\forall$(f:D(∞-Grpd).Hom X Y),$\forall$(g:D(∞-Grpd).Hom X Y),($\forall$(n:Nat),(Π_n f = Π_n g)) $\to$ f = g

$\forall$(X:D(∞-Grpd$_{-1}$)),$\forall$(Y:D(∞-Grpd$_{-1}$)),$\forall$(f:D(∞-Grpd$_{-1}$).Hom X Y),$\forall$(g:D(∞-Grpd$_{-1}$).Hom X Y),($\forall$(n:Nat),(π_n f = π_n g)) $\to$ f = g

Plans to prove the Whitehead theorem
in Lean 4, with extensive use of Mathlib 4

We wish to acknowledge the collaborative efforts of E. Dean Young and Jiazhen Xia. Dean Young initially formulated the introduction with twelve goals, posting them on the Lean Zulip in August of 2023. Together the authors are pursuing these plans as a long term project.

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2. Introduction

The main goal of this repository is to prove the Whitehead theorem in Lean 4 using Mathlib 4's homotopy groups. Two other subsequent goals are to state and prove two variations of the Whitehead theorem. It is important that initial pull requests stemming from our work remain basic and accessible; we hope to make progress which is gradual and incremental.

Besides this goal, we have three others. Here are the four heorems which will form our main goals:

(a) (The Whitehead theorem for based ∞-groupoids) $\forall$(E:D(∞-Grpd$_{-1}$)),$\forall$(B:D(∞-Grpd$_{-1}$)),$\forall$(F:D(∞-Grpd$_{-1}$).Hom E B),$\forall$(G:D(∞-Grpd$_{-1}$).Hom E B),($\forall$(n:Nat),(π_n F $=$ π_n G)) $\rightarrow$ F $=$ G, where π_n is notation for π n, where π n : Functor D(∞-Grpd$_{-1}$) Set.

(b) (The Whitehead theorem for ∞-groupoids) $\forall$(E:D(∞-Grpd)),$\forall$(B:D(∞-Grpd)),$\forall$(F:D(∞-Grpd).Hom E B),$\forall$(G:D(∞-Grpd).Hom E B),($\forall$(n:Nat),(Π_n F $=$ Π_n G)) $\rightarrow$ F $=$ G, where Π_n is notation for Π n, where Π n : Functor D(∞-Grpd) Set.

(c) (The Whitehead theorem for based) $\forall$(E:D(∞-Cat$_{-1}$)),$\forall$(B:D(∞-Cat$_{-1}$)),$\forall$(F:D(∞-Cat$_{-1}$).Hom E B),$\forall$(G:D(∞-Cat$_{-1}$).Hom E B),($\forall$(n:Nat),($\vec{\pi}_n$ F $=$ $\vec{\pi}_n$ G)) $\rightarrow$ F $=$ G, where $\vec{\pi}_n$ is notation for $\vec{\pi}$ n, where $\vec{\pi}$ n : Functor D(∞-Cat$_{-1}$) Set.

(d) (The Whitehead theorem for ∞-categories) $\forall$(E:D(∞-Cat)),$\forall$(B:D(∞-Cat)),$\forall$(F:D(∞-Cat).Hom E B),$\forall$(G:D(∞-Cat).Hom E B),($\forall$(n:Nat),($\vec{\Pi}_n$ F $=$ $\vec{\Pi}_n$ G)) $\rightarrow$ F $=$ G, where $\vec{\Pi}_n$ is notation for $\vec{\Pi}$ n, where $\vec{\Pi}$ n : Functor D(∞-Cat$_{-1}$) Set.

(a) in the above reflects the known Whitehead theorem, which dates back to Whitehead's two papers titled 'Combinatorial Homotopy I' and 'Combinatorial Homotopy II'. There is also a fourth Whitehead theorem.

We will use two models of each of the following categories in the theorems above:

(i) We model ∞-Grpd$_{-1}$: Cat using based CW-complexes.

(ii) We model ∞-Grpd : Cat using CW-complexes.

(iii) We model ∞-Cat$_{-1}$: Cat using based directed CW-complexes.

(iv) We model ∞-Cat : Cat using directed CW-complexes.

This choice accords with the standard approach to the third theorem, in which one typically chooses both a combinatorial and point-set model, with the former featuring a geometric realization functor into the latter.

We will use `Mathlib 4`'s category theory, particularly

1. Categories (see Mathlib's `Category X`)
2. Functors (see Mathlib's `Functor C D`)
3. Natural transformations (see Mathlib's `NatTrans F G`)
4. Equations between natural transformations (see Mathlib's `NatExt` here; these are related to our `equation`)

While the functors π_n occuring in the main theorems above are already defined in `Mathlib 4` for the desired point-set model, the functors π_n, Π_n, and $\vec{\Pi}_n$ are not. The existence of a base point makes π_n relatively straightforward to define. Here are their types:

(i) π_n : Functor D(∞-Grpd$_{-1}$) Set (???)

(ii) $\vec{\pi}_n$: Functor D(∞-Cat$_{-1}$) Set

(iii) Π_n : Functor D(∞-Grpd) Set

(iv) $\vec{\Pi}_n$: Functor D(∞-Cat) Set

We also form their derived functors:

(i) D(π_n) : Functor D(∞-Grpd$_{-1}$) Set

(ii) D($\vec{\pi}_n$) : Functor D(∞-Cat$_{-1}$) Set

(iii) D(Π_n) : Functor D(∞-Grpd) (Glb $\to$_(Cat) Sets)

(iv) D($\vec{\Pi}_n$) : Functor D(∞-Cat) (Glb $\to$_(Cat) Sets)

Where Glb $\to$_(Cat) Sets is the category of globular sets. In the course of the repository we will need the directed path space, path space, and loop space functors as well, which fit with the analogy formed by the Whitehead theorem and its two variations:

1. Ω : Functor ∞-Grpd$_{-1}$ ∞-Grpd$_{-1}$
2. $\vec{\Omega}$: Functor ∞-Cat$_{-1}$ ∞-Cat$_{-1}$

γ,- : Functor ∞-Grpd ∞-Grpd

$\vec{\gamma}$,- : Functor ∞-Grpd ∞-Grpd

Where γ is the unit interval and $\vec{\gamma}$ is the directed unit interval.

	Enriched		Internal	
`Strict Unitial`	enriched category	7 entries	internal category	13 entrie
`Strict Actional`	enriched presheaf	3 entries	internal presheaf	5 entries
$\mathtt{A}^{\infty}$ `Unitial`	enriched A^{∞}-operoid	7 entries	internal A^{∞}-operoid action	13 entrie
$\mathtt{A}^{\infty}$ `Actional`	enriched A^{∞}-operoid action	3 entries	internal A^{∞}-operad action	5 entries

3. Unicode

Here is a list of the unicode characters we will use:

Symbol	Unicode	VSCode shortcut	Use
		Lean's Kernel	
$\times$	2A2F	\times	Product of types
$\rightarrow$	2192	\rightarrow	Hom of types
$\langle,\rangle$	27E8,27E9	\langle,\rangle	Product term introduction
$\mapsto$	21A6	\mapsto	Hom term introduction
$\wedge$	2227	\wedge	Conjunction
$\vee$	2228	\vee	Disjunction
$\forall$	2200	\forall	Universal quantification
$\exists$	2203	\exists	Existential quantification
$\neg$	00AC	\neg	Negation
		Variables and Constants	
$^{a},^{b},^{c},\ldots,^{z}$	1D52,1D56		Variables and constants
$^{0},^{1},^{2},^{3},^{4},^{5},^{6},^{7},^{8},^{9}$	1D52,1D56		Variables and constants
$^{-}$	207B		Variables and constants
$_{0},_{1},_{2},_{3},_{4},_{5},_{6},_{7},_{8},_{9}$	2080 - 2089	\0-\9	Variables and constants
$\mathbb{A},\ldots,\mathbb{Z}$	1D538		
$\mathbb{a},\ldots,\mathbb{z}$	1D552		
$\mathbf{A},\ldots,\mathbf{Z}$	1D41A		
$\mathbf{a},\ldots,\mathbf{z}$	1D41A		
α-ω,Α-Ω	03B1-03C9		Variables and constants
		Categories	
$\mathbb{1}$	1D7D9	\b1	The identity morphism
$\circ$	2218	\circ	Composition
		Bicategories	
$\bullet$	2022	\smul	Horizontal composition of objects
		Adjunctions	
$\rightleftarrows$	21C4	\rightleftarrows	Adjunctions
$\leftrightarrows$	21C6	\leftrightarrows	Adjunctions
.	1BC94		Right adjoints
·	0971		Left adjoints
$\dashv$	22A3	\dashv	The condition that two functors are adjoint
		Monads and Comonads	
?,¿	003F, 00BF	?,\?	The corresponding (co)monad of an adjunction
!,¡	0021, 00A1	!, \!	The (co)-Eilenberg-(co)-Moore adjunction
ꜝ,ꜞ	A71D, A71E		The (co)exponential maps
		Miscellaneous	
$\sim$	223C	\sim	Homotopies
$\simeq$	2243	\equiv	Equivalences
$\cong$	2245	\cong	Isomorphisms
$\perp$	22A5	\bot	The overobject classifier
∞	221E	\infty	Infinity categories and infinity groupoids
$\leftrightarrow$	20D7		Homotopical operations on ∞-categories
$\rightarrow$	20E1		Homotopical operations on ∞-groupoids

Of these, the characters !, ¡, ., ˙, →, and ↔ do not have VSCode shortcuts.

4. Introduction to Lean 4

The main way to tell Lean 4 what something means is with `def`, which defines a term in dependent type theory. Much in the same way as other computer languages, we then supply the type of the term (e.g. `Int` for integer), followed by the formula itself:

Lean 1

```
def zero : Nat := 0
```

Here we have introduced a natural number `n` using the type `Nat` that comes with Lean 4.

As a beginner, it's normal to take some time to get comfortable with Lean and formal proof systems. It's a journey that requires practice and patience. Lean has an active community that provides support and resources to help you along the way.

Constituents of `x, y : X` of types `X` can also stand to be equal or unequal, written `x = y`, and it is the properties of equality which in addition to the dependent type theory make a type behave like a set. Equality satisfies the three properties of an equivalence relation, which we cover presently. Consider first the reflexivity property of equality:

Lean 2

```
def reflexivity {X : Type} {x : X} : x = x := Eq.refl
↪ x
```

This command defines a function called reflexivity that proves the reflexivity property of equality. The function takes two type parameters: `X` represents the type of the elements being compared, and `x` represents an element of type `X`. It also takes an argument ω which is a proof that `x` is equal to itself (`x = x`). The function body states that the result of `reflexivity` is the proof ω itself using the `Eq.refl` constructor, which indicates that `x` is equal to itself.

In `Lean 4`, `{x : X}` represents an implicit argument, where Lean will attempt to infer the value of `x` based on the context. `(x : X)` represents an explicit argument, requiring the value of `x` to be provided explicitly when using the function or definition.

Lean 3

```
def symmetry {X : Type} {x : X} {y : X}  (p : x = y)
↪  := Eq.symm p
```

This command defines a function called symmetry that proves the symmetry property of equality. It takes three type parameters: `X` represents the type of the elements being compared, and `x` and `y` represent elements of type `X`. The function also takes an argument ω which is a proof that `x` is equal to `y` (`x=y`). The function body states that the result of `symmetry` is the proof ω itself using the `Eq.symm` constructor, which allows you to reverse an equality proof.

Lean 4

```
def transitivity {X : Type} {x : X} {y : X} {z : X}
↪ (p : x = y) (q : y = z) := Eq.trans p q
```

This command defines a function called transitivity that proves the transitivity property of equality. It takes four type parameters: `X` represents the type of the elements being compared, and `x`, `y`, and `z` represent elements of type `X`. The function also takes two arguments `p` and `q`. `p` is a proof that `x` is equal to `y` (`x = y`), and `q` is a proof that `y` is equal to `z` (`y = z`). The function body states that the result of transitivity is the proof of the composition of ω and `q` using the `Eq.trans` constructor, which allows you to combine two equality proofs to obtain a new one.

These `Lean` commands define functions that prove fundamental properties of equality: reflexivity (every element is equal to itself), symmetry (equality is symmetric), and transitivity (equality is transitive). These properties are essential for reasoning about equality in mathematics and formal proofs.

We must also require that functions satisfy extensionality:

Lean 5

```
def extensionality (f g : X → Y) (p : (x:X) → f x =
↪ g x) : f = g := funext p
```

Extensionality, a key characteristic of sets and types, asserts that functions which are equal on all values are themselves equal, and it is featured prominently in what is perhaps the most well known mathematical foundations of ZFC.

There are several other features of equality with respect to functions which we should be aware of:

Lean 6

```
def equal_arguments {X : Type} {Y : Type} {a : X} {b
↪ : X} (f : X → Y) (p : a = b) : f a = f b :=
↪ congrArg f p

def equal_functions {X : Type} {Y : Type} {f₁ : X →
↪ Y} {f₂ : X → Y} (p : f₁ = f₂) (x : X) : f₁ x =
↪ f₂ x := congrFun ω x

def pairwise {A : Type} {B : Type} (a₁ : A) (a₂ : A)
↪ (b₁ : B) (b₂ : B) (p : a₁ = a₂) (q : b₁ = b₂) :
↪ (a₁,b₁)=(a₂,b₂) := (congr ((congrArg Prod.mk) p)
↪ q)
```

Here are some introductions to Lean 4 and Mathlib 4:

1. The tutorial here gives an introduction to using the dependent type theory in Lean.

2.

PART 1: BASED ∞-GROUPOIDS

In this first section we prove the standard Whitehead theorem.

Chapter 1: ∞-Grpd [1]

Definition 1 (CW-complex). Given a well order ω, a CW-complex ...

Implementation Progress

Lean 7

```
/-- A relative CW-complex contains an expanding
 ↪  sequence of subspaces `sk i`
(called the `i`-skeleta) for `i   -1`, where `sk
 ↪  (-1)` is an arbitrary topological space,
isomorphic to `A`, and each `sk (n+1)` is obtained
 ↪  from `sk n` by attaching (n+1)-disks. -/
structure RelativeCWComplex (A : TopCat) where
  /-- Skeleta -/
  sk : ℤ → TopCat
  /-- A is isomorphic to the (-1)-skeleton. -/
  iso_sk_neg_one : A ≅ sk (-1)
  /-- The (n+1)-skeleton is obtained from the
   ↪  n-skeleton by attaching (n+1)-disks. -/
  attach_cells : (n : ℤ) → CWComplex.AttachCells (sk
   ↪  n) (sk (n + 1)) n

/-- A CW-complex is a relative CW-complex whose
 ↪  (-1)-skeleton is empty. -/
abbrev CWComplex := RelativeCWComplex (TopCat.of
 ↪  Empty)
```

Lean 8

```
/-- The topology on a relative CW-complex -/
def toTopCat {A : TopCat} (X : RelativeCWComplex A) :
↪  TopCat :=
  Limits.colimit (colimitDiagram X)

instance : Coe CWComplex TopCat where coe X :=
↪  toTopCat X
```

Lean 9

```
def IsCWComplex (X : TopCat) : Prop := ∃ Y :
↪  CWComplex, Nonempty (↑Y ≅ X)

def CWComplexCat := FullSubcategory IsCWComplex
```

Writing Progress

Here we define CW-complexes, as well as relative CW-complexes, and also the derived categories $D(\infty\text{-Grpd}_0)$ of connected based ∞-groupoids and $D(\infty\text{-Grpd}_0/G_0)$, made from CW-complexes.

5. D(∞-Grpd $_1$)

In this section, we construct the homotopy category of based ∞-groupoids D(∞-Grpd$_{-1}$) as the category of (CW) complexes with homotopy classes of maps. A CW-complex, which we here refer to as a complex.

Lean 10

6. $\mathrm{D}(\infty\text{-}\mathrm{Grpd}_1/X)$

The derived category of an overcategory of based ∞-groupoids...

7. Σ : ∞-Grpd $_1$ $\rightleftarrows$ ∞-Grpd $_1$: Ω

The loop space functor Ω : Functor ∞-Grpd $_1$ ∞-Grpd $_1$ is right adjoint to the (based) suspension functor Σ : Functor ∞-Grpd $_1$ ∞-Grpd $_1$.

8. σ f : ∞-Grpd $_1$/B $\rightleftarrows$ ∞-Grpd $_1$/E : ω f, f : ∞-Grpd $_1$

The homotopy fiber
Based homotopy pushout

9. π_n : Functor ∞-Grpd 1 Set

The homotopy groups can be first understood as functors into `Set`, only later adding in the fact that π_n factors through `InternalGroup` $\bullet \cdots \bullet$ `InternalGroup Set` $\simeq$ `InternalAbelianG` for `n` 2, and `InternalGroup Set` $\simeq$ `Group` for `n = 1`.

Chapter 2: The Whitehead Theorem

The proof of the Whitehead theorem divides into REP (replacement for based ∞-groupoids $\mathrm{X} : \infty\text{-Grpd}_{-1}$) and HEP (the homotopy extension property for weak equivalent maps of based ∞-groupoids). The replacement functor $\infty\texttt{-Grpd}_0$ can be constructed using globular sets.

Globular sets are not a rich enough invariant for homotopy, but maps of globular sets bear a criticall difference because of

$$\forall\texttt{(E:D(}\infty\texttt{-Grpd }_1\texttt{))},\forall\texttt{(B:D(}\infty\texttt{-Grpd }_1\texttt{))},\forall\texttt{(f:D(}\infty\texttt{-Grpd }_1\texttt{).Hom E B)},\forall\texttt{(G:D(}\infty\texttt{-Grpd }_1\texttt{).Hom E B)},(\forall\texttt{(n:Nat)},(\pi_n\texttt{ F = }\pi_n\texttt{ G)}) \longrightarrow \texttt{F = G}$$

10. Globular Sets

The globe category $\mathbb{G}$ is the category

$$\bullet \overset{\sigma_0}{\underset{\tau_0}{\rightrightarrows}} \bullet \overset{\sigma_1}{\underset{\tau_1}{\rightrightarrows}} \bullet \overset{\sigma_2}{\underset{\tau_2}{\rightrightarrows}} \bullet \overset{\sigma_3}{\underset{\tau_3}{\rightrightarrows}} \bullet \rightrightarrows \cdots$$

Globular sets are functors from the opposite category of the globe category $\mathbb{G}$ into the category of sets, and maps of globular sets are natural transformations between them.

In this chapter we prove the following (which we have called Whitehead Theorem (c)): $\forall$(E:D(∞-Grpd$_0$)),$\forall$(B:D(∞-Grpd$_0$)),$\forall$(f:E $\longrightarrow$ B),$\forall$(G:E $\longrightarrow$ B),($\forall$(n:Nat),(π_n F = π_n G)) $\longrightarrow$ F = G, where π_n is notation for π n.

This can be shown using CW-replacement and induction on n. Fibrant replacement of an object X entails replacing an object in ∞-Grpd$_0$ with a CW-object (an object made by successively glueing in higher and higher simplices along their boundaries obtaining a sequence X$_n$). Given an equality $\pi_{n+1}(f)$ =$\pi_{n+1}(g)$ and a homotopy equivalence h$_n$: $\Delta^1 \times$ X$_n$ $\longrightarrow$ Y between $f|_{X_n}, g|_{X_n}$: X$_n$ $\longrightarrow$ Y, we construct an extension of the homotopy equivalence $\Delta^1 \times X_{n+1} \longrightarrow$ Y.

Spheres and balls Next we turn to defining spheres and balls:

	Spheres and Balls	
Name of the `X` value	∂X $\cong$ S^n	X $\cong$ D^n
p-norm unit ball for p = 1	∂B(1,n)	B(1,n)
p-norm unit ball for 1 < p < 2	∂B(p,n)	B(p,n)
p-norm unit ball for p = 2	∂B(2,n)	B(2,n)
p-norm unit ball for 2 < p < ∞	∂B(p,n)	B(p,n)
p-norm unit ball for p = ∞	∂B(∞,n)	B(∞,n)
The n-simplex	$\partial\Delta^n$	Δ^n

While each of the above unit balls are homeomorphic, so that one has a choice of p-norm, the unit balls in [$\mathbb{N}$,$\mathbb{R}$] for different norms are not homeomorphic. Here are two lemmas we have for the 2-norm and ∞-norm unit balls in [$\mathbb{N}$,$\mathbb{R}$]:

Theorem 1. I $\times$ B $\cong$ B, where B is the unit ball in l$_2$ under the 2-norm.

Proof. ... $\square$

Theorem 2. $\mathtt{I} \times [\mathbb{N},\mathtt{I}] \cong [\mathbb{N},\mathtt{I}]$, where B is the unit ball in l_2 under the 2-norm.

Proof. ... $\square$

Definition 2 ((a)). ...

Theorem 3. $i^1 : S^0 \longrightarrow D^1$

Theorem 4. $D^n \times D^1 \longrightarrow D^{n+1}$

Definition 3. $D^n \longrightarrow D^m$

Theorem 5. Fix n : $\mathbb{N}$ and let $\partial^n : S^n \longrightarrow D^{n+1}$ be the inclusion. The pushout of the following diagram is isomorphic to S^{n+1}:

$$\begin{tikzcd}
S^n \arrow[r] \arrow[d] & D^{n+1} \\
D^{n+1} &
\end{tikzcd}$$

Proof. $\square$

Theorem 6. Define a function $||\text{-}||_2 : D^n \longrightarrow I$ sending $(x_1,...,x_n)$ to $\sqrt{\sum_{i=1}^{n} x_i^2}$, and write $||\text{-}||_2$

Proof. ... $\square$

11. HEP for based connected ∞-groupoids

In this section we prove the homotopy extension property for based ∞-groupoids, which we model as CW-complexes.

Jar filling Next we turn to defining 'jar shapes' $\mathtt{J}^{\mathtt{n}}$, which include into $\mathtt{D}^{\mathtt{n}} \times \mathtt{I}$ $\mathtt{i}_n : \mathtt{J}^{\mathtt{n}} \longrightarrow \mathtt{D}^{\mathtt{n}} \times \mathtt{I}$, after which we 'fill' them (i.e. demonstrate that any continuous map $f : J^n \longrightarrow X$ extends to a continuous map $g : D^n \times I \longrightarrow X$).

Definition 4. We define the n-jar $\mathtt{J}_n$:= pushout $(\mathtt{S}^{\mathtt{n}} \times \mathtt{d}_0)\ \partial\mathtt{D}^{\mathtt{n}}$, where $\mathtt{d}_0 : * \longrightarrow \mathtt{I}$ sends the unique point $* : *$ to 0. There is a continuous function $\mathtt{j}_n$ from $\mathtt{J}_n$ to $\mathtt{D}^{\mathtt{n}}$ arising from the functions $\partial\mathtt{D}^{\mathtt{n}} \times \mathtt{I} : \mathtt{S}^{\mathtt{n}} \times \mathtt{I} \longrightarrow \mathtt{D}^{\mathtt{n}} \times \mathtt{I}$ and

$\mathtt{j}_n$ in the above is injective. In the case where $\mathtt{n} = 3$ we can depict it as the inclusion of the 'empty jar shape' into the 'filled jar shape' of $\mathtt{D}^2 \times \mathtt{I}$. 'Jar filling' then asserts that any continuous function $f : J_n \longrightarrow X$ extends to a continuous function $g : D^2 \times I \longrightarrow X$:

Theorem 7 (Jar filling). $\forall(\mathtt{f} : \mathtt{J}_n \longrightarrow \mathtt{X}),\exists(\mathtt{g} : \mathtt{D}^{\mathtt{n}} \times \mathtt{I}),\mathtt{g} \bullet \mathtt{j}_n = \mathtt{f}$.

The first approach I cover here involves 'shining a light ray down from above the jar', i.e. projection. This divides into two steps, where in the first we define the projection onto the sides and bottom (seperately), and in the second we show that these continuous functions match on S^{n-1} and that they assemble into a continuous function $proj_n$ from $D^n \times I$ to J_n.

Theorem 8. There is a continuous function $\mathtt{proj}_{\mathtt{n}} : \mathtt{D}^{\mathtt{n}} \times \mathtt{I} \to \mathtt{J}_n$ such that .

After this, we show that $\mathtt{proj}_n \bullet \mathtt{j}_n = 1_{\mathtt{J}_n}$.

Change of Base Change of base demonstrates that π_n(X,x) is isomorphic to π_n(X,y) for a connected CW-complex X and two different points x and y in X given a path between them. This isomorphism depends on the choice of path.

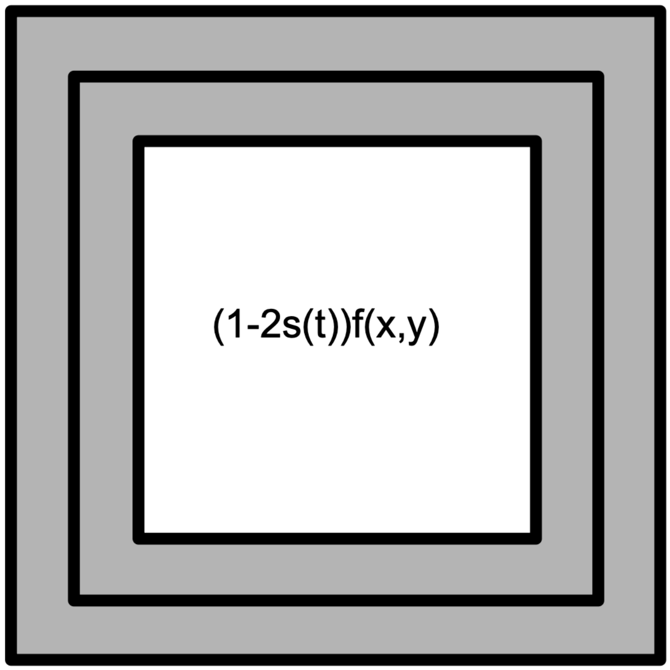
Definition 5. Let X be a connected CW-complex and let n : Nat be a natural number. The transport function $\mathtt{trans\ n\ X} : (\mathtt{f} : [\mathtt{I},\mathtt{X}]) \to \pi_n\ (\mathtt{X},\mathtt{ev\ f\ 0}) \longrightarrow \pi_n\ (\mathtt{X},\ \mathtt{ev\ f\ 1})$ ...

Theorem 9. Let X_{-1} be a connected CW-complex and let $f : I \longrightarrow X_{-1}$ be a path, so that $(\mathtt{trans\ n\ X}_1\ \mathtt{f}^{-1}) \bullet (\mathtt{trans\ n\ X}_1\ \mathtt{f})$ has type $\pi_n\ (\mathtt{f\ 0}) \longrightarrow \pi_n\ (\mathtt{f\ 0})$. Then

$$\texttt{(trans n X}_1\ \texttt{f}^{-1}\texttt{)} \bullet \texttt{(trans n X}_1\ \texttt{f) = 1\_(}\pi_n\ \texttt{(f 0))}$$

Proof. ... □

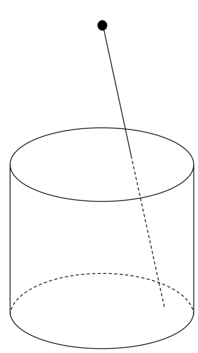
The proof in the above can be depicted like so, as a 'painting with two concentric frames':



that the based CW-complexes (X,x) and (X,y) are

Theorem 10.

Proof. ... □



the homotopy extension property

Lemma 1. Given based complexes `X, Y : ∞-Grpd 1` and continuous maps `f, g : ∞-Grpd 1.Hom X` and a continuous function `∞-Grpd 1.Hom I pushout d0 d1` such that , ...

Proof. `f • (¬ g)`... □

Note: the 'little lines' operad and its algebras arise in the study of how .

Theorem 11 (HEP)**.** Given a based complex `X` and a space `Y` with π_0`.obj X ≅ *` and a continuous maps `f : ∞-Grpd 1.Hom X (Ω.obj Y)` which is weak equivalent to 0, there is a homotopy `f ≃ *` between `f` and the constant function `* : ∞-Grpd 1 X Y`.

Proof. For each `n : Nat`, let X_n be the nth space in the complex `X`, and let α_n `: ∞-Grpd 1.Hom (`A_n `× S` be the attaching map. Let f_n be the precomposition of `f` with the map `∞-Grpd 1.Hom` X_n `X`. We construct a homotopy `f ≃ *` inductively. □

12. REP for based connected ∞-groupoids

In this section we use the notion of globular sets to replace a topological space with a CW-complex. Together with HEP (homotopy extension), this will complete the proof of the Whitehead theorem.

In fact, we will construct more than this: an adjunction $F \dashv G$ between globular sets and topological spaces. For this we continue with the construction of G:

Definition 6 (The object component of the functor from topological spaces to globular sets)**.** Fix a topological space X, and to form the object component of G.obj X, Gobj X : TGlb, we define (Gobj X).obj n to Top.Hom D^{n} X. Defining Gobj X on morphisms is not much more difficult, and involves composition $\sigma_n, \tau_n : D^{n} \longrightarrow D^{n+1}$.

Definition 7 (The morphism component of the functor from topological spaces to globular sets)**.** ...

Definition 8 (Proving the identity for the functor from topological spaces to globular sets)**.** ...

Definition 9 (Proving the compositionality law for the functor from topological spaces to globular sets)**.** ...

To construct F, we first construct a term of the CW-complex structure built from a globular set Φ.

13. The Whitehead theorem

Theorem 12 (HEP)**.** Given based complexes X and Y with $\pi_0\texttt{.obj X} \cong *$ and continuous maps ϕ : ???.Hom S^n X, f : pushout ∂D^{n+1} $\alpha \longrightarrow$ Y, g : pushout ∂D^{n+1} $\alpha \longrightarrow$ Y, H : homotopy (f • ?, g • ?), then f and g are homotopic.

Proof. For $\texttt{n = 0}$, $\square$

Here we show the Whitehead theorem using the homotopy extension property and replacement (REP).

Chapter 3: The Category of Maps

In this section I cover the category of maps `Map C` in a category `C`. After this I inductively form `n 1 C` as `Map ( n C)` and `inf C` as the colimit of `n C`.

PART 2: ∞-GROUPOIDS

The Whitehead theorem is about the ways that spheres get trapped in spaces (higher homotopy groups), and the last section established how these higher homotopy groups relate to maps in the homotopy category of based CW-complexes.

Chapter 9: ∞-Grpd

14. - $\times$ I : ∞-Grpd $\rightleftarrows$ ∞-Grpd : [I,-]

Our choice of symbols refects our choice of three variations of the Whitehead theorem and three Puppe sequences. $\overleftrightarrow{\Omega}$, the analogue of loop space, is the internal hom functor [I,-] : ∞-Grpd $\longrightarrow$ ∞-Grpd. This is not hard to construct, with the main lemma being that the path space of a quasicategory has the quasicategory lifting conditon.

We will be interested in one formal model of D(∞-Cat) which consists of formal compositions $f_1 \bullet g_1 \bullet f_2 \bullet g_2 \bullet \cdots \bullet f_n \bullet g_n$, where $g_n : Dom(f_{n+1}) \longrightarrow$??? is a weak equivalence, and something similar for D(∞-Cat). However, it is still vital to have the replacement functor `repl`, which ensures the Whitehead theorem for particular ∞-categories which are constructed out of attaching maps.

15. ??? : $\rightleftarrows$: ???

$\vec{\Omega}$ is to internal categories as $\vec{\omega}$ is to internal G-actions. It is also called directed homotopy pullback. These functors will later be used to produce functors $\vec{\mathrm{P}}$: D(∞-Grpd) $\longrightarrow$ InternalCategory D(∞-Grpd) and $\vec{\mathrm{p}}$: D(∞-Grpd/C) $\longrightarrow$ InternalPresheaf ($\vec{\mathrm{P}}$ G) D(∞-Grpd/G).

16. Π_n : Functor ∞-Grpd Set

Chapter 10: The Whitehead Theorem for ∞-Groupoids

17. Cubical Complexes

...

1. Defining `repl`
2.

18. REP

We have divided the work of proving Whitehead theorem (a) into two steps: REP and HEP. In this section, we construct a functor `repl : ∞-Cat ⟶ ∞-Cat` along with a natural transformation `weak_equivalence : repl ⟶ (𝟙 ∞-Cat)`. To construct `repl`

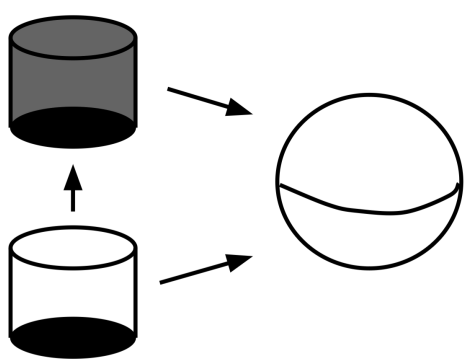
19. HEP

Consider the context of , supposing that we have constructed a homotopy ... This gives a picture that is a bit like “filling up a jar“: a homotopy h : of f, g : $\partial\Delta^2 \longrightarrow Y$, along with the value of g on Δ^2, produces a “jar” shape in Y, which can be “filled up” to produce a homotopy $h : \Delta^1 \times \Delta^2 \longrightarrow Y$. This is easier for simplicial-based approaches than for point-set topological approaches, the latter of which needs extra steps that deform a map into a cellular map.

This construction, in the case of point set topology, often involves first deforming maps so as to be cellular; however our analogue of CW complexes allows us to skip this step.

This construction (HEP for quasicategories) may even be equivalent to the quasicategory lifting condition if we are lucky. It is also the main technical device allowing for our concrete choice of model (quasicategories).

In this section, we demonstrate this extension property and use it to conclude the Whitehead theorem for ∞-categories stated above.

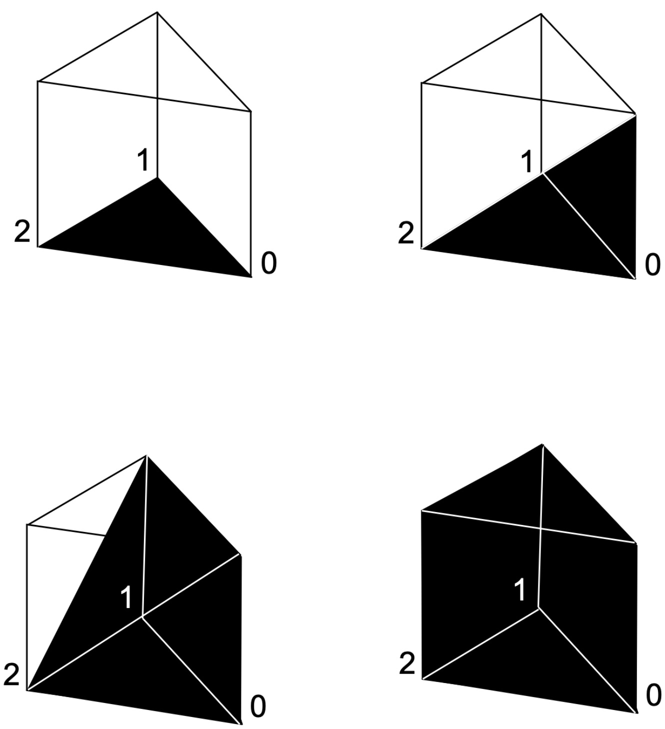


Prism Filling (PF) Let Y be a quasicategory, and let $f, g : \partial\Delta^n \longrightarrow Y$. A homotopy $h : \partial\Delta^n \times \Delta^1 \longrightarrow Y$ between $f, g : \partial\Delta^n \longrightarrow Y$ extends to a map $H : \Delta^n \times \Delta^1 \longrightarrow Y$;

this follows from the condition that Y be a quasicategory. H(-,1) and g match on $\partial\Delta^n$, producing a map f : X $\longrightarrow$ Y, where X consists of two copies of Δ^n glued together at the boundary. Consider a space X' formed as a quotient of $\Delta^n \times \Delta^1$ by $\partial\Delta^n \times \Delta^1$. There is a map ϕ : X $\longrightarrow$ X'. An induction hypothesis on f and g involving π_n ensures that the aparent map X $\longrightarrow$ Y lifts along ϕ, producing a map from $\Delta^n \times \Delta^1$ which is constant on $\partial\Delta^n \times \Delta^1$. Stacking this on top of H can be done using an isomorphism between Δ^1 and Δ^1 glued with itself along different endpoints. Altogether this produces a homotopy between f and g.

Directed prism filling may combine fruitfully with the yoneda lemma and/or the fact that simplicial sets are determined by the sets [Δ^n,X] along with combinatorial information (face and degeneracy maps).

Decomposing $\Delta^n \times \Delta^1$ into a colimit involving n+1 Δ^{n+1}'s ...



In the above, it may be easier if we make use of sub-simplicial sets and prove the theorem using that colimit applied to a natural isomorphism of diagrams products an isomorphism.

The decomposition

A definition of $\vec{\Pi}_n$ which is consistent with our goals of Wa and Pa is one as a certain pushout involving ($\vec{\Omega}^n$ X)– one which amounts to taking an equivalence relation by paths in $\vec{\Omega}^n$ X which restrict to constant paths along the face maps f⍰ : $\vec{\Omega}^{n-1}$ X $\longrightarrow$ $\vec{\Omega}^n$ X. Here, $\vec{\Omega}$ is easy to define in the model of quasi-categories, and it amounts . Besides fullfilling our goal of the first Whitehead theorem and puppe sequence, this definition of $\vec{\Pi}_n$ strikes me as elegant because it uses all of the ways for $\vec{\Omega}^n$ X to map into $\vec{\Omega}^{n+1}$ X.

The next symbols in the project's "periodic table" that we construct, after $\vec{\Omega}$ and $\vec{\Pi}_n$, will be $\vec{B}$ and $\vec{E}$, which we feature in the chapter on Puppe sequence (a).

A useful thing for us to construct first is the boundary of a product of Δ^1's and the boundary of a directed simplex. We might even like to expand on this later, but for now just consider for a moment how each might be made out of a glueing construction involving face maps.

Even though the $\vec{\Pi}_n$'s can be defined using $\vec{\Omega}^n$ X and various face maps f_(n, b) : $\vec{\Omega}^{n-1}$ X $\longrightarrow$ $\vec{\Omega}^n$ X for $b : \{0, 1\}$, it may be nice to have this as a result, with the definition one featuring two cubes glued together along their boundary.

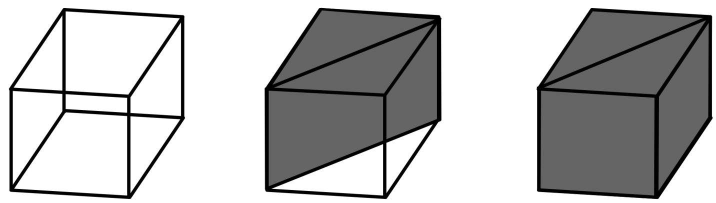
This means that we want directed box filling in addition to directed prism filling (but which also uses directed prism filling in its proof).

Box Filling (BF) Let Y be a quasicategory, and let f, g : $\partial\Delta^n \longrightarrow$ Y. A homotopy h : $\partial\Delta^n \times \Delta^1 \longrightarrow$ Y between f, g : $\partial\Delta^n \longrightarrow$ Y extends to a map H : $\Delta^n \times \Delta^1 \longrightarrow$ Y; this follows from the condition that Y be a quasicategory. H(-,1) and g match on $\partial\Delta^n$, producing a map f : X $\longrightarrow$ Y, where X consists of two copies of Δ^n glued together at the boundary. Consider a space X' formed as a quotient of $\Delta^n \times \Delta^1$ by $\partial\Delta^n \times \Delta^1$. There is a map ϕ : X $\longrightarrow$ X'. An induction hypothesis on f and g involving π_n ensures that the aparent map X $\longrightarrow$ Y lifts along ϕ, producing a map from $\Delta^n \times \Delta^1$ which is constant on $\partial\Delta^n \times \Delta^1$. Stacking this on top of H can be done using an isomorphism between Δ^1 and Δ^1 glued with itself along different endpoints. Altogether this produces a homotopy between f and g.

This goes hand-in-hand with a definition of $\vec{\Pi}_n$ which suits (I) and (II) in the introduction to chapter (3). If we make sure to prove lemmas...

The box filling and prism filling HEPs can be extended to the case of attaching all cells of a particular fixed dimension and as indexed by simplicial set arising from a set (or Lean 4 `Type`). That is, we might like to extend $\times$ () (or possibly somehow a `Set` as well), and that we may find an interest in the following two definitions of $\vec{\Pi}_n$, which are designed to fullfill both (I) and (II) in the chapter's introduction.

Breaking down BF further can be done conveniently using sub-simplicial sets, just like we used in the proof of prism filling.



Decomposing $(\Delta^1)^n$ into a colimit involving n! Δ^n's Consider the face maps f? : $\Delta^n \longrightarrow \Delta^{n+1}$

The decomposition The box filling lemma allows us to prove HEP:

The HEP in the last

..H(-,1) and g match on $\partial\Delta^n$, producing a map f : X $\longrightarrow$ Y, where X consists of two copies of Δ^n glued together at the boundary. Consider a space X' formed as a quotient of $\Delta^n \times \Delta^1$ by $\partial\Delta^n \times \Delta^1$. There is a map ϕ : X $\longrightarrow$ X'. An induction hypothesis on f and g involving π_n ensures that the aparent map X $\longrightarrow$ Y lifts along ϕ, producing a map from $\Delta^n \times \Delta^1$ which is constant on $\partial\Delta^n \times \Delta^1$. Stacking this on top of H can be done using an isomorphism between Δ^1 and Δ^1 glued with itself along different endpoints. Altogether this produces a homotopy between f and g.

Imagine

Chapter 11: The Category of Maps of ∞-Groupoids

PART 3: BASED ∞-CATEGORIES

In this first section we prove the standard Whitehead theorem.

Chapter 1: ∞-Cat [1]

Implementation Progress

Lean 11

```lean
/-- A relative CW-complex contains an expanding sequence of subspaces `sk i`
(called the `i`-skeleta) for `i   -1`, where `sk (-1)` is an arbitrary topological space,
isomorphic to `A`, and each `sk (n+1)` is obtained from `sk n` by attaching (n+1)-disks. -/
structure RelativeCWComplex (A : TopCat) where
  /-- Skeleta -/
  sk : ℤ → TopCat
  /-- A is isomorphic to the (-1)-skeleton. -/
  iso_sk_neg_one : A ≅ sk (-1)
  /-- The (n+1)-skeleton is obtained from the n-skeleton by attaching (n+1)-disks. -/
  attach_cells : (n : ℤ) → CWComplex.AttachCells (sk n) (sk (n + 1)) n

/-- A CW-complex is a relative CW-complex whose (-1)-skeleton is empty. -/
abbrev CWComplex := RelativeCWComplex (TopCat.of Empty)
```

Lean 12

```
/-- The topology on a relative CW-complex -/
def toTopCat {A : TopCat} (X : RelativeCWComplex A) :
↪  TopCat :=
  Limits.colimit (colimitDiagram X)

instance : Coe CWComplex TopCat where coe X :=
↪  toTopCat X
```

Lean 13

```
def IsCWComplex (X : TopCat) : Prop := ∃ Y :
↪  CWComplex, Nonempty (↑Y ≅ X)

def CWComplexCat := FullSubcategory IsCWComplex
```

Writing Progress

Here we define CW-complexes, as well as relative CW-complexes, and also the derived categories $D(\infty\text{-Grpd}_0)$ of connected based ∞-groupoids and $D(\infty\text{-Grpd}_0/G_0)$, made from CW-complexes.

20. D(∞-Grpd$_0$)

Symbol	Unicode	VSCode shortcut	Use
		Lean's Kernel	
$\times$	2A2F	\times	Product of types
$\to$	2192	\rightarrow	Hom of types
$\dashv$	22A3	\dashv	The condition that two functors are adjoint
?,¿	003F, 00BF	?,\?	The corresponding (co)monad of an adjunction
$\sim$	223C	\sim	Homotopies

Lean 14

21. $\mathtt{D}(\infty\text{-}\mathtt{Grpd}_0/\mathtt{X}_0)$

The derived category of BASED ∞-groupoids over X_0.

22. $\Omega : \infty\text{-}\mathtt{Grpd}_0 \longrightarrow \infty\text{-}\mathtt{Grpd}$

23. $\Sigma : \infty\text{-}\mathtt{Grpd}_0 \longrightarrow \infty\text{-}\mathtt{Grpd}_0$

The based suspension functor

$$24.\ \ \omega\ \mathtt{f} : \infty\text{-}\mathtt{Grpd}/\mathtt{D}_0 \longrightarrow \infty\text{-}\mathtt{Grpd}/\mathtt{C}_0$$

The homotopy fiber

$$25.\ \ \sigma\, \mathtt{f} : \infty\text{-}\mathtt{Grpd}_0/\mathtt{C}_0 \longrightarrow \infty\text{-}\mathtt{Grpd}_0/\mathtt{D}_0$$

Based homotopy pushout

26. $\pi_n : \infty\text{-}\mathtt{Grpd}_0 \longrightarrow \mathtt{Set}$

The connected components functors

Chapter 2: The Whitehead Theorem

The proof of the Whitehead theorem divides into REP (replacement for BASED ∞-groupoids X : ∞-Grpd$_0$) and HEP (the homotopy extension property for weak equivalent maps of based ∞-groupoids). The replacement functor ∞-Grpd$_0$ can be constructed using globular sets.

Globular sets are not a rich enough invariant for homotopy, but maps of globular sets bear a criticall difference because of

$$\forall(\mathtt{E}:\mathtt{D}(\infty\text{-}\mathtt{Grpd}_0)),\forall(\mathtt{B}:\mathtt{D}(\infty\text{-}\mathtt{Grpd}_0)),\forall(\mathtt{f}:\mathtt{E}\longrightarrow\mathtt{B}),\forall(\mathtt{G}:\mathtt{E}\longrightarrow\mathtt{B}),(\forall(\mathtt{n}:\mathtt{Nat}),(\pi_n\ \mathtt{F}=\pi_n\ \mathtt{G}))\longrightarrow\mathtt{F}=\mathtt{G}$$

27. Globular Sets

The globe category $\mathbb{G}$ is the category

$$\begin{tikzcd} \bullet \arrow[r, shift left, "\sigma_0"] \arrow[r, shift right, "\tau_0"'] & \bullet \arrow[r, shift left, "\sigma_1"] \arrow[r, shift right, "\tau_1"'] & \bullet \arrow[r, shift left, "\sigma_2"] \arrow[r, shift right, "\tau_2"'] & \bullet \arrow[r, shift left, "\sigma_3"] \arrow[r, shift right, "\tau_3"'] & \bullet \arrow[r, shift left] \arrow[r, shift right] & \cdots \end{tikzcd}$$

Globular sets are functors from the opposite category of the globe category $\mathbb{G}$ into the category of sets, and maps of globular sets are natural transformations between them.

In this chapter we prove the following (which we have called Whitehead Theorem (c)): $\forall$(E:D(∞-Grpd$_0$)),$\forall$(B:D(∞-Grpd$_0$)),$\forall$(f:E $\longrightarrow$ B),$\forall$(G:E $\longrightarrow$ B),($\forall$(n:Nat),(π_n F $= \pi_n$ G)) $\longrightarrow$ F = G, where π_n is notation for π n.

This can be shown using CW-replacement and induction on n. Fibrant replacement of an object X entails replacing an object in ∞-Grpd$_0$ with a CW-object (an object made by successively glueing in higher and higher simplices along their boundaries obtaining a sequence X_n). Given an equality $\pi_{n+1}(f) = \pi_{n+1}(g)$ and a homotopy equivalence $h_n : \Delta^1 \times X_n \longrightarrow Y$ between $f|_{X_n}, g|_{X_n} : X_n \longrightarrow Y$, we construct an extension of the homotopy equivalence $\Delta^1 \times X_{n+1} \longrightarrow Y$.

Spheres and balls Next we turn to defining spheres and balls:

	Spheres and Balls	
Name of the X value	$\partial X \cong S^n$	$X \cong D^n$
p-norm unit ball for p = 1	∂B(1,1)	B(1,1)
p-norm unit ball for 1 < p < 2	∂B(p,1)	B(p,1)
p-norm unit ball for p = 2	∂B(2,1)	B(2,1)
p-norm unit ball for 2 < p < ∞	∂B(p,1)	B(p,1)
p-norm unit ball for p = ∞	∂B(∞,1)	B(∞,1)
The n-simplex	$\partial\Delta^n$	Δ^n

Definition 10. ...

Theorem 13. $i^1 : S^0 \longrightarrow D^1$

Theorem 14. $D^n \times D^1 \longrightarrow D^{n+1}$

Definition 11. $\mathtt{D}^{\mathtt{n}} \longrightarrow \mathtt{D}^{\mathtt{m}}$

Theorem 15. Fix n : $\mathbb{N}$ and let $\partial^{\mathtt{n}} : \mathtt{S}^{\mathtt{n}} \longrightarrow \mathtt{D}^{\mathtt{n+1}}$ be the inclusion. The pushout of the following diagram is isomorphic to $\mathtt{S}^{\mathtt{n+1}}$:

$$\begin{tikzcd}
S^n \arrow[r] \arrow[d] & D^{n+1} \\
D^{n+1} &
\end{tikzcd}$$

Proof. □

Theorem 16. Define a function $||\text{-}||_2 : \mathtt{D}^{\mathtt{n}} \longrightarrow \mathtt{I}$ sending $(x_1,...,x_n)$ to $\sqrt{\sum_{i=1}^{n} x_i^2}$, and write $||\text{-}||_2$

Proof. ... □

28. HEP for BASED ∞-groupoids

In this section we prove the homotopy extension property for BASED ∞-groupoids, which we here model as CW-complexes.

Jar filling Next we turn to defining 'jar shapes' $\mathtt{J^n}$, which include into $\mathtt{D^n} \times \mathtt{I}$ $\mathtt{i}_n : \mathtt{J^n} \longrightarrow \mathtt{D^n} \times \mathtt{I}$, after which we 'fill' them (i.e. demonstrate that any continuous map $\mathrm{f} : \mathrm{J^n} \longrightarrow \mathrm{X}$ extends to a continuous map $\mathrm{g} : \mathrm{D^n} \times \mathrm{I} \longrightarrow \mathrm{X}$).

The first and most common approach involves 'shining a light ray down from above the jar', i.e. projection. We obtain a formula for .

The second way to fill the jar

Change of Base Jar filling leaves the question

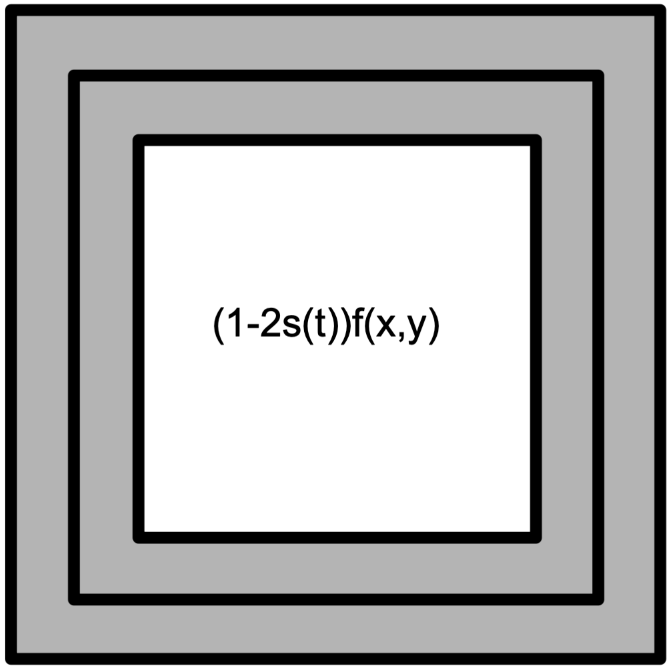
Definition 12. Let X_{-1} be a connected CW-complex and let n : Nat be a natural number. The transport function $\mathtt{trans\ n\ X}_{\mathtt{1}} : (\mathtt{f} : [\mathtt{I},\mathtt{X}_{\mathtt{1}}]) \to \pi_n\ (\mathtt{f\ 0}) \longrightarrow \pi_n\ (\mathtt{f\ 1})$ is

Theorem 17. Let X_{-1} be a connected CW-complex and let $\mathrm{f} : \mathrm{I} \longrightarrow \mathrm{X}_{-1}$ be a path, so that $(\mathtt{trans\ n\ X}_{\mathtt{1}}\ \mathtt{f}^{-1}) \bullet (\mathtt{trans\ n\ X}_{\mathtt{1}}\ \mathtt{f})$ has type $\pi_n\ (\mathtt{f\ 0}) \longrightarrow \pi_n\ (\mathtt{f\ 0})$. Then

$$(\mathtt{trans\ n\ X}_{\mathtt{1}}\ \mathtt{f}^{-1}) \bullet (\mathtt{trans\ n\ X}_{\mathtt{1}}\ \mathtt{f}) = \mathtt{1\_}(\pi_n\ (\mathtt{f\ 0}))$$

Proof. ... $\square$

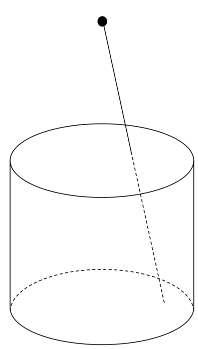
The proof in the above can be depicted like so, as a 'painting with two concentric frames':



that the based CW-complexes (X_{-1},x) and (X_{-1},y) are

Theorem 18.

Proof. ... □



29. REP for based ??? ∞-categories

In this section we use the notion of globular sets to replace a topological space with a CW-complex. Together with HEP (homotopy extension), this will complete the proof of the Whitehead theorem.

30. The Whitehead theorem

Here we show the Whitehead theorem.

Chapter 3: The Category of Maps

In this section I would like to

PART 4: ∞-CATEGORIES

Chapter 13: ∞-Cat

This chapter and the next chapter are more technical and difficult than the rest of the book.

1. Defining D(∞-Cat) by formally inverting weak equivalences.
2. Defining D(∞-Cat/C) by formally inverting weak equivalences.
3. Defining a fibrant replacement functor for ∞-Cat
4. Defining a fibrant replacement functor for ∞-Cat/C
5. We first construct both the category D(∞-Cat) and, for each C : D(∞-Cat), the category D(∞-Cat/C) by formally inverting weak equivalences in the category of quasicategories and the category of quasicategories over C.

31. Ω

Our choice of symbols refects our choice of three variations of the Whitehead theorem and three Puppe sequences. $\vec{\Omega}$, the analogue of loop space, is the internal hom functor $[\Delta^1,\text{-}] : \infty\text{-Cat} \longrightarrow \infty\text{-Cat}$. This is not hard to construct, with the main lemma being that the path space of a quasicategory has the quasicategory lifting conditon.

We will be interested in one formal model of D(∞-Cat) which consists of formal compositions $f_1 \bullet g_1 \bullet f_2 \bullet g_2 \bullet \cdots \bullet f_n \bullet g_n$, where $g_n : Dom(f_{n+1}) \longrightarrow$??? is a weak equivalence, and something similar for D(∞-Cat). However, it is still vital to have the replacement functor `repl`, which ensures the Whitehead theorem for particular ∞-categories which are constructed out of attaching maps.

32. ω

$\vec{\Omega}$ is to internal categories as $\vec{\omega}$ is to internal C-presheaves. It is also called directed homotopy pullback. These functors will later be used to produce functors $\vec{\mathrm{P}} : \mathrm{D}(\infty\text{-Cat}) \longrightarrow \mathrm{InternalCategory}\ \mathrm{D}(\infty\text{-Cat})$ and $\vec{\mathrm{p}} : \mathrm{D}(\infty\text{-Cat}/\mathrm{C}) \longrightarrow \mathrm{InternalPresheaf}\,(\vec{\mathrm{P}}\ \mathrm{C})\ \mathrm{D}(\infty\text{-Cat}/\mathrm{C})$.

33. Π_n

The mentioned functors $\vec{\Pi}_n$ are designed with both Whitehead theorem (a) and Puppe sequence (a) in mind.

Chapter 14: The Whitehead Theorem for ∞-Categories

34. Directed Cubical Complexes

...

In this chapter, we take on the objective of Whitehead theorem (a), out of which we will prove the other more concrete Whitehead theorems:

$$\forall(\mathrm{E{:}D}(\infty\text{-Cat})),\forall(\mathrm{B{:}D}(\infty\text{-Cat})),\forall(\mathrm{F{:}E} \longrightarrow \mathrm{B}),\forall(\mathrm{G{:}E} \longrightarrow \mathrm{B}),(\forall(\mathrm{n{:}Nat}),(\vec{\Pi}_n\ \mathrm{F} = \vec{\Pi}_n\ \mathrm{G})) \longrightarrow \mathrm{F} = \mathrm{G}$$

We can attempt to form a slightly different category, much like the above, called $\mathcal{D}(\infty$-Cat), at first, and in a formal way, so as to create a category whose object component $\mathcal{D}(\infty$-Cat).α matches the object component ∞-Cat.α while featuring the above theorem in a formal way. However, with this as our model of D(∞-Cat), we may then also be interested in the establishment of a model in which the Whitehead theorem is demonstrated, with the main idea being to prove two complementary concepts:

1. (REP) Establish a kind of "weak equivalent fibrant replacement" R : ∞-Cat.$\alpha \longrightarrow \infty$-Cat.$\alpha$ (.α gives the object component in Mathlib's category theory library), analogous to CW-complex replacement in Whitehead's original paper. It's especially nice if R forms the object component of a functor F : ∞-Cat $\longrightarrow \infty$-Cat. D(F) : D(∞-Cat) $\longrightarrow$ D(∞-Cat) should be a categorical equivalence, and that is what we will do.

2. (HEP) For the object R X, demonstrate that any F,G : (R X) $\longrightarrow$ Y such that $\forall$(n:Nat),($\vec{\Pi}_n$ F = $\vec{\Pi}_n$ G), there is a directed homotopy equivalence between F and G. Note that "directed homotopy equivalence" consists of a composible sequence of simple directed homotopies H⍰ : $\Delta^1 \times$ (R X) $\longrightarrow$ Y, 1 ⍰ i ⍰ n, with even H⍰ running reverse to the odd H⍰.

Both of these will use induction on Lean's `Nat`. The first of these could be called a REP (for REplacement Property, but this isn't usual terminology), and the second typically uses induction and a HEP (Homotopy Extension Property). Our REPa will consist of objects made out of particular kinds of pushouts called attaching maps, and can be made functorial. Proving the HEPa can be done by well-order induction on the attaching maps present in our choice of R, thereby reducing to the case of extending a homotopy along a single attachment.

Our HEPa (directed box filling) is similar to the HEP shown in Whitehead's original paper, and to the approach detailed in Hatcher's textbook, though no doubt modified to suit our two goals:

(I) The analogue of the Puppe sequence on the front cover needs to hold.

(II) The first Whitehead theorem on the front cover needs to hold.

These two considerations determine our choice of $\vec{\Pi}_n$, $\vec{\Omega}$, and $\vec{\omega}$. We take $\vec{\Omega}$ to be (simply) the internal hom functor [Δ^1,-] (which requires showing that $\vec{\Omega}$X has the inner-horn filling condition). $\vec{\omega}$ is then defined as a certain pullback of $\vec{\Omega}$, and $\vec{\Pi}_n$ is designed to produce a Puppe sequence with a meaningful notion of exactness by which we can demonstrate the goal of recognition theorems (i) and (ii). Specifically, it makes sense to use cubes in our definition of $\vec{\Pi}_n$ because of how they are representing objects of $\vec{\Omega}^n$. Meanwhile, it is also clear that the quotient producing $\vec{\Pi}_n$ is subtle in exactly how it requires fixing the endpoints of a sequence of alternating directed homotopies. We will define $\vec{\Pi}_n$'s by identifying those objects x, y: $\vec{\Omega}^n$ X which are homotopic by a homotopy which restricts to a constant along the face maps f⍰ : $\vec{\Omega}^{n-1}$ X $\longrightarrow$ $\vec{\Omega}^{n-1}$ X (which correspond to Maps (n,b), where b : Bool).

Imagine for a moment the picture of a square shaped cusion; we might make such a cusion by first soeing together 6 squares of cloth and filling it with material, then "soeing the walls down to a square". Here we go with this:

1. Define a n-cubical cusion using the boundary of an n-1 cube times Δ^1, i.e. the quotient of $(\Delta^1)^{n-1} \times \Delta^1$ by an equivalence relation, but we have to start our model somewhere), or perhaps more easily the pushout of f : $\Delta^1 \times (\partial((\Delta^1)^n))$ $\longrightarrow (\Delta^1)^{n+1}$ by the projection map $\Delta^1 \times (\partial((\Delta^1)^n)) \longrightarrow \partial((\Delta^1)^n)$

2. Define a simplicial cusion using the boundary of an n-1 simplex times Δ^1, i.e. the quotient of (Δ^1) by an equivalence relation, or perhaps more easily the pushout of f : $\Delta^1 \times (\partial(\Delta^n)) \longrightarrow (\Delta^1) \times \Delta^n$ by the projection map $\Delta^1 \times (\partial((\Delta^1)^n)) \longrightarrow$ $\partial(\Delta^n)$

The boundary of a cusion is a pouch, isomorphic to a pushout of two cubes glued together at their boundaries:

1. Define a n-cubical pouch as the pushout of two boundary maps $\partial((\Delta^1)^n) \longrightarrow$ $(\Delta^1)^n$

2. Define a simplicial pouch as the pushout of two boundary maps $\partial(\Delta^n) \longrightarrow \Delta^n$

Notice that paths in $\vec{\Omega}^n$X produce paths in $\vec{\Omega}^{n-1}$X in as many ways as there are face maps $(\Delta^1)^{n-1} \longrightarrow \Delta^{1n}$, these could be called restrictions and are no doubt related to the pouches and cusions we just defined. The cartesian closed structure on simplicial sets with the lifting condition clarifies the relationship between the two available definitions of $\vec{\Pi}_n$:

1. Homotopies of maps from a cube which are constant on the boundary
2. Paths of maps in $\vec{\Omega}^{n-1}$X which produce constant maps under the mentioned restritions.
3. Maps from a pouch mod an equivalence relation (really we phrase this as a pushout!), namely the equivalence relation in which any two maps from a pouch that extend to maps from a cusion are identified.

After we construct $\vec{\Pi}_n$ in the first section, we will be in a place to demonstrate that the natural transformation `weak_equivalence : repl` $\longrightarrow$ `(𝟙 ∞-Cat)` consists of weak equivalences (a fact which we call REP, which is short for REplacement Principal). This is covered in the section titled REP, which also constructs `repl` and `weak_requivalence`.

In sum, the goal of the present chapter is to use similar insights to the proof of the Whitehead theorem featured Hatcher's textbook to prove Wa and Pa for the model of quasicategories, using Mathlib's predefined horns and simplices in its simplicial sets section. The main difference is that our work must take care to respect the directed nature of quasicategories.

1. Defining `repl`
2.

35. REP

We have divided the work of proving Whitehead theorem (a) into two steps: REP and HEP. In this section, we construct a functor `repl : ∞-Cat ⟶ ∞-Cat` along with a natural transformation `weak_equivalence : repl ⟶ (𝟙 ∞-Cat)`. To construct `repl`

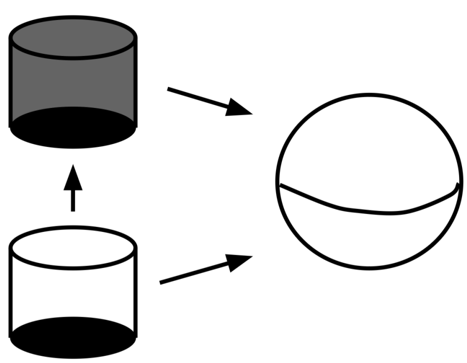
36. HEP

Consider the context of , supposing that we have constructed a homotopy ... This gives a picture that is a bit like "filling up a jar": a homotopy h : of f, g : $\partial\Delta^2 \longrightarrow$ Y, along with the value of g on Δ^2, produces a "jar" shape in Y, which can be "filled up" to produce a homotopy $h : \Delta^1 \times \Delta^2 \longrightarrow Y$. This is easier for simplicial-based approaches than for point-set topological approaches, the latter of which needs extra steps that deform a map into a cellular map.

This construction, in the case of point set topology, often involves first deforming maps so as to be cellular; however our analogue of CW complexes allows us to skip this step.

This construction (HEP for quasicategories) may even be equivalent to the quasicategory lifting condition if we are lucky. It is also the main technical device allowing for our concrete choice of model (quasicategories).

In this section, we demonstrate this extension property and use it to conclude the Whitehead theorem for ∞-categories stated above.

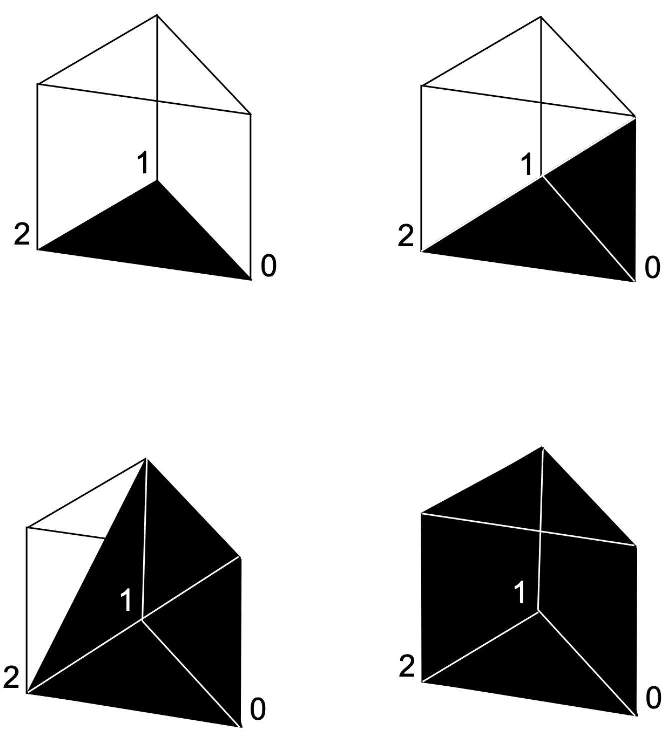


Prism Filling (PF) Let Y be a quasicategory, and let $f, g : \partial\Delta^n \longrightarrow Y$. A homotopy $h : \partial\Delta^n \times \Delta^1 \longrightarrow Y$ between $f, g : \partial\Delta^n \longrightarrow Y$ extends to a map $H : \Delta^n \times \Delta^1 \longrightarrow Y$;

this follows from the condition that Y be a quasicategory. H(-,1) and g match on $\partial\Delta^{n}$, producing a map f : X $\longrightarrow$ Y, where X consists of two copies of Δ^{n} glued together at the boundary. Consider a space X' formed as a quotient of $\Delta^{n} \times \Delta^{1}$ by $\partial\Delta^{n} \times \Delta^{1}$. There is a map ϕ : X $\longrightarrow$ X'. An induction hypothesis on f and g involving π_n ensures that the aparent map X $\longrightarrow$ Y lifts along ϕ, producing a map from $\Delta^{n} \times \Delta^{1}$ which is constant on $\partial\Delta^{n} \times \Delta^{1}$. Stacking this on top of H can be done using an isomorphism between Δ^{1} and Δ^{1} glued with itself along different endpoints. Altogether this produces a homotopy between f and g.

Directed prism filling may combine fruitfully with the yoneda lemma and/or the fact that simplicial sets are determined by the sets [Δ^{n},X] along with combinatorial information (face and degeneracy maps).

Decomposing $\Delta^{n} \times \Delta^{1}$ into a colimit involving n+1 Δ^{n+1}'s ...



In the above, it may be easier if we make use of sub-simplicial sets and prove the theorem using that colimit applied to a natural isomorphism of diagrams products an isomorphism.

The decomposition

A definition of $\vec{\Pi}_n$ which is consistent with our goals of Wa and Pa is one as a certain pushout involving ($\vec{\Omega}^{n}$ X)– one which amounts to taking an equivalence relation by paths in $\vec{\Omega}^n$ X which restrict to constant paths along the face maps f⍰ : $\vec{\Omega}^{n-1}$ X $\longrightarrow$ $\vec{\Omega}^n$ X. Here, $\vec{\Omega}$ is easy to define in the model of quasi-categories, and it amounts . Besides fullfilling our goal of the first Whitehead theorem and puppe sequence, this definition of $\vec{\Pi}_n$ strikes me as elegant because it uses all of the ways for $\vec{\Omega}^n$ X to map into $\vec{\Omega}^{n+1}$ X.

The next symbols in the project's "periodic table" that we construct, after $\vec{\Omega}$ and $\vec{\Pi}_n$, will be $\vec{B}$ and $\vec{E}$, which we feature in the chapter on Puppe sequence (a).

A useful thing for us to construct first is the boundary of a product of Δ^1's and the boundary of a directed simplex. We might even like to expand on this later, but for now just consider for a moment how each might be made out of a glueing construction involving face maps.

Even though the $\vec{\Pi}_n$'s can be defined using $\vec{\Omega}^n$ X and various face maps f_(n, b) : $\vec{\Omega}^{n-1}$ X $\longrightarrow$ $\vec{\Omega}^n$ X for $b : \{0, 1\}$, it may be nice to have this as a result, with the definition one featuring two cubes glued together along their boundary.

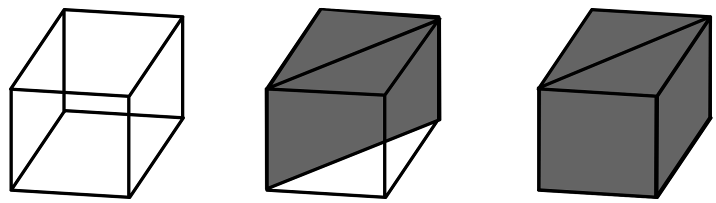
This means that we want directed box filling in addition to directed prism filling (but which also uses directed prism filling in its proof).

Box Filling (BF) Let Y be a quasicategory, and let f, g : $\partial\Delta^n \longrightarrow$ Y. A homotopy h : $\partial\Delta^n \times \Delta^1 \longrightarrow$ Y between f, g : $\partial\Delta^n \longrightarrow$ Y extends to a map H : $\Delta^n \times \Delta^1 \longrightarrow$ Y; this follows from the condition that Y be a quasicategory. H(-,1) and g match on $\partial\Delta^n$, producing a map f : X $\longrightarrow$ Y, where X consists of two copies of Δ^n glued together at the boundary. Consider a space X' formed as a quotient of $\Delta^n \times \Delta^1$ by $\partial\Delta^n \times \Delta^1$. There is a map ϕ : X $\longrightarrow$ X'. An induction hypothesis on f and g involving π_n ensures that the aparent map X $\longrightarrow$ Y lifts along ϕ, producing a map from $\Delta^n \times \Delta^1$ which is constant on $\partial\Delta^n \times \Delta^1$. Stacking this on top of H can be done using an isomorphism between Δ^1 and Δ^1 glued with itself along different endpoints. Altogether this produces a homotopy between f and g.

This goes hand-in-hand with a definition of $\vec{\Pi}_n$ which suits (I) and (II) in the introduction to chapter (3). If we make sure to prove lemmas...

The box filling and prism filling HEPs can be extended to the case of attaching all cells of a particular fixed dimension and as indexed by simplicial set arising from a set (or Lean 4 `Type`). That is, we might like to extend $\times$ () (or possibly somehow a `Set` as well), and that we may find an interest in the following two definitions of $\vec{\Pi}_n$, which are designed to fullfill both (I) and (II) in the chapter's introduction.

Breaking down BF further can be done conveniently using sub-simplicial sets, just like we used in the proof of prism filling.



Decomposing $(\Delta^1)^n$ into a colimit involving $n!$ Δ^n's Consider the face maps f? : $\Delta^n \longrightarrow \Delta^{n+1}$

The decomposition The box filling lemma allows us to prove HEP:

37. The Whitehead Theorem for ∞-Cat

The HEP in the last

..H(-,1) and g match on $\partial\Delta^n$, producing a map $f : X \longrightarrow Y$, where X consists of two copies of Δ^n glued together at the boundary. Consider a space X' formed as a quotient of $\Delta^n \times \Delta^1$ by $\partial\Delta^n \times \Delta^1$. There is a map $\phi : X \longrightarrow X'$. An induction hypothesis on f and g involving π_n ensures that the aparent map $X \longrightarrow Y$ lifts along ϕ, producing a map from $\Delta^n \times \Delta^1$ which is constant on $\partial\Delta^n \times \Delta^1$. Stacking this on top of H can be done using an isomorphism between Δ^1 and Δ^1 glued with itself along different endpoints. Altogether this produces a homotopy between f and g.

Imagine

Chapter 15: The Category of Maps of ∞-Categories

…

PART 5: CATEGORIES AND E^k-CATEGORIES

In this section we establish the following notions:

internal operad, enriched operad, internal operoid, and enriched operoid. These structures pertain to categories in which one can

Strict/Lax	Category	Operoid		
Internal/Enriched	Internal	Enriched	Internal	Enriched
C.Obj ≅ terminal_object ℂ				
C.Obj ≅ terminal_object ℂ				

PART 6: MODEL STRUCTURES

38. ... ∞-Grpd

1. γ_(Cat) $\to$_(Cat) - : Cat.Hom Cat Cat is an endofunctor of Cat.
2. The colimit of Φ_n := (γ_(Cat) $\to$_(Cat) -)$^{\mathrm{n}}$ under the inclusions which use identity maps produces a category C, and the functor form C to the colimit of a natural transformation from Φ_n to itself is γ_(Cat) $\to$_(Cat) C.
3. Call the new category ⍰ C
4. There is a functor from based objects in C to ⍰ C which is the composition of * C $\to$ Maps C $\to$ Maps ⍰ C $\to$ ⍰ C
5. There is a functor from Maps C
6. There is a functor from
7. There is a functor from
8. The category of presheaves in ∞-Grpd out of the infinite box (Nat $\to$_(Cat) γ_(Cat)) is
9. (Nat $\to$_(Cat) γ_(Cat) $\to$_(Cat) ∞-Grpd...
10.

39. ... ∞-Cat

1. (Nat $\rightarrow$_(Cat) γ_(Cat)) $\rightarrow$_(Cat) ∞-Cat

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10.

Some lectures, videos, and Stackexchange questions:

1. `https://www.youtube.com/watch?v=Ob9tOgWumPI`

2. `https://www.youtube.com/watch?v=xYenPIeX6MY`

3. `https://mathoverflow.net/questions/5901/do-the-signs-in-puppe-sequences-matter`

Ideas for future applications:

1. `https://arxiv.org/pdf/2206.13563.pdf`

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