Three Whitehead Theorems and Three Puppe Sequences

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Whitehead Theorem

```
∀X : D(∞-Grpd₀), ∀Y : D(∞-Grpd₀),
∀f : X → Y, ∀g : X → Y,
(∀n : Nat, πₙ f = πₙ g) → f = g
```

Puppe Sequence

$$\cdots \to \pi_1.\mathrm{obj}\ E_0 \to \pi_1.\mathrm{obj}\ B_0 \to$$
$$\pi_0.\mathrm{obj}\ ((\mathbb{1}\ B_0) \bullet ((\omega.\mathrm{hom}\ (\mathbb{1}\ B_0)).\mathrm{hom}\ f)) \to$$
$$\pi_0.\mathrm{obj}\ (E_0) \to \pi_0.\mathrm{obj}(B_0)$$

- The homotopy fiber

Definition of ∞-category

```
def horn_filling_condition (X : SSet) (n i : Nat): Prop :=
  ∀ f : Λ[n, i] → X, ∃ g : Δ[n] → X,
  f = SSet.hornInclusion n i ≫ g

/-- A simplicial set is called an ∞-category
if it has the extension property for all inner horn inclusions
`Λ[n, i] → Δ[n]`, n ≥ 2, 0 < i < n. -/
def InfCategory := {X : SSet //
  ∀ (n i : Nat),
  n ≥ 2 ∧ 0 < i ∧ i < n → horn_filling_condition X n i}

#check (inferInstance : Category SSet) -- OK
#check (inferInstance : Category InfCategory) -- fail
```

Automatic typeclass inference?

```
-- instance : Category InfCategory := inferInstance -- ?

-- instance : Category InfCategory := by -- ?
--   dsimp only [InfCategory]
--   infer_instance

instance : Category InfCategory where
  Hom X Y := NatTrans X.1 Y.1
  id X := NatTrans.id X.1
  comp α β := NatTrans.vcomp α β

#check (inferInstance : Category SSet) -- OK
#check (inferInstance : Category InfCategory) -- OK
```

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