

# The Whitehead Theorem and Two Variations

$$\forall (C:D(\infty\text{-Cat})), \forall (D:D(\infty\text{-Cat})), \forall (F:C \rightarrow D), \forall (G:C \rightarrow D), (\forall (n:\text{Nat}), (\vec{\pi}_n F = \vec{\pi}_n G)) \rightarrow F = G$$
$$\forall (X:D(\infty\text{-Grpd})), \forall (Y:D(\infty\text{-Grpd})), \forall (f:X \rightarrow Y), \forall (g:X \rightarrow Y), (\forall (n:\text{Nat}), (\vec{\pi}_n f = \vec{\pi}_n g)) \rightarrow f = g$$
$$\forall (X:D(\infty\text{-Grpd}_0)), \forall (Y:D(\infty\text{-Grpd}_0)), \forall (f:X \rightarrow Y), \forall (g:X \rightarrow Y), (\forall (n:\text{Nat}), (\pi_n f = \pi_n g)) \rightarrow f = g$$

Plans to prove three variations of the  
Whitehead theorem of homotopy groups in  
Lean 4, with extensive use of Mathlib 4







We wish to acknowledge the collaborative efforts of E. Dean Young and Jiazhen Xia. Dean Young initially formulated the introduction with twelve goals, posting them on the Lean Zulip in August of 2023. Together the authors are pursuing these plans as a long term project.

# 1. Contents

The table of contents below reflects the tentative long-term goals of the authors, with the main goal the pursuit of the Whitehead theorem for a point-set model involving Mathlib's predefined homotopy groups.

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Unfinished	
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$??? : \infty\text{-Cat}_0 \rightarrow \infty\text{-Cat}_0$	The based suspension functor
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$??? f : \infty\text{-Cat}_0/C_0 \rightarrow \infty\text{-Cat}_0/D_0$	Based homotopy pushout
$??? : \infty\text{-Cat}_0 \rightarrow \text{Set}$	The connected components functors
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???	...
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REP for based connected $\infty$ -categories	The replacement functor on $\infty\text{-Cat}_0$
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$\bar{\Sigma} : \infty\text{-Grpd} \rightarrow \infty\text{-Grpd}$	The unbased suspension functor
$\vec{\omega} f : \infty\text{-Grpd}/D \rightarrow \infty\text{-Grpd}/C$	The directed homotopy pullback functor
$\vec{\sigma} f : \infty\text{-Grpd}/C \rightarrow \infty\text{-Grpd}/D$	Homotopy pushout with a point
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$\bar{\Omega} : \infty\text{-Cat} \rightarrow \infty\text{-Cat}$	The directed path space functor
$\bar{\Sigma} : \infty\text{-Cat} \rightarrow \infty\text{-Cat}$	The directed unbased suspension
$\vec{\omega} f : \infty\text{-Cat}/D \rightarrow \infty\text{-Cat}/C$	The directed homotopy pullback functor
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OperadicPresheaf : $\mathbf{Cat} \rightarrow \mathbf{Cat}$	The category of operadic presheaves
The P and O Functors	$P = ? \bullet 0$
The p and o Functors	$p = ? \bullet o$



## 2. Introduction

The main goal of this repository is to prove the Whitehead theorem in Lean 4 using Mathlib 4's homotopy groups. Two other subsequent goals are to state and prove two variations of the Whitehead theorem. It is important that initial pull requests stemming from our work remain basic and accessible; we hope to make progress which is gradual and incremental.

Besides this goal, we have two others. Here are the three Whitehead Theorems which form our main three goals:

- (a) (The Whitehead theorem)  $\forall(E:D(\infty\text{-Grpd}_0)), \forall(B:D(\infty\text{-Grpd}_0)), \forall(F:E \longrightarrow B), \forall(G:E \longrightarrow B), (\forall(n:\text{Nat}), (\pi_n F = \pi_n G)) \rightarrow F = G$ , where  $\pi_n$  is notation for  $\pi_n$ .
- (b) (The Whitehead theorem for  $\infty$ -groupoids)  $\forall(E:D(\infty\text{-Grpd})), \forall(B:D(\infty\text{-Grpd})), \forall(F:E \longrightarrow B), \forall(G:E \longrightarrow B), (\forall(n:\text{Nat}), (\vec{\pi}_n F = \vec{\pi}_n G)) \rightarrow F = G$ , where  $\vec{\pi}_n$  is notation for  $\vec{\pi}_n$ .
- (c) (The Whitehead theorem for  $\infty$ -categories)  $\forall(E:D(\infty\text{-Cat})), \forall(B:D(\infty\text{-Cat})), \forall(F:E \longrightarrow B), \forall(G:E \longrightarrow B), (\forall(n:\text{Nat}), (\vec{\pi}_n F = \vec{\pi}_n G)) \rightarrow F = G$ , where  $\vec{\pi}_n$  is notation for  $\vec{\pi}_n$ .

Here are a few comments on these theorems:

1. The first Whitehead theorem reflects the known one, which dates back to a paper that can be found in defined on Mathlib 4's homotopy groups, which make use of cubes  $I^n$  and their boundaries  $\partial I^n$ . There is also a fourth Whitehead theorem concerning  $\infty\text{-Cat}_0$ , but which is omitted; it is less general than the third while the first is present for its use of Mathlib 4's predefined  $\pi_n$ 's.
2. The unicode symbol  $\longrightarrow$  (U+2B62) reflects a notation not present in Mathlib 4
- 3.

These are stated in an order reversed from the order of its implementation. We have omitted in this project the proof that CW-complexes are compactly generated weak hausdorff topological spaces and that the category of compactly generated weak hausdorff topological spaces is cartesian closed with internal hom given by the compact-open topology. The compact-open topology will instead be covered in a future repository concerning geometric maps.

We will use two models of each of the following categories in the theorems above:

- (i) We model  $\infty\text{-Cat} : \text{Cat}$  firstly as the category of categories enriched over a convenient category of topological spaces, and secondly as the category of quasicategories.
- (ii) We model  $\infty\text{-Grpd} : \text{Cat}$  firstly as a convenient category of topological spaces, and secondly as the category of Kan complexes.
- (iii) We model  $\infty\text{-Grpd}_0 : \text{Cat}$  firstly as the based connected objects of a convenient category of topological spaces, and secondly as the category of based connected Kan complexes.

This choice accords with the standard approach to the third theorem, in which one typically chooses both a combinatorial and point-set model, with the former featuring a geometric realization functor into the latter (`Mathlib` already has this).

We will make heavy use of `Mathlib` 4's material on category theory, particularly their categories, functors, and natural transformations:

1. Categories (see `Mathlib's Category` X here; these can be bundled into `category`)
2. Functors (see `Mathlib's Functor` C D here; these can be bundled into `functor`)
3. Natural transformations (see `Mathlib's NatTrans` F G here; these can be bundled into `natural_transform`)
4. Equations between natural transformations (see `Mathlib's NatExt` here; these are related to our `equation`)

While the functors  $\pi_n$  occurring in the main theorems above are already defined in `Mathlib` for the desired point-set model, the functors  $\vec{\pi}_n$  and  $\overleftarrow{\pi}_n$  are not, and their definition will require great care. Here are their types:

- (i)  $\vec{\pi}_n : \text{Functor } \infty\text{-Cat } \text{Set}$
- (ii)  $\overleftarrow{\pi}_n : \text{Functor } \infty\text{-Grpd } \text{Set}$
- (iii)  $\pi_n : \text{Functor } \infty\text{-Grpd}_0 \text{ Set}$

We may wish to modify these types out of convenience and to accord with the pre-existing functors  $\pi_n$  in `Mathlib` 4.

The existence of a base point makes  $\pi_n$  relatively straightforward to define, while  $\vec{\pi}_n$  and  $\overleftarrow{\pi}_n$  'grow' as  $n$  does. We also form their derived functors:

- (i)  $D(\vec{\pi}_n) : D(\infty\text{-Cat}) \longrightarrow D(\text{Set}) \simeq \text{Set}$
- (ii)  $D(\overleftarrow{\pi}_n) : D(\infty\text{-Grpd}) \longrightarrow D(\text{Set}) \simeq \text{Set}$
- (iii)  $D(\pi_n) : D(\infty\text{-Grpd}_0) \longrightarrow D(\text{Set}) \simeq \text{Set}$

In the course of the repository we will need the directed path space, path space, and loop space functors as well, which fit with the analogy formed by the Whitehead theorem and its two variations:

1.  $\vec{\Omega} : \infty\text{-Cat} \longrightarrow \infty\text{-Cat}$  sends
2.  $\vec{\Omega} : \infty\text{-Grpd} \longrightarrow \infty\text{-Grpd}$  is ...
3.  $\Omega$  is the loop space functor

We develop two models of  $(\infty, 1)$ -categories, which we call model  $\vec{A}$  and model  $\vec{B}$ . Model  $\vec{B}$  features a deformation retraction  $\vec{\gamma} \rightrightarrows \vec{\gamma}^2$ .

The third theorem (c), is the one from Whitehead's original papers.

With the choice of quasicategories as a combinatorial model, we hope to give good integration with Mathlib's existing features (though technically only the inner horns and simplices are defined, not even the category of quasicategories itself).

In the directed context, a homotopy between two maps in  $\infty\text{-Cat}/\mathcal{C}$  consists of a sequence of compatible directed homotopies with the odd morphisms in the sequence formed from reversed copies of  $\Delta^1$ . Really we have two such categories, one of which consists of formal words, and another which involves  $\infty$ -categories and  $\infty$ -functors in the image of `repl`).

The main technical feature in the proofs of these theorems concerns a lifting property which successively lifts a homotopy along a single attachment of  $\Delta^n$  along its boundary  $\partial\Delta^n$ . A homotopy  $h : \partial\Delta^n \times \Delta^1 \longrightarrow Y$  between  $f, g : \partial\Delta^n \longrightarrow Y$  extends to a map  $H : \Delta^n \times \Delta^1 \longrightarrow Y$ . The directed case requires an extra technical feature.  $H(-, 1)$  and  $g$  match on  $\partial\Delta^n$ , producing a map  $f : X \longrightarrow Y$ , where  $X$  consists of two copies of  $\Delta^n$  glued together at the boundary.

Consider a space  $X'$  formed as a quotient of  $\Delta^n \times \Delta^1$  by  $\partial\Delta^n \times \Delta^1$ . There is a map  $\phi : X \longrightarrow X'$ . An induction hypothesis on  $f$  and  $g$  involving  $\pi_n$  ensures that the aparent map  $X \longrightarrow Y$  lifts along  $\phi$ , producing a map from  $\Delta^n \times \Delta^1$  which is constant on  $\partial\Delta^n \times \Delta^1$ . Stacking this on top of  $H$  can be done using an isomorphism between  $\Delta^1$  and  $\Delta^1$  glued with itself along different endpoints. Altogether this produces a homotopy between  $f$  and  $g$ .

We will define three different kinds of derived category:

1.  $D(\infty\text{-Cat}) : \text{Cat}$  (the directed derived category of  $\infty$ -categories)
2.  $D(\infty\text{-Grpd}) : \text{Cat}$  (the derived category of  $\infty$ -groupoids)

3.  $D(\infty\text{-Grpd}_0) : \text{Cat}$  (the derived category of based  $\infty$ -groupoids)

We then create the second kind of derived category, one for each of the objects in the respective categories above:

1. For  $C : D(\infty\text{-Cat})$ , a category  $D(\infty\text{-Cat}/C) : \text{Cat}$
2. For  $G : D(\infty\text{-Grpd})$ , a category  $D(\infty\text{-Grpd}/G) : \text{Cat}$
3. For  $G_0 : D(\infty\text{-Grpd}_0)$ , a category  $D(\infty\text{-Grpd}_0/G_0) : \text{Cat}$

For the model built on simplicial sets,  $\vec{\Omega}$  will be representable by  $\Delta^1$  with respect to an internal hom, and  $\vec{\Omega}$  will be representable by a model of the unit interval  $I := [0,1]$ .

We will use six (strict) “internal” structures in addition to the standard structures in category theory:

- (i)  $\text{InternalCategory} : \text{Cat} \rightarrow \text{Cat}$
- (ii)  $\text{InternalPresheaf} : (X : \text{Cat}) \rightarrow (C : (\text{InternalCategory } X)) \rightarrow \text{Cat}$
- (iii)  $\text{InternalGroupoid} : \text{Cat} \rightarrow \text{Cat}$
- (iv)  $\text{InternalGroupoidAction} : (X : \text{Cat}) \rightarrow (G : (\text{InternalGroupoid } X)) \rightarrow \text{Cat}$
- (v)  $\text{InternalGroup} : \text{Cat} \rightarrow \text{Cat}$
- (vi)  $\text{InternalGroupAction} : (X : \text{Cat}) \rightarrow (G_0 : (\text{InternalGroup } X)) \rightarrow \text{Cat}$

The book “Galois theories” by Borceux and Janelidze contains the internal structures (iii), (iv), (v), and (vi), and the first two internal structures have fewer entries. That book details how to think about Galois theory using internal groupoids, internal G-presheaves, monadicity, comonadicity, and the constructions involved in Eilenberg-Moore theory.

Some previous work done on these structures can be found at the thread [here](#).

The six internal structures above arise here in relation to six functors:

- (I)  $\vec{\Omega} : \infty\text{-Cat} \rightarrow \infty\text{-Cat}$  (notation for the directed path space functor, related to  $[\Delta^1, -]$ ).  $D(\vec{\Omega})$  factors through internal categories in  $D(\infty\text{-Cat})$  by a categorical equivalence  $D(\infty\text{-Cat}) \cong \text{InternalCategory } D(\infty\text{-Cat})$  (internal categories in  $D(\infty\text{-Cat})$ )
- (II)  $\vec{\omega}(\mathbb{1} C) : \infty\text{-Cat}/C \rightarrow \infty\text{-Cat}/C$ , the derived directed homotopy pullback with  $\mathbb{1} C$ .  $D(\vec{\omega}(\mathbb{1} C))$  factors through a categorical equivalence between  $D(\infty\text{-Cat}/C)$  and internal  $\vec{P}C$ -presheaves in  $D(\infty\text{-Cat}/C)$ .

- (III)  $\tilde{\Omega} : \infty\text{-Grpd} \longrightarrow \infty\text{-Grpd}$  (notation for the path space functor  $[I, -]$ ), the derived homotopy pullback of an  $\infty$ -groupoid with itself.  $D(\tilde{\Omega})$  factors through a categorical equivalence between  $D(\infty\text{-Grpd})$  and internal groupoids in  $D(\infty\text{-Grpd})$
- (IV)  $\tilde{\omega}(\mathbb{1} X) : \infty\text{-Grpd}/X \longrightarrow \infty\text{-Grpd}/X$ , the derived homotopy pullback with  $\mathbb{1} X$ .  $D(\tilde{\omega}(\mathbb{1} X))$  factors through internal  $\tilde{P}X$
- (V)  $\Omega : \infty\text{-Grpd}_0 \longrightarrow \infty\text{-Grpd}_0$ , the loop space functor.  $D(\Omega)$  factors through a categorical equivalence between  $D(\infty\text{-Grpd}_0)$  and internal groups in  $D(\infty\text{-Grpd}_0)$  (the loop space functor on connected based  $\infty$ -groupoids)
- (VI)  $\omega(\mathbb{1} X) : \infty\text{-Grpd}_0/X_0 \longrightarrow \infty\text{-Grpd}_0/X_0$ , the homotopy pullback with the base of  $X_0$ .  $D(\omega(\mathbb{1} X))$  factors through internal  $PX_0$ -actions in based connected spaces over  $X_0$ .

(v) in the above is shown here and (vi) in the above is shown in a typical exposition of  $G$ -principal bundles.

The functors  $\tilde{\omega}(\mathbb{1} C)$ ,  $\tilde{\omega}(\mathbb{1} X)$ , and  $\omega(\mathbb{1} C)$  in the above ensue from a more general construction:

1. For  $C, D : D(\infty\text{-Cat})$ , and  $f : C \longrightarrow D$ ,  $\vec{\omega} f : D(\infty\text{-Cat}/D) \longrightarrow D(\infty\text{-Cat}/C)$  (derived directed homotopy pullback)
2. For  $B, E : D(\infty\text{-Grpd})$ , and  $f : E \longrightarrow B$ ,  $\vec{\omega} f : D(\infty\text{-Grpd}/B) \longrightarrow D(\infty\text{-Grpd}/E)$  (derived homotopy pullback)
3. For  $B_0, E_0 : D(\infty\text{-Grpd}_0)$ , and  $f : E_0 \longrightarrow B_0$ ,  $\omega f : D(\infty\text{-Grpd}_0/B_0) \longrightarrow D(\infty\text{-Grpd}_0/E_0)$  (homotopy pullback with the base)

These six factored functors  $\vec{P}$ ,  $\tilde{P}$ ,  $P : D(\infty\text{-Grpd}_0)$ ,  $\vec{p}(\mathbb{1} C)$ ,  $\tilde{p}(\mathbb{1} X)$ ,  $p$  are each fully faithful and produce categorical equivalences; we later construct functors  $\vec{B}$ ,  $\tilde{B}$ ,  $B$ ,  $\vec{b}$ ,  $\tilde{b}$ ,  $b$  defined on the essential image of these six, which are inverse to them up to natural isomorphism.

We obtain six categorical equivalences witnessed by these twelve functors (along with twelve natural isomorphisms). Here are the types of  $\vec{P}$ ,  $\tilde{P}$ ,  $P : D(\infty\text{-Grpd}_0)$ ,  $\vec{p}(\mathbb{1} C)$ ,  $\tilde{p}(\mathbb{1} X)$ ,  $p$ :

1. The directed path space, the path space, and loop space form components of the functors  $\vec{P}$ ,  $\tilde{P}$ , and  $P$ , which are valued in internal categories, internal groupoids, and internal groups respectively.
  - (a)  $\vec{P} : D(\infty\text{-Cat}) \longrightarrow \text{Cat } D(\infty\text{-Cat})$
  - (b)  $\tilde{P} : D(\infty\text{-Grpd}) \longrightarrow \text{Grpd } D(\infty\text{-Grpd})$

- (c)  $P : D(\infty\text{-Grpd}_0) \longrightarrow \text{Grp } D(\infty\text{-Grpd})$  (see here)
2. The directed homotopy pullback, the homotopy pullback, and the homotopy pullback with the base form components of the functors  $\text{Alg}(\text{Mon}(\vec{\omega}))$ ,  $\text{Alg}(\text{Mon}(\vec{\omega}))$ , and  $\text{Alg}(\text{Mon}(p))$ , respectively.
- (a)  $\vec{P}(\mathbb{1} C) : D(\infty\text{-Cat}/C) \longrightarrow \text{InternalPresheaf } D(\infty\text{-Cat}/C) \vec{P}.\text{obj } C$
- (b)  $\vec{P}(\mathbb{1} X) : D(\infty\text{-Grpd}/X) \longrightarrow \text{InternalGroupoidAction } D(\infty\text{-Grpd}/X) \vec{P}.\text{obj } X$
- (c)  $p(\mathbb{1} X_0) : D(\infty\text{-Grpd}_0/X_0) \longrightarrow \text{InternalGroupAction } D(\infty\text{-Grpd}_0/X_0) P.\text{obj } X_0$

Above, the functors  $\vec{P}$ ,  $\vec{P}$ ,  $P$ ,  $\vec{p}$ ,  $\vec{p}$ , and  $p$  feature  $\vec{\Omega}$ ,  $\vec{\Omega}$ ,  $\Omega$ ,  $\vec{\omega}$ ,  $\vec{\omega}$ , and  $\omega$  in their components, and can be related to them using constructions from Eilenberg-Moore theory.

These six new functors combine with the functors below to form categorical equivalences:

1. The directed homotopy colimit of a point with an internal category in  $D(\infty\text{-Cat})$  as a diagram, the homotopy colimit of a constant functor with an internal internal group as a diagram
- (a)  $\vec{B} : \text{essential\_image } \vec{P} \longrightarrow D(\infty\text{-Cat})$
- (b)  $\vec{B} : \text{essential\_image } \vec{P} \longrightarrow D(\infty\text{-Grpd})$
- (c)  $B : \text{essential\_image } P \longrightarrow D(\infty\text{-Grpd}_0)$  (see here)
2. The clutching functors are inverse to the above functors up to natural isomorphism:
- (a)  $\vec{b} : \text{essential\_image } \vec{p} \longrightarrow D(\infty\text{-Cat}/C)$
- (b)  $\vec{b} : \text{essential\_image } \vec{p} \longrightarrow D(\infty\text{-Cat}/C)$
- (c)  $b : \text{essential\_image } p \longrightarrow D(\infty\text{-Grpd}_0/X_0)$

We will show six categorical equivalences featuring these:

1.  $\vec{P} \bullet \vec{B} \cong \mathbb{1} (D\text{lpInternalCategory } D(\infty\text{-Cat}))$  and  $\vec{B} \bullet \vec{P} \cong \mathbb{1} D(\infty\text{-Cat})$
2.  $\vec{P} \bullet \vec{B} \cong \mathbb{1} (D\text{lpInternalGroupoid } D(\infty\text{-Grpd}))$  and  $\vec{B} \bullet \vec{P} \cong \mathbb{1} D(\infty\text{-Grpd})$
3.  $P \bullet B \cong \mathbb{1} (D\text{lpInternalGroup } D(\infty\text{-Grpd}_0))$  and  $B \bullet P \cong \mathbb{1} D(\infty\text{-Grpd}_0)$  (see here)
4.  $\vec{p} \bullet \vec{b} \cong \mathbb{1} (D\text{lpInternalPresheaf } D(\infty\text{-Cat}/C) \vec{P}C)$  and  $\vec{b} \bullet \vec{p} \cong \mathbb{1} D(\infty\text{-Cat}/C)$
5.  $\vec{p} \bullet \vec{b} \cong \mathbb{1} (D\text{lpInternalGroupoidAction } D(\infty\text{-Cat}/C) \vec{P}X)$  and  $\vec{b} \bullet \vec{p} \cong \mathbb{1} D(\infty\text{-Cat}/C)$

6.  $p \bullet b \cong \mathbb{1}$  ( $\text{DlpInternalGroupAction } D(\infty\text{-Grpd}_0/X_0) \text{ } PX_0$ ) and  $b \bullet p \cong \mathbb{1}$  ( $D(\infty\text{-Grpd}_0/X_0)$ ) (see here)

Take special note that each of these six involves a condition ensuring that the functor  $\vec{B}$  be well defined. Consider the functors:

1.  $D(\text{InternalCategory } \infty\text{-Cat}) \longrightarrow \text{InternalCategory } D(\infty\text{-Cat})$
2.  $D(\text{InternalGroupoid } \infty\text{-Grpd}) \longrightarrow \text{InternalGroupoid } D(\infty\text{-Grpd})$
3.  $D(\text{InternalGroup } \infty\text{-Grpd}_0) \longrightarrow \text{InternalGroup } D(\infty\text{-Grpd}_0)$
4.  $D(\text{InternalPresheaf } \infty\text{-Cat}/C) \vec{P}C \longrightarrow \text{InternalPresheaf } D(\infty\text{-Cat}/C) \vec{P}C$
5.  $D(\text{InternalGroupoidAction } \infty\text{-Cat}/C) \vec{P}X \longrightarrow \text{InternalGroupoidAction } D(\infty\text{-Cat}/C) \vec{P}X$
6.  $D(\text{InternalGroupAction } \infty\text{-Grpd}_0/X_0) \text{ } PX_0 \longrightarrow \text{InternalGroupAction } D(\infty\text{-Grpd}_0/X_0) \text{ } PX_0$

It may happen that a given object in the codomain of these six functors lies in their essential image. In this case, any of the six of  $\vec{B}$ ,  $\vec{B}$ ,  $B$ ,  $\vec{b}$ ,  $\vec{b}$ ,  $b$  can sometimes but not always be obtained as a quotient of six functors  $\vec{E}$ ,  $\vec{E}$ ,  $E$ ,  $\vec{e}$ ,  $\vec{e}$ ,  $e$ , respectively:

1.  $\vec{E} : \text{InternalCategory } \infty\text{-Cat} \longrightarrow \infty\text{-Cat}$
2.  $\vec{E} : \text{InternalGroupoid } \infty\text{-Grpd} \longrightarrow \infty\text{-Grpd}$
3.  $E : \text{InternalGroup } \infty\text{-Grpd}_0 \longrightarrow \infty\text{-Grpd}_0$
4.  $\vec{e} : \text{PreShf } \infty\text{-Cat}/C \vec{P}.\text{obj } C \longrightarrow \infty\text{-Cat}/C \vec{P}.\text{obj } C$
5.  $\vec{e} : \text{InternalGroupoidAction } \infty\text{-Cat}/C \vec{P}.\text{obj } X \longrightarrow \infty\text{-Cat}/C \vec{P}.\text{obj } X$
6.  $e : \text{InternalGroupAction } \infty\text{-Grpd}_0/X_0 \text{ } P.\text{obj } X_0 \longrightarrow \infty\text{-Grpd}_0/X_0 \text{ } P.\text{obj } X_0$

The functors above will be defined as certain homotopy colimits, themselves certain coequalizers. On the condition that an internal category is internally filtered and internally cofiltered, we can further construct the  $\vec{B}$ .

We will make extensive use of Mathlib’s bicategory of categories and material on simplicial sets. We further use Mathlib’s pullbacks and categorical products, as well as their Eilenberg-Moore theory constructions. I’d like to extend my appreciation to Scott Morison, Eric Wieser, Floris Van Doorn, and all the contributors who have put their efforts into creating these robust features for Mathlib 4.

Altogether, the project gets the following “periodic table” of 30 functors featured on the front cover:

$D(\infty\text{-Cat})$	$\vec{\Sigma}$	$\vec{\Omega}$	$\vec{P}$	$\vec{B}$	$\vec{E}$	$D(\infty\text{-Cat}/C)$	$\vec{\sigma}$	$\vec{\omega}$	$\vec{b}$	$\vec{p}$	$\vec{e}$
$D(\infty\text{-Grpd})$	$\vec{\Sigma}$	$\vec{\Omega}$	$\vec{P}$	$\vec{B}$	$\vec{E}$	$D(\infty\text{-Grpd}/G)$	$\vec{\sigma}$	$\vec{\omega}$	$\vec{b}$	$\vec{p}$	$\vec{e}$
$D(\infty\text{-Grpd}_0)$	$\Sigma$	$\Omega$	$P$	$B$	$E$	$D(\infty\text{-Grpd}_0/G_0)$	$\sigma$	$\omega$	$b$	$p$	$e$

Here are the names of the symbols featured above:

Suspensional	Deductive	Remembrant	Delooping	Free
$\vec{\Sigma}$ (Directed suspension)	$\vec{\Omega}$ (Directed path space)	$\vec{P}$ (Remembrant derived directed path space)	$\vec{B}$ (Classifying space for internal categories)	$\vec{E}$
$\vec{\Sigma}$ (Suspensionoid)	$\vec{\Omega}$ (Path space)	$\vec{P}$ (Remembrant derived path space)	$\vec{B}$ (Classifying space for internal groupoids)	$\vec{E}$
$\Sigma$ (Suspension)	$\Omega$ (Loop space)	$P$ (Remembrant derived loop space)	$B$ (Classifying space for internal groups)	$E$
$\vec{\sigma}$	$\vec{\omega}$ (Directed homotopy pushout with a point)	$\vec{p}$ (Remembrant derived directed homotopy pullback)	$\vec{b}$ (Classifying space for internal presheaves)	$\vec{e}$
$\vec{\sigma}$	$\vec{\omega}$ (Homotopy pushout with a point)	$\vec{p}$ (Remembrant derived homotopy pullback)	$\vec{b}$ (Classifying space for internal groupoid actions)	$\vec{e}$
$\sigma$	$\omega$ (Homotopy fiber)	$p$ (Remembrant derived homotopy fiber)	$b$ (Classifying space for internal group actions)	$e$

The term “remembrant” in the above is not common terminology. It is intended to mean that the second column features functors which are valued in categories of internal objects whereas the left column forms particular components of those structures.

The notation here is both an attempt to make the three-fold division of the project (three Whitehead theorems, three Puppe sequences, etc.) manifest while sticking to the standard notation for the established theorems ( $\Sigma$ ,  $\Omega$ ,  $B$ ,  $E$ ). In the above,  $P$  could be said to stand for “(remembrant) path space” and  $p$  for “(remembrant) pullback”, while at the same time this matches the theme that our capital letters reflect various internal structures and that their lower-case forms reflect the corresponding actions.

The mentioned “recognition theorems”, which identify inverses to the remembrant functors *on their essential image*, form important consequences of the three Whitehead theorems. All in all, there are twelve important theorems we want to show:



### Twelve Goals

- (I) Define and inhabit the `whitehead_theorem_for_categories` : `Type`.
- (II) Define the Puppe sequence for  $\infty$ -categories and prove its exactness.
- (III) Define and inhabit the `internal_category_delooping_principal` : `Type`.
- (IV) Define and inhabit the `internal_sheaf_delooping_principal` : `Type`.
- (V) Define and inhabit the `whitehead_theorem_for_groupoids` : `Type`.
- (VI) Define the Puppe sequence for  $\infty$ -groupoids and prove its exactness
- (VII) Define and inhabit the `internal_groupoid_delooping_principal` : `Type`.
- (VIII) Define and inhabit the `internal_groupoid_action_delooping_principal` : `Type`.
- (IX) Define and inhabit the `whitehead_theorem` : `Type`.
- (X) Define the Puppe sequence for based connected  $\infty$ -groupoids and prove its exactness
- (XI) Define and inhabit the `internal_group_delooping_principal` : `Type`.
- (XII) Define and inhabit the `internal_group_action_delooping_principal` : `Type`.

None of these theorems are currently contained in Mathlib. The last four are famously known as:

1. The Whitehead theorem
2. The Puppe sequence and its exactness
3. The theorem that  $D(P) \bullet D(B)$  is naturally isomorphic to the identity functor on the essential image of  $D(P)$ , and that  $D(B) \bullet D(P)$  is naturally isomorphic to the identity functor on  $D(\infty\text{-Grpd}_0)$
4. The theorem that BG classifies G-principal bundles

These four established theorems and the eight novel ones form a good vignette for our project. Where previous approaches have considered comparisons between models such as the ones we have developed here, the goals above provide a unification which I would like to discuss in the section on pairs.

In the work that ensues, we plan to take an approach which establishes the known results before the original ones, taking advantage of the predefined  $\pi_n$  functors in Mathlib 4 in the process. This decision will also help to take an approach which is more gradual and incremental, and to start with smaller pull requests.

Twelve Structures			
Strict		Lax	
Unitial	Actional	Unitial	Actional
InternalCategory	InternalPresheaf	OperadicCategory	OperadicPresheaf
InternalGroupoid	InternalGroupoidAction	OperadicGroupoid	OperadicGroupoidAction
InternalGroup	InternalGroupAction	OperadicGroup	OperadicGroupAction

$\bar{\Omega} : \infty\text{-Cat} \longrightarrow \infty\text{-Cat}$	$\bar{\omega} : (C : \infty\text{-Cat}) \rightarrow (D : \infty\text{-Cat}) \rightarrow (F : \infty\text{-Cat.hom } C \ D) \rightarrow (\infty\text{-Cat}/D) \longrightarrow \infty\text{-Cat}$
$\bar{O} : \infty\text{-Cat} \longrightarrow \text{OperadicCategory } \infty\text{-Cat}$	$\bar{o} : (C : \infty\text{-Cat}) \rightarrow (D : \infty\text{-Cat}) \rightarrow (F : \infty\text{-Cat.hom } C \ D) \rightarrow (\infty\text{-Cat}/D) \longrightarrow \text{OperadicCategory } \infty\text{-Cat}$
$\bar{P} : \infty\text{-Cat} \longrightarrow \text{InternalCategory } D(\infty\text{-Cat})$	$\bar{p} : (C : \infty\text{-Cat}) \rightarrow (D : \infty\text{-Cat}) \rightarrow (F : \infty\text{-Cat.hom } C \ D) \rightarrow (\infty\text{-Cat}/D) \longrightarrow \text{InternalCategory } D(\infty\text{-Cat})$
$\bar{\Omega} : \infty\text{-Grpd} \longrightarrow \infty\text{-Grpd}$	$\bar{\omega} : (X : \infty\text{-Grpd}) \rightarrow (Y : \infty\text{-Grpd}) \rightarrow (F : \infty\text{-Cat.hom } X \ Y) \rightarrow (\infty\text{-Grpd}/Y) \longrightarrow \infty\text{-Grpd}$
$\bar{O} : \infty\text{-Grpd} \longrightarrow \text{OperadicGroupoid } \infty\text{-Grpd}$	$\bar{o} : (X : \infty\text{-Grpd}) \rightarrow (Y : \infty\text{-Grpd}) \rightarrow (F : \infty\text{-Cat.hom } X \ Y) \rightarrow (\infty\text{-Grpd}/Y) \longrightarrow \text{OperadicGroupoid } \infty\text{-Grpd}$
$\bar{P} : \infty\text{-Grpd} \longrightarrow \text{InternalGroupoid } D(\infty\text{-Grpd})$	$\bar{p} : (X : \infty\text{-Grpd}) \rightarrow (Y : \infty\text{-Grpd}) \rightarrow (F : \infty\text{-Cat.hom } C \ D) \rightarrow (\infty\text{-Grpd}/Y) \longrightarrow \text{InternalGroupoid } D(\infty\text{-Grpd})$
$\Omega : \infty\text{-Grpd}_{-1} \longrightarrow \infty\text{-Grpd}_{-1}$	$\omega : (X_{-1} : \infty\text{-Grpd}_{-1}) \rightarrow (Y_{-1} : \infty\text{-Grpd}_{-1}) \rightarrow (F : \infty\text{-Grpd}_{-1}.\text{hom } X_{-1} \ Y_{-1}) \rightarrow (\infty\text{-Grpd}_{-1}/Y_{-1}) \longrightarrow \infty\text{-Grpd}_{-1}$
$O : \infty\text{-Grpd}_{-1} \longrightarrow \text{OperadicGroup } \infty\text{-Grpd}_{-1}$	$o : (X_{-1} : \infty\text{-Grpd}_{-1}) \rightarrow (Y_{-1} : \infty\text{-Grpd}_{-1}) \rightarrow (F : \infty\text{-Grpd}_{-1}.\text{hom } X_{-1} \ Y_{-1}) \rightarrow (\infty\text{-Grpd}_{-1}/Y_{-1}) \longrightarrow \text{OperadicGroup } \infty\text{-Grpd}_{-1}$
$P : \infty\text{-Grpd}_{-1} \longrightarrow \text{InternalGroup } D(\infty\text{-Grpd}_{-1})$	$p : (X_{-1} : \infty\text{-Grpd}_{-1}) \rightarrow (Y_{-1} : \infty\text{-Grpd}_{-1}) \rightarrow (F : \infty\text{-Grpd}_{-1}.\text{hom } X_{-1} \ Y_{-1}) \rightarrow (\infty\text{-Grpd}_{-1}/Y_{-1}) \longrightarrow \text{InternalGroup } D(\infty\text{-Grpd}_{-1})$

### 3. Unicode

Here is a list of the unicode characters we will use:

Symbol	Unicode	VSCode shortcut	Use
Lean's Kernel			
$\times$	2A2F	<code>\times</code>	Product of types
$\rightarrow$	2192	<code>\rightarrow</code>	Hom of types
$\langle, \rangle$	27E8, 27E9	<code>\langle \rangle</code> , <code>\rangle \langle</code>	Product term introduction
$\mapsto$	21A6	<code>\mapsto</code>	Hom term introduction
$\wedge$	2227	<code>\wedge</code>	Conjunction
$\vee$	2228	<code>\vee</code>	Disjunction
$\forall$	2200	<code>\forall</code>	Universal quantification
$\exists$	2203	<code>\exists</code>	Existential quantification
$\neg$	00AC	<code>\neg</code>	Negation
Variables and Constants			
$a, b, c, \dots, z$	1D52, 1D56		Variables and constants
$0, 1, 2, 3, 4, 5, 6, 7, 8, 9$	1D52, 1D56		Variables and constants
$\sim$	207B		Variables and constants
$0.1.2.3.4.5.6.7.8.9$	2080 - 2089	<code>\0-\9</code>	Variables and constants
$\mathbb{A}, \dots, \mathbb{Z}$	1D538		
$\mathbb{Q}, \dots, \mathbb{Z}$	1D552		
$\mathbb{A}, \dots, \mathbb{Z}$	1D41A		
$\mathbb{a}, \dots, \mathbb{z}$	1D41A		
$\alpha, \omega, \mathbb{A}, \Omega$	03B1-03C9		Variables and constants
Categories			
$\mathbb{1}$	1D7D9	<code>\b1</code>	The identity morphism
$\circ$	2218	<code>\circ</code>	Composition
Bicategories			
$\bullet$	2022	<code>\smul</code>	Horizontal composition of objects
Adjunctions			
$\rightrightarrows$	21C4	<code>\rightrightarrows</code>	Adjunctions
$\leftrightharpoons$	21C6	<code>\leftrightharpoons</code>	Adjunctions
$\cdot$	1BC94		Right adjoints
$\cdot$	0971		Left adjoints
$\dashv$	22A3	<code>\dashv</code>	The condition that two functors are adjoint
Monads and Comonads			
$?_{\circ}$	003F, 00BF	<code>?, \?</code>	The corresponding (co)monad of an adjunction
$!_{\circ}$	0021, 00A1	<code>!, \!</code>	The (co)-Eilenberg-(co)-Moore adjunction
$!_{\circ}$	A71D, A71E		The (co)exponential maps
Miscellaneous			
$\sim$	223C	<code>\sim</code>	Homotopies
$\simeq$	2243	<code>\equiv</code>	Equivalences
$\cong$	2245	<code>\cong</code>	Isomorphisms
$\perp$	22A5	<code>\bot</code>	The overobject classifier
$\infty$	221E	<code>\infty</code>	Infinity categories and infinity groupoids
$\leftrightarrow$	20D7		Homotopical operations on $\infty$ -categories
$\rightarrow$	20E1		Homotopical operations on $\infty$ -groupoids

Of these, the characters `'`, ```, `.`, `;`, `→`, and `↔` do not have VSCode shortcuts.

## 4. Introduction to Lean 4

The main way to tell Lean 4 what something means is with `def`, which defines a term in dependent type theory. Much in the same way as other computer languages, we then supply the type of the term (e.g. `Int` for integer), followed by the formula itself:

Lean 1
<pre>def zero : Nat := 0</pre>

Here we have introduced a natural number `n` using the type `Nat` that comes with Lean 4.

As a beginner, it's normal to take some time to get comfortable with Lean and formal proof systems. It's a journey that requires practice and patience. Lean has an active community that provides support and resources to help you along the way.

Constituents of  $x, y : X$  of types  $X$  can also stand to be equal or unequal, written  $x = y$ , and it is the properties of equality which in addition to the dependent type theory make a type behave like a set. Equality satisfies the three properties of an equivalence relation, which we cover presently. Consider first the reflexivity property of equality:

Lean 2
<pre>def reflexivity {X : Type} {x : X} : x = x := Eq.refl   ↪ x</pre>

This command defines a function called `reflexivity` that proves the reflexivity property of equality. The function takes two type parameters:  $X$  represents the type of the

elements being compared, and  $x$  represents an element of type  $X$ . It also takes an argument  $\omega$  which is a proof that  $x$  is equal to itself ( $x = x$ ). The function body states that the result of `reflexivity` is the proof  $\omega$  itself using the `Eq.refl` constructor, which indicates that  $x$  is equal to itself.

In Lean 4,  $\{x : X\}$  represents an implicit argument, where Lean will attempt to infer the value of  $x$  based on the context.  $(x : X)$  represents an explicit argument, requiring the value of  $x$  to be provided explicitly when using the function or definition.

Lean 3

```
def symmetry {X : Type} {x : X} {y : X} (p : x = y)
  ↪ := Eq.symm p
```

This command defines a function called `symmetry` that proves the symmetry property of equality. It takes three type parameters:  $X$  represents the type of the elements being compared, and  $x$  and  $y$  represent elements of type  $X$ . The function also takes an argument  $\omega$  which is a proof that  $x$  is equal to  $y$  ( $x=y$ ). The function body states that the result of `symmetry` is the proof  $\omega$  itself using the `Eq.symm` constructor, which allows you to reverse an equality proof.

Lean 4

```
def transitivity {X : Type} {x : X} {y : X} {z : X}
  ↪ (p : x = y) (q : y = z) := Eq.trans p q
```

This command defines a function called `transitivity` that proves the transitivity property of equality. It takes four type parameters:  $X$  represents the type of the elements being compared, and  $x$ ,  $y$ , and  $z$  represent elements of type  $X$ . The function also takes two arguments  $p$  and  $q$ .  $p$  is a proof that  $x$  is equal to  $y$  ( $x = y$ ), and  $q$  is a proof that  $y$  is equal to  $z$  ( $y = z$ ). The function body states that the result of `transitivity` is the proof of the composition of  $\omega$  and  $q$  using the `Eq.trans` constructor, which allows you to combine two equality proofs to obtain a new one.

These Lean commands define functions that prove fundamental properties of equality: reflexivity (every element is equal to itself), symmetry (equality is symmetric), and transitivity (equality is transitive). These properties are essential for reasoning about equality in mathematics and formal proofs.

We must also require that functions satisfy extensionality:

## Lean 5

```
def extensionality (f g : X → Y) (p : (x:X) → f x =
  → g x) : f = g := funext p
```

Extensionality, a key characteristic of sets and types, asserts that functions which are equal on all values are themselves equal, and it is featured prominently in what is perhaps the most well known mathematical foundations of ZFC.

There are several other features of equality with respect to functions which we should be aware of:

## Lean 6

```
def equal_arguments {X : Type} {Y : Type} {a : X} {b
  → : X} (f : X → Y) (p : a = b) : f a = f b :=
  → congrArg f p

def equal_functions {X : Type} {Y : Type} {f1 : X →
  → Y} {f2 : X → Y} (p : f1 = f2) (x : X) : f1 x =
  → f2 x := congrFun ω x

def pairwise {A : Type} {B : Type} (a1 : A) (a2 : A)
  → (b1 : B) (b2 : B) (p : a1 = a2) (q : b1 = b2) :
  → (a1, b1) = (a2, b2) := (congr ((congrArg Prod.mk) p)
  → q)
```

The tutorial here provides a great introduction to using the dependent type theory in Lean.

PART 1: BASED CONNECTED  
 $\infty$ -GROUPOIDS



# Chapter 1: $\infty\text{-Grpd}_0$

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Implementation Progress

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Writing Progress

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Here we define CW-complexes, as well as relative CW-complexes, and also the derived categories  $D(\infty\text{-Grpd}_0)$  of connected based  $\infty$ -groupoids and  $D(\infty\text{-Grpd}_0/G_0)$ , made from CW-complexes.

5.  $D(\infty\text{-Grpd}_0)$

Symbol	Unicode	VSCode shortcut	Use
Lean's Kernel			
$\times$	2A2F	<code>\times</code>	Product of types
$\rightarrow$	2192	<code>\rightrightarrows</code>	Hom of types
$\dashv$	22A3	<code>\dashv</code>	The condition that two functors are adjoint
$?_L$	003F, 00BF	<code>?, \?</code>	The corresponding (co)monad of an adjunction
$\sim$	223C	<code>\sim</code>	Homotopies

Lean 7

## 6. $D(\infty\text{-Grpd}_0/X_0)$

The derived category of based connected  $\infty$ -groupoids over  $X_0$ .

$$7. \quad \Omega : \infty\text{-Grpd}_0 \longrightarrow \infty\text{-Grpd}$$

$$8. \quad \Sigma : \infty\text{-Grpd}_0 \longrightarrow \infty\text{-Grpd}_0$$

The based suspension functor

$$9. \quad \omega f : \infty\text{-Grpd}/D_0 \longrightarrow \infty\text{-Grpd}/C_0$$

The homotopy fiber

$$10. \quad \sigma \circ f : \infty\text{-Grpd}_0/C_0 \longrightarrow \infty\text{-Grpd}_0/D_0$$

Based homotopy pushout

$$11. \quad \pi_n : \infty\text{-Grpd}_0 \longrightarrow \mathbf{Set}$$

The connected components functors



## Chapter 2: The Whitehead Theorem

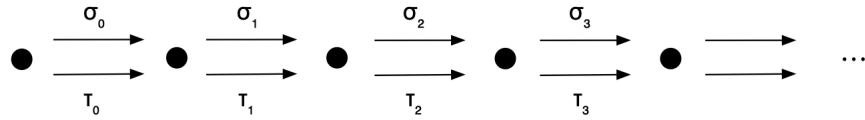
The proof of the Whitehead theorem divides into REP (replacement for based connected  $\infty$ -groupoids  $X : \infty\text{-Grpd}_0$ ) and HEP (the homotopy extension property for weak equivalent maps of based  $\infty$ -groupoids). The replacement functor  $\infty\text{-Grpd}_0$  can be constructed using globular sets.

Globular sets are not a rich enough invariant for homotopy, but maps of globular sets bear a critical difference because of

$$\forall (E : D(\infty\text{-Grpd}_0)), \forall (B : D(\infty\text{-Grpd}_0)), \forall (f : E \longrightarrow B), \forall (G : E \longrightarrow B), (\forall (n : \text{Nat}), (\pi_n F = \pi_n G)) \longrightarrow F = G$$

## 12. Globular Sets

The globe category  $\mathbb{G}$  is the category



Globular sets are functors from the opposite category of the globe category  $\mathbb{G}$  into the category of sets, and maps of globular sets are natural transformations between them.

In this chapter we prove the following (which we have called Whitehead Theorem (c)):  $\forall(E:D(\infty\text{-Grpd}_0)), \forall(B:D(\infty\text{-Grpd}_0)), \forall(f:E \rightarrow B), \forall(G:E \rightarrow B), (\forall(n:\text{Nat}), (\pi_n F = \pi_n G)) \rightarrow F = G$ , where  $\pi_n$  is notation for  $\pi$  n.

This can be shown using CW-replacement and induction on n. Fibrant replacement of an object  $X$  entails replacing an object in  $\infty\text{-Grpd}_0$  with a CW-object (an object made by successively glueing in higher and higher simplices along their boundaries obtaining a sequence  $X_n$ ). Given an equality  $\pi_{n+1}(f) = \pi_{n+1}(g)$  and a homotopy equivalence  $h_n : \Delta^1 \times X_n \rightarrow Y$  between  $f|_{X_n}, g|_{X_n} : X_n \rightarrow Y$ , we construct an extension of the homotopy equivalence  $\Delta^1 \times X_{n+1} \rightarrow Y$ .

Spheres and balls Next we turn to defining spheres and balls:

	Spheres and Balls	
Name of the X value	$\partial X \cong S^n$	$X \cong D^n$
p-norm unit ball for $p = 1$	$\partial B(1,1)$	$B(1,1)$
p-norm unit ball for $1 < p < 2$	$\partial B(p,1)$	$B(p,1)$
p-norm unit ball for $p = 2$	$\partial B(2,1)$	$B(2,1)$
p-norm unit ball for $2 < p < \infty$	$\partial B(p,1)$	$B(p,1)$
p-norm unit ball for $p = \infty$	$\partial B(\infty,1)$	$B(\infty,1)$
The n-simplex	$\partial \Delta^n$	$\Delta^n$

Definition 1. ...

Theorem 1.  $i^1 : S^0 \rightarrow D^1$

Theorem 2.  $D^n \times D^1 \rightarrow D^{n+1}$

Definition 2.  $D^n \longrightarrow D^m$

Theorem 3. Fix  $n : \mathbb{N}$  and let  $\partial^n : S^n \longrightarrow D^{n+1}$  be the inclusion. The pushout of the following diagram is isomorphic to  $S^{n+1}$ :

$$\begin{array}{ccc} S^n & \longrightarrow & D^{n+1} \\ \downarrow & & \\ D^{n+1} & & \end{array}$$

*Proof.*

□

Theorem 4. Define a function  $\|-\|_2 : D^n \longrightarrow I$  sending  $(x_1, \dots, x_n)$  to  $\sqrt{\sum_{i=1}^n x_i^2}$ , and write  $\|-\|_2$

*Proof.* ...

□

### 13. HEP for based connected $\infty$ -groupoids

In this section we prove the homotopy extension property for based connected  $\infty$ -groupoids, which we here model as CW-complexes.

Jar filling Next we turn to defining ‘jar shapes’  $J^n$ , which include into  $D^n \times I$   $i_n : J^n \rightarrow D^n \times I$ , after which we ‘fill’ them (i.e. demonstrate that any continuous map  $f : J^n \rightarrow X$  extends to a continuous map  $g : D^n \times I \rightarrow X$ ).

The first and most common approach involves ‘shining a light ray down from above the jar’, i.e. projection. We obtain a formula for .

The second way to fill the jar

Change of Base Jar filling leaves the question

Definition 3. Let  $X_{-1}$  be a connected CW-complex and let  $n : \text{Nat}$  be a natural number. The transport function  $\text{trans } n \ X_{-1} : (f : [I, X_{-1}]) \rightarrow \pi_n (f \ 0) \rightarrow \pi_n (f \ 1)$  is

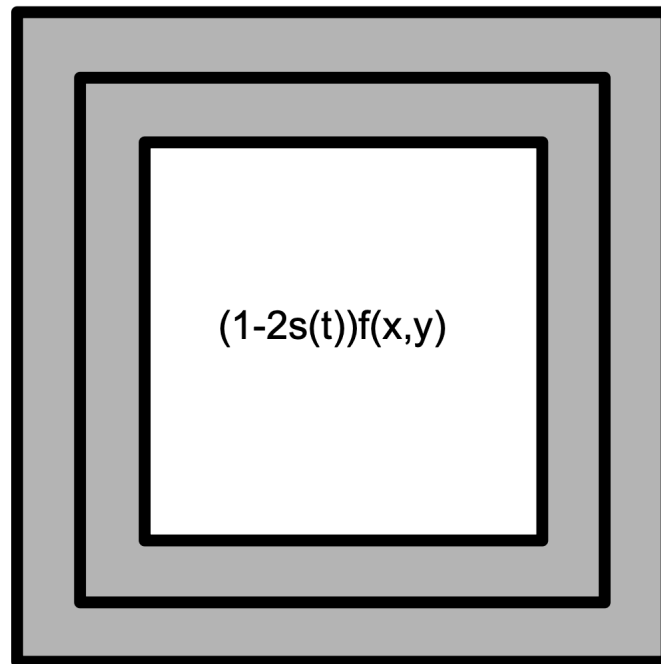
Theorem 5. Let  $X_{-1}$  be a connected CW-complex and let  $f : I \rightarrow X_{-1}$  be a path, so that  $(\text{trans } n \ X_{-1} \ f^{-1}) \bullet (\text{trans } n \ X_{-1} \ f)$  has type  $\pi_n (f \ 0) \rightarrow \pi_n (f \ 0)$ . Then

$$(\text{trans } n \ X_{-1} \ f^{-1}) \bullet (\text{trans } n \ X_{-1} \ f) = 1_{\pi_n (f \ 0)}$$

*Proof.* ...

□

The proof in the above can be depicted like so, as a ‘painting with two concentric frames’:

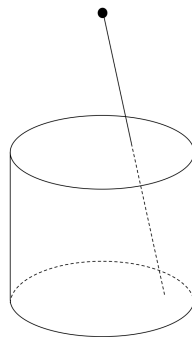


that the based CW-complexes  $(X_{-1}, x)$  and  $(X_{-1}, y)$  are

Theorem 6.

*Proof.* ...

□



## 14. REP for based connected $\infty$ -groupoids

In this section we use the notion of globular sets to replace a topological space with a CW-complex. Together with HEP (homotopy extension), this will complete the proof of the Whitehead theorem.

## 15. The Whitehead theorem

Here we show the Whitehead theorem.

In proving the Whitehead theorem formally, *we make use of a line* as well as, in essence, the structure of a *groupoid*  $G_X$  where  $\text{Hom}_G(x, x)$  is taken to be  $\pi_n(X, x)$ . This interesting feature of the proof relates to the other two Whitehead theorems featuring categories  $\vec{\pi}_n$  and groupoids  $\tilde{\pi}_n$ .

## Chapter 3: The Category of Pairs

In this section I would like to



## Chapter 4: Internal and Operadic Groups

## 16. InternalGroup

Lean 8

```
/-  
structure internal_group ... where  
  Obj := D(Γ).Obj  
  -- Dom :=  
  -- Cod  
  -- Idn  
  -- Fst  
  -- Snd  
  -- Cmp  
  -- Id1  
  -- Id2  
  -- Ass  
  -- Com  
-/
```

## 17. InternalGroupAction $G$

Here we define internal group actions. These will be important when we talk about  $G$ -principal bundles (themselves defined as internal group actions in the derived category of an overcategory).

## 18. OperadicGroup

## 19. OperadicGroupActions

## 20. The $\Omega$ and $P$ Functors

The internal group principal stems from the simple observation that the loop space forms a component of an internal group.

## 21. The $o$ and $p$ Functors

The internal group actions principal stems from the simple observation that the homotopy fiber forms a component of an internal group action.

PART 2: BASED CONNECTED  
 $\infty$ -MONOIDS



## Chapter 5: $\infty\text{-Cat}_0$

Here we define the categories  $D(\infty\text{-Grpd}_0)$  of connected based  $\infty$ -groupoids and  $D(\infty\text{-Grpd}_0/G_0)$  using CW-complexes.

22.  $D(\infty\text{-Cat}_0)$

Lean 9

## 23. $D(\infty\text{-Grpd}_0/X_0)$

The derived category of based connected  $\infty$ -groupoids over  $X_0$ .

$$24. \quad \Omega : \infty\text{-Grpd}_0 \longrightarrow \infty\text{-Grpd}$$

$$25. \quad \Sigma : \infty\text{-Grpd}_0 \longrightarrow \infty\text{-Grpd}_0$$

The based suspension functor

$$26. \quad \omega f : \infty\text{-Grpd}/D_0 \longrightarrow \infty\text{-Grpd}/C_0$$

The homotopy fiber

$$27. \quad \sigma \circ f : \infty\text{-Grpd}_0/C_0 \longrightarrow \infty\text{-Grpd}_0/D_0$$

Based homotopy pushout

$$28. \quad \pi_n : \infty\text{-Grpd}_0 \longrightarrow \mathbf{Set}$$

The connected components functors



## Chapter 6: The Whitehead Theorem for ???

## 29. ???

Globular sets

REP for based connected  $\infty$ -groupoids & The replacement functor on  $\infty\text{-Grpd}_0$

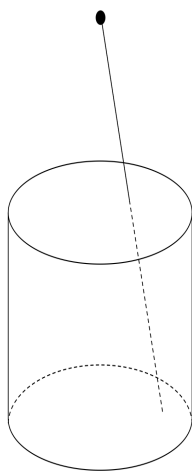
HEP for based connected  $\infty$ -groupoids & The homotopy extension property for  $\infty\text{-Grpd}_0$

In this chapter we prove the following (which we have called Whitehead Theorem (c)):  $\forall(E:D(\infty\text{-Grpd}_0)), \forall(B:D(\infty\text{-Grpd}_0)), \forall(f:E \rightarrow B), \forall(G:E \rightarrow B), (\forall(n:\text{Nat}), (\pi_n F = \pi_n G)) \rightarrow F = G$ , where  $\pi_n$  is notation for  $\pi$  n.

This can be shown using CW-replacement and induction on n. Fibrant replacement of an object X entails replacing an object in  $\infty\text{-Grpd}_0$  with a CW-object (an object made by successively glueing in higher and higher simplices along their boundaries obtaining a sequence  $X_n$ ). Given an equality  $\pi_{n+1}(f) = \pi_{n+1}(g)$  and a homotopy equivalence  $h_n : \Delta^1 \times X_n \rightarrow Y$  between  $f|_{X_n}, g|_{X_n} : X_n \rightarrow Y$ , we construct an extension of the homotopy equivalence  $\Delta^1 \times X_{n+1} \rightarrow Y$ .

### 30. HEP for ???

Can the attaching map be



### 31. The Whitehead theorem for ???

Here we show the Whitehead theorem proper.

# Chapter 7: The Category of Pairs

In this section I would like to

## Chapter 8: Internal and Operadic Groups

## 32. InternalMonoid

Lean 10

```
/-  
structure internal_group ... where  
  Obj := D(Γ).Obj  
  -- Dom :=  
  -- Cod  
  -- Idn  
  -- Fst  
  -- Snd  
  -- Cmp  
  -- Id1  
  -- Id2  
  -- Ass  
  -- Com  
-/
```

### 33. InternalMonoidAction G

Here we define internal group actions. These will be important when we talk about G-principal bundles (themselves defined as internal group actions in the derived category of an overcategory).



## 34. OperadicMonoid

## 35. OperadicMonoidActions

## 36. The $\mathcal{H}^*$ and $\mathcal{H}_*$ Functors

## 37. The $\mathcal{H}^*$ and $\mathcal{H}_*$ Functors

## PART 3: $\infty$ -GROUPOIDS

The Whitehead theorem is about the ways that spheres can get stuck in spaces (higher homotopy groups), and the last section established how these higher homotopy groups relate to maps in the homotopy category of based connected CW-complexes. There is the nuance that the base .

## Chapter 9: $\infty$ -Grpd

## 38. $\Omega$

Our choice of symbols reflects our choice of three variations of the Whitehead theorem and three Puppe sequences.  $\tilde{\Omega}$ , the analogue of loop space, is the internal hom functor  $[I, -] : \infty\text{-Grpd} \rightarrow \infty\text{-Grpd}$ . This is not hard to construct, with the main lemma being that the path space of a quasicategory has the quasicategory lifting condition.

We will be interested in one formal model of  $D(\infty\text{-Cat})$  which consists of formal compositions  $f_1 \bullet g_1 \bullet f_2 \bullet g_2 \bullet \cdots \bullet f_n \bullet g_n$ , where  $g_n : \text{Dom}(f_{n+1}) \rightarrow ???$  is a weak equivalence, and something similar for  $D(\infty\text{-Cat})$ . However, it is still vital to have the replacement functor  $\text{repl}$ , which ensures the Whitehead theorem for particular  $\infty$ -categories which are constructed out of attaching maps.

## 39. $\omega$

$\vec{\Omega}$  is to internal categories as  $\vec{\omega}$  is to internal G-actions. It is also called directed homotopy pullback. These functors will later be used to produce functors  $\vec{P} : D(\infty\text{-Grpd}) \rightarrow \text{InternalCategory } D(\infty\text{-Grpd})$  and  $\vec{p} : D(\infty\text{-Grpd}/C) \rightarrow \text{InternalPresheaf } (\vec{P} \text{ } G) D(\infty\text{-Grpd}/G)$ .



## 40. $\pi_n$

The mentioned functors  $\vec{\pi}_n$  are designed with both Whitehead theorem (a) and Puppe sequence (a) in mind.

# Chapter 10: The Whitehead Theorem for $\infty$ -Groupoids

## 41. Cubical Complexes

...

In this chapter, we take on the objective of Whitehead theorem (a), out of which we will prove the other more concrete Whitehead theorems:

$$\forall(E:D(\infty\text{-Cat})), \forall(B:D(\infty\text{-Cat})), \forall(F:E \longrightarrow B), \forall(G:E \longrightarrow B), (\forall(n:\text{Nat}), (\vec{\pi}_n F = \vec{\pi}_n G)) \longrightarrow F = G$$

We can attempt to form a slightly different category, much like the above, called  $\mathcal{D}(\infty\text{-Cat})$ , at first, and in a formal way, so as to create a category whose object component  $\mathcal{D}(\infty\text{-Cat}).\alpha$  matches the object component  $\infty\text{-Cat}.\alpha$  while featuring the above theorem in a formal way. However, with this as our model of  $D(\infty\text{-Cat})$ , we may then also be interested in the establishment of a model in which the Whitehead theorem is demonstrated, with the main idea being to prove two complementary concepts:

1. (REP) Establish a kind of “weak equivalent fibrant replacement”  $R : \infty\text{-Cat}.\alpha \longrightarrow \infty\text{-Cat}.\alpha$  ( $\alpha$  gives the object component in Mathlib’s category theory library), analogous to CW-complex replacement in Whitehead’s original paper. It’s especially nice if  $R$  forms the object component of a functor  $F : \infty\text{-Cat} \longrightarrow \infty\text{-Cat}$ .  $D(F) : D(\infty\text{-Cat}) \longrightarrow D(\infty\text{-Cat})$  should be a categorical equivalence, and that is what we will do.
2. (HEP) For the object  $R X$ , demonstrate that any  $F, G : (R X) \longrightarrow Y$  such that  $\forall(n:\text{Nat}), (\vec{\pi}_n F = \vec{\pi}_n G)$ , there is a directed homotopy equivalence between  $F$  and  $G$ . Note that “directed homotopy equivalence” consists of a composable sequence of simple directed homotopies  $H[i] : \Delta^1 \times (R X) \longrightarrow Y$ ,  $1 \leq i \leq n$ , with even  $H[i]$  running reverse to the odd  $H[i]$ .

Both of these will use induction on Lean’s  $\text{Nat}$ . The first of these could be called a REP (for REplacement Property, but this isn’t usual terminology), and the second typically uses induction and a HEP (Homotopy Extension Property). Our REP will consist of objects made out of particular kinds of pushouts called attaching maps, and can be made functorial. Proving the HEP can be done by well-order induction on the attaching maps present in our choice of  $R$ , thereby reducing to the case of extending a homotopy along a single attachment.

Our HEPa (directed box filling) is similar to the HEP shown in Whitehead's original paper, and to the approach detailed in Hatcher's textbook, though no doubt modified to suit our two goals:

- (I) The analogue of the Puppe sequence on the front cover needs to hold.
- (II) The first Whitehead theorem on the front cover needs to hold.

These two considerations determine our choice of  $\vec{\pi}_n$ ,  $\vec{\Omega}$ , and  $\vec{\omega}$ . We take  $\vec{\Omega}$  to be (simply) the internal hom functor  $[\Delta^1, -]$  (which requires showing that  $\vec{\Omega}X$  has the inner-horn filling condition).  $\vec{\omega}$  is then defined as a certain pullback of  $\vec{\Omega}$ , and  $\vec{\pi}_n$  is designed to produce a Puppe sequence with a meaningful notion of exactness by which we can demonstrate the goal of recognition theorems (i) and (ii). Specifically, it makes sense to use cubes in our definition of  $\vec{\pi}_n$  because of how they are representing objects of  $\vec{\Omega}^n$ . Meanwhile, it is also clear that the quotient producing  $\vec{\pi}_n$  is subtle in exactly how it requires fixing the endpoints of a sequence of alternating directed homotopies. We will define  $\vec{\pi}_n$ 's by identifying those objects  $x, y: \vec{\Omega}^n X$  which are homotopic by a homotopy which restricts to a constant along the face maps  $f[\square]: \vec{\Omega}^{n-1} X \longrightarrow \vec{\Omega}^{n-1} X$  (which correspond to pairs  $(n, b)$ , where  $b: \text{Bool}$ ).

Imagine for a moment the picture of a square shaped cushion; we might make such a cushion by first soeing together 6 squares of cloth and filling it with material, then "soeing the walls down to a square". Here we go with this:

1. Define a n-cubical cushion using the boundary of an n-1 cube times  $\Delta^1$ , i.e. the quotient of  $(\Delta^1)^{n-1} \times \Delta^1$  by an equivalence relation, but we have to start our model somewhere), or perhaps more easily the pushout of  $f: \Delta^1 \times (\partial((\Delta^1)^n)) \longrightarrow (\Delta^1)^{n+1}$  by the projection map  $\Delta^1 \times (\partial((\Delta^1)^n)) \longrightarrow \partial((\Delta^1)^n)$
2. Define a simplicial cushion using the boundary of an n-1 simplex times  $\Delta^1$ , i.e. the quotient of  $(\Delta^1)$  by an equivalence relation, or perhaps more easily the pushout of  $f: \Delta^1 \times (\partial(\Delta^n)) \longrightarrow (\Delta^1) \times \Delta^n$  by the projection map  $\Delta^1 \times (\partial((\Delta^1)^n)) \longrightarrow \partial(\Delta^n)$

The boundary of a cushion is a pouch, isomorphic to a pushout of two cubes glued together at their boundaries:

1. Define a n-cubical pouch as the pushout of two boundary maps  $\partial((\Delta^1)^n) \longrightarrow (\Delta^1)^n$
2. Define a simplicial pouch as the pushout of two boundary maps  $\partial(\Delta^n) \longrightarrow \Delta^n$

Notice that paths in  $\vec{\Omega}^n X$  produce paths in  $\vec{\Omega}^{n-1} X$  in as many ways as there are face maps  $(\Delta^1)^{n-1} \longrightarrow \Delta^{1n}$ , these could be called restrictions and are no doubt related to the pouches and cushions we just defined. The cartesian closed structure on simplicial sets with the lifting condition clarifies the relationship between the two available definitions of  $\vec{\pi}_n$ :

1. Homotopies of maps from a cube which are constant on the boundary
2. Paths of maps in  $\vec{\Omega}^{n-1}X$  which produce constant maps under the mentioned restrictions.
3. Maps from a pouch mod an equivalence relation (really we phrase this as a pushout!), namely the equivalence relation in which any two maps from a pouch that extend to maps from a cushion are identified.

After we construct  $\vec{\pi}_n$  in the first section, we will be in a place to demonstrate that the natural transformation  $\text{weak\_equivalence} : \text{repl} \longrightarrow (\mathbb{1} \infty\text{-Cat})$  consists of weak equivalences (a fact which we call REP, which is short for REplacement Principal). This is covered in the section titled REP, which also constructs  $\text{repl}$  and  $\text{weak\_requivalence}$ .

In sum, the goal of the present chapter is to use similar insights to the proof of the Whitehead theorem featured Hatcher's textbook to prove  $\text{Wa}$  and  $\text{Pa}$  for the model of quasicategories, using Mathlib's predefined horns and simplices in its simplicial sets section. The main difference is that our work must take care to respect the directed nature of quasicategories.

1. Defining  $\text{repl}$
- 2.

## 42. REP

We have divided the work of proving Whitehead theorem (a) into two steps: REP and HEP. In this section, we construct a functor  $\text{repl} : \infty\text{-Cat} \rightarrow \infty\text{-Cat}$  along with a natural transformation  $\text{weak\_equivalence} : \text{repl} \rightarrow (\mathbb{1} \infty\text{-Cat})$ . To construct  $\text{repl}$

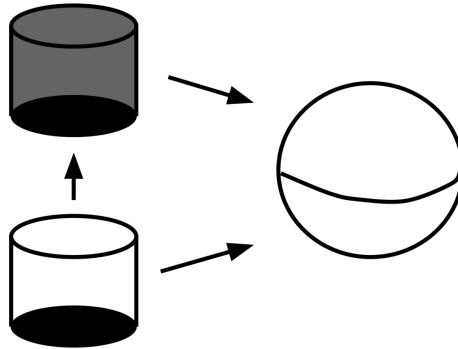
## 43. HEP

Consider the context of , supposing that we have constructed a homotopy ... This gives a picture that is a bit like “filling up a jar”: a homotopy  $h$  : of  $f, g : \partial\Delta^2 \rightarrow Y$ , along with the value of  $g$  on  $\Delta^2$ , produces a “jar” shape in  $Y$ , which can be “filled up” to produce a homotopy  $h : \Delta^1 \times \Delta^2 \rightarrow Y$ . This is easier for simplicial-based approaches than for point-set topological approaches, the latter of which needs extra steps that deform a map into a cellular map.

This construction, in the case of point set topology, often involves first deforming maps so as to be cellular; however our analogue of CW complexes allows us to skip this step.

This construction (HEP for quasicategories) may even be equivalent to the quasi-category lifting condition if we are lucky. It is also the main technical device allowing for our concrete choice of model (quasicategories).

In this section, we demonstrate this extension property and use it to conclude the Whitehead theorem for  $\infty$ -categories stated above.

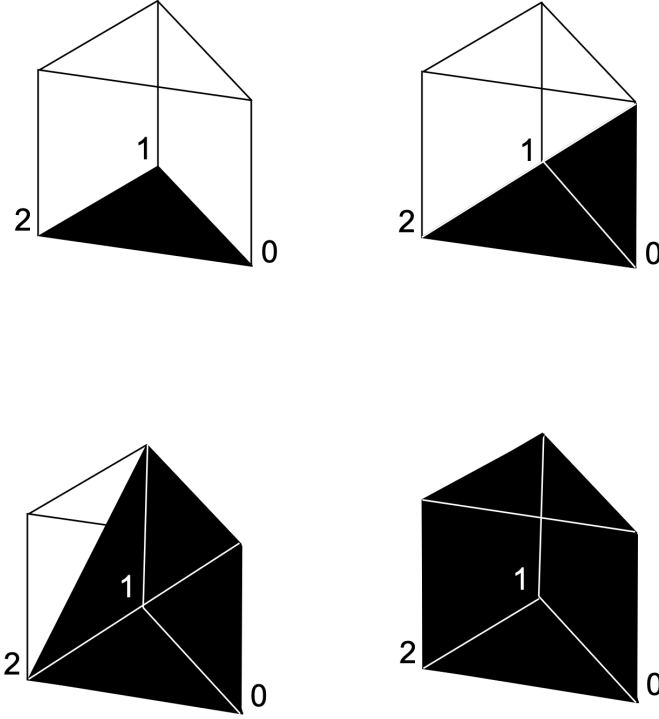


**Prism Filling (PF)** Let  $Y$  be a quasicategory, and let  $f, g : \partial\Delta^n \rightarrow Y$ . A homotopy  $h : \partial\Delta^n \times \Delta^1 \rightarrow Y$  between  $f, g : \partial\Delta^n \rightarrow Y$  extends to a map  $H : \Delta^n \times \Delta^1 \rightarrow Y$ ;

this follows from the condition that  $Y$  be a quasicategory.  $H(-,1)$  and  $g$  match on  $\partial\Delta^n$ , producing a map  $f : X \rightarrow Y$ , where  $X$  consists of two copies of  $\Delta^n$  glued together at the boundary. Consider a space  $X'$  formed as a quotient of  $\Delta^n \times \Delta^1$  by  $\partial\Delta^n \times \Delta^1$ . There is a map  $\phi : X \rightarrow X'$ . An induction hypothesis on  $f$  and  $g$  involving  $\pi_n$  ensures that the apparent map  $X \rightarrow Y$  lifts along  $\phi$ , producing a map from  $\Delta^n \times \Delta^1$  which is constant on  $\partial\Delta^n \times \Delta^1$ . Stacking this on top of  $H$  can be done using an isomorphism between  $\Delta^1$  and  $\Delta^1$  glued with itself along different endpoints. Altogether this produces a homotopy between  $f$  and  $g$ .

Directed prism filling may combine fruitfully with the yoneda lemma and/or the fact that simplicial sets are determined by the sets  $[\Delta^n, X]$  along with combinatorial information (face and degeneracy maps).

Decomposing  $\Delta^n \times \Delta^1$  into a colimit involving  $n+1$   $\Delta^{n+1}$ 's ...



In the above, it may be easier if we make use of sub-simplicial sets and prove the theorem using that colimit applied to a natural isomorphism of diagrams products an isomorphism.

The decomposition

A definition of  $\vec{\pi}_n$  which is consistent with our goals of  $W_a$  and  $P_a$  is one as a certain pushout involving  $(\vec{\Omega}^n X)$ —one which amounts to taking an equivalence relation



by paths in  $\vec{\Omega}^n X$  which restrict to constant paths along the face maps  $f_{[i]} : \vec{\Omega}^{n-1} X \rightarrow \vec{\Omega}^n X$ . Here,  $\vec{\Omega}$  is easy to define in the model of quasi-categories, and it amounts to . Besides fulfilling our goal of the first Whitehead theorem and puppe sequence, this definition of  $\vec{\pi}_n$  strikes me as elegant because it uses all of the ways for  $\vec{\Omega}^n X$  to map into  $\vec{\Omega}^{n+1} X$ .

The next symbols in the project’s “periodic table” that we construct, after  $\vec{\Omega}$  and  $\vec{\pi}_n$ , will be  $\vec{B}$  and  $\vec{E}$ , which we feature in the chapter on Puppe sequence (a).

A useful thing for us to construct first is the boundary of a product of  $\Delta^1$ ’s and the boundary of a directed simplex. We might even like to expand on this later, but for now just consider for a moment how each might be made out of a glueing construction involving face maps.

Even though the  $\vec{\pi}_n$ ’s can be defined using  $\vec{\Omega}^n X$  and various face maps  $f_{-(n,b)} : \vec{\Omega}^{n-1} X \rightarrow \vec{\Omega}^n X$  for  $b : \{0, 1\}$ , it may be nice to have this as a result, with the definition one featuring two cubes glued together along their boundary.

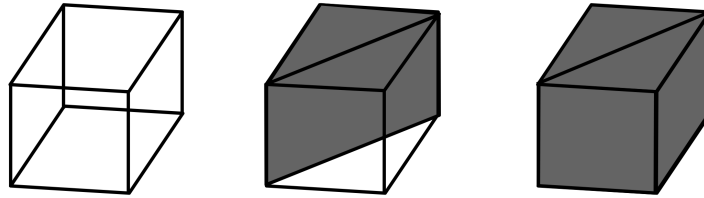
This means that we want directed box filling in addition to directed prism filling (but which also uses directed prism filling in its proof).

**Box Filling (BF)** Let  $Y$  be a quasicategory, and let  $f, g : \partial\Delta^n \rightarrow Y$ . A homotopy  $h : \partial\Delta^n \times \Delta^1 \rightarrow Y$  between  $f, g : \partial\Delta^n \rightarrow Y$  extends to a map  $H : \Delta^n \times \Delta^1 \rightarrow Y$ ; this follows from the condition that  $Y$  be a quasicategory.  $H(-, 1)$  and  $g$  match on  $\partial\Delta^n$ , producing a map  $f : X \rightarrow Y$ , where  $X$  consists of two copies of  $\Delta^n$  glued together at the boundary. Consider a space  $X'$  formed as a quotient of  $\Delta^n \times \Delta^1$  by  $\partial\Delta^n \times \Delta^1$ . There is a map  $\phi : X \rightarrow X'$ . An induction hypothesis on  $f$  and  $g$  involving  $\pi_n$  ensures that the aparent map  $X \rightarrow Y$  lifts along  $\phi$ , producing a map from  $\Delta^n \times \Delta^1$  which is constant on  $\partial\Delta^n \times \Delta^1$ . Stacking this on top of  $H$  can be done using an isomorphism between  $\Delta^1$  and  $\Delta^1$  glued with itself along different endpoints. Altogether this produces a homotopy between  $f$  and  $g$ .

This goes hand-in-hand with a definition of  $\vec{\pi}_n$  which suits (I) and (II) in the introduction to chapter (3). If we make sure to prove lemmas...

The box filling and prism filling HEPs can be extended to the case of attaching all cells of a particular fixed dimension and as indexed by simplicial set arising from a set (or Lean 4 Type). That is, we might like to extend  $\times ()$  (or possibly somehow a  $\text{Set}$  as well), and that we may find an interest in the following two definitions of  $\vec{\pi}_n$ , which are designed to fulfill both (I) and (II) in the chapter’s introduction.

Breaking down BF further can be done conveniently using sub-simplicial sets, just like we used in the proof of prism filling.



Decomposing  $(\Delta^1)^n$  into a colimit involving  $n!$   $\Delta^n$ 's Consider the face maps  $f_i : \Delta^n \rightarrow \Delta^{n+1}$

The decomposition The box filling lemma allows us to prove HEP:

The HEP in the last

.. $H(-,1)$  and  $g$  match on  $\partial\Delta^n$ , producing a map  $f : X \rightarrow Y$ , where  $X$  consists of two copies of  $\Delta^n$  glued together at the boundary. Consider a space  $X'$  formed as a quotient of  $\Delta^n \times \Delta^1$  by  $\partial\Delta^n \times \Delta^1$ . There is a map  $\phi : X \rightarrow X'$ . An induction hypothesis on  $f$  and  $g$  involving  $\pi_n$  ensures that the apparent map  $X \rightarrow Y$  lifts along  $\phi$ , producing a map from  $\Delta^n \times \Delta^1$  which is constant on  $\partial\Delta^n \times \Delta^1$ . Stacking this on top of  $H$  can be done using an isomorphism between  $\Delta^1$  and  $\Delta^1$  glued with itself along different endpoints. Altogether this produces a homotopy between  $f$  and  $g$ .

Imagine

# Chapter 11: The Category of Pairs of $\infty$ -Groupoids

# Chapter 12: Internal and Operadic Groupoids and their Actions

In this chapter, we discuss internal categories and internal presheaves in a pullback system. We may keep in mind that internal categories and internal presheaves can be formed in any category with pullbacks, even though we focus on the case of pullback systems because of our interest in Whitehead theorem (a).

After defining the category of internal categories  $D(\Gamma)$ , we proceed to observe how, for  $C, D : D(\Gamma)$ ,  $F : C \longrightarrow D$ ,  $(\vec{\omega} F).obj F$  forms an internal category. Further, in considering internal  $(\vec{P}_-(\Gamma) F)$ -presheaves for  $C, D : D(\Gamma)$ ,  $F : C \longrightarrow D$ , we proceed to make observations about  $(\vec{\omega} F).obj G$ .

Section	Description
<code>InternalGroupoid : Cat</code>	Internal categories
<code>InternalGroupoidAction G : Cat</code>	Internal C-presheaves
The internal category principal	$\mathbf{f} \times_{\mathbf{B}} \mathbf{f}$ forms an internal category
The internal presheaf principal	$\mathbf{f} \times_{\mathbf{B}} \mathbf{f}$ forms an internal presheaf
$\vec{P} C : \text{InternalCategory } D(\infty\text{-Cat})$	$\vec{\Omega} C$ forms a component of an internal category
$\vec{p} (\mathbb{1} C) D : \text{InternalPresheaf } D(\infty\text{-Cat}/C) (\vec{P} C)$	$\vec{\omega} (\mathbb{1} C) D$ forms a component of an internal C-presheaf

## 44. InternalGroupoid $\Gamma$

In this chapter I define an internal category. Internal categories are most commonly defined on categories with enough pullbacks, but here I may also like to keep in mind that it is valuable to be able to iterate InternalGroupoid in the way of composition.

Lean 11

```
-- definition of an internal category in a pullback
-- ↪ system
/-
structure internal_groupoid ( $\Gamma$  : Cat) where
  Obj : .Obj
  Mor : .Obj
  Dom : .Hom Mor Obj
  Cod : .Hom Mor Obj
  Idn : .Hom Obj Mor
  Fst : .Cmp Obj Mor Obj Idn Dom =  $\mathbb{1}_\Gamma$  (Obj) Obj
  Snd : .Cmp Obj Mor Obj Idn Cod =  $\mathbb{1}_\Gamma$  (Obj) Obj
-- Cmp :  $D(\Gamma).$   $\Gamma$ .PullObj ...
-- Id1 :  $D(\Gamma).$ 
-- Id2 :  $D(\Gamma).$ 
-- Ass :  $D(\Gamma).$ 
- /
```

The internal functor structure combines with the internal category structure to give a category of internal categories in a pullback system.

## Lean 12

```

-- definition of an internal functor in a pullback
  ↪ system
structure internal_functor ( $\Gamma$  : pullback_system) (C :
  ↪ internal_category  $\Gamma$ ) (D : internal_category  $\Gamma$ )
  ↪ where
    Obj : D( $\Gamma$ ).Hom C.Obj D.Obj
-- Mor : D( $\Gamma$ ).
-- Fst : D( $\Gamma$ ).
-- Snd : D( $\Gamma$ ).
-- Idn : D( $\Gamma$ ).
-- Cmp : D( $\Gamma$ ).

```

## Lean 13

```

-- definition of the identity internal functor in a
  ↪ pullback system
def InternalCategoryIdn ( $\Gamma$  : pullback_system) (C :
  ↪ internal_category  $\Gamma$ ) : (internal_functor  $\Gamma$  C C)
  ↪ := sorry

```

## Lean 14

```

-- definition of the composition of internal
  ↪ functors in a pullback system
def InternalCategoryCmp ( $\Gamma$  : pullback_system) (C :
  ↪ internal_category  $\Gamma$ ) (D : internal_category  $\Gamma$ ) (E
  ↪ : internal_category  $\Gamma$ ) (F : internal_functor  $\Gamma$  C
  ↪ D) (G : internal_functor  $\Gamma$  D E) :
  ↪ (internal_functor  $\Gamma$  C E) := sorry

```

## Lean 15

```

-- proving the the first identity law for internal
  ↪ categories in a pullback system
def InternalCategoryId1 ( $\Gamma$  : pullback_system) (X :
  ↪ internal_category  $\Gamma$ ) (Y : internal_category  $\Gamma$ ) (f
  ↪ : internal_functor  $\Gamma$  X Y) : InternalCategoryCmp  $\Gamma$ 
  ↪ X Y Y f (InternalCategoryIdn  $\Gamma$  Y) = f := sorry

```

## Lean 16

```

-- proving the second identity law for internal
  ↳ categories in a pullback system
def InternalCategoryId2 (Γ : pullback_system) (X :
  ↳ internal_category Γ) (Y : internal_category Γ) (f
  ↳ : internal_functor Γ X Y) : (InternalCategoryCmp
  ↳ Γ X X Y (InternalCategoryIdn Γ X) f = f) := sorry

```

## Lean 17

```

-- proving the associativity law for internal
  ↳ categories in a pullback system
def InternalCategoryAss (Γ : pullback_system) (W :
  ↳ internal_category Γ) (X : internal_category Γ) (Y
  ↳ : internal_category Γ) (Z : internal_category Γ)
  ↳ (f : internal_functor Γ W X) (g :
  ↳ internal_functor Γ X Y) (h : internal_functor Γ Y
  ↳ Z) : InternalCategoryCmp Γ W X Z f
  ↳ (InternalCategoryCmp Γ X Y Z g h) =
  ↳ InternalCategoryCmp Γ W Y Z (InternalCategoryCmp
  ↳ Γ W X Y f g) h := sorry

```

## Lean 18

```

/-
def InternalCategory (Γ : pullback_system) :
  ↳ Cat.Obj := {Obj := internal_category Γ, Hom :=
  ↳ internal_functor Γ, Idn := InternalCategoryIdn
  ↳ Γ, Cmp := InternalCategoryCmp Γ, Id1 :=
  ↳ InternalCategoryId1 Γ, Id2 :=
  ↳ InternalCategoryId2 Γ, Ass :=
  ↳ InternalCategoryAss Γ}
-/

```

## Lean 19

```

-- notation : 2000 "Cat_(" Γ ")" => InternalCategory
  ↳ Γ

```

## 45. InternalGroupoidAction $\Gamma$ C

The mentioned book *Galois Theories* by Janelidze and Borceux features a definition of internal presheaves for an internal groupoid in chapter 7 which makes a good reference for the present discussion.

Lean 20

```
-- internal C-presheaves
-- def internal_presheaf (C : (InternalCategory
  ↪ C).Obj) : Type := sorry
```

Lean 21

```
-- defining an internal functor between internal
  ↪ C-presheaves
/-
def ShfHom (C : (InternalCategory  $\Gamma$ ).Obj) (F :
  ↪ internal_presheaf  $\Gamma$  C) (G : internal_presheaf
  ↪  $\Gamma$  C) : Type := sorry
-/-
```

Lean 22

```
-- defining the identity internal functor of an
  ↪ internal C-sheaf
/-
def Shfidn ( $\Gamma$  : pullback_system) (C :
  ↪ (InternalCategory  $\Gamma$ ).Obj) (F :
  ↪ internal_presheaf  $\Gamma$  C) : ShfHom  $\Gamma$  C F F :=
  ↪ sorry
-/-
```



## Lean 23

```

-- defining the composition of internal functors
def Shfcmp ( $\Gamma$  : pullback_system) (C :
   $\rightarrow$  (InternalCategory  $\Gamma$ ).Obj) (F : internal_presheaf
   $\rightarrow$   $\Gamma$  C) (G : internal_presheaf  $\Gamma$  C) (H :
   $\rightarrow$  internal_presheaf  $\Gamma$  C) (f : ShfHom  $\Gamma$  C F G) (g :
   $\rightarrow$  ShfHom  $\Gamma$  C G H) : ShfHom  $\Gamma$  C F H := sorry

```

## Lean 24

```

-- proving the first identity law for internal
   $\rightarrow$  functors
/-
def Shf... ( $\Gamma$  : pullback_system) (C :
   $\rightarrow$  (InternalCategory  $\Gamma$ ).Obj) (X :
   $\rightarrow$  internal_presheaf  $\Gamma$  C) (Y : internal_presheaf
   $\rightarrow$   $\Gamma$  C) (f : ShfHom  $\Gamma$  C X Y) : ((ShfCmp  $\Gamma$  C X Y
   $\rightarrow$  Y f (ShfIdn  $\Gamma$  C Y)) = f) := sorry
-/

```

## Lean 25

```

-- proving the second identity law for internal
   $\rightarrow$  functors
/-
def ShfId2 ( $\Gamma$  : pullback_system) (C :
   $\rightarrow$  (InternalCategory  $\Gamma$ ).Obj) (X :
   $\rightarrow$  internal_presheaf  $\Gamma$  C) (Y : internal_presheaf
   $\rightarrow$   $\Gamma$  C) (f : ShfHom  $\Gamma$  C X Y) : ((ShfCmp  $\Gamma$  C X X
   $\rightarrow$  Y (ShfIdn  $\Gamma$  C X) f) = f) := sorry
-/

```

## Lean 26

```

-- proving the associativity law for internal
  → functors
/-
def ShfAss (Γ : pullback_system) (C :
  → (InternalCategory Γ).Obj) (W :
  → internal_presheaf Γ C) (X : internal_presheaf
  → Γ C) (Y : internal_presheaf Γ C) (Z :
  → internal_presheaf Γ C) (f : ShfHom Γ C W X)
  → (g : ShfHom Γ C X Y) (h : ShfHom Γ C Y Z) :
  → (ShfCmp Γ C) W X Z f ((ShfCmp Γ C) X Y Z g
  → h) = (ShfCmp Γ C) W Y Z ((ShfCmp Γ C) W X Y
  → f g) h := sorry
-/

```

## Lean 27

```

/-
def InternalPresheaf (Γ : pullback_system) (C :
  → (InternalCategory Γ).Obj) : Cat.Obj := {Obj :=
  → internal_presheaf Γ C, Hom := ShfHom Γ C, Idn
  → := ShfIdn Γ C, Cmp := ShfCmp Γ C, Id1 :=
  → ShfId1 Γ C, Id2 := ShfId2 Γ C, Ass := ShfAss
  → Γ C}
-/

```

## Lean 28

```

/-
notation : 2000 "Shf_ (\" Γ \")" => InternalPresheaf
  → Γ
-/

```

Next we approach the internal category principal and internal presheaf principals, which concern how (directed) homotopy pullback can produce internal categories and internal presheaves.

## 46. Operadic Groupoid

## 47. Operadic Groupoid Action

## 48. The $P$ and $0$ Functors

In this section we mention the internal category principal...

## 49. The $p$ and $o$ Functors

Next we mention the internal presheaf principal, which says that the pullback of any morphism with another forms a component of an internal presheaf in any category with pullbacks. Just as is the case for the last theorem, the most general form of this idea works for non-commutative analogues of pullback, whereas the case of pullback gives an internal groupoid action.

## PART 4: $\infty$ -CATEGORIES

## Chapter 13: $\infty$ -Cat

This chapter and the next chapter are more technical and difficult than the rest of the book.

1. Defining  $D(\infty\text{-Cat})$  by formally inverting weak equivalences.
2. Defining  $D(\infty\text{-Cat}/C)$  by formally inverting weak equivalences.
3. Defining a fibrant replacement functor for  $\infty\text{-Cat}$
4. Defining a fibrant replacement functor for  $\infty\text{-Cat}/C$
5. We first construct both the category  $D(\infty\text{-Cat})$  and, for each  $C : D(\infty\text{-Cat})$ , the category  $D(\infty\text{-Cat}/C)$  by formally inverting weak equivalences in the category of quasicategories and the category of quasicategories over  $C$ .



## 50. $\Omega$

Our choice of symbols reflects our choice of three variations of the Whitehead theorem and three Puppe sequences.  $\vec{\Omega}$ , the analogue of loop space, is the internal hom functor  $[\Delta^1, -] : \infty\text{-Cat} \rightarrow \infty\text{-Cat}$ . This is not hard to construct, with the main lemma being that the path space of a quasicategory has the quasicategory lifting condition.

We will be interested in one formal model of  $D(\infty\text{-Cat})$  which consists of formal compositions  $f_1 \bullet g_1 \bullet f_2 \bullet g_2 \bullet \cdots \bullet f_n \bullet g_n$ , where  $g_n : \text{Dom}(f_{n+1}) \rightarrow ???$  is a weak equivalence, and something similar for  $D(\infty\text{-Cat})$ . However, it is still vital to have the replacement functor  $\text{repl}$ , which ensures the Whitehead theorem for particular  $\infty$ -categories which are constructed out of attaching maps.

## 51. $\omega$

$\vec{\Omega}$  is to internal categories as  $\vec{\omega}$  is to internal  $\mathbf{C}$ -presheaves. It is also called directed homotopy pullback. These functors will later be used to produce functors  $\vec{P} : D(\infty\text{-Cat}) \longrightarrow \text{InternalCategory } D(\infty\text{-Cat})$  and  $\vec{p} : D(\infty\text{-Cat}/\mathbf{C}) \longrightarrow \text{InternalPresheaf } (\vec{P} \mathbf{C})$   $D(\infty\text{-Cat}/\mathbf{C})$ .

## 52. $\pi_n$

The mentioned functors  $\vec{\pi}_n$  are designed with both Whitehead theorem (a) and Puppe sequence (a) in mind.

# Chapter 14: The Whitehead Theorem for $\infty$ -Categories

## 53. Directed Cubical Complexes

...

In this chapter, we take on the objective of Whitehead theorem (a), out of which we will prove the other more concrete Whitehead theorems:

$$\forall(E:D(\infty\text{-Cat})), \forall(B:D(\infty\text{-Cat})), \forall(F:E \longrightarrow B), \forall(G:E \longrightarrow B), (\forall(n:\text{Nat}), (\vec{\pi}_n F = \vec{\pi}_n G)) \longrightarrow F = G$$

We can attempt to form a slightly different category, much like the above, called  $\mathcal{D}(\infty\text{-Cat})$ , at first, and in a formal way, so as to create a category whose object component  $\mathcal{D}(\infty\text{-Cat}).\alpha$  matches the object component  $\infty\text{-Cat}.\alpha$  while featuring the above theorem in a formal way. However, with this as our model of  $D(\infty\text{-Cat})$ , we may then also be interested in the establishment of a model in which the Whitehead theorem is demonstrated, with the main idea being to prove two complementary concepts:

1. (REP) Establish a kind of “weak equivalent fibrant replacement”  $R : \infty\text{-Cat}.\alpha \longrightarrow \infty\text{-Cat}.\alpha$  ( $\alpha$  gives the object component in Mathlib’s category theory library), analogous to CW-complex replacement in Whitehead’s original paper. It’s especially nice if  $R$  forms the object component of a functor  $F : \infty\text{-Cat} \longrightarrow \infty\text{-Cat}$ .  $D(F) : D(\infty\text{-Cat}) \longrightarrow D(\infty\text{-Cat})$  should be a categorical equivalence, and that is what we will do.
2. (HEP) For the object  $R X$ , demonstrate that any  $F, G : (R X) \longrightarrow Y$  such that  $\forall(n:\text{Nat}), (\vec{\pi}_n F = \vec{\pi}_n G)$ , there is a directed homotopy equivalence between  $F$  and  $G$ . Note that “directed homotopy equivalence” consists of a composable sequence of simple directed homotopies  $H[i] : \Delta^1 \times (R X) \longrightarrow Y$ ,  $1 \leq i \leq n$ , with even  $H[i]$  running reverse to the odd  $H[i]$ .

Both of these will use induction on Lean’s  $\text{Nat}$ . The first of these could be called a REP (for REplacement Property, but this isn’t usual terminology), and the second typically uses induction and a HEP (Homotopy Extension Property). Our REP will consist of objects made out of particular kinds of pushouts called attaching maps, and can be made functorial. Proving the HEP can be done by well-order induction on the attaching maps present in our choice of  $R$ , thereby reducing to the case of extending a homotopy along a single attachment.

Our HEPa (directed box filling) is similar to the HEP shown in Whitehead's original paper, and to the approach detailed in Hatcher's textbook, though no doubt modified to suit our two goals:

- (I) The analogue of the Puppe sequence on the front cover needs to hold.
- (II) The first Whitehead theorem on the front cover needs to hold.

These two considerations determine our choice of  $\vec{\pi}_n$ ,  $\vec{\Omega}$ , and  $\vec{\omega}$ . We take  $\vec{\Omega}$  to be (simply) the internal hom functor  $[\Delta^1, -]$  (which requires showing that  $\vec{\Omega}X$  has the inner-horn filling condition).  $\vec{\omega}$  is then defined as a certain pullback of  $\vec{\Omega}$ , and  $\vec{\pi}_n$  is designed to produce a Puppe sequence with a meaningful notion of exactness by which we can demonstrate the goal of recognition theorems (i) and (ii). Specifically, it makes sense to use cubes in our definition of  $\vec{\pi}_n$  because of how they are representing objects of  $\vec{\Omega}^n$ . Meanwhile, it is also clear that the quotient producing  $\vec{\pi}_n$  is subtle in exactly how it requires fixing the endpoints of a sequence of alternating directed homotopies. We will define  $\vec{\pi}_n$ 's by identifying those objects  $x, y: \vec{\Omega}^n X$  which are homotopic by a homotopy which restricts to a constant along the face maps  $f[\vec{a}]: \vec{\Omega}^{n-1} X \rightarrow \vec{\Omega}^{n-1} X$  (which correspond to pairs  $(n, b)$ , where  $b: \text{Bool}$ ).

Imagine for a moment the picture of a square shaped cushion; we might make such a cushion by first soeing together 6 squares of cloth and filling it with material, then "soeing the walls down to a square". Here we go with this:

1. Define a n-cubical cushion using the boundary of an n-1 cube times  $\Delta^1$ , i.e. the quotient of  $(\Delta^1)^{n-1} \times \Delta^1$  by an equivalence relation, but we have to start our model somewhere), or perhaps more easily the pushout of  $f: \Delta^1 \times (\partial((\Delta^1)^n)) \rightarrow (\Delta^1)^{n+1}$  by the projection map  $\Delta^1 \times (\partial((\Delta^1)^n)) \rightarrow \partial((\Delta^1)^n)$
2. Define a simplicial cushion using the boundary of an n-1 simplex times  $\Delta^1$ , i.e. the quotient of  $(\Delta^1)$  by an equivalence relation, or perhaps more easily the pushout of  $f: \Delta^1 \times (\partial(\Delta^n)) \rightarrow (\Delta^1) \times \Delta^n$  by the projection map  $\Delta^1 \times (\partial((\Delta^1)^n)) \rightarrow \partial(\Delta^n)$

The boundary of a cushion is a pouch, isomorphic to a pushout of two cubes glued together at their boundaries:

1. Define a n-cubical pouch as the pushout of two boundary maps  $\partial((\Delta^1)^n) \rightarrow (\Delta^1)^n$
2. Define a simplicial pouch as the pushout of two boundary maps  $\partial(\Delta^n) \rightarrow \Delta^n$

Notice that paths in  $\vec{\Omega}^n X$  produce paths in  $\vec{\Omega}^{n-1} X$  in as many ways as there are face maps  $(\Delta^1)^{n-1} \rightarrow \Delta^{1n}$ , these could be called restrictions and are no doubt related to the pouches and cushions we just defined. The cartesian closed structure on simplicial sets with the lifting condition clarifies the relationship between the two available definitions of  $\vec{\pi}_n$ :

1. Homotopies of maps from a cube which are constant on the boundary
2. Paths of maps in  $\vec{\Omega}^{n-1}X$  which produce constant maps under the mentioned restrictions.
3. Maps from a pouch mod an equivalence relation (really we phrase this as a pushout!), namely the equivalence relation in which any two maps from a pouch that extend to maps from a cushion are identified.

After we construct  $\vec{\pi}_n$  in the first section, we will be in a place to demonstrate that the natural transformation  $\text{weak\_equivalence} : \text{repl} \longrightarrow (\mathbb{1} \infty\text{-Cat})$  consists of weak equivalences (a fact which we call REP, which is short for REplacement Principal). This is covered in the section titled REP, which also constructs  $\text{repl}$  and  $\text{weak\_requivalence}$ .

In sum, the goal of the present chapter is to use similar insights to the proof of the Whitehead theorem featured Hatcher's textbook to prove  $\text{Wa}$  and  $\text{Pa}$  for the model of quasicategories, using Mathlib's predefined horns and simplices in its simplicial sets section. The main difference is that our work must take care to respect the directed nature of quasicategories.

1. Defining  $\text{repl}$
- 2.

## 54. REP

We have divided the work of proving Whitehead theorem (a) into two steps: REP and HEP. In this section, we construct a functor  $\text{repl} : \infty\text{-Cat} \longrightarrow \infty\text{-Cat}$  along with a natural transformation  $\text{weak\_equivalence} : \text{repl} \longrightarrow (\mathbb{1} \infty\text{-Cat})$ . To construct  $\text{repl}$



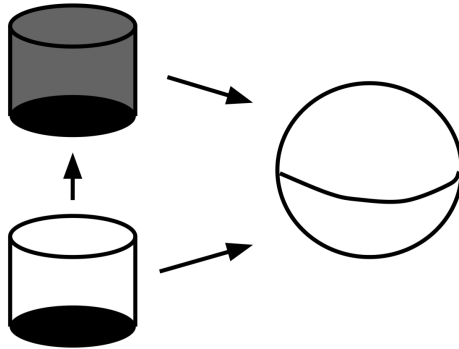
## 55. HEP

Consider the context of , supposing that we have constructed a homotopy ... This gives a picture that is a bit like “filling up a jar”: a homotopy  $h$  : of  $f, g : \partial\Delta^2 \rightarrow Y$ , along with the value of  $g$  on  $\Delta^2$ , produces a “jar” shape in  $Y$ , which can be “filled up” to produce a homotopy  $h : \Delta^1 \times \Delta^2 \rightarrow Y$ . This is easier for simplicial-based approaches than for point-set topological approaches, the latter of which needs extra steps that deform a map into a cellular map.

This construction, in the case of point set topology, often involves first deforming maps so as to be cellular; however our analogue of CW complexes allows us to skip this step.

This construction (HEP for quasicategories) may even be equivalent to the quasi-category lifting condition if we are lucky. It is also the main technical device allowing for our concrete choice of model (quasicategories).

In this section, we demonstrate this extension property and use it to conclude the Whitehead theorem for  $\infty$ -categories stated above.

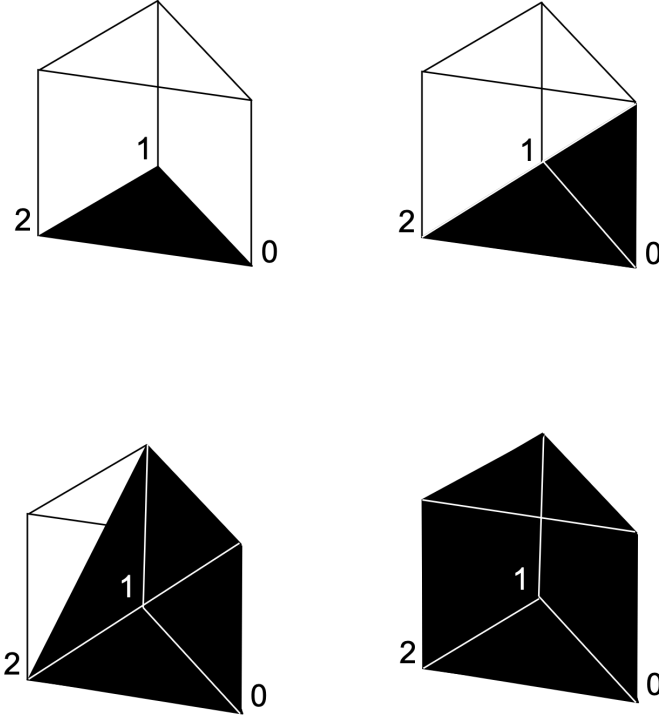


**Prism Filling (PF)** Let  $Y$  be a quasicategory, and let  $f, g : \partial\Delta^n \rightarrow Y$ . A homotopy  $h : \partial\Delta^n \times \Delta^1 \rightarrow Y$  between  $f, g : \partial\Delta^n \rightarrow Y$  extends to a map  $H : \Delta^n \times \Delta^1 \rightarrow Y$ ;

this follows from the condition that  $Y$  be a quasicategory.  $H(-,1)$  and  $g$  match on  $\partial\Delta^n$ , producing a map  $f : X \rightarrow Y$ , where  $X$  consists of two copies of  $\Delta^n$  glued together at the boundary. Consider a space  $X'$  formed as a quotient of  $\Delta^n \times \Delta^1$  by  $\partial\Delta^n \times \Delta^1$ . There is a map  $\phi : X \rightarrow X'$ . An induction hypothesis on  $f$  and  $g$  involving  $\pi_n$  ensures that the apparent map  $X \rightarrow Y$  lifts along  $\phi$ , producing a map from  $\Delta^n \times \Delta^1$  which is constant on  $\partial\Delta^n \times \Delta^1$ . Stacking this on top of  $H$  can be done using an isomorphism between  $\Delta^1$  and  $\Delta^1$  glued with itself along different endpoints. Altogether this produces a homotopy between  $f$  and  $g$ .

Directed prism filling may combine fruitfully with the yoneda lemma and/or the fact that simplicial sets are determined by the sets  $[\Delta^n, X]$  along with combinatorial information (face and degeneracy maps).

Decomposing  $\Delta^n \times \Delta^1$  into a colimit involving  $n+1$   $\Delta^{n+1}$ 's ...



In the above, it may be easier if we make use of sub-simplicial sets and prove the theorem using that colimit applied to a natural isomorphism of diagrams products an isomorphism.

The decomposition

A definition of  $\vec{\pi}_n$  which is consistent with our goals of  $W_a$  and  $P_a$  is one as a certain pushout involving  $(\vec{\Omega}^n X)$ —one which amounts to taking an equivalence relation

by paths in  $\vec{\Omega}^n X$  which restrict to constant paths along the face maps  $f_{[i]} : \vec{\Omega}^{n-1} X \rightarrow \vec{\Omega}^n X$ . Here,  $\vec{\Omega}$  is easy to define in the model of quasi-categories, and it amounts to . Besides fulfilling our goal of the first Whitehead theorem and puppe sequence, this definition of  $\vec{\pi}_n$  strikes me as elegant because it uses all of the ways for  $\vec{\Omega}^n X$  to map into  $\vec{\Omega}^{n+1} X$ .

The next symbols in the project’s “periodic table” that we construct, after  $\vec{\Omega}$  and  $\vec{\pi}_n$ , will be  $\vec{B}$  and  $\vec{E}$ , which we feature in the chapter on Puppe sequence (a).

A useful thing for us to construct first is the boundary of a product of  $\Delta^1$ ’s and the boundary of a directed simplex. We might even like to expand on this later, but for now just consider for a moment how each might be made out of a glueing construction involving face maps.

Even though the  $\vec{\pi}_n$ ’s can be defined using  $\vec{\Omega}^n X$  and various face maps  $f_{-(n,b)} : \vec{\Omega}^{n-1} X \rightarrow \vec{\Omega}^n X$  for  $b : \{0, 1\}$ , it may be nice to have this as a result, with the definition one featuring two cubes glued together along their boundary.

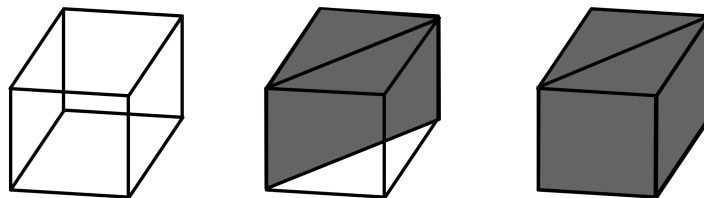
This means that we want directed box filling in addition to directed prism filling (but which also uses directed prism filling in its proof).

**Box Filling (BF)** Let  $Y$  be a quasicategory, and let  $f, g : \partial\Delta^n \rightarrow Y$ . A homotopy  $h : \partial\Delta^n \times \Delta^1 \rightarrow Y$  between  $f, g : \partial\Delta^n \rightarrow Y$  extends to a map  $H : \Delta^n \times \Delta^1 \rightarrow Y$ ; this follows from the condition that  $Y$  be a quasicategory.  $H(-, 1)$  and  $g$  match on  $\partial\Delta^n$ , producing a map  $f : X \rightarrow Y$ , where  $X$  consists of two copies of  $\Delta^n$  glued together at the boundary. Consider a space  $X'$  formed as a quotient of  $\Delta^n \times \Delta^1$  by  $\partial\Delta^n \times \Delta^1$ . There is a map  $\phi : X \rightarrow X'$ . An induction hypothesis on  $f$  and  $g$  involving  $\pi_n$  ensures that the aparent map  $X \rightarrow Y$  lifts along  $\phi$ , producing a map from  $\Delta^n \times \Delta^1$  which is constant on  $\partial\Delta^n \times \Delta^1$ . Stacking this on top of  $H$  can be done using an isomorphism between  $\Delta^1$  and  $\Delta^1$  glued with itself along different endpoints. Altogether this produces a homotopy between  $f$  and  $g$ .

This goes hand-in-hand with a definition of  $\vec{\pi}_n$  which suits (I) and (II) in the introduction to chapter (3). If we make sure to prove lemmas...

The box filling and prism filling HEPs can be extended to the case of attaching all cells of a particular fixed dimension and as indexed by simplicial set arising from a set (or Lean 4 Type). That is, we might like to extend  $\times ()$  (or possibly somehow a  $\text{Set}$  as well), and that we may find an interest in the following two definitions of  $\vec{\pi}_n$ , which are designed to fullfill both (I) and (II) in the chapter’s introduction.

Breaking down BF further can be done conveniently using sub-simplicial sets, just like we used in the proof of prism filling.



Decomposing  $(\Delta^1)^n$  into a colimit involving  $n!$   $\Delta^n$ 's Consider the face maps  $f_i : \Delta^n \rightarrow \Delta^{n+1}$

The decomposition The box filling lemma allows us to prove HEP:

## 56. The Whitehead Theorem for $\infty$ -Cat

The HEP in the last

..H(-,1) and g match on  $\partial\Delta^n$ , producing a map  $f: X \rightarrow Y$ , where X consists of two copies of  $\Delta^n$  glued together at the boundary. Consider a space  $X'$  formed as a quotient of  $\Delta^n \times \Delta^1$  by  $\partial\Delta^n \times \Delta^1$ . There is a map  $\phi: X \rightarrow X'$ . An induction hypothesis on f and g involving  $\pi_n$  ensures that the aparent map  $X \rightarrow Y$  lifts along  $\phi$ , producing a map from  $\Delta^n \times \Delta^1$  which is constant on  $\partial\Delta^n \times \Delta^1$ . Stacking this on top of H can be done using an isomorphism between  $\Delta^1$  and  $\Delta^1$  glued with itself along different endpoints. Altogether this produces a homotopy between f and g.

Imagine

# Chapter 15: The Category of Pairs of $\infty$ -Categories

...

# Chapter 16: Internal and Operadic Categories and their Presheaves

In this chapter, we discuss internal categories and internal presheaves in a pullback system. We may keep in mind that internal categories and internal presheaves can be formed in any category with pullbacks, even though we focus on the case of pullback systems because of our interest in Whitehead theorem (a).

After defining the category of internal categories  $D(\Gamma)$ , we proceed to observe how, for  $C, D : D(\Gamma)$ ,  $F : C \longrightarrow D$ ,  $(\vec{\omega} F).obj F$  forms an internal category. Further, in considering internal  $(\vec{P}_-(\Gamma) F)$ -presheaves for  $C, D : D(\Gamma)$ ,  $F : C \longrightarrow D$ , we proceed to make observations about  $(\vec{\omega} F).obj G$ .

Section	Description
<b>InternalCategory</b> $\Gamma : Cat$	Internal categories
<b>InternalPresheaf</b> $\Gamma C : Cat$	Internal C-presheaves
The internal category principal	$f \times_{-}(B) f$ forms an internal category
The internal presheaf principal	$f \times_{-}(B) g$ forms an internal presheaf
$\vec{P} C : \text{InternalCategory } D(\infty\text{-Cat})$	$\vec{\Omega} C$ forms a component of an internal category
$\vec{p} (\mathbb{1} C) D : \text{InternalPresheaf } D(\infty\text{-Cat}/C) (\vec{P} C)$	$\vec{\omega} (\mathbb{1} C) D$ forms a component of an internal C-presheaf

## 57. InternalCategory $\Gamma$

In this chapter I define an internal category. Internal categories are most commonly defined on categories with enough pullbacks, but here I may also like to keep in mind that it is valuable to be able to iterate InternalCategory in the way of composition.

Lean 29

```
-- definition of an internal category in a pullback
-- ↪ system
/-
structure internal_category ( $\Gamma$  : Cat) where
  Obj : .Obj
  Mor : .Obj
  Dom : .Hom Mor Obj
  Cod : .Hom Mor Obj
  Idn : .Hom Obj Mor
  Fst : .Cmp Obj Mor Obj Idn Dom =  $\mathbb{1}_\Gamma$  Obj Obj
  Snd : .Cmp Obj Mor Obj Idn Cod =  $\mathbb{1}_\Gamma$  Obj Obj
-- Cmp :  $D(\Gamma).$  PulObj ...
-- Id1 :  $D(\Gamma).$ 
-- Id2 :  $D(\Gamma).$ 
-- Ass :  $D(\Gamma).$ 
- /
```

The internal functor structure combines with the internal category structure to give a category of internal categories in a pullback system.



## Lean 30

```

-- definition of an internal functor in a pullback
  ↪ system
structure internal_functor ( $\Gamma$  : pullback_system) (C :
  ↪ internal_category  $\Gamma$ ) (D : internal_category  $\Gamma$ )
  ↪ where
    Obj : D( $\Gamma$ ).Hom C.Obj D.Obj
-- Mor : D( $\Gamma$ ).
-- Fst : D( $\Gamma$ ).
-- Snd : D( $\Gamma$ ).
-- Idn : D( $\Gamma$ ).
-- Cmp : D( $\Gamma$ ).

```

## Lean 31

```

-- definition of the identity internal functor in a
  ↪ pullback system
def InternalCategoryIdn ( $\Gamma$  : pullback_system) (C :
  ↪ internal_category  $\Gamma$ ) : (internal_functor  $\Gamma$  C C)
  ↪ := sorry

```

## Lean 32

```

-- definition of the composition of internal
  ↪ functors in a pullback system
def InternalCategoryCmp ( $\Gamma$  : pullback_system) (C :
  ↪ internal_category  $\Gamma$ ) (D : internal_category  $\Gamma$ ) (E
  ↪ : internal_category  $\Gamma$ ) (F : internal_functor  $\Gamma$  C
  ↪ D) (G : internal_functor  $\Gamma$  D E) :
  ↪ (internal_functor  $\Gamma$  C E) := sorry

```

## Lean 33

```

-- proving the the first identity law for internal
  ↪ categories in a pullback system
def InternalCategoryId1 ( $\Gamma$  : pullback_system) (X :
  ↪ internal_category  $\Gamma$ ) (Y : internal_category  $\Gamma$ ) (f
  ↪ : internal_functor  $\Gamma$  X Y) : InternalCategoryCmp  $\Gamma$ 
  ↪ X Y Y f (InternalCategoryIdn  $\Gamma$  Y) = f := sorry

```

## Lean 34

```

-- proving the second identity law for internal
  ↳ categories in a pullback system
def InternalCategoryId2 (Γ : pullback_system) (X :
  ↳ internal_category Γ) (Y : internal_category Γ) (f
  ↳ : internal_functor Γ X Y) : (InternalCategoryCmp
  ↳ Γ X X Y (InternalCategoryIdn Γ X) f = f) := sorry

```

## Lean 35

```

-- proving the associativity law for internal
  ↳ categories in a pullback system
def InternalCategoryAss (Γ : pullback_system) (W :
  ↳ internal_category Γ) (X : internal_category Γ) (Y
  ↳ : internal_category Γ) (Z : internal_category Γ)
  ↳ (f : internal_functor Γ W X) (g :
  ↳ internal_functor Γ X Y) (h : internal_functor Γ Y
  ↳ Z) : InternalCategoryCmp Γ W X Z f
  ↳ (InternalCategoryCmp Γ X Y Z g h) =
  ↳ InternalCategoryCmp Γ W Y Z (InternalCategoryCmp
  ↳ Γ W X Y f g) h := sorry

```

## Lean 36

```

/-
def InternalCategory (Γ : pullback_system) :
  ↳ Cat.Obj := {Obj := internal_category Γ, Hom :=
  ↳ internal_functor Γ, Idn := InternalCategoryIdn
  ↳ Γ, Cmp := InternalCategoryCmp Γ, Id1 :=
  ↳ InternalCategoryId1 Γ, Id2 :=
  ↳ InternalCategoryId2 Γ, Ass :=
  ↳ InternalCategoryAss Γ}
-/

```

## Lean 37

```

-- notation : 2000 "Cat_(" Γ ")" => InternalCategory
  ↳ Γ

```

## 58. InternalPresheaf $\Gamma$ C

The mentioned book *Galois Theories* by Janelidze and Borceux features a definition of internal presheaves for an internal groupoid in chapter 7 which makes a good reference for the present discussion.

Lean 38

```
-- internal C-presheaves
-- def internal_presheaf (C : (InternalCategory
  ↪ C).Obj) : Type := sorry
```

Lean 39

```
-- defining an internal functor between internal
  ↪ C-presheaves
/-
def ShfHom (C : (InternalCategory  $\Gamma$ ).Obj) (F :
  ↪ internal_presheaf  $\Gamma$  C) (G : internal_presheaf
  ↪  $\Gamma$  C) : Type := sorry
-/-
```

Lean 40

```
-- defining the identity internal functor of an
  ↪ internal C-sheaf
/-
def Shfidn ( $\Gamma$  : pullback_system) (C :
  ↪ (InternalCategory  $\Gamma$ ).Obj) (F :
  ↪ internal_presheaf  $\Gamma$  C) : ShfHom  $\Gamma$  C F F :=
  ↪ sorry
-/-
```

## Lean 41

```

-- defining the composition of internal functors
def Shfcmp ( $\Gamma$  : pullback_system) (C :
  → (InternalCategory  $\Gamma$ ).Obj) (F : internal_presheaf
  →  $\Gamma$  C) (G : internal_presheaf  $\Gamma$  C) (H :
  → internal_presheaf  $\Gamma$  C) (f : ShfHom  $\Gamma$  C F G) (g :
  → ShfHom  $\Gamma$  C G H) : ShfHom  $\Gamma$  C F H := sorry

```

## Lean 42

```

-- proving the first identity law for internal
  → functors
/-
def Shf... ( $\Gamma$  : pullback_system) (C :
  → (InternalCategory  $\Gamma$ ).Obj) (X :
  → internal_presheaf  $\Gamma$  C) (Y : internal_presheaf
  →  $\Gamma$  C) (f : ShfHom  $\Gamma$  C X Y) : ((ShfCmp  $\Gamma$  C X Y
  → Y f (ShfIdn  $\Gamma$  C Y)) = f) := sorry
-/

```

## Lean 43

```

-- proving the second identity law for internal
  → functors
/-
def ShfId2 ( $\Gamma$  : pullback_system) (C :
  → (InternalCategory  $\Gamma$ ).Obj) (X :
  → internal_presheaf  $\Gamma$  C) (Y : internal_presheaf
  →  $\Gamma$  C) (f : ShfHom  $\Gamma$  C X Y) : ((ShfCmp  $\Gamma$  C X X
  → Y (ShfIdn  $\Gamma$  C X) f) = f) := sorry
-/

```

## Lean 44

```

-- proving the associativity law for internal
  ↪ functors
/-
def ShfAss (Γ : pullback_system) (C :
  ↪ (InternalCategory Γ).Obj) (W :
  ↪ internal_presheaf Γ C) (X : internal_presheaf
  ↪ Γ C) (Y : internal_presheaf Γ C) (Z :
  ↪ internal_presheaf Γ C) (f : ShfHom Γ C W X)
  ↪ (g : ShfHom Γ C X Y) (h : ShfHom Γ C Y Z) :
  ↪ (ShfCmp Γ C) W X Z f ((ShfCmp Γ C) X Y Z g
  ↪ h) = (ShfCmp Γ C) W Y Z ((ShfCmp Γ C) W X Y
  ↪ f g) h := sorry
-/

```

## Lean 45

```

/-
def InternalPresheaf (Γ : pullback_system) (C :
  ↪ (InternalCategory Γ).Obj) : Cat.Obj := {Obj :=
  ↪ internal_presheaf Γ C, Hom := ShfHom Γ C, Idn
  ↪ := ShfIdn Γ C, Cmp := ShfCmp Γ C, Id1 :=
  ↪ ShfId1 Γ C, Id2 := ShfId2 Γ C, Ass := ShfAss
  ↪ Γ C}
-/

```

## Lean 46

```

/-
notation : 2000 "Shf_ (\" Γ \")" => InternalPresheaf
  ↪ Γ
-/

```

Next we approach the internal category principal and internal presheaf principals, which concern how (directed) homotopy pullback can produce internal categories and internal presheaves.

## 59. The $P$ and $O$ Functors

In this section we mention the internal category principal, which says that the pullback of any morphism with itself forms a component of an internal category in any category in which this pullback exists. In fact, the most general form of the theorem works for a noncommutative analogue of pullback.

## 60. The $p$ and $o$ Functors

Next we mention the internal presheaf principal, which says that the pullback of any morphism with another forms a component of an internal presheaf in any category with pullbacks. Just as is the case for the last theorem, the most general form of this idea works for non-commutative analogues of pullback, whereas the case of pullback gives an internal groupoid action.

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### Further reading:

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2. Saunders Mac Lane, “Categories for the Working Mathematician,” Graduate Texts in Mathematics, vol. 5, Springer-Verlag, New York, 1971.
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8. A comment thread concerning Jacob Lurie’s breakthrough prize and different approaches to homotopy on the computer
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### Some lectures, videos, and Stackexchange questions:

1. <https://www.youtube.com/watch?v=0b9t0gWumPI>
2. <https://www.youtube.com/watch?v=xYenPIeX6MY>
3. <https://mathoverflow.net/questions/5901/do-the-signs-in-puppe-sequences-matter>

### Ideas for future applications:

1. <https://arxiv.org/pdf/2206.13563.pdf>

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