

Geometric height on flag varieties

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November 12, 2024

Abstract

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1 Introduction

Under construction In this paper, we aim to provide a clear description of the height function and height filtration on a flag bundle over a curve, without making any assumptions about the characteristic of the base field, thereby addressing all characteristics. The results have been proven under the assumption of characteristic 0 in [FLQ24].

2 Height filtration and Successive minima

2.1 Strongly cononical reduction

Definition 2.1. A principal G -bundle F on C is called semistable if for any parabolic subgroup P , any reduction F_P of F to P and any dominant character λ of P which is trivial on $Z(G)$, we have $\langle \deg(F_P), \lambda \rangle \leq 0$. We say F is strongly semistable if for any non-constant finite morphism $f : C' \rightarrow C$, the pullback f^*F is semistable.

Definition 2.2. Let F be a principal G -bundle on C . A reduction F_Q of F to a parabolic subgroup Q is called canonical if the following two conditions hold:

- The principal L_Q bundle $F_Q \times_Q L_Q$ is semistable, where L_Q is the Levi subgroup of Q .
- For any non-trivial character λ of Q which is non-negative linear combination of simple roots, $\langle \deg(F_Q), \lambda \rangle > 0$.

We say F_Q is strongly canonical if for any non-constant finite morphism $f : C' \rightarrow C$, the pullback $f^*(F_Q)$ is canonical.

If the base field is of characteristic 0, then semistability (*resp.* canonical reduction) and strongly semistability (*resp.* strongly canonical reduction) are equivalent. But in positive characteristic, the notion of strongly canonical is necessary, which we shall see afterwards. Under the assumption of characteristic 0, the canonical reduction exists uniquely [Beh00]. But in positive characteristic, strongly canonical reduction does not exist in general.

When $G = \mathrm{GL}_n$, this semistability amounts to the semistability of the associated vector bundle with the natural representation; the canonical reduction amounts to the Harder-Narasimhan filtration of vector bundles with the natural representation.

2.2 Height function

Here we mainly follow [FLQ24].

Let F be a principal G -bundle with strongly canonical reduction F_Q to some parabolic subgroup $Q \subseteq G$. Let $P \subseteq G$ be a parabolic subgroup. Set $\mathcal{X} = F/P$ and $X = \mathcal{X}_K$.

A character $\lambda : P \rightarrow \mathbb{G}_m$ is called strictly anti-dominant if the natural pairing $\langle \alpha^\vee, \lambda \rangle < 0$ for any $\alpha \in \Delta \setminus \Delta_P$. Let $\lambda : P \rightarrow \mathbb{G}$ be a strictly anti-dominant character. Then the line bundle $M_\lambda = G \times_P k_\lambda$ on G/P is ample. Therefore $\mathcal{L}_\lambda = F \times_G M_\lambda$ is a relatively ample line bundle on $\mathcal{X} = F \times_G G/P$ and induces a height function

$$h_{\mathcal{L}_\lambda} : X(\bar{K}) \rightarrow \mathbb{Q}, \quad x \mapsto \frac{\mathcal{L}_\lambda \cdot \overline{\{x\}}}{\deg(x)}$$

where $\overline{\{x\}}$ is the Zariski closure of x in \mathcal{X} . For $x \in X(K)$, let $\sigma_x : C \rightarrow \mathcal{X}$ be the section induced by $x : \mathrm{Spec}(K) \rightarrow X$ via valuative criterion, and let $F_{P,x} = \sigma_x^* F$ be the corresponding reduction to P .

Lemma 2.3. *For $x \in X(K)$, $h_{\mathcal{L}_\lambda}(x) = \langle \deg(F_{P,x}), \lambda \rangle$.*

Proof. By definition, $h_{\mathcal{L}_\lambda}(x)$ is the degree of $\sigma_x^*(\mathcal{L}_\lambda)$ and $\langle \deg(F_{P,x}), \lambda \rangle$ is the degree of $F_{P,x} \times_P k_\lambda$. We have equality of line bundles

$$\sigma_x^*(\mathcal{L}_\lambda) = \sigma_x^*(F \times_P k_\lambda) = (\sigma_x^* F) \times_P k_\lambda = F_{P,x} \times_P k_\lambda$$

and the lemma follows by taking degree. \square

2.3 A height lower bound in Schubert cells

For $w \in W_Q \setminus W/W_P$, write $\mathcal{C}_w = F_Q \times_Q QwP/P$, $\mathcal{X}_w = F_Q \times_Q \overline{QwP}/P$, $C_w = \mathcal{C}_{w,K}$ and $X_w = \mathcal{X}_{w,K}$. We shall show that

Proposition 2.4. *For any $x \in C_w(\bar{K})$, $h_{\mathcal{L}_\lambda}(x) \geq \langle \deg F_Q, w\lambda \rangle$.*

Note that for any $w' \in W_Q$ and $\lambda \in X(T)$, we have $w'\lambda - \lambda \in \mathbb{Z}[\Delta_Q]$ and consequently $\langle \deg(F_Q), w'\lambda \rangle = \langle \deg(F_Q), \lambda \rangle$. Note also that for any $\lambda \in X(P)$ and $w \in W_P$, $w\lambda = \lambda$. So the number $\langle \deg(F_Q), w\lambda \rangle$ is well-defined for any $\lambda \in X(P)$ and $w \in W_Q \setminus W/W_P$.

Definition 2.5 (relative position). A reduction F_P of F to P is called in relative position $w \in W_Q \setminus W/W_P$ with respect to F_Q if the image of the natural map

$$F_Q \times_C F_P \rightarrow G_C, \quad (a, b) \mapsto a^{-1}b$$

lies in QwP_C . Here we implicitly apply the injections

$$F_P \hookrightarrow F_P \times_P G \cong F, \quad F_Q \hookrightarrow F_Q \times_Q G \cong F$$

Note that by the universal property of the quotient stacks $[G//Q \times P]$ and $[QwP//Q \times P]$, this definition coincides with the one in [SS15, 4.1].

Lemma 2.6. *When $x \in C_w(K)$, $F_{P,x}$ is of relative position w with respect to F_Q .*

Proposition 2.7 ([SS15, Proposition 4.6]). *If F_P is in relative position w with respect to F_Q , then $\langle \deg(F_P), \lambda \rangle \geq \langle \deg(F_Q), w\lambda \rangle$ for any anti-dominant character λ .*

2.4 Height filtration and successive minima

We compute the essential minimum of X_w .

Lemma 2.8 ([Jan96], Part I, §5.18). *On $\mathcal{X}_w = F_Q \times_Q \overline{QwP}/P \subseteq \mathcal{X}$, we have $\pi_* (\mathcal{L}_\lambda|_{\mathcal{X}_w}) = F_Q \times_Q H^0(\overline{QwP}/P, M_\lambda)$.*

Let V be any Q -representation of highest weight $\lambda \in X^*(T)$ and let $V = \bigoplus_\nu V[\nu]$ be its weight decomposition. Furthermore, let F_Q be the strongly canonical reduction of F . We define a filtration V_\bullet on the vector space V : For any rational number $q \in \mathbb{Q}$, we define the subspace V_q as the sum of weight spaces

$$V_q := \bigoplus_{\langle \deg F_Q, \nu \rangle \geq q} V[\nu]$$

Clearly, $V_{q'} \subseteq V_q$ whenever $q' \geq q$. We will consider subspaces V_q only for the finitely many $q \in \mathbb{Q}$ where a jump occurs, that is, only for those q such that $V_{q'} \subsetneq V_q$ for all $q' > q$. Let q_0 be the smallest and q_1 the largest rational number occurring among such q . Then V_{q_1} is the smallest non-zero filtration step and V_{q_0} equals V .

Then, by twisting the Q -subrepresentations V_q above by F_P , we obtain a filtration V_{\bullet, F_Q} of the vector bundle $V_{F_G} = F_G \times^G V$ by subbundles

$$0 \neq V[\lambda + \mathbb{Z}\Delta_M]_{F_M} = \left(\bigoplus_{\nu \in \lambda + \mathbb{Z}\Delta_M} V[\nu] \right)_{F_M} = V_{q_1, F_Q} \subsetneq \cdots \subsetneq V_{q, F_Q} \subsetneq \cdots \subsetneq V_{q_0, F_Q} = V_{F_G}.$$

Improving the result of Schieder¹, we proved that:

Proposition 2.9. *Assume the reduction F_Q is the strongly canonical reduction, then the filtration V_{\bullet, F_Q} of the vector bundle V_{F_G} is the Harder-Narasimhan filtration of V_{F_G} .*

We will need the following well-known theorem.

Theorem 2.10 ([RR84], Theorem 3.23). *If F is a strongly semistable G -bundle and $\rho : G \rightarrow H$ is a homomorphism that maps the connected component of the center of G into that of H , then $F \times^G H$ is a semistable H -bundle.*

Proof of Proposition 2.9. It will suffice to check two assertions: all graded pieces are semistable vector bundles and their slopes are decreasing.

As shown in [SS15] Proposition 5.1, we have

$$\mathrm{gr}_q(V_{\bullet, F_P}) = (\mathrm{gr}_q V_{\bullet})_{F_M}.$$

And its slope is computed as

$$\mu((\mathrm{gr}_q V_{\bullet})_{F_M}) = q.$$

Therefore, the slopes of the graded pieces are decreasing by definition.

To prove the semistability, consider the semisimplification W of $(\mathrm{gr}_q V_{\bullet})$ as a representation of M . Now apply [SS15] Proposition 3.2(a), we see that for each simple direct summand W_i of W the associated bundle W_{i, F_M} has slope q . Thus $(\mathrm{gr}_q V_{\bullet})_{F_M}$ admits a filtration by subbundles such that each graded piece is a semistable vector bundle of slope q , where the semistability is obtained by applying Theorem 2.10 to $M \rightarrow \mathrm{GL}(M_i)$. Then $(\mathrm{gr}_q V_{\bullet})_{F_M}$ is also semistable of slope q by the following lemma. \square

Lemma 2.11. *If E is a vector bundle on C that admits a filtration of subbundles*

$$0 \subsetneq E_1 \subsetneq \cdots \subsetneq E_r = E,$$

and all graded piece are semistable of slope equals to λ , so is E .

¹In [SS15], The case of characteristic 0 and a specific case of positive characteristic have been proven.

Proof. By induction on r it will suffice to prove the case $r = 2$, which amounts to the following assertion: Assume that in the short exact sequence of vector bundles

$$0 \rightarrow E' \rightarrow E \rightarrow E'' \rightarrow 0,$$

both E' and E'' are semistable of slope equal to λ , then so is E . It is easy to see $\mu(E) = \lambda$. To prove E is semistable, let F is a subbundle of E . Then we have the short exact sequence

$$0 \rightarrow F \cap E' \rightarrow F \rightarrow \bar{F} \rightarrow 0,$$

where \bar{F} is the image of F in E'' . Since E' and E'' are semistable, $F \cap E'$ and \bar{F} has slopes smaller than $\mu(E)$, and so is F . \square

Corollary 2.12. *Assume F_Q is the strongly canonical reduction of F . The Harder-Narasimhan filtration of $F_Q \times_Q H^0(\overline{QwP}/P, M_\lambda)$ is $F_Q \times_Q H^0(\overline{QwP}/P, M_\lambda)_{\bullet, \deg(F_Q)}$. Moreover, the maximal slope is $\langle \deg(F_Q), w\lambda \rangle$.*

Proof. The first assertion is obtained by taking V to be $H^0(\overline{QwP}/P, M_\lambda)$ in Proposition 2.9. For the second assertion, it will suffice to show the highest weights in $H^0(\overline{QwP}/P, M_\lambda)$ belong to $w\lambda + \mathbb{Z}[\Delta_Q]$. The argument in [FLQ24] is also valid without any assumption on the characteristic of base field.

Some informal explanation: We can apply 14.12 in [Jan96] also in positive char, since in general our schubert variety is normal by [RR85]. \square

Corollary 2.13. *Under the assumption that F has strongly canonical reduction, the essential minimum $\zeta_1(h_{\mathcal{L}_\lambda}, X_w)$ of $h_{\mathcal{L}_\lambda}$ on X_w is $\langle \deg(F_Q), w\lambda \rangle$.*

Proof. By Balla's theorem [Bal21, 2],

$$\zeta_1(h_{\mathcal{L}_\lambda}, X_w) = \lim_{n \rightarrow \infty} \frac{\mu_{\max}(\pi_* \mathcal{L}_{n\lambda}|_{X_w})}{n} = \lim_{n \rightarrow \infty} \frac{n \langle \deg(F_Q), w\lambda \rangle}{n} = \langle \deg(F_Q), w\lambda \rangle.$$

\square

Theorem 2.14. *Assume the strongly canonical reduction of F exists. The height filtration of $h_{\mathcal{L}_\lambda}$ on F/P is given by successively deleting Schubert cells $C_w = (F_Q \times_Q QwP/P)_K$ for $w \in W_Q \setminus W/W_P$, i.e.*

$$Z_t = X \setminus \coprod_{\langle \deg(F_Q), w\lambda \rangle \geq t} C_w = \coprod_{\langle \deg(F_Q), w\lambda \rangle < t} C_w.$$

In particular, successive minima are $\zeta_w = \langle \deg(F_Q), w\lambda \rangle$ and Zhang's successive minima are $e_i = \min \{\zeta_w : \ell(w) = \dim G/P - i + 1\}$ where

$$\ell(w) = \max_{\sigma \in W_Q w W_P} \min_{\tau \in \sigma W_P} \ell(\tau).$$

Proof. On X_w , $\zeta_1(h_{\mathcal{L}_\lambda}, X_w) = \zeta_w$ and for any $x \in C_w$, $h(x) \geq \zeta_w$. Thus $Z_{\zeta_w}(X_w) \subseteq X_w \setminus C_w = \bigcup_{w' < w} X_{w'}$. For a closed subvariety Y of X , let $Z_t(Y)$ be the Zariski closure in Y of $\{y \in Y : h(y) < t\}$. We claim that $Z_{\zeta_w}(X_w) = X_w \setminus C_w = \bigcup_{w' < w} X_{w'}$.

In fact, suppose conversely that $Z_{\zeta_w}(X_w) \subsetneq X_w \setminus C_w$. Then there exists $X_{w'} \subsetneq X_w$ such that $Z_{\zeta_w}(X_w) \cap X_{w'} \subsetneq X_{w'}$. This forces $Z_{\zeta_w}(X_{w'}) \subseteq Z_{\zeta_w}(X_w) \cap X_{w'} \subsetneq X_{w'}$. But on $X_{w'}$, the essential minimum is $\zeta_{w'} < \zeta_w$, so $Z_{\zeta_w}(X_{w'}) = X_{w'}$. We get contradiction and we have thus proved that on X_w , the set $Z_t(X_w) = X_w$ when $t > \zeta_w$ and $Z_t(X_w) = X_w \setminus C_w$ when $t \leq \zeta_w$.

Let $w_0 \in W_Q \setminus W/W_P$ be the longest element. On $X = X_{w_0}$, when $t > \zeta_{w_0}$, $Z_t = X_{w_0}$ and when $t \leq \zeta_{w_0}$, $Z_t = X_{w_0} \setminus C_{w_0} = \bigcup_{w' < w_0} X_{w'}$ with $w' < w_0$ and $\ell(w') = \ell(w_0) - 1$. Repeating this procedure on each smaller $X_{w'}$, we can complete the proof by induction. \square

Now we treat the case F may not have a strongly canonical reduction.

Theorem 2.15. *The height filtration of $h_{\mathcal{L}_\lambda}$ on F/P is given by the image of the height filtration of the height filtration of $(\mathrm{Fr}^n)^*F/P$, the successive minima are $\frac{1}{p^n}$ of the successive minima of $(\mathrm{Fr}^n)^*F/P$.*

Proof. Theorem 5.1 of [Lan05] shows for a principal G -bundle F (might not admits a strongly canonical reduction), the strongly canonical reduction exists for $(\mathrm{Fr}^n)^*F$, when n is sufficiently large. We have the following cartesian diagram

$$\begin{array}{ccc} (\mathrm{Fr}_C^n)^*F/P & \xrightarrow{\phi} & F/P \\ \downarrow & & \downarrow \\ C & \xrightarrow{\mathrm{Fr}^n} & C \end{array}$$

with its generic fibre

$$\begin{array}{ccc} \tilde{X} = ((\mathrm{Fr}_K^n)^*F/P)_K & \xrightarrow{\phi_K} & X = (F/P)_K \\ \downarrow & & \downarrow \\ \mathrm{Spec} K & \xrightarrow{\mathrm{Fr}^n} & \mathrm{Spec} K \end{array}$$

Here Fr_C is the absolute Frobenius on C . Suppose n is large enough such that $(\mathrm{Fr}^n)^*F$ has strongly canonical reduction. Note that ϕ is purely inseparable and we have the equality

$$h_{\phi^*\mathcal{L}_\lambda}(x) = h_{\mathcal{L}_\lambda}(\phi_K(x))$$

for all $x \in \tilde{X}$. Our theorem follows. \square

2.5 Base loci and movable cones

Let k be a field and $Y \xrightarrow{\pi} \mathrm{Spec} k$ be a normal projective variety. L is a line bundle on Y .

Definition 2.16. For every k -subspace V of $H^0(Y, L)$, the base locus $Bs(V)$ of V is defined as the closed subscheme of Y associated to the image of $\pi^*V \times L^\vee$ under the map

$$H^0(Y, L) \times L^\vee \rightarrow \mathcal{O}_Y$$

The stable base locus of L is defined as $B(L) = \cap_{n \geq 1} Bs(H^0(Y, nL))$. For any ample line bundle A , the augmented base locus of L is defined as $B_+(L) = \cap_{n \geq 1} B(nL - A)$. This is independent of the choice of A .

For a closed point $p_0 \in C$, we set $f = \pi^*(\mathcal{O}_C(p_0))$. Then we want to compute the augmented base locus $B_+(\mathcal{L}_\lambda - tf)$.

Lemma 2.17. ([FLQ24], lemma 4.4)

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