Geometric height on flag varieties

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Abstract

This is the outcome of my 2024 Algebra and Number Theory Summer School at Peking University. In this article, we show the proof of computation of geometric height on flag varieties in charatersitc p.

Contents

1 Introduction

1.1 Height

Let us start with some introduction on geometric Height functions and height filtration. Let C be a curve over a field k and K be its function field. We are interested in the height defined on varieties over K. In practice, we consider the following diagram: Let $\mathcal{X} \longrightarrow C$ be a projective flat morphism with $\mathcal{X}$ integral, and let $\mathcal{L}$ be a line bundle on $\mathcal{X}$. Consider the following diagram

$$\begin{array}{ccc} X & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ \operatorname{Spec} K & \longrightarrow & C \end{array}$$

and define a height function $h_{\mathcal{L}} : X(\overline{K}) \to \mathbb{R}$ associated to $\mathcal{L}$ by

$$x \mapsto \frac{\overline{\{x\}} \cdot \mathcal{L}}{\deg(x)},$$

where $\overline{\{x\}}$ is the cloure of x in $\mathcal{X}$ and $(- \cdot -)$ means taking the intersection number. Actually, in the famous paper of Shouwu Zhang, he defined so-called adelic line bundle and used it to derive a height function, which is applicable for both number field and function field. In the number field case, it does not make any sense to talk about such integral model and then it is necesssary to work with adelic line bundle. In the function field case, by virtue of the existence of the integral model $\mathcal{X}$, we are able to simplify to above definition. In another word, we can restrict ourselves into the world of algebraic geometry.

Now assume $\mathcal{L}$ is ample. Let $Z_t(X, h_{\overline{L}})$ be the Zariski closure of the set $\{x \in X(\overline{K}) : h_{\overline{L}}(x) < t\}$. Note that

- $t \longmapsto Z_t(X, h_{\overline{L}})$ is an increasing filtration of Zariski closed subsets.
- $Z_t(X, h_{\overline{L}}) = X$ when $t \gg 0$ and $Z_t(X, h_{\overline{L}}) = \varnothing$ when $t \ll 0$.

Since Zariski topology on X is Noetherian, the filtration $Z_t(X, h_{\overline{L}})$ gives a finite filtration $X_0 = X \supsetneq X_1 \supsetneq X_2 \supsetneq \cdots \supsetneq X_r = \varnothing$, the *height filtration.* Its jumping points $\zeta_i(X, h_{\overline{L}}) = \inf\{t : Z_t(X, h_{\overline{L}}) = X_{i-1}\}$ are called the *successive minima.*

Height and height filtration are hard to compute in general. There are only two available case in the literature, namely Neron-Tate height on abelian varieties and toric height function on toric vaieties. In the paper of [FLQ24], they compute a new case: geometric height on flag varieties, under assumption of characteristic 0. We aim at the characteristic p case.

1.2 An Illustrating Example

Before I present the main result, I would like to present a toy example, which essentially shows how situation differs in char p and char 0. Take $G = \mathrm{GL}_n$ and P to be $(n-1)\times(n-1)$ and 1×1 block-wise upper triangle matrix. Now $\mathcal{X}$ is nothing but a relative projective space $\mathbb{P}(E)$ where E is a vector bundle on C and $\mathcal{L}$ can be taken as the relative ample line bundle $\mathcal{O}_{\mathbb{P}(E)}(1)$ and $X = \mathbb{P}(V)$ where $V =$ is the generic fibre of E. An crucial result is due to Ballay, which connects the jumping point with the notion of slopes and stability.

Theorem 1.1 (Ballay)**.** *Let $\zeta_{ess} := \inf_{t\in\mathbb{R}}\{Z_t = X\}$. Then*

$$\zeta_{ess} = \lim_n \frac{\mu_{max}(\pi_*\mathcal{L}^{\otimes n})}{n} (= \lim_n \frac{\mu_{max}(\mathrm{Sym}^n E)}{n}).$$

where $\mu_{\max}$ is the slope of the maximal destabilized subsheaf.

Fact 1.2. When the characteristic of base field is 0, any symmetric power of semistable vector bundle is always semistable.

This implies the sequence is a constant sequence in this case. Therefore, we have the following result.

Theorem 1.3. *The height filtration of h is $\mathbb{P}(V) \supsetneq \mathbb{P}(V/V_1) \supsetneq \mathbb{P}(V/V_2)\cdots \supsetneq \mathbb{P}(V/V_r) = \emptyset$, and the jumping points (successive minima) are $\mu(E_i/E_{i-1})$, where $\{E_i\}$ is the Harder-Narasimhan filtration of E and $V_i = E_i \otimes K$.*

Heuristically, the moduli interpretation of this filtration explains that they parametrize line bundle quotients with smaller degrees as filtration goes down. This idea will still be useful in arbitrary flag varieties. Unfortunately, this fact does not hold in char p. However, a theorem by Ramanan and Ramanathan save us by considering Frobenius twists.

Theorem 1.4 (Ramanan, Ramanathan, [RR85])**.** *For above E semistable, there exists N large enough such that for $n \geq N$, any symmetric power of $\mathrm{Fr}_C^{n,*}E$ is semistable, where Fr_C is the absolute Frobenius of C.*

It suggests us to consider the following Cartesian diagram

$$\begin{tikzcd}
\mathbb{P}((\mathrm{Fr}^n)^*V) \arrow[r, "f"] \arrow[d] & \mathbb{P}(V) \arrow[d] \\
\mathrm{Spec}\ (K) \arrow[r, "\mathrm{Fr}_K"] & \mathrm{Spec}\ (K).
\end{tikzcd}$$

Theorem 1.5 (Baby-version)**.** *The height filtration of h is nothing but the image of the height filtration of $\mathbb{P}(f^*V)$. The jumping points are $\frac{1}{p^n}$ of the slopes of quotients of HN filtration of f^*E.*

A bypoduct would be the following equality:

$$L_{max} = \lim_n \frac{\mu_{max}(\mathrm{Sym}^n E)}{n},$$

where $L_{max} := \max_{f:Y\to C}\{\mu_{max}(f^*E)\}$. Note that $L_{max} = \mu_{max}$ in char 0, and $L_{max} = \max_n\{\mu_{max}(Fr_C^{n,*}E)\}$. It would be in particular interesting to have a purely vector bundle theoretical proof of this equality.

1.3 Main result

Now let G/k be a reductive group and $F \longrightarrow C$ be a principal G-bundle. Let $P \subseteq$ be a parabolic subgroup and $\lambda : P \longrightarrow \mathbb{G}_m$ be a strictly anti-dominant character, if the natural pairing $\langle\alpha^\vee, \lambda\rangle < 0$ for any $\alpha \in \Delta\backslash\Delta_P$. Then $F/P \longrightarrow C$ is a G/P-bundle and $\mathcal{L}_\lambda = F\times_P k_\lambda$ is a line bundle on F/P and induces height function $h = h_{\mathcal{L}_\lambda} : X(\overline{K}) \longrightarrow \mathbb{R}$.

$$\begin{tikzcd}
X = (F/P)_K \arrow[r] \arrow[d] & \mathcal{X} = F/P \arrow[d] \\
\mathrm{Spec}(K) \arrow[r] & C
\end{tikzcd}$$

Definition 1.6. Let F be a principal G-bundle on C. A reduction F_Q of F to a parabolic subgroup Q is called canonical if the following two conditions hold:

- Principal L_Q-bundle $F_Q \times_Q L_Q$ is semistable, where L_Q is the Levi subgroup of Q.
- For any non-trivial character λ of Q which is non-negative linear combination of simple roots, $\langle\deg(F_Q), \lambda\rangle > 0$.

We say F_Q is strongly canonical if for any non-constant finite morphism $f : C' \to C$, the pullback $f^*(F_Q)$ is canonical.

Remark 1.7. Being strongly canonical amounts to say the pullback of HN filtration is HN filtration of pullback.

Theorem 1.8. *Assume the strongly canonical reduction of F exists. The height filtration of $h_{\mathcal{L}_\lambda}$ on F/P is given by successively deleting Schubert cells $C_w = (F_Q \times_Q QwP/P)_K$ for $w \in W_Q\backslash W/W_P$, i.e.*

$$Z_t = X \setminus \coprod_{\langle\deg(F_Q), w\lambda\rangle \geq t} C_w = \coprod_{\langle\deg(F_Q), w\lambda\rangle < t} C_w.$$

In particular, successive minima are $\zeta_w = \langle\deg(F_Q), w\lambda\rangle$.

In general, a theorem by Langer [Lan05] shows for a principal G-bundle F (might not admits a strongly canonical reduction), the strongly canonical reduction exists for $(\mathrm{Fr}^n)^*F$, when n is sufficiently large.

We have a cartesian diagram

$$\begin{tikzcd}
(\mathrm{Fr_C}^n)^*F/P \arrow[r, "\phi"] \arrow[d] & F/P \arrow[d] \\
C \arrow[r, "\mathrm{Fr}^n"] & C
\end{tikzcd}$$

with its generic fibre

$$\begin{tikzcd}
\tilde{X} = ((\mathrm{Fr_K}^n)^*F/P)_K \arrow[r, "\phi_K"] \arrow[d] & X = (F/P)_K \arrow[d] \\
\mathrm{Spec}\ K \arrow[r, "\mathrm{Fr}^n"] & \mathrm{Spec}\ K
\end{tikzcd}$$

Here Fr is the absolute Frobenius on C. Suppose n is large enough such that $(\mathrm{Fr}^n)^*F$ has strongly canonical reduction, then we have:

Theorem 1.9. *The height filtration of $h_{\mathcal{L}_\lambda}$ on F/P is given by the image of the height filtration of the height filtration of $(\mathrm{Fr^n})^*F/P$, the successive minima are $\frac{1}{p^n}$ of the succesive minima of $(\mathrm{Fr^n})^*F/P$.*

2 Height filtration and Successive minima

2.1 Strongly cononical reduction

Definition 2.1. A principal G-bundle F on C is called semistable if for any parabolic subgroup P, any reduction F_P of F to P and any dominant character λ of P which is trivial on $Z(G)$, we have $\langle \deg(F_P), \lambda\rangle \leq 0$. We say F is strongly semistable if for any non-constant finite morphism $f : C' \to C$, the pullback f^*F is semistable.

Definition 2.2. Let F be a principal G-bundle on C. A reduction F_Q of F to a parabolic subgroup Q is called canonical if the following two conditions hold:

- The principal L_Q bundle $F_Q \times_Q L_Q$ is semistable, where L_Q is the Levi subgroup of Q.
- For any non-trivial character λ of Q which is non-negative linear combination of simple roots, $\langle \deg(F_Q), \lambda\rangle > 0$.

We say F_Q is strongly canonical if for any non-constant finite morphism $f : C' \to C$, the pullback $f^*(F_Q)$ is canonical.

If the base field is of charateristic 0, then semistability (*resp.* canonical reduction) and strongly semistability (*resp.* strongly canonical reduction) are equivalent. But in positive characteristic, the notion of strongly canonical is necessary, which we shall see afterwards. Under the assumption of characteristic 0, the canonical reduction exists uniquely [Beh00]. But in positive characteristic, strongly canonical reduction does not exist in general.

When $G = \mathrm{GL}_n$, this semistability amounts to the semistability of the associated vector bundle with the natural representation; the canonical reduction amounts to the Harder-Narasimhan filtration of vector bundles with the natural representation.

2.2 Height function on flag varieties

Here we mainly follow [FLQ24].

Let F be a principal G-bundle with strongly canonical reduction F_Q to some parabolic subgroup $Q \subseteq G$. Let $P \subseteq G$ be a parabolic subgroup. Set $\mathcal{X} = F/P$ and $X = \mathcal{X}_K$.

A character $\lambda : P \longrightarrow \mathbb{G}_m$ is called strictly anti-dominant if the natural pairing $\langle \alpha^\vee, \lambda\rangle < 0$ for any $\alpha \in \Delta\backslash\Delta_P$. Let $\lambda : P \longrightarrow \mathbb{G}$ be a stricly anti-dominant character. Then the line bundle $M_\lambda = G \times_P k_\lambda$ on G/P is ample. Therefore $\mathcal{L}_\lambda = F \times_G M_\lambda$ is a relatively ample line bundle on $\mathcal{X} = F \times_G G/P$ and induces a height function

$$h_{\mathcal{L}_\lambda} : X(\bar{K}) \longrightarrow \mathbb{Q}, \quad x \longmapsto \frac{\mathcal{L}_\lambda \cdot \overline{\{x\}}}{\deg(x)}$$

where $\overline{\{x\}}$ is the Zariski closure of x in $\mathcal{X}$. For $x \in X(K)$, let $\sigma_x : C \longrightarrow \mathcal{X}$ be the section induced by $x : \mathrm{Spec}(K) \longrightarrow X$ via valuative criterion, and let $F_{P,x} = \sigma_x^* F$ be the corresponding reduction to P.

Lemma 2.3. *For $x \in X(K), h_{\mathcal{L}_\lambda}(x) = \langle \deg(F_{P,x}), \lambda\rangle$.*

Proof. By definition, $h_{\mathcal{L}_\lambda}(x)$ is the degree of $\sigma_x^*(\mathcal{L}_\lambda)$ and $\langle \deg(F_{P,x}), \lambda\rangle$ is the degree of $F_{P,x} \times {}_P k_\lambda$. We have equality of line bundles

$$\sigma_x^*(\mathcal{L}_\lambda) = \sigma_x^*(F \times_P k_\lambda) = (\sigma_x^* F) \times_P k_\lambda = F_{P,x} \times_P k_\lambda$$

and the lemma follows by taking degree. □

2.3 A height lower bound in Schubert cells

For $w \in W_Q \backslash W / W_P$, write $\mathcal{C}_w = F_Q \times_Q QwP/P, \mathcal{X}_w = F_Q \times_Q \overline{QwP}/P, C_w = \mathcal{C}_{w,K}$ and $X_w = \mathcal{X}_{w,K}$. We shall show that

Proposition 2.4. *For any $x \in C_w(\bar{K}), h_{\mathcal{L}_\lambda}(x) \geq \langle \deg F_Q, w\lambda \rangle$.*

Note that for any $w' \in W_Q$ and $\lambda \in X(T)$, we have $w'\lambda - \lambda \in \mathbb{Z}[\Delta_Q]$ and consequently $\langle \deg(F_Q), w'\lambda \rangle = \langle \deg(F_Q), \lambda \rangle$. Note also that for any $\lambda \in X(P)$ and $w \in W_P, w\lambda = \lambda$. So the number $\langle \deg(F_Q), w\lambda \rangle$ is well-defined for any $\lambda \in X(P)$ and $w \in W_Q \backslash W / W_P$.

Definition 2.5 (relative position)**.** A reduction F_P of F to P is called in relative position $w \in W_Q \backslash W / W_P$ with respect to F_Q if the image of the natural map

$$F_Q \times_C F_P \longrightarrow G_C, \quad (a, b) \longmapsto a^{-1}b$$

lies in QwP_C. Here we implicitly apply the injections

$$F_P \hookrightarrow F_P \times_P G \cong F, \quad F_Q \hookrightarrow F_Q \times_Q G \cong F$$

Note that by the universal property of the quotient stacks $[G//Q \times P]$ and $[QwP//Q \times P]$, this definition coincides with the one in [SS15, 4.1].

Lemma 2.6. *When $x \in C_w(K), F_{P,x}$ is of relative position w with respect to F_Q.*

Proposition 2.7 ([SS15], Proposition 4.6)**.** *If F_P is in relative position w with respect to F_Q, then $\langle \deg(F_P), \lambda \rangle \geq \langle \deg(F_Q), w\lambda \rangle$ for any anti-dominant character λ.*

2.4 Height filtration and successive minima

We compute the essential minimum of X_w.

Lemma 2.8 ([Jan96], Part I, §5.18)**.** *On $\mathcal{X}_w = F_Q \times_Q \overline{QwP}/P \subseteq \mathcal{X}$, we have $\pi_*\left(\mathcal{L}_\lambda|_{\mathcal{X}_w}\right) = F_Q \times_Q \mathrm{H}^0\left(\overline{QwP}/P, M_\lambda\right)$.*

Let V be any Q-representation of highest weight $\lambda \in X^*(T)$and let $V = \bigoplus_\nu V[\nu]$ be its weight decomposition. Furthermore, let F_Q be the strongly canonical reduction of F. We define a filtration $V_\bullet$ on the vector space V: For any rational number $q \in \mathbb{Q}$, we define the subspace V_q as the sum of weight spaces

$$V_q := \bigoplus_{\langle \deg F_Q, \nu \rangle \geq q} V[\nu]$$

Clearly, $V_{q'} \subseteq V_q$ whenever $q' \geq q$. We will consider subspaces V_q only for the finitely many $q \in \mathbb{Q}$ where a jump occurs, that is, only for those q such that $V_{q'} \subsetneq V_q$ for all $q' > q$. Let q_0 be the smallest and q_1 the largest rational number occurring among such q. Then V_{q_1} is the smallest non-zero filtration step and V_{q_0} equals V.

Then, by twisting the Q-subrepresentations V_q above by F_P, we obtain a filtration $V_{\bullet, F_Q}$ of the vector bundle $V_{F_G} = F_G \times^G V$ by subbundles

$$0 \neq V[\lambda + \mathbb{Z}\Delta_M]_{F_M} = (\bigoplus_{\nu \in \lambda + \mathbb{Z}\Delta_M} V[\nu])_{F_M} = V_{q_1, F_Q} \subsetneq \cdots \subsetneq V_{q, F_Q} \subsetneq \cdots \subsetneq V_{q_0, F_Q} = V_{F_G}.$$

Improving the result of Schieder[1], we proved that:

Proposition 2.9. *Assume the redution F_Q is the strongly canonical reduction, then the filtration $V_{\bullet, F_Q}$ of the vector bundle V_{F_G} is the Harder-Narasimhan filtration of V_{F_G}.*

[1] In [SS15], The case of characteristic 0 and a specific case of positive characteristic have been proven.

We will need the following well-known theorem.

Theorem 2.10 ([RR84], Theorem 3.23)**.** *If F is a strongly semistable G-bundle and $\rho : G \to H$ is a homomorphism that maps the connected component of of the center of G into that of H, then $F \times^G H$ is a semistable H-bundle.*

Proof of Proposition 2.9. It will suffice to check two assertions: all qraded pieces are semistable vector bundles and their slopes are decreasing.

As shown in [SS15] Proposition 5.1, we have

$$\mathrm{gr}_q(V_{\bullet,F_P}) = (\mathrm{gr}_q V_\bullet)_{F_M}.$$

And its slope is computed as

$$\mu((\mathrm{gr}_q V_\bullet)_{F_M}) = q.$$

Therefore, the slopes of the graded pieces are decreasing by definition.

To prove the semistability, consider the semisimplification W of $(\mathrm{gr}_q V_\bullet)$ as a representation of M. Now apply [SS15] Proposition 3.2(a), we see that for each simple direct summand W_i of W the associated bundle W_{i,F_M} has slope q. Thus $(\mathrm{gr}_q V_\bullet)_{F_M}$ admits a filtration by subbundles such that each graded piece is a semistable vector bundle of slope q, where the semistability is obtained by applying Theorem 2.10 to $M \to \mathrm{GL}(M_i)$. Then $(\mathrm{gr}_q V_\bullet)_{F_M}$ is also semistable of slope q by the following lemma. □

Lemma 2.11. *If E is a vector bundle on C that admits a filtration of subbundles*

$$0 \subsetneq E_1 \subsetneq \cdots \subsetneq E_r = E,$$

and all graded piece are semistable of slope equals to λ, so is E.

Proof. By induction on r it will suffice to prove the case $r = 2$, which amounts to the following assertion: Assume that in the short exact sequence of vector bundles

$$0 \to E' \to E \to E'' \to 0,$$

both E' and E'' are semistable of slope equal to λ, then so is E. It is easy to see $\mu(E) = \lambda$. To prove E is semistable, let F is a subbundle of E. Then we have the short exact sequence

$$0 \to F \cap E' \to F \to \bar{F} \to 0,$$

where $\bar{F}$ is the image of F in E''. Since E' and E'' are semistable, $F \cap E'$ and $\bar{F}$ has slopes smaller than $\mu(E)$, and so is F. □

Corollary 2.12. *Assume F_Q is the strongly canonical reduction of F. The Harder-Narasimhan filtration of $F_Q \times_Q H^0\left(\overline{QwP}/P, M_\lambda\right)$ is $F_Q \times_Q H^0\left(\overline{QwP}/P, M_\lambda\right)_{\bullet,\deg(F_Q)}$. Moreover, the maximal slope is $\langle \deg(F_Q), w\lambda\rangle$.*

Proof. The first assertion is obtained by taking V to be $H^0\left(\overline{QwP}/P, M_\lambda\right)$ in Proposition 2.9. For the second assertion, it will suffice to show the highest weights in $H^0\left(\overline{QwP}/P, M_\lambda\right)$ belong to $w\lambda + \mathbb{Z}[\Delta_Q]$. The argument in [FLQ24] is also valid without any assumption on the characteristic of base field.

Some informal explaination: We can apply 14.12 in [Jan96] also in positive char, since in general our schubert variety is normal by [RR85]. □

Corollary 2.13. *Under the assumption that F has strongly canonical reduction, the essential minimum $\zeta_1(h_{\mathcal{L}_\lambda}, X_w)$ of $h_{\mathcal{L}_\lambda}$ on X_w is $\langle \deg(F_Q), w\lambda\rangle$.*

Proof. By Ballaÿ's theorem [Bal21, 2],

$$\zeta_1(h_{\mathcal{L}_\lambda}, X_w) = \lim_{n\to\infty} \frac{\mu_{\max}(\pi_*\mathcal{L}_{n\lambda}|_{X_w})}{n} = \lim_{n\to\infty} \frac{n\langle \deg(F_Q), w\lambda\rangle}{n} = \langle \deg(F_Q), w\lambda\rangle.$$

□

Theorem 2.14. *Assume the strongly canonical reduction of F exists. The height filtration of $h_{\mathcal{L}_\lambda}$ on F/P is given by successively deleting Schubert cells $C_w = (F_Q \times_Q QwP/P)_K$ for $w \in W_Q \backslash W / W_P$, i.e.*

$$Z_t = X \setminus \coprod_{\langle \deg(F_Q), w\lambda \rangle \geq t} C_w = \coprod_{\langle \deg(F_Q), w\lambda \rangle < t} C_w.$$

In particular, successive minima are $\zeta_w = \langle \deg(F_Q), w\lambda \rangle$ and Zhang's successive minima are $e_i = \min\{\zeta_w : \ell(w) = \dim G/P - i + 1\}$ where

$$\ell(w) = \max_{\sigma \in W_Q w W_P} \min_{\tau \in \sigma W_P} \ell(\tau).$$

Proof. On $X_w, \zeta_1(h_{\mathcal{L}_\lambda}, X_w) = \zeta_w$ and for any $x \in C_w, h(x) \geq \zeta_w$. Thus $Z_{\zeta_w}(X_w) \subseteq X_w \backslash C_w = \bigcup_{w' < w} X_{w'}$. For a closed subvariety Y of X, let $Z_t(Y)$ be the Zariski closure in Y of $\{y \in Y : h(y) < t\}$. We claim that $Z_{\zeta_w}(X_w) = X_w \backslash C_w = \bigcup_{w' < w} X_{w'}$

In fact, suppose conversely that $Z_{\zeta_w}(X_w) \subsetneq X_w \backslash C_w$. Then there exists $X_{w'} \subsetneq X_w$ such that $Z_{\zeta_w}(X_w) \cap X_{w'} \subsetneq X_{w'}$. This forces $Z_{\zeta_w}(X_{w'}) \subseteq Z_{\zeta_w}(X_w) \cap X_{w'} \subsetneq X_{w'}$. But on $X_{w'}$, the essential minimum is $\zeta_{w'} < \zeta_w$, so $Z_{\zeta_w}(X_{w'}) = X_{w'}$. We get contradiction and we have thus proved that on X_w, the set $Z_t(X_w) = X_w$ when $t > \zeta_w$ and $Z_t(X_w) = X_w \backslash C_w$ when $t \leq \zeta_w$.

Let $w_0 \in W_Q \backslash W / W_P$ be the longest element. On $X = X_{w_0}$, when $t > \zeta_{w_0}$, $Z_t = X_{w_0}$ and when $t \leq \zeta_{w_0}, Z_t = X_{w_0} \backslash C_{w_0} = \bigcup_{w'} X_{w'}$ with $w' < w_0$ and $\ell(w') = \ell(w_0) - 1$. Repeating this procedure on each smaller $X_{w'}$, we can complete the proof by induction. □

Now we treat the case F may not have a strongly canonical reduction.

Theorem 2.15. *The height filtration of $h_{\mathcal{L}_\lambda}$ on $(F/P)_K$ is given by the image of the height filtration of the height filtration of $((\mathrm{Fr^n})^* F/P)_K$, the successive minima are $\frac{1}{p^n}$ of the successive minima of $((\mathrm{Fr^n})^* F/P)_K$.*

Proof. Theorem 5.1 of [Lan05] shows for a principal G-bundle F (might not admits a strongly canonical reduction), the strongly canonical reduction exists for $(\mathrm{Fr}^n)^* F$, when n is sufficiently large. We have the following cartesian diagram

$$\begin{array}{ccc} (\mathrm{Fr_C}^n)^* F/P & \xrightarrow{\phi} & F/P \\ \downarrow & & \downarrow \\ C & \xrightarrow{\mathrm{Fr}^n} & C \end{array}$$

with its generic fibre

$$\begin{array}{ccc} \tilde{X} = ((\mathrm{Fr_K}^n)^* F/P)_K & \xrightarrow{\phi_K} & X = (F/P)_K \\ \downarrow & & \downarrow \\ \mathrm{Spec}\ K & \xrightarrow{\mathrm{Fr}^n} & \mathrm{Spec}\ K \end{array}$$

Here $\mathrm{Fr_C}$ is the absolute Frobenius on C. Suppose n is large enough such that $(\mathrm{Fr}^n)^* F$ has strongly canonical reduction. Note that ϕ is purely inseparable and we have the equality

$$h_{\phi^* \mathcal{L}_\lambda}(x) = h_{\mathcal{L}_\lambda}(\phi_K(x))$$

for all $x \in \tilde{X}$. Our theorem follows. □

References

[Bal21] François Ballaÿ, *Successive minima and asymptotic slopes in Arakelov geometry*, Compos. Math. **157** (2021), no. 6, 1302–1339. MR 4271920

[Beh00] Kai A. Behrend, *Semi-stability of reductive group schemes over curves*, Journal of Algebraic Geometry **9** (2000), no. 3, 347–367.

[FLQ24] Yangyu Fan, Wenbin Luo, and Binggang Qu, *Height filtrations and base loci on flag bundles over a curve*, 2024.

[Jan96] Jens Jantzen, *Representations of algebraic groups*, Mathematical Surveys and Monographs, vol. 107, American Mathematical Society, 1996.

[Lan05] Adrian Langer, *Semistable principal G-bundles in positive characteristic*, Duke Mathematical Journal **128** (2005), no. 3, 511 – 540.

[RR84] S. Ramanan and A. Ramanathan, *Some remarks on the instability flag*, Tohoku Mathematical Journal **36** (1984), no. 2.

[RR85] S. Ramanan and A. Ramanathan, *Projective normality of flag varieties and schubert varieties*, Inventiones mathematicae **79** (1985), 217–224.

[SS15] Schieder and Simon, *The harder–narasimhan stratification of the moduli stack of (g)-bundles via drinfeld's compactifications*, Selecta Mathematica **21** (2015), no. 3, 763–831.