Notes on lecture series of L. Fargues

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Abstract

This is a note of a series of lectures given by L. Fargues at MCM, August 2024.

Contents

0 Overview

This is on the lecture given by Prof. Fargues at the 1st Chen Jingrun Prize, which is a overview of the following lectures.

1 Lecture 1

1.1 The curve

We first recall the construction of the Fargues-Fontaine curve.

Let E be a finite extension over $\mathbb{Q}_p$ with residue cardinality q. We denote $\mathrm{Perf}_{\bar{\mathbb{F}}_q}$ the category of perfectoid spaces over $\bar{\mathbb{F}}_q$. Note that there is no final object in this category, but we put $\mathrm{Spa}(\bar{\mathbb{F}}_q, \bar{\mathbb{F}}_q)$ into this category and consider it as the final object. An object S in $\mathrm{Perf}_{\bar{\mathbb{F}}_q}$ is locally of the form

$$\mathrm{Spa}(A, A^+),$$

where A is a perfect $\bar{\mathbb{F}}_q$-Banach algebra and A^+ is an integrally closed subring that contains all topologically nilpotent elements $A^{\circ\circ}$ and is contained by the subring of power bounded elements A°. We usually call it a perfect Huber pair over $\bar{\mathbb{F}}_q$, and call an invertible element $\varpi \in A^{\circ\circ}$ a pseudo-uniformizer of A.

Example 1.1. A basic example is $\mathrm{Spa}(\widehat{\bar{\mathbb{F}}_q((\varpi^{\frac{1}{p^\infty}}))}, \widehat{\bar{\mathbb{F}}_q[\![\varpi^{\frac{1}{p^\infty}}]\!]})$.

Definition 1.2. A collection of morphisms $(T_i \to S)_I$ is called a v-covering if for any open qc subset U of S, there exists a finite index subset $J \subset I$ and open qc $V_j \subseteq T_j$ for any $j \in J$ such that $|U| = \bigcup_{j\in J} \mathrm{Im}(|V_j| \to |S|)$. We call the category $\mathrm{Perf}_{\bar{\mathbb{F}}_q}$ equipped with the v-topology the big v-site. We will called a sheaf (*resp.* stack) on v-site a v-sheaf (*resp.* v-stack).

Remark 1.3. This is actually an analogue of fpqc topology in p-adic world. Even if there is nothing to do with flatness, it turns out to be the right one due to some special feature of adic spaces.

Remark 1.4. In the category of adic spaces, fibre product does NOT exist in general, but it does in $\mathrm{Perf}_{\bar{\mathbb{F}}_q}$. Thus, we are safe to talk about v-site.

Recall that for each $S \in \mathrm{Perf}_{\bar{\mathbb{F}}_q}$, one can construct the family of Fargues-Fontaine curve parametrized by S as if

$$\text{“}X = X \times S''.$$

This is an E-adic space defined by

$$X_S = Y_S/\varphi^{\mathbb{Z}},$$

where the action of φ is properly discontinuously without fixed points.

Assume $S = \mathrm{Spa}(A, A^+)$ is affinoid perfedctoid, define

$$Y_S = \mathrm{Spa}(\mathbb{W}_{\mathcal{O}_E}(A^+), \mathbb{W}_{\mathcal{O}_E}(A^+)) \backslash V(\pi[\varpi]),$$

where $\mathbb{W}_{\mathcal{O}_E}(A^+)$ is the ramified Witt vectors, namely the unique π-adic complete lift of A^+ flat over $\mathcal{O}_E$. Note that Y_S is Stein, that is, it is completely determined by the E-Frechet algebra $\mathcal{O}(Y_S)$. It is an analogue of holomorphic functions on punctured open unit disk (under this analogue we identify $\mathbb{W}_{\mathcal{O}_E}(A^+)[\frac{1}{\pi[\varpi]}]$ as the subring of meromorphic functions along the boundry). Another equivalent description is to regard $\mathcal{O}(Y_S)$ as the Frechet completion of $\mathbb{W}_{\mathcal{O}_E}(A^+)$.

Fact 1.5 (Important)**.** Y_S is a sous-perfectoid, namely it is covered by $\mathrm{Spa}(R, R^+)$ where the algebra $R\hat{\otimes}_E\hat{\bar{E}}$ is perfectoid. A basic example of sousperfectoid spaces is $\mathbb{Q}_p\langle T^{\frac{1}{p^\infty}}\rangle$.

Theorem 1.6. *Any subperfectoid space is sheafy, that is, its structure presheaf is a sheaf.*

Corollary 1.7. *Y_S and X_S are both (sheafy) E-adic spaces.*

Remark 1.8. IF $S = \mathrm{Spa}(F, F^+)$ where F is a perfectoid field, then X_S is a Neotherian adic space regular of dimension 1, *i.e.*, it is covered by $\mathrm{Spa}(R, R^+)$ where R is a Banach E-algebra that is a PID. But there are no such finiteness properties in families.

Note that the construction $S \mapsto X_S$ is functorial in S, namely if there is a map $T \to S$ in $\mathrm{Perf}_{\bar{\mathbb{F}}_q}$, then there exists a natural map

$$X_T \to X_S,$$

as if “$X = X \times S''$ again, or “$X_S = (\text{family } X_{k(s),k(s)^+})''_{s\in S}$.

Warning 1.9. In order to construct X_S for general S, we can no longer glue “$X_S \to S''$ because there is no such map, but $|X_S| \to |S|$ exists along which we can glue up to a global construction; see [FS24, Ch II].

Remark 1.10. Note again that X_S is a E-adic space of characteristic 0, therefore, there does NOT exist a map $X_S \to S$. To see the structure map $X_S \to \mathrm{Spa}(E)$, we have

$$\mathcal{O}(X_S) = \mathcal{O}(Y_S)^{\varphi=\mathrm{id}} = \mathcal{C}(|S|, E).$$

Nevertheless, there exists "a version of the curve"

$$\mathrm{Div}^1_S := \mathrm{Div}^1 \times S,$$

where $\mathrm{Div}^1 := \mathrm{Spa}(E)^\diamondsuit/\varphi^{\mathbb{Z}}$. Here $(-)^\diamondsuit$ is the diamond functor, that is, for any analytic adic space Y, the corresponding diamond $Y^\diamondsuit$ is a v-sheaf sending $S \in \mathrm{Perf}_{\bar{\mathbb{F}}_q}$ to the set of triples $(S^\sharp, \iota, f)$, where $S^\sharp$ is an untilt of S with $\iota : S^{\sharp\flat} \simeq S$ and $f : S^\sharp \to Y$ is a morphism of adic spaces.

More precisely, we have

$$X_S^\diamondsuit = (S \times \mathrm{Spa}(E)^\diamondsuit)/\varphi^{\mathbb{Z}} \times \mathrm{id},\ \mathrm{Div}^1_S = (S \times \mathrm{Spa}(E)^\diamondsuit)/\mathrm{id} \times \varphi^{\mathbb{Z}}.$$

Here, $Y_S^\diamondsuit = S \times \mathrm{Spa}(E)^\diamondsuit$.

Being sousperfectoid also provides a good notion of vector bundles on X_S. For affinoid sousperfectoid $U \subset X_S$, we have the equivalence between the category of finite projective R-modules and the category of locally free $\mathcal{O}_U$-modules, given by

$$M \mapsto M \otimes_R \mathcal{O}_U.$$

Theorem 1.11. *The functor* $\mathrm{Bun}_{\mathrm{X}} : \mathrm{Perf}_{\bar{\mathbb{F}}_q} \to \mathrm{Groupoid}$*, sending S to the groupoid of vector bundles on X_S is a v-stack.*

Proof. TBA. See [FS24, Ch II]. □

1.2 Banach-Colmez Space

Let S be an object in $\mathrm{Perf}_{\bar{\mathbb{F}}_q}$, and $\mathcal{E}$ be a vector bundle over X_S. If S is affinoid perfectoid, then one can compute the cohomology of $\mathcal{E}$ by the cohomology on Y_S and group cohomology, namely

$$R\Gamma(X_S, \mathcal{E}) = [\Gamma(Y_S, \mathcal{E}|_{Y_S}) \xrightarrow{\mathrm{id}-\varphi} \Gamma(Y_S, \mathcal{E}|_{Y_S})].$$

Definition 1.12 (Banach-Colmez space)**.** We define $\mathrm{BC}(\mathcal{E})$ to be the v-sheaf sending T/S to $\mathrm{H}^0(X_T, \mathcal{E}_{X_T})$, and define $\mathrm{BC}(\mathcal{E}[1])$ to be the sheafification of presheaf $T/S \mapsto \mathrm{H}^1(X_T, \mathcal{E}_{X_T})$.

Theorem 1.13. *Both* $\mathrm{BC}(\mathcal{E}) \to *$ *and* $\mathrm{BC}(\mathcal{E}[1]) \to *$ *are relatively representable in locally spatial diamonds.*

Recall that Diamonds are an analogue of algebraic spaces on $\mathrm{Perf}_{\bar{\mathbb{F}}_q}$ for proétale topology. To be more precise, we need to define proétale topology. Proétale topology is as what its name suggests, we need to consider some reasonable limits of étale covers. Contrary to the category of Neotherian adic spaces, it admits any cofiltered limits in the category of affinoid perfectoid spaces, in fact

$$\varprojlim \mathrm{Spa}(A_i, A_i^+) = \mathrm{Spa}(A_\infty, A_\infty^+),$$

where A_∞^+ is the ϖ-adic completion of $\varinjlim A_i$ and $A_\infty = A_\infty^+[\frac{1}{\varpi}]$.

Definition 1.14. A morphism of perfectoid spaces is called proétale if it is locally both on source and target of the form

$$\varprojlim \mathrm{Spa}(A_i, A_i^+) \to \mathrm{Spa}(A_0, A_0^+)$$

with $\mathrm{Spa}(A_i, A_i^+) \to \mathrm{Spa}(A_0, A_0^+)$ étale.

Warning 1.15. Note that proétale does NOT imply openness, for example, let $(\kappa(s), \kappa(s)^+)$ be the completed residue of a point $s \in S$, then the inclusion $\mathrm{Spa}(\kappa(s), \kappa(s)^+) \to S$ is proétale because

$$\mathrm{Spa}(\kappa(s), \kappa(s)^+) = \varprojlim_{s \in U} U.$$

This is a major difference between schemes and analytic adic spaces, that is, in the world of schemes, we have

$$\varprojlim_{s \in U} U = \mathrm{Spec}\ (\mathcal{O}_{S,s}),$$

while in the world of analytic adic spaces, the limit turns out to be the completed residue field at $s \in S$ as above.

Proposition 1.16. *If $f : X \to Y$ is a morphism of analytic adic spaces, then $|f| : |X| \to |Y|$ is generalizing,* i.e., *, if $x \mapsto y$, then there exists the following diagram*

$$\begin{tikzcd} \kappa(y)^+ \arrow[r] \arrow[d, hook] & \kappa(x)^+ \arrow[d, hook] \\ \kappa(y) \arrow[r] & \kappa(x). \end{tikzcd}$$

Definition 1.17. A collection of maps $(T_i \to S)$ in $\mathrm{Perf}_{\bar{\mathbb{F}}_q}$ is called proétale if it is a v-cover and each $T_i \to S$ is proétale.

Definition 1.18. A diamond is a proétale sheaf X on $\mathrm{Perf}_{\bar{\mathbb{F}}_q}$ such that $X \cong \widetilde{X}/R$, where $\widetilde{X}$ is a perfectoid and R is an equivalent relation in $\widetilde{X} \times \widetilde{X}$ such that two projections $R \rightrightarrows X$ are proétale.

2 Lecture 2

Recall that last time we reviewed the construction of the FF curve X_S for $S \in \mathrm{Perf}_{\bar{\mathbb{F}}_q}$ and gave the definition of BC spaces. We now continue to explain what is a locally spatial diamond and prove Theorem 1.13.

2.1 Diamond

We recall the definition of diamond. The main reference of this section is [Sch22].

Definition 2.1. A diamond is a proétale sheaf X on $\mathrm{Perf}_{\bar{\mathbb{F}}_q}$ such that $X \cong \widetilde{X}/R$, where $\widetilde{X}$ is a perfectoid and R is an equivalent relation in $\widetilde{X} \times \widetilde{X}$ such that two projections $R \rightrightarrows X$ are proétale.

Theorem 2.2. *Any diamond is a v-sheaf.*

Remark 2.3. This is an analogue of a theorem by Gabber, which says that any algebraic spaces is an fpqc sheaf.

Remark 2.4. Just as algebraic spaces, for a diamond X, we defined the underlying topological spaces as the following:

$$|X| := \coprod_{(K,K^+)} X(K, K^+)/\sim,$$

where (K, K^+) runs through all affinoid fields and we say $x \sim x'$ if and only if there exists (L, L^+) filling up the diagram

$$\begin{tikzcd} \mathrm{Spa}(L, L^+) \arrow[r] \arrow[d] & x' = \mathrm{Spa}(K', K'^+) \arrow[d] \\ x = \mathrm{Spa}(K, K^+) \arrow[r] & X, \end{tikzcd}$$

Open subsets of X are defined to be

$$\{|U| \subset |X| \text{ where } U \subset X \text{ open subdiamond}\}.$$

Lemma 2.5. *If $X = \widetilde{X}/R$ as in the definition of diamond, then*

$$|X| = |\widetilde{X}|/\mathrm{Im}(|R| \to |\widetilde{X} \times \widetilde{X}| \to |\widetilde{X}| \times |\widetilde{X}|).$$

Example 2.6. If τ is a topological space, then one can associate it with the corresponding diamond $\underline{\tau}$, which sends any $S \in \mathrm{Perf}_{\overline{\mathbb{F}}_q}$ to the set $\mathcal{C}(|S|, \tau)$ of all continuous functions from the underlying space $|S|$ to τ, where $|S|$ is locally spectual. If $T \to S$ is a v-cover of perfectoid spaces, then

$$|S| \cong \mathrm{Coeq}(|T| \times_{|S|} |T| \rightrightarrows |S|).$$

If Γ is a profinite set and $S = \mathrm{Spa}(A, A^+)$ is affinoid perfectoid, then

$$\underline{\Gamma} \times S = \mathrm{Spa}(\mathcal{C}(\Gamma, A), \mathcal{C}(\Gamma, A^+)).$$

Since in $\mathrm{Perf}_{\overline{\mathbb{F}}_q}$ we can rise arbitrary cofiltered limits, in particular for any profinite set $\Gamma = \varprojlim_I \Gamma_i$ as in the definition of profinite set, we have $\underline{\Gamma} = \varprojlim_I \underline{\Gamma_i}$, where $\underline{\Gamma_i}$ denotes the usual constant sheaf with finite stalks and the limit is taken in $\mathrm{Perf}_{\overline{\mathbb{F}}_q}$.

Let T be a compact Hausdorff topological space. Conisder the Stone-Čech compactification[1] βT^{disc} of T^{disc}, which is heomomorphic to $\mathrm{Spm}(k^T)$.

Proposition 2.7. *Any compact Hausdorff T is isomorphic to the coequilizer of*

$$\beta T^{\mathrm{disc}} \times_T \beta T^{\mathrm{disc}} \rightrightarrows \beta T^{\mathrm{disc}},$$

and the corresponding v-sheaf $\underline{T}$ is isomorphic to the coequilizer of

$$\underline{\beta T^{\mathrm{disc}} \times_T \beta T^{\mathrm{disc}}} \rightrightarrows \underline{\beta T^{\mathrm{disc}}}.$$

In particular, $\underline{T}$ is a diamond. The morphism $\underline{T} \to *$ is representable in diamond.[2]

Proposition 2.8. *The functor $T \mapsto \underline{T} \times \mathrm{Spa}(C, C^+)$ is a fully faithful functor from the category of compact Hausdorff spaces to the category of diamonds over $\mathrm{Spa}(C, C^+)$.*

Remark 2.9. This is the beginning of condensed mathematics.

However, there are too many objects in our category of "algebraic spaces". Therefore, we introduce the following definition to characterize all objects that do look like analytic adic spaces.

Definition 2.10. A diamond X is spatial if

1. X is qcqs.

2. Any point of $|X|$ has a basis formed by quasi-compact open subsets.

Theorem 2.11. *If Y is a spatial diamond, then $|Y|$ is spectral. Therefore, it is qcqs sober and any point has a basis of qc open subset, which is equivalent to a pro-object in the category of finite ordered sets.*

It does look like analytic adic spaces in the following sense:

[1]This construction is the left adjoint of the natural inclusion of compact Hausdorff spaces into the all topological spaces. More precisely, it satisfies the following universal property: For a topological space X, the Stone-Čech compactification is initial for all maps $X \to S$, where S is compact Hausdorff. For a discrete E, $\beta E = \varprojlim_{E \to E'} E'$, where the limit runs through all finite quotients E' of E.

[2]Note that $\underline{T}$ itself is not necessarily representable.

Proposition 2.12. *If $X \to Y$ is a morphism between locally spatial diamond, then $|f| : |X| \to |Y|$ is generalizing and any $x \in |X|$, the set of generalization $|X|_x$ is a totally ordered chain under specialization.*

Example 2.13. Let X be an analytic adic space, we define the correspoding diamond $X^\diamond$ to be the v-sheaf sending $S \in \mathrm{Perf}_{\bar{\mathbb{F}}_q}$ to the set of triples $(S^\sharp, \iota, x)$, where $\iota : S^{\sharp\flat} \simeq S$ fixes an untilt of S and $x \in X(S^\sharp)$.

Remark 2.14.

1. $X^\diamond$ is locally spatial diamond with the underlying topological space
$$|X^\diamond| = |X|.$$
2. If X is neotherian, then $X_{\acute{e}t} \cong X^\diamond_{\acute{e}t}$.

Example 2.15.

1. If X is an analytic adic space over $\bar{\mathbb{F}}_q$, then $X^\diamond = \varprojlim_\varphi X = X^{1/p^\infty} \in \mathrm{Perf}_{\bar{\mathbb{F}}_q}$, which locally is the completion of perfection.
2. If E is a finite extension of $\mathbb{Q}_p$, then
$$\mathrm{Spa}(E)^\diamond = \mathrm{Spa}(\widehat{\overline{E}})^\flat / \underline{\mathrm{Gal}(\overline{E}/E)},$$
which parametrize all untilts over E.

Theorem 2.16. *Let X be an analytic adic space, then*
$$(-)^\diamond : \widetilde{\mathrm{Perf}/X} \to \widetilde{\mathrm{Perf}_{\bar{\mathbb{F}}_q}}/X^\diamond$$
induces an equivalence of v-topos. In particular, if $X = \mathrm{Spa}(\mathbb{Q}_p)$, then we have
$$\widetilde{\mathrm{Perf}_{\mathbb{Q}_p}} \simeq \widetilde{\mathrm{Perf}_{\bar{\mathbb{F}}_q}}/\mathrm{Spa}(\mathbb{Q}_p)^\diamond.$$

2.2 Back to Banach-Colmez

Warning 2.17. In Theorem 1.13, we did not claim that our Banach-Colmez spaces are diamonds, but our assertion is that $\mathrm{BC}(\mathcal{E}) \to *$ and $\mathrm{BC}(\mathcal{E}[1]) \to *$ is relatively representable in locally spatial diamonds. In fact, they are NOT diamonds. For example,
$$\mathrm{BC}(\mathcal{O}(1)) \simeq \mathrm{Spd}(\mathbb{F}_q[\![X^{\frac{1}{p^\infty}}]\!]),$$
but $\mathbb{F}_q[\![X^{\frac{1}{p^\infty}}]\!]$ is not analytic since it does not admit a(n invertible) pseudo-uniformizer.

Proof of Theorem 1.13. **Unfinished**

Recall the line bundle $\mathcal{O}_{X_S}(1)$ is ample, thus for any $\mathcal{E}$ a vector bundle of finite rank over X_S and $m \gg 0$, the vector bundle $\check{\mathcal{E}}(m)$ is generated by global sections, therefore we obtain a presentation
$$0 \to \mathcal{E} \to \mathcal{O}(N)^d \to \mathcal{O}(N')^{d'},$$
and so as the Banach-Colmez spaces
$$0 \to \mathrm{BC}(\mathcal{E}) \to \mathrm{BC}(\mathcal{O}(N)^d) \to \mathrm{BC}(\mathcal{O}(N')^{d'}).$$
We claim that this shows $\mathrm{BC}(\mathcal{E})$ is a closed subspace of $\mathrm{BC}(\mathcal{O}(N)^d)$. Assuming this, we reduce ourselves to the case $\mathcal{E} = \mathcal{O}_{X_S}(N)$. See [FS24, II.2.2]. $\square$

Example 2.18. The Banach-Colmez space $\mathrm{BC}(\mathcal{O}(1))$ is represented by the universil cover of $\widehat{\mathbb{G}}_m$ over S, more precisely, we have
$$\mathrm{BC}(\mathcal{O}(1)) \simeq \mathrm{Spd}(\mathbb{F}_q[\![X^{\frac{1}{p^\infty}}]\!]).$$
The Banach-Colmez space $\mathrm{BC}(\mathcal{O}_{X_S}(-1)[1])$ is isomorphic to $(\mathbb{G}_{a,S^\sharp})^\diamond/\underline{E}$, where $S^\sharp$ is an arbitrary untilt of S.

2.3 Absolute BC spaces

Fix a collection of slopes $(\lambda_i)_I \in \mathbb{Q}^I$, we obtain a vector bundle

$$\bigoplus_{i\in I} \mathcal{O}(\lambda_i)$$

on X_S for any S. The Banach-Colmez spaces associate with such a vector bundle is called an *absolute* one. Even though our Banach-Colmez spaces are not diamonds, one can consider the punctured one, which is indeed spatial diamonds.

Theorem 2.19. *The punctured spaces* $\mathrm{BC}(\bigoplus_{i\in I}\mathcal{O}(\lambda_i))\backslash\{0\}$ *and* $\mathrm{BC}(\bigoplus_{i\in I}\mathcal{O}(\lambda_i)[1])\backslash\{0\}$ *are locally spatial diamonds.*

Example 2.20. Let $(\lambda_i) \in (]0,1] \cap \mathbb{Q})^I$, then

$$\mathrm{BC}(\bigoplus_{i\in I}\mathcal{O}(\lambda_i))\backslash\{0\} \simeq \mathrm{Spa}(\bar{\mathbb{F}}_q[\![x_1^{1/p^\infty}, \cdots, x_d^{1/p^\infty}]\!])\backslash V(x_1, \cdots, x_d)$$

is a spatial diamond, moreover, a qcqs perfectoid space.

Theorem 2.21 (Artin's criterion)**.** *Let X be a small v-sheaf in* $\mathrm{Perf}_{\bar{\mathbb{F}}_q}$*, then X is a spatial diamond if and only if*

1. *X is a spatial sheaf,* i.e.*, X is qcqs and $|X|$ is spatial.*
2. *For any $x \in X$, $X_x := \varprojlim_{x\in U} U$ is a diamond.*

Example 2.22. Let X be a spatial v-sheaf such that there exists

$$|X| = \bigcup_i |Z_i|,$$

where Z_i is pro-constructable generalizing and each Z_i is a diamond. Then X is a spatial diamond. Note that for any subset $|Z| \subset |X|$, there are a natural sub v-sheaf Z defined by

$$Z(S) := \{f \in X(S) \big| \operatorname{Im}(|f|) \subset |Z|\}.$$

This construction is used in the theory of $\mathbb{B}_{\mathrm{dR}}$-affine Grassmannian to produce Schubert cells; see [SW20].

3 Lecture 3

3.1 Absolute (punctured) BC spaces

Let E be a p-adic field with uniformizer π and residue field $\mathcal{O}_E/\pi \cong \mathbb{F}_q$. For any $S \in \mathrm{Perf}_{\bar{\mathbb{F}}_q}$ perfectoid characteristic p, one can construct the relative FF curve X_S parametrized by S. Fix a interger $d \in \mathbb{Z}$, there exists a line bundle $\mathcal{O}(d)$ over X_S, which is given by the descent data on Y_S: it is given by gluing the trivial bundle on Y_S and automorphic factor

$$\varphi \mapsto \pi^{-d} \in \mathcal{O}(Y_S)^*.$$

Definition 3.1. We define a v-sheaf $\mathbb{B} : S \mapsto \mathcal{O}(Y_S)$ on $\mathrm{Perf}_{\bar{\mathbb{F}}_q}$, therefore

$$\mathrm{H}^0(X_S, \mathcal{O}(d)) = \mathbb{B}(S)^{\varphi=\pi^d}.$$

In particular, $\mathrm{BC}(\mathcal{O}(1))$ is represented by a formal group.[3] Let $\mathcal{G}$ be the Lubin-Tate π-divisible formal group over $\bar{\mathbb{F}}_q$, there exists a period isomorphism

$$\varprojlim_\pi \mathcal{G} \simeq \mathbb{B}^{\varphi=\pi},$$

[3] There is a nice explanation of this in [FS24, Ch II]

where the left-hand side is the universal cover of $\mathcal{G}$ and is isomorphic to $\mathrm{Spf}(\bar{\mathbb{F}}_q[\![T^{\frac{1}{p^\infty}}]\!])$. More concretely, if $\mathcal{F}$ is a 1-dimensional formal group law over $\mathcal{O}_E$ with a logarithm

$$\log_{\mathcal{F}} : \mathcal{F}\hat{\otimes}_{\mathcal{O}_E} E \simeq \widehat{\mathbb{G}}_{a,E}$$

given by $\log_{\mathcal{F}}(T) = \sum_{n\geq 0} \frac{T^{q^n}}{\pi^n}$, then for any $\mathrm{Spa}(R, R^+) \in \mathrm{Perf}_{\mathbb{F}_q}$, there is an isomorphism

$$(R^{\circ\circ}, +_{\mathcal{F}}) \simeq \mathbb{B}(R, R^+)^{\varphi=\pi},\ x \mapsto \sum_{n\in\mathbb{Z}} [x^{q^{-n}}]\pi^n \in \mathcal{O}(Y_S).$$

Thus, the v-sheaf $\mathbb{B}^{\varphi=\pi}$ is "represented" by

$$\mathrm{Spa}(\bar{\mathbb{F}}_q[\![T^{1/p^\infty}]\!], \bar{\mathbb{F}}_q[\![T^{1/p^\infty}]\!]),$$

whereas this space is in fact not perfectoid. However, the v-sheaf $\mathbb{B}^{\varphi=\pi}\backslash\{0\}$ is represented by $\mathrm{Spa}(\bar{\mathbb{F}}_q(\!(T^{1/p^\infty})\!), \bar{\mathbb{F}}_q[\![T^{1/p^\infty}]\!])$, which is affinoid perfectoid.

Now consider the $\mathcal{O}_E^\times$-extension $\breve{E}_\infty$ over $\breve{E}$, where $\breve{E}$ is the completion of maximal unramified extension of E and $\breve{E}_\infty$ is the completion of Lubin-Tate extension of $\breve{E}$. Note that $\breve{E}_\infty$ is a perfectoid field with its tilting $\bar{\mathbb{F}}_q(\!(T^{1/p^\infty})\!)$, actually we obtain an isomorphism

$$\mathbb{B}^{\varphi=\pi}\backslash\{0\} = \mathrm{Spa}(\breve{E}_\infty^\flat)$$

which is compatible with the action $\underline{\mathcal{O}_E^\times}$ and identify the action of π on the left-hand side with the action of φ on the right-hand side.

Definition 3.2. Define v-sheaf Div^1 sending each S to the sets of equivalent classes of pairs $(\mathcal{L}, u)$, where $\mathcal{L}$ is a line bundle on X_S of degree 1 and $u \in \mathrm{H}^0(X_S, \mathcal{L})$ such that for any $s \in S$, the stalk of u at s does not vanish, *i.e.*, $u|_{X_{\kappa(s),\kappa(s)^+}} \neq 0$. Two pairs $(\mathcal{L}, u)$ and $(\mathcal{L}', u')$ are equivalent if there is an isomorphism $\mathcal{L} \simeq \mathcal{L}'$ that sends u to u'.

Theorem 3.3. *1. Let $\mathcal{L}$ be a line bundle over X_S, the function* $\deg : |S| \to \mathbb{Z}$ *given by*

$$s \mapsto \deg(\mathcal{L}|_{\mathrm{Spa}(\kappa(s),\kappa(s)^+)})$$

is locally constant.

2. If it is constant with value $d \in \mathbb{Z}$, then there exists a proétale cover S' of S such that $\mathcal{L}|_{X_{S'}} \simeq \mathcal{O}_{X_{S'}}(d)$.

Corollary 3.4. *The Picard stack, sending each S to the groupoid of line bundles on X_S, admits a stratification by degree,* i.e.,

$$\mathcal{P}ic = \coprod_{d\in\mathbb{Z}} \mathcal{P}ic^d,$$

where each $\mathcal{P}ic^d$, sending S to the groupoid of line bundles of degree d on X_S, is an open and closed substack of $\mathcal{P}ic$. Moreover, for each $d \in \mathbb{Z}$, there is a natural isomorphism

$$\mathcal{P}ic^d \simeq [*/\underline{E^\times}],\ \mathcal{L} \mapsto \underline{\mathrm{Isom}}(\mathcal{O}(d), \mathcal{L})$$

where the right-hand side is the classifying stack of pro'etale $\underline{E^\times}$torsors.

Corollary 3.5. *Let G be a locally profinite group, and S be a perfectoid space. It is an equivalence between the groupoid of proétale $\underline{G}$-torsors and the groupoid of v-$\underline{G}$-torsors given by the obvious map,* i.e.,

$$\mathrm{H}^1_{pro\acute{e}t}(S, \underline{G}) \simeq \mathrm{H}^1_v(S, \underline{G}).$$

Proof. It suffices to prove any $\underline{G}$-torsor $\underline{T}$ can be trivialized by a proétale cover. Note that $\underline{T} \simeq \underline{K}\backslash\underline{T}$ where this limit runs through all compact open subsets K of G. Furthermore, $\underline{K}\backslash\underline{T}$ is v-locally isomorphic to $S \times K\backslash G$.

Theorem 3.6. *Separated étale morphism of perfectoid spaces satisfies v -descent.*

By virtue of this theorem, We see that $\underline{K}\backslash\underline{T}$ is represented by a separated étale perfectoid space over S. After taking limits, we see that $\underline{T}$ is represented by a proétale perfectoid space over S. $\square$

Back to Div^1, this is the space parametrizing "degree 1 over the curve", with an analogue of Abel-Jacobi map

$$\mathrm{Div}^1 \to \mathcal{P}ic^1,\ (\mathcal{L}, u) \mapsto \mathcal{L}.$$

Thus, one obtains the following Cartesian diagram

$$\begin{tikzcd} \mathbb{B}^{\varphi=\pi} \arrow[r] \arrow[d] \arrow[loop left, "\underline{E^\times}"] & * \arrow[d] \\ \mathrm{Div}^1 \arrow[r, "AJ^1"] & \mathcal{P}ic^1 = [*/\underline{E^\times}]. \end{tikzcd}$$

Corollary 3.7. *One has the following isomorphisms*

$$\mathrm{Div}^1 \simeq (\mathbb{B}^{\varphi=\pi}\backslash\{0\})/\underline{E^\times},\ \mathrm{Div}^d \simeq (\mathbb{B}^{\varphi=\pi^d}\backslash\{0\})/\underline{E^\times}.$$

Now, let us look at this from another point of view. Denote $\breve{E}$ the maximal unramified extension of E and $\breve{E}_\infty$ the Lubin-Tate extension of $\breve{E}$.From Lubin-Tate theory, we see that $\breve{E}_\infty$ is a (infinite) Galois extension over $\breve{E}$ with Galois group isomorphic to $\mathcal{O}_E^\times$. Previously, we have the isomorphism

$$\mathbb{B}^{\varphi=\pi}\backslash\{0\} = \mathrm{Spa}(\breve{E}_\infty^\flat),$$

therefore if we quotient out this $\underline{E^\times}$-action on both hand side, we have

$$\mathrm{Div}^1 \simeq \mathrm{Spa}(\breve{E})^\diamond/\varphi^{\mathbb{Z}} \simeq \mathrm{Spa}(\breve{E}_\infty^\flat)/\underline{\mathcal{O}_E^\times} \times \varphi^{\mathbb{Z}}.$$

Theorem 3.8. *Let $\lambda = \frac{d}{h}$ be a rational number that lies in the interval $]0,1]$. Let $\mathcal{O}_{X_S}(\lambda)$ be the vector bundle associated with λ. Then we have*

$$\mathrm{BC}(\mathcal{O}(\lambda)) \simeq \widetilde{\mathcal{G}},$$

where $\widetilde{\mathcal{G}} \simeq \varprojlim_{\times\pi} \mathcal{G}$ is the universal cover of $\mathcal{G}$ and $\mathcal{G}$ is the formal π-divisible $\mathcal{O}_E$-module over $\mathbb{F}_q$ isoclinic of slope λ. More explicitly, one has the following isomorphism

$$\mathrm{BC}(\mathcal{O}(\lambda)) \simeq \mathrm{Spf}(\mathbb{F}_q[\![x_1^{1/p^\infty}, \cdots, x_d^{1/p^\infty}]\!]).$$

In particular, The punctured BC space $\mathrm{BC}(\mathcal{O}(\lambda))\backslash\{0\} \simeq \mathbb{B}^{\varphi^h=\pi^d}$ is represented by a qcqs perfectoid space

$$\mathrm{Spa}(\mathbb{F}_q[\![x_1^{1/p^\infty}, \cdots, x_d^{1/p^\infty}]\!])\backslash V(x_1, \cdots, x_d),$$

For any $S = \mathrm{Spa}(R, R^+) \in \mathrm{Perf}_{\mathbb{F}_q}$, we have

$$(R^{\circ\circ})^d \simeq \mathbb{B}(R, R^+)^{\varphi=\pi^d},\ (x_0, \cdots, x_{d-1}) \mapsto \sum_{k\in\mathbb{Z}}\sum_{i=0}^{d-1}[x^{q^{-k}}]\pi^{kd+i},$$

where the left hand side is equipped with addition given by a d-dimensional formal group law.

Now we explain the formula

$$\mathrm{Div}^1 \simeq \mathrm{Spa}(\breve{E})^\diamond/\varphi^{\mathbb{Z}}.$$

Note that $\mathrm{Spa}(\breve{E})^\diamond$ is the moduli of untilts over E, *i.e.*, For any $S \in \mathrm{Perf}_{\overline{\mathbb{F}}_q}$ the S-valued point of $\mathrm{Spa}(\breve{E})^\diamond$ are in bijection with the set of all untilts of S over E. On the other hand, by the universal property of Witt vectors, one always has a quotient map

$$\theta : \mathbb{W}_{\mathcal{O}_E}(A^+) \to A^{\sharp,+},\ \sum_{m\geq 0}[a_m]\pi^m \mapsto \sum_{m\geq 0} a_m^\sharp \pi^m.$$

The kernel of this map is always a principal ideal generated by ξ, which is called distinguished of degree 1 in [FFC18], that is, if $\xi = \sum_{m\geq 0}[a_m]\pi^m$, then a_0 is invertible topologically nilpotent in A and a_1 is invertible in A^+. Also, the element ξ lies in $\mathcal{O}(Y_S)^+$. The map θ gives a Cartier divisor $S^\sharp \hookrightarrow Y_S$ in the sense of [SW20, 5.3]. This will furthermore give an effective Cartier divisor of degree 1 of X_S by the composition map $S^\sharp \hookrightarrow Y_S \to X_S$. The formal completion of X_S along $S^\sharp$ is nothing but $\mathrm{Spf}(\mathbb{B}^+_{dR}(R^\sharp, R^{\sharp,+}))$. This defines a map

$$\mathrm{Spa}(\breve{E})^\diamondsuit \to \mathrm{Div}^1.$$

A Frobenius twist of an untilt will not change the associated Cartier divisor of X_S and all Cartier divisors rise in this manner. Thus, this map induces an isomorphism

$$\mathrm{Spa}(\breve{E})^\diamondsuit/\varphi^{\mathbb{Z}} \simeq \mathrm{Div}^1.$$

Remark 3.9. Note that $\mathrm{Spa}(\breve{E})^\diamondsuit \simeq \mathrm{Spa}(E)^\diamondsuit \times_{\mathrm{Spa}(\mathbb{F}_q)} \mathrm{Spa}(\overline{\mathbb{F}}_q)$. Consider the tower

$$\begin{tikzcd} \mathrm{Spa}(\widehat{\overline{E}})^\flat \arrow[r, "I_E"] \arrow[rr, bend right, "W_E"'] & \mathrm{Spa}(\breve{E})^\diamondsuit \arrow[r, "\mathbb{Z}"] & \mathrm{Spa}(\breve{E})^\diamondsuit/\varphi^{\mathbb{Z}}. \end{tikzcd}$$

Therefore, we can see roughly that $\pi_1^{\acute{e}t}(\mathrm{Div}^1) = W_E$. The $\underline{W_E}$-torsor $\mathrm{Spa}(\widehat{\overline{E}})^\flat \simeq \mathrm{Spa}(\mathbb{C}_p^\flat)$ over Div^1 gives a map $\mathrm{Div}^1 \to [*/\underline{W_E}]$. In [FS24], they defines for each object $\mathbb{W} \in \mathrm{Rep}_\Lambda({}^LG)$ a functor

$$T_{\mathbb{W}} : D_{lis}(\mathrm{Bun}_G, \Lambda) \to D_{lis}(\mathrm{Bun}_G \times (\mathrm{Div}^1)^I, \Lambda).$$

3.2 Quasi-proétale morphisms of perfectoid spaces

Definition 3.10. A morphism $f : X \to Y$ of perfectoid spaces is quasi-proétale if there exists a proétale covering $\tilde{Y} \to Y$ such that $X \times_Y \tilde{Y} \to \tilde{Y}$ is proétale.

The reason we do need this definition is that being proétale is not local, *i.e.*, there exists morphisms that are proétale on each geometric point but not proétale, see [SW20, 9.1.5] for an example.

Recall that S is a spectral spaces $S \twoheadrightarrow \pi_0(S)$, where $\pi_0(S)$ is equipped with the quotient topology. This is a profinite set. Spectral spaces are directed inverse limits of finite sober topological spaces, [Sta18, Tag 09XX]. If a spectral space S can be written as $\varprojlim_i S_i$ where S_i finite sober, then $\pi_0(S) \simeq \varprojlim_i \pi_0(S_i)$ with $\pi_0(S_i)$ finite discrete.

If X is qcqs perfectoid space with $X \twoheadrightarrow \pi_0(X)$, then each connected component is a perfectoid space proétale over X. Indeed, a connected component C is isomorphic to

$$\varprojlim_{C\subset U, U \text{clopen}} \hookrightarrow X.$$

Proposition 3.11. *The following are equivalent:*

1. *Any connected component of X has a unique closed point* i.e., *is isomorphic to* $\mathrm{Spa}(K, K^+)$ *where (K, K^+) is affinoid perfectoid.*

2. *Any cover of X splits,* i.e., *any cover $\{U_i\}$ of X, there exists a split $X \to \coprod_I U_i$.*

Definition 3.12. We call perfectoid spaces as above *totally disconnected.*

Proposition 3.13. *Any X qcqs perfectoid space can be covered by a totally disconnected cover $\tilde{X}$, with $\tilde{X} \to X$ is proétale.*

Theorem 3.14. *If X is qcqs totally disconnected and $f: Y \to X$ is separated with Y qcqs such that each rank 1 geometric point $\bar{x} = \mathrm{Spa}(C, \mathcal{O}_C)$ of X, the fibre product $Y \times_X \bar{x}$ is profinite,* i.e., *is isomorphic to $\mathrm{Spa}(C, \mathcal{O}_C) \times \underline{P}$ with a profinite set P, then f is proétale.*

Corollary 3.15. *Let $f: X \to Y$ be a morphism of perfectoid spaces. Then f is quasi-proétale if and only if there exists a cover $\{U_i\}_I$ of X such that for any i, the map $f|_{U_i}: U_i \to Y$ has profinite rank 1 geometric fibre.*

Now let us repeat the example from [SW20]. Consider the natural map

$$\mathrm{Spa}(K\langle T^{1/2p^\infty}\rangle) \to \mathrm{Spa}(K\langle T^{1/p^\infty}\rangle),$$

where K is a perfectoid field with character not equal to 2. It is a quasi-proétale morphism but not proétale. The ramification at 0 is killed by taking some proétale cover.

Theorem 3.16 (Fargues)**.** *For $d \geq 1$, the map $(\mathrm{Div}^1)^d \to \mathrm{Div}^d$ sending $(D_1, \cdots, D_d) \to \sum_i D_i$ is quasi-proétale, and $(\mathrm{Div}^1)^d/\mathfrak{S}_d \to (\mathrm{Div}^d)$ is proétale.*

Example 3.17. Assume $p \neq 2$. Something similar happens in the example of quasi-proétale map

$$f: \mathbb{B} = \mathrm{Spa}(K\langle T^{1/2p^\infty}\rangle) \to \mathbb{B} = \mathrm{Spa}(K\langle T^{1/p^\infty}\rangle),$$

which induces an isomorphism

$$\mathbb{B}/\mu_2 \xrightarrow{\sim} \mathbb{B},$$

where $\mu_2 = \{1, -1\}$ and the left-hand side is a proétale sheaf quotient.

4 Lecture 4

4.1 Last time

Let $\mathcal{G}$ be the Lubin-Tate formal $\mathcal{O}_E$-module, we have the period isomorphism

$$\varprojlim_{\times\pi} \mathcal{G}_{\bar{\mathbb{F}}_q} \simeq \mathbb{B}^{\varphi=\pi},$$

which is compatible the structure of E-vector space. Once fix a coordinate of $\mathcal{G}$, one obtain an isomorphism

$$\mathbb{B}^{\varphi=\pi}\backslash\{0\} \simeq \mathrm{Spa}(\bar{\mathbb{F}}_q(\!(T^{1/p^\infty})\!)).$$

The right-hand side admits a $\mathcal{O}_E$-action, which is inherited from the action

$$[-]: \mathcal{O}_E \to \mathrm{End}(\mathcal{G}).$$

This action makes the isomorphism equivariant with the $\mathcal{O}_E$-action on the left. And the $\pi = \varphi$-action on the left hand side is identified as the relative Frobenius action on the right. We also talk about the formula

$$\mathrm{Div}^1 = \mathbb{B}^{\varphi=\pi}\backslash\{0\}/\underline{E^\times},$$

which is obtained by identifying $\mathrm{Spa}(\bar{\mathbb{F}}_q(\!(T^{1/p^\infty})\!))$ as the tilting of the completion $\breve{E}_\infty$ of Lubin-Tate tower over the maximal unramified $\breve{E}$.

Moreover, we have the fundamental exact sequence in p-adic Hodge theory

$$0 \to V_p(\mathcal{G}) \to \mathbb{B}(\breve{E}_\infty)^{\varphi=\pi} \xrightarrow{\theta} \breve{E}_\infty \to 0,$$

where $V_p(\mathcal{G})$ is the rational Tate module. For any nonzero element of $V_p(\mathcal{G})$, it determines an isomorphism of $\mathrm{Spa}(\breve{E}_\infty)^\flat \to \mathbb{B}^{\varphi=\pi}\backslash\{0\}$. Let $\epsilon^{(n)}$ be the primitive element lies in $\mathcal{G}[\pi^n](\bar{E})$ with $[\pi](\epsilon^{(n+1)}) = \epsilon^{(n)}$. This implies that

$$(\epsilon^{(n+1)})^q \equiv \epsilon^{(n)} \mod \pi.$$

We denote $T = (\epsilon^{(n)} \mod \pi)_{n\geq 0}$ and $V_p(\mathcal{G})$ is generated by $\epsilon = \varprojlim_n \epsilon^{(n)}$ as a $\mathbb{Q}_p$-vector space. Also, one have the following diagram

$$\begin{tikzcd}
\mathrm{Spa}(\breve{E}_\infty) \arrow[r, "\sim"] \arrow[d, "\underline{\mathcal{O}_E^\times}"] \arrow[dd, bend right=60, "W_E"'] & \mathbb{B}^{\varphi=\pi}\backslash\{0\} \\
\mathrm{Spa}(\breve{E})^\diamond \arrow[d, "\varphi^{\mathbb{Z}}"] & \\
\mathrm{Spa}(\breve{E})^\diamond/\varphi^{\mathbb{Z}} &
\end{tikzcd}$$

Theorem 4.1. *Let $f : X \to Y$ be a separated morphism between perfectoid spaces. The following are equivalent:*

1. *for all C algebraic closed perfectoid field and all geometric point $\mathrm{Spa}(C, \mathcal{O}_C) \xrightarrow{\bar{y}} Y$, the fibre product $X_{\bar{y}} = X \times_Y \bar{y}$ is profinite,* i.e., *isomorphic to $\underline{P} \times \mathrm{Spa}(C, \mathcal{O}_C)$ for a profinite set P.*
2. *there eixsts a proétale cover $\tilde{Y}$ of Y such that $X \times_Y \tilde{Y} \to \tilde{Y}$ is proétale.*

Corollary 4.2. *Let $f : X \to Y$ be a morphism satisfying above equivalent conditions. Assume further that for each rank 1 geometric point $\bar{y}$ of Y, $X_{\bar{y}}$ is not empty. Then $X \to T$ is an epimorphism of proétale sheaves, and Y is a proétale quotient of X,* i.e.,

$$X/(X \times_Y X) \simeq Y.$$

4.2 Application

With this preparation, we are able to deduce Theorem 3.16 from the original work of [FFC18]. For $d > 0$, the proétale sheaf $\mathbb{B}^{\varphi=\pi^d}\backslash\{0\}$ is a spatial diamond, and

$$(B^{\varphi=\pi}\backslash\{0\})^d/\underline{\triangle_d \rtimes \mathfrak{S}_d} \simeq \mathbb{B}^{\varphi=\pi^d},\ (t_1, \cdots, t_d) \mapsto \prod_i t_i.$$

Here $\triangle_d = \{(\lambda_1, \cdots, \lambda_d) \in (E^\times)^d \mid \prod_{i=1}^d \lambda_i = 1\}$. To prove the assertion, it will suffice to check on each geometric point.

Theorem 4.3 (Fargues-Fontaine)**.** *Let C be an algebraic closed perfectoid field over $\mathbb{F}_q$ with a ring of integers C^+. Then the graded ring*

$$P := \bigoplus_{d\geq 0} \mathbb{B}^{\varphi=\pi^d}$$

is factorial with an irreducible element of degree 1. And $(\coprod_{d>0} P_d\backslash\{0\})/E^\times$ is freely generated by degree 1 elements.

Therefore, all geometric fibers are $\triangle_d \rtimes \mathfrak{S}_d$.

Fact 4.4. For $1 \leq d \leq [E : \mathbb{Q}_p]$, the diamond $\mathbb{B}^{\varphi=\pi^d}\backslash\{0\}$ is a qcqs perfectoid space. In fact, the construction of the courbe has two inputs, that is, a perfectoid space S of characteristic p and a finite extension E of $\mathbb{Q}_p$. For different choices of the second input, we have

$$X_{S,E} \cong X_{S,\mathbb{Q}_p} \times_{\mathbb{Q}_p} E,$$

and the projection $\pi_{E/\mathbb{Q}_p} : X_{S,E} \to X_{S,\mathbb{Q}_p}$ is finite étale. Then the push forward $\pi_{E/\mathbb{Q}_p,*}\mathcal{O}_{X_{S,E}}(d)$ is isomorphic to $\mathcal{O}_{X_{S,\mathbb{Q}_p}}(\frac{d}{[E:\mathbb{Q}_p]})^{\oplus(d,[E:\mathbb{Q}_p])}$. Therefore, we reduce to the case of Theorem 3.8.

4.3 The mod p Riemann-Hilbert correspondence

Let us recall the usual RH correspondence fist. Let X be a $\mathcal{C}^\infty$-manifold. Then there is a natural equivalence between the category of local systems of $\mathbb{R}$-vector spaces on X and the category of integral connections on X. If $\mathcal{F}$ is a local system, then correspondent connection is $(\mathcal{F}\otimes_{\underline{\mathbb{R}}}\mathcal{C}^\infty, \mathrm{id}\otimes d)$. If $(\mathcal{E},\nabla)$ is an integral connection, then the correspondent local system is the solution $\mathcal{E}^{\nabla=0}$.

This correspondence extends naturally to smooth projective varieties over $\mathbb{C}$, *i.e.*, we have an equivalence of category

$$\mathrm{Perv}(X(\mathbb{C})) \xrightarrow{\sim} \Big\{\text{holomonic regular } D\text{−modules}\Big\}/X.$$

Modular p. The mod p Riemann-Hilbert correspondence is firstly constructed by Katz. Let X be a $\mathbb{F}_q$-scheme. There is an equivalence of categories

$$\mathrm{LocSys}(X_{\acute{e}t},\mathbb{F}_q) \simeq \Big\{(\mathcal{E},\Phi)\Big|\ \mathcal{E} \text{ v.b.}, \Phi : \mathrm{Frob}^*\mathcal{E} \simeq \mathcal{E}\Big\}.$$

Let me explain the notation here. The left-hand side the category of $\mathbb{F}_q$-étale local system over X. On the right-hand side, $\mathcal{E}$ is a vector bundle on X and Φ is a Frobenius semilinear isomorphism. The map is defined in a manner similar to the classical case, where Frobenius map plays the role of differential operator. The map from the left to the right is defined by

$$\mathcal{F} \mapsto (\mathcal{F}\otimes_{\underline{\mathbb{F}_q}}\mathcal{O}_X, \mathrm{id}\otimes\mathrm{Frob}_q),$$

which defines (a priori) a étale locally free $\mathcal{O}_X$-module, and étale descents to a vector bundle. The inverse map is defined by

$$\tilde{\mathcal{E}}^{\varphi=\mathrm{id}} \leftarrow\!\shortmid (\mathcal{E},\Phi),$$

where $\tilde{\mathcal{E}}$ is the associated étale sheaf of $\mathcal{E}$ and $\varphi : \mathcal{E}\to\mathcal{E}$ is the linearization of Φ. The local picture of this is Artin-Schrier theory. Locally, let R be a $\mathbb{F}_q$-algebra. Then Φ amounts to an element in $\mathrm{GL}_n(R)$. The solution of the equation

$$(x_1^q,\cdots,x_n^q) = (x_1,\cdots,x_n)A$$

is represented by Spec $(R[x_1,\cdots,x_n]/(X_i^q - \sum_j a_{ij}X_j))$ where $A = (a_{ij})_{i,j}$. The natural map Spec $(R[x_1,\cdots,x_n]/(X_i^q-\sum_j a_{ij}X_j)) \to$ Spec R is finite étale of degree q^n.

Bhatt-Lurie. A level-up is that through a RH functor defined by Bhatt-Lurie, the category $\mathrm{LocSys}(X_{\acute{e}t},\mathbb{F}_q)$ is embeded fully faithfully into a new category. The object of this category is a pair (M,φ), where M is a quasi-coherent $\mathcal{O}_X$-module and $\varphi : M\to M$ is simply a semi-linear isomorphism. This RH functor is left adjoint to the solution functor

$$\mathrm{Sol}(M,\varphi) = \tilde{M}^{\varphi=\mathrm{id}},$$

where $\tilde{M}$ is again the associated étale sheaf of M. Moreover, if $\mathcal{F}$ is an étale local system, then

$$\mathrm{RH}(\mathcal{F}) = \varinjlim_{\varphi}\mathrm{Katz}(\mathcal{F}).$$

Example 4.5. Let $j : \mathbb{A}^1_{\mathbb{F}_q}\setminus\{0\} \hookrightarrow \mathbb{A}^1_{\mathbb{F}_q}$ be the natural immersion and its complement $i : \{0\} \hookleftarrow \mathbb{A}^1_{\mathbb{F}_q}$. One can form the following exact sequence of étale sheaves

$$0\to j_!\underline{\mathbb{F}_q}\to\underline{\mathbb{F}_q}\to i_*\underline{\mathbb{F}_q}\to 0,$$

then the RH functor gives us an exact sequence of quasi-coherent sheaves

$$0\to T^{1/p^\infty}\mathbb{F}_q[T^{1/p^\infty}]\to\mathbb{F}_q[T^{1/p^\infty}]\to\mathbb{F}_q\to 0.$$

4.4 The case of Div^1

Proposition 4.6. *There exists a RH functor that induces an equivalence between the category of étale local systems on* Div^1 *and the category of locally free* $\mathcal{O}_v$*-modules on* Div^1*, where* $\mathcal{O}_v$ *means the structure sheaf in* v*-topology.*

5 Lecture 5

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