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Exercise 1 (2.6.1)**.** Let X be a scheme satisfying (*), then show that $X \times \mathbb{P}^n$ also satisfying (*) and $\mathrm{Cl}(X \times \mathbb{P}^n) \cong \mathrm{Cl}(X) \times \mathbb{Z}$.

Proof. Since X is neotherian, for every open affine Spec A, noticing that $X \times \mathbb{P}^n$ is glued up locally by $A[\frac{x_0}{x_i}, \cdots, \frac{x_n}{x_i}]$, so we have $X \times \mathbb{P}^n$ is neotherian and integral. Since separated morphism is stable under base change, a composition of separated morphism is separated and $\mathbb{P}^n$ is separated, we have $X \times \mathbb{P}^n$ is separated. Regularity of codimension 1 is a local property, and by induction we may assume $X = \mathrm{Spec}\ A$ and $Y = \mathrm{Spec}\ A[X]$, it's enough to prove that Y is regular of codimension 1. Let P be any prime ideal of height 1, $\mathfrak{p} = A \cap P$ is zero ideal or of height 1: if $\mathfrak{p} = (0)$ then $B_{\mathfrak{p}} = k[X]$ where k is the fraction field of A, so B_P is a localization of this polynomial ring at a prime of height 1, hence regular; if $ht(\mathfrak{p}) = 1$, then easy to verify that $\mathfrak{p}B$ is of height 1, so we have $\mathfrak{p}B = P$, so $P/P^2 \cong \mathfrak{p}B/(\mathfrak{p}B)^2 = \mathfrak{p}[X]/\mathfrak{p}^2[X] \cong \mathfrak{p}/\mathfrak{p}^2$, hence regular. So above all, $X \times \mathbb{P}^n$ satisfy (*).

Now we prove that $\mathrm{Cl}(X \times \mathbb{P}^n) \cong \mathrm{Cl}(X) \times \mathbb{Z}$, consider proposition 6.5 we have the exact sequence

$$0 \to \mathbb{Z} \overset{1 \mapsto Z}{\to} \mathrm{Cl}(X \times \mathbb{P}^n) \overset{j}{\to} \mathrm{Cl}(X \times \mathbb{A}^n) \to 0.$$

where $Z = V(x_0)$ is the hyperplane at infinite defined by $x_1 = 0$ and we have $D(x_0) \cong X \times \mathbb{A}^n$, the proposition 6.5 keeps the exactness of last two position and we prove the first one. denote K be the function field of X and $L = K(t_1, \cdots, t_n)$ be function field of $X \times \mathbb{P}^n$ where $t_i = \frac{x_i}{x_0}$. Suppose $nZ = 0$ i.e. there exists $f \in L$ such that $(f) = nZ$ i.e. $v_Z(f) = n$ and $V_Y(f) = 0$ for any other Y. Therefore, we can write f into $f = x_0^n \frac{g(x_0, \cdots, x_n)}{h(x_0, \cdots, x_n)}$ where $g, h \in K[x_0, \cdots, x_n]$ is homogenous and $d + \deg(g) = \deg(h)$ and both of them has no factor of x_i. Since degree of g and h is different, they can't define same divisor, it's a contradiction. So the sequence is exact everywhere. And we also know that $\mathrm{Cl}(X \times \mathbb{A}^n) \cong \mathrm{Cl}(X)$ by proposition 6.6.

Last, I claim that the sequence is split, so we have $\mathrm{Cl}(X \times \mathbb{P}^n) \cong \mathrm{Cl}(X) \times \mathbb{Z}$. Recall that $i : \sum n_i Y_i \mapsto \sum n_i \pi^{-1}(Y_i) = \sum n_i (Y_i \times \mathbb{A}^n)$ defines an isomorphism between $\mathrm{Cl}(X) \cong \mathrm{Cl}(X \times \mathbb{A}^n)$, enough to define a map $k : \mathrm{Cl}(X) \to \mathrm{Cl}(X \times \mathbb{P}^n)$ such that the diagram is commutative:

$$\begin{tikzcd} \mathrm{Cl}(X \times \mathbb{P}^n) \arrow[r, "j"] & \mathrm{Cl}(X \times \mathbb{A}^n) \arrow[d, "i"] \\ & \mathrm{Cl}(X) \arrow[ul, "k"] \end{tikzcd}$$

Obviously, we define $k : \sum n_i Y_i \mapsto \sum n_i (Y_i \times \mathbb{P}^n)$ and if it's well-defined from $\mathrm{Cl}(X) \to \mathrm{Cl}(X \times \mathbb{P}^n)$ then the diagram is commutative. Let $f \in K$ and we have $k((f)) = \sum v_Y(f)(Y \times \mathbb{P}^n)$ which is exactly defined by $f \in K \subset L$. Now we have the exact sequence is exact and split and The result follows. $\square$

Exercise 2 (2.6.3(*Cones*))**.** In this exercise we compare the class group of a projective variety V to the class group of its cone(I, Ex.2.10). So let V be a projective variety in $\mathbb{P}^n$, which is of dimension ≥ 1 and nonsingular in codimension 1. Let $X = C(V)$ be the affine cone over V in A^{n+1}, and let $\bar{X}$ be its projective closure in $\mathbb{P}^{n+1}$. Let $P \in X$ be the vertex of the cone.

(a) Let $\pi : \bar{X} - P \to V$ be the projection map. Show that V can be covered by open subsets U_i such that $\pi^{-1}(U_i) \cong U_i \times \mathbb{A}^1$ for each i, and then show as in (6.6) that $\pi^* : \mathrm{Cl}(V) \to \mathrm{Cl}(\bar{X} - P)$ is as isomorphism. Since $\mathrm{Cl}(\bar{X}) \cong \mathrm{Cl}(\bar{X} - P)$, we have also $\mathrm{Cl}(V) \cong \mathrm{Cl}(\bar{X})$.

(b) We have $V \subset \bar{X}$ as the hyperplane section at infinity. Show that the class of the divisor V in $\mathrm{Cl}(\bar{X})$ is equal to π^* (class of $V.H$) where H is any hyperplane of $\mathbb{P}^n$ not containing V. Thus conclude using (6.5) that there is an exact sequence

(c) Let $S(V)$ be the homogeneous coordinate ring of V (which is also the affine coordinate ring of X). Show that $S(V)$ is a UFD if and only if (1) V is projectively normal(5.14) and (2)$\mathrm{Cl}(V) \cong \mathbb{Z}$ and is generated by the class of $V.H$.

$$0 \to \mathbb{Z} \to \mathrm{Cl}(V) \to \mathrm{Cl}(X) \to 0,$$

where the first arrow sends $1 \to V.H$, and the second is π^* followed by the restriction to $X - P$ and inclusion in$X$. (the injective of the first arrow follows from the previous exercises.)

(d) Let $\mathcal{O}_P$ be the local ring of P on X. Show that the natural restriction map induces an isomorphism $\mathrm{Cl}X \to \mathrm{Cl}(\mathrm{Spec}\ \mathcal{O}_P)$.

Proof. (a) Suppose V is defined by I, I claim that $\bar{X}$ is also defined by I but forget the graded structure. In fact, Let arbitrary $f \in I$ we have X vanishes f, consider $X \subset \mathbb{A}^{n+1}$ which is an affine chart of $\mathbb{P}^{n+1}$, we denote the coordinates of $\mathbb{A}^{n+1}$ be $(\frac{X_1}{X_0}, \cdots, \frac{X_{n+1}}{X_0})$ and $\mathbb{P}^{n+1} = \mathrm{Proj}\ \mathbb{Z}[X_0, \cdots, X_{n+1}]$, then we have $\bar{X}$ vanishes $x^{\deg f} f(\frac{X_1}{X_0}, \cdots, \frac{X_n}{X_0}) = f$ since f is homogeneous. Above all, $\bar{X}$ is defined by the ideal $I \subset \mathbb{Z}[X_0, \cdots, X_{n+1}]$. In $H := V(X_0) \cong \mathbb{P}^n$ there's $V = \bar{X} \cap H$. So choose U_i as standard affine cover of $\mathbb{P}^n$ intersecting with V. Notice that the projection $\pi : \bar{X} - P \to V$ is defined by $\pi(a_0, \cdots, a_{n+1}) = (a_1, \cdots, a_{n+1})$, so define $U_i \times \mathbb{A}^1 \to \pi^{-1}(U_i), ((a_0, \cdots, a_n), t) \mapsto (t, a_1, \cdots, a_n)$ and it's clearly a isomorphism. Note that $U_i \times \mathbb{A}^1 \cong \pi^{-1}(U_i)$ but $V \times \mathbb{A}^1$ is not necessarily isomorphic to π^{-1}(V) which is a (nontrivial) line bundle over V.

We similarly define $\pi^* : \sum n_i Y_i \mapsto \sum n_i \pi^{-1}(Y_i)$, it's well-defined as a morphism of divisor, as proposition 6.6 show, it's also well-defined on divisor class. Now check it's injective, if any divisor D such that $\pi^* D = (f)$ where f is in the function field of $\bar{X}$. By definition of π^* we can see X_0 can't appear in f, because locally f must be Type 1 which is mentioned in proof of proposition 6.6, so f must be in function field of V, and this implies $D = (f)$.

Now check it's surjective, it's sufficient to prove that any prime divisor of type 2 is linear equivalent to type 1, let D be such a prime divisor, and localize at the generic point of V in $\bar{X}$ we have a prime divisor of Spec $K[t]$ where K denotes the function field of V and it's also principle, saying generated by f. Now we can see that $(f) - D$ is of type 1, now we proved $\mathrm{Cl}(V) \cong \mathrm{Cl}(\bar{X})$.

Notice that $\dim V \geq 1$ so $\mathrm{codim} P \geq 2$ implies $\mathrm{Cl}(V) \cong \mathrm{Cl}(\bar{X} - P) \cong \mathrm{Cl}(\bar{X})$.

(b) Let $H_0 = V(X_0) \subset \mathbb{P}^{n+1}$ we have $V = H_0 \cap \bar{X}$, and let $H = V(g) \subset \mathbb{P}^n$ where $g = \sum_0^n a_i x_i$, assume V is defined by $(f_1, \cdots, f_m)$, then $V.H$ is defined by $(f_1, \cdots, f_m, g)$, hence $\pi^*(V.H) = V((f_1, \cdots, f_m, g)) \subset \mathbb{P}^{n+1}$, consider h be the image in $K(\bar{X})$ of $\frac{g}{X_0} \in K(\mathbb{P}^{n+1})$, we know $(h) = \pi^*(V.H) - V$, so we are done. this implies

$$0 \to \mathbb{Z} \stackrel{1 \mapsto V.H}{\to} \mathrm{Cl}(V) \stackrel{\pi^*}{\to} \mathrm{Cl}(X) \to 0$$

is exact.

(c) '$\Rightarrow$': Since UFD is always integrally closed, hence by definition V id projectively normal. And since $X = \mathrm{Spec}\ S(V)$ which don't consider the graded structure, so by proposition 6.2 $\mathrm{Cl}(X) = 0$, by the exact sequence in (b), we have $\mathrm{Cl}(V) \cong \mathbb{Z}$ and is generated by $V.H$.

'$\Leftarrow$': Since $S(V)$ is integrally closed, then by the property of localization, for any prime ideal $\mathfrak{p}$, i.e. every local field of X is normal i.e. X is normal. and by the exact sequence in (b), we have $\mathrm{Cl}(X) = 0$, by proposition 6.2 again the coordinate ring $S(V)$ of X is UFD.

(d) The maximal ideal of P is $m = (X_0, \cdots, X_n)$, and the projection morphism $\pi : \mathrm{Spec}\ \mathcal{O}_P \to X$ is induced by $S(V) \to S(V)_m$ so $\pi^* : \mathrm{Cl}(X) \to \mathrm{Cl}(\mathrm{Spec}\ \mathcal{O}_P)$ is defined as usual and it's well-defined on divisor class by the property of localization. Observation is: the prime divisors of Spec $\mathcal{O}_P$ is one-to-one correspond to the prime divisor of X also contain P. First, for any prime divisor Y not contain P since Y is defined by a principle ideal, say (f), we have f is a unit in $S(V)_m$, so $\pi^*(Y) = 0$. Hence, we may assume any Y contains P. Surjective is clear, now prove injective. If not, then exists $D_1 = \sum n_i \pi^* Y_i = (g)$ where g is an element of function field of X as the same as Spec $\mathcal{O}_P$, we denote D_2 be the divisor g defined in X, we see $D_1 - D_2$ is a sum of finite prime divisor containing P which is principle, the injective follows. □

Exercise 3 (6.4)**.** Let k be a field of char $\neq 2$. Let $f \in k[x_1, \dots, x_n]$ be a square-free nonconstant polynomial. Let $A = k[x_1, \dots, x_n, z]/(z^2 - f)$. Show that A is an integrally closed ring.

Proof. Following the hint, we have $K = \mathrm{Frac}(A) = k(x_1, \dots, x_n)[z]/(z^2 - f)$. It's a quadratic extension and any element can be written as $\alpha = g + hz$, where $g, h \in k(x_1, \dots, x_n)$. The minimal polynomial of such an α is $X^2 - 2gX + (g^2 - h^2 f)$. Notice that α is integral iff all coefficients are in $k[x_1, \dots, x_n]$ and since f is square-free this is equivalent to $g, h \in k[x_1, \dots, x_n]$. That is to say the ingrally closure of $k[x_1, \dots, x_n]$ is just A. □

Exercise 4 (6.5(Quadric Hypersurfaces))**.** Let char $k \neq 2$, and let X be affine quadratic hypersurface Spec $k[x_0, \dots, x_n]/(x_0^2 + \cdots + x_r^2)$.

(a) Show that X is normal if $r \geq 2$.

(b) Show by a suitable linear change of coordinates that the equation of X could be written as $x_0 x_1 = x_2^2 + \cdots + x_r^2$. Now imitate the method of (6.5.2) to show that:
  (1) if $r = 2$ then $\mathrm{Cl}(X) \cong \mathbb{Z}/2\mathbb{Z}$;
  (2) If $r = 3$ then $\mathrm{Cl}(X) \cong \mathbb{Z}$;
  (3) If $r \geq 4$ then $\mathrm{Cl}(X) \cong 0$.

(c) Now let Q be the projective quadratic hypersurfaces in $\mathbb{P}_k^n$ defined by the same equation. Show that:
  (1) If $r = 2$, then $\mathrm{Cl}(X) \cong \mathbb{Z}$;
  (2) if $r = 3$, then $\mathrm{Cl}(X) \cong \mathbb{Z} \oplus \mathbb{Z}$;
  (3) If $r \geq 4$ then $\mathrm{Cl}(X) = 0$.

(d) Prove Klein's theorem, which says that if $r \geq 4$, and if Y is an irreducible subvariety of codimension 1 on Q, then there is an irreducible hypersurface $V \subset \mathbb{P}^n$ such that $V \bigcap Q = Y$. with multiplicity one. In other word, Y is complete intersection.

Proof. (a) We know that when $r \geq 2$ we have f is irreducible, so by exercise 6.4 $k[x_0, \dots, x_n]/(x_0^2 + \cdots + x_r^2)$ is normal.

(b) Maybe we need a little more assumption to make $x_0 \to \frac{(x_0 - x_1)^2}{2}, x_1 \to \frac{(x_0 + x_1)^2}{2\sqrt{-1}}$ work, say k is algebraically closed.

(1) From Example 6.5.2 we have $\mathrm{Cl}(\mathrm{Spec}\ k[x, y, z]/(z^2 - xy)) \cong \mathbb{Z}/2\mathbb{Z}$, so if $r = 2$ we have $X \cong \mathrm{Spec}\ k[x, y, z]/(z^2 - xy) \times_k \mathbb{A}_k^{n-2}$, so $\mathrm{Cl}(X) = \mathbb{Z}/2\mathbb{Z}$.

(2) If $r = 3$, make the transform for both x_0, x_1 and x_2, x_3 we have $X \cong \mathrm{Spec}\ k[x, y, z, w]/(xy - zw)$, by exercise 6.1 and 6.3 we have $\mathrm{Cl}(X) = \mathrm{Cl}(Cone(\mathrm{Proj}\ k[x, y, z, w]/(xy - zw))) = \mathbb{Z}$.

(3) Let $U = D(x_0)$ we have $U \cong \mathrm{Spec}\ k[x_0, \frac{1}{x_0}, x_2, \dots, x_n]$ and it's a UFD so we have following exact sequence
$$\mathbb{Z} \to \mathrm{Cl}(X) \to \mathrm{Cl}(U) \to 0.$$
Enough to prove $Y = V(x_0)$ is a principle divisor. At the local ring of Y, since $g = x_2^2 + \cdots + x_n^2$ is irreducible then $v_Y(x_0) = v_Y(\frac{g}{x_1}) = 1$ so Y is principle.

(c) From example 6.6.2 we know the answer when $r = 3$. When $r \geq 4$, we have exact sequence $0 \to \mathbb{Z} \to \mathrm{Cl}(X) \to \mathrm{Cl}(\mathrm{Spec}\ k[x_0, \dots, x_n]/(x_0^2 + \cdots + x_r^2)) \to 0$. By (b) we have $\mathrm{Cl}(X) \cong \mathbb{Z}$. When $r = 2$ we have the same exact sequence $0 \to \mathbb{Z} \to \mathrm{Cl}(X) \to \mathbb{Z}/2\mathbb{Z} \to 0$ where the first map is $1 \mapsto X.H$, so after tensoring $\mathbb{Q}$ we have $\mathrm{Cl}(X) \cong \mathbb{Z} \oplus T$ where T is either trivial or torsion. Then tensoring $\mathbb{Z}/p\mathbb{Z}$ where $p \neq 2$ we have $\mathbb{Z}/p\mathbb{Z} \to \mathrm{Cl}(X) \otimes \mathbb{Z}/p\mathbb{Z} \to 0$, clearly if T not trivial the only possible situation is $T \cong \oplus \mathbb{Z}/p2\mathbb{Z}$. Notice that the generator of $X = \mathrm{Spec}\ k[x_0, \dots, x_n]/(x_0^2 + \cdots + x_r^2)$ is

$Y = V(x_0, x_2)$ or $Y' = V(x_0, x_1)$, the preimage of Y, Y' is $S = [0, a, 0, ...]$ and $S' = [0, 0, a, ...]$. And we have $Q.H = Q \bigcap V(x_0) = S + S'$. So if T exists, then by the exact sequence S, S' must be torsion element, this is a contradiction, so $\mathrm{Cl}(X) \cong \mathbb{Z}$. it also prove that $Q.H$ is twice generator.

(d) We already know that $S(Q)$ is UFD by previous exercise. Since Y is a subvariety of codimension 1 on Q, then the corresponding prime ideal in $S(Q)$ is principle, say $\mathfrak{p} = (f)$, just let $V = V(\bar{f})$ where $\bar{f}$ is the preimage of f in $k[x_0, \cdots, x_n]$, then we have $V \bigcap Q = Y$ and multiplicity is 1. □

Exercise 5 (6.6)**.** Let X be the nonsingular plane cubic curve $y^2 z = x^3 - xz^2$.

- (a) Show that three point P, Q, R of X are collinear if and only if $P + Q + R = 0$ in the group law on X.
- (b) A point $P \in X$ has order 2 in the group law on X if and only if the tangent line at P passes through P_0.
- (c) A point $P \in X$ has order 3 in the group law on X if and only if P is an inflection point.
- (d) Let $k = \mathbb{C}$. Show that the points of X with coordinates in $\mathbb{Q}$ form a subgroup of the group X. Can you determine the structure of this subgroup explicitly?

Proof. (a) If $P + Q + R = 0$ then we take L to be the line cross P, Q, by Bezout theorem L intersect with X on another point T, which is must be R since L makes $P + Q + T = 0$.

If P, Q, R is collinear then let L be the line across them and on $D(z)$ it's defined by $f \in \Gamma(D(z), \mathcal{O}_X)$. Consider divisor defined by f which is $P + Q + R$ so $P + Q + R = 0$ in $\mathrm{Cl}^\circ(X)$.

(b) If P has order 2 i.e. there exists f such that $(f) = 2(P - P_0)$, consider tangent line $L = V(ax + by + cz) = \{P, P, T\} \subset X$ and let $g = \frac{ax+by+cz}{z}$ so we have $(g) = 2P + T - 3P_0$. Now we have $(\frac{g}{f}) = T - P_0$ which is contradict to X is not birational.

If tangent line of P pass through P_0, then $L = V(ax + by + cz) = \{P.P.P_0\}$, so $(\frac{ac+by+cz}{z}) = 2P - 2P_0 = 2(P - P_0)$, implies P is order 2.

(c) If P is of order 3, then $3(P - P_0) = (f)$ where $f \in K(X)$. On affine open set $D(z)$ we have $V(f)$ intersect with intersection multiplicity 3 i.e. P is an inflection point. If $L = V(ac + by + cz)$ is the tangent line of P with multiplicity ≤ 3, then $(\frac{ax+by+cz}{z}) = 3(P - P_0)$ i.e. $3P = 0$.

(d) This use some high school math, Let $P, Q \in X(\mathbb{Q})$ and $R = P + Q$, we want to prove $R \in X(\mathbb{Q})$ and $P^{-1} \in X(\mathbb{Q})$. Let $L = V(ax + by + cz)$ pass through P, Q where we can assume $a, b, c \in \mathbb{Q}$ since $P, Q \in X(\mathbb{Q})$. Then $-R$ will be the public solution of $y^2 z = x^3 - xz^2$ and $ax + by + cz = 0$ and this induces a root of polynomial with rational coefficients of which other two roots is rational, so by Wieda' theorem, $-R \in X(\mathbb{Q})$, then replace P, Q by $-R, P_0$, it's done. □

Exercise 6 (6.7)**.** Let X be the nodal cubic curve $y^2 z = x^3 + x^2 z$ in $\mathbb{P}^2$. Imitate (6.11.4) and show that the group of Cartier divisors of degree 0, CaCl°X, is naturally isomorphic to the multiplicative group $\mathbb{G}_m$.

Proof. First we need to give a map $\deg : \mathrm{CaCl}(X) \to \mathbb{Z}$. For any element $D \in \mathrm{CaCl}(X)$, D is linear equivalent to a divisor which near $Z = (0, 0, 1)$ is invertible. So this defines a Weil divisor on $X - Z$, let $\deg \mathrm{CaCl}(X) := \deg \mathrm{Cl}(X)$ and denote $\mathrm{CaCl}^\circ(X)$ to be the kernel of this degree map. Now we define map from closed point to $\mathrm{CaCl}^\circ(X)$. For any closed point $P \in X - Z$, we associate the Cartier divisor D_P to be 1 in the neighbourhood of Z and $P - P_0$ on X_Z, This is injective because X is not birational and it's surjective because we can let $D = \sum n_i(P_i - P_0)$ and make all $n_i \geq 0$ and finally use induction. Now we have a group variety stucture on $X - Z$. Define $\mathbb{G}_m \to X - Z$ by $t \mapsto (1 - t, 1 + t, \frac{(1-t)^3}{4t})$, then a lot of elementary calculation to prove it's a morphism of group variety... □

Exercise 7 (6.8)**.** .

- (a) Let $f : X \to Y$ be a morphism of schemes. Show that $\mathscr{L} \to f^*\mathscr{L}$ induces a homomorphism of Picard groups, $f^* : \mathrm{Pic}Y \to \mathrm{Pic}X$.
- (b) If f is a finite morphism of nonsingular curves, show that this homomorphism correspondent to the homomorphism $f^* : \mathrm{Cl}\ Y \to \mathrm{Cl}\ X$.

(c) If X is a locally factorial integral closed subscheme of $\mathbb{P}^n_k$, and if $f : X \to \mathbb{P}^n_k$ is the inclusion map, then f^* on Pic agrees with the homomorphism on divisor class groups defined in (Ex.6.2).

Proof. (a) We may assume $V \cong \text{Spec } B \subset X$ and $U \cong \text{Spec } A \subset Y$, $f^*\mathscr{L}|_V \cong (\widetilde{M \otimes_A B})$ so it's locally free. By property of tensor product, f^* is a homomorphism.

(b) First we notice that induced f^* on Pic and Cartier divisor is same by definition, so we may check the f^* is same on Cartier divisor and normal one. Let $\varphi : \text{Pic } Y \to \text{Pic } X$ be the morphism induced by Cartier divisor and $\psi : \text{Cl}(Y) \to \text{Cl}(X)$ be the morphism induced by definition. Sufficient to check it's compatible on prime divisor. Assume that P is a prime (Weil) divisor of Y and let $\{(U_i, g_i)\}$ be corresponding Cartier divisor. Then $\psi(P) = \sum_{f(Q)=P} v_Q(t)Q$ where t is a local parameter of $\mathcal{O}_Q$ and $\varphi(Q) = \sum v_Q(f_* g_i)Q = \sum v_P(g_i)v_Q(t)Q$ where $f_* : K(Y) \to K(X)$ so by the way we define $\{(U_i, g_i)\}$ we know they are the same.

(c) Let V be any prime divisor of $\mathbb{P}^n_k$ Let U be an affine subset of $\mathbb{P}^n_k$ such that $U \bigcap V \neq \varnothing$ then V is defined by $f \in \Gamma(U, \mathcal{O}_{\mathbb{P}^n_k})$ and let $\bar{f}$ be the image of f in $\Gamma(U \bigcap X, \mathcal{O}_X)$ we have the image in $\text{Cl}(X)$ is $\sum v_Y(\bar{f})Y$ which is exactly the definition of Ex6.2. □

Exercise 8 (6.10(The Grothendieck Group $K(X)$))**.** Let X be a neotherian scheme. We define $K(X)$ be the Grothendieck Group of X by... If $\mathscr{F}$ is a coherent sheaf, we denote by $\gamma(\mathscr{F})$ its image in $K(X)$.

(a) If $X = \mathbb{A}^1_k$, then $K(X) \cong \mathbb{Z}$.
(b) If X is any integral scheme, and $\mathscr{F}$ a coherent sheaf, we define the rank of $\mathscr{F}$ to be $\dim_K \mathscr{F}_\zeta$ where ζ is the generic point of X, and $K = \mathcal{O}_\zeta$ is the function field of X. Show the rank function defines a surjective homomorphism rank: $K(X) \to \mathbb{Z}$
(c) If Y is a closed subscheme of X, there is an exact sequence

$$K(Y) \to K(X) \to K(X - Y) \to 0,$$

where the first map is extension by zero , and the second is restriction.

Proof. (a) Since any coherent sheaf over X can be represented as a cokernel of morphism between two free sheaves, so $K(X)$ is generated by $\mathcal{O}_X$, so it's isomorphic to $\mathbb{Z}$.

(b) First check it's well-defined. Let $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ be an exact sequence of coherent sheaves over X, so we take the stalk at ζ we have $0 \to \mathscr{F}'_\zeta \to \mathscr{F}_\zeta \to \mathscr{F}''_\zeta \to 0$ as exact sequence of vector space over K so $\dim \mathscr{F}' - \dim \mathscr{F} + \dim \mathscr{F}'' = 0$, so the map is well-defined. And obviously the map keeps addition and $\mathcal{O}^{\oplus n}$ maps to n implies surjective.

(c) Surjective is directly from exercise 5.15, say every coherent sheaf over $X - Y$ can be gained by restriction from a coherent sheaf on X. Now prove the exactness of the second place. Let $\mathscr{F}$ be a coherent sheaf over Y after extended by zero on X and restrict to $X - Y$, it's obviously a zero sheaf. Let $\mathscr{G}$ be any coherent sheaf on X such that restricts to $X - Y$ is a zero sheaf i.e. $\mathscr{G}$ have its support inside of Y. On every affine subset $U = \text{Spec } A \subset X$ we have $Y \bigcap U \cong \text{Spec } A/I$ for some ideal $I \subset A$. Consider the natural map $\mathscr{F} \to i_* i^* \mathscr{F}$ where $i_* i^* \mathscr{F}|_U \cong \widetilde{M/IM}$, we may set $\mathscr{F}_0 = \mathscr{F}$ and $\mathscr{F}_i = \ker(\mathscr{F}_{i-1} \to i_* i^* \mathscr{F}_{i-1})$. Locally, since M is finite generated and X is neotherian, we have for a sufficiently large n, then $\mathscr{F}_n = 0$. Now, by definition $\mathscr{F}_i/\mathscr{F}_{i+1}$ is a coherent sheaf extended zero outside Y, so $\gamma(\mathscr{F}) = \sum \gamma(\mathscr{F}_i/\mathscr{F}_{i+1})$, so the second place is exact. □

Exercise 9 (6.11(The Grothendieck Group of a Nonsingular Curve).)**.** Let X be a nonsingular curve over an algebraically closed k. We will show that $K(X) \cong \text{Pic } X \oplus \mathbb{Z}$, in several steps.

(a) For any divisor $D = \sum n_i P_i$ on X, let $\psi(D) = \sum n_i \gamma(k(P_i)) \in K(X)$, where $k(P_i)$ is the skyscraper sheaf k at P_i and 0 elsewhere. If D is an effective divisor, let $\mathcal{O}_D$ be the structure sheaf of the associate subscheme of codimension 1, and show that $\psi(D) = \gamma(\mathcal{O}_D)$. Then use (6.18) to show that any D, $\psi(D)$ depends only on the linear equivalent class of D, so ψ defines a homomorphism $\psi : \text{Cl}(X) \to K(X)$.
(b) For any coherent sheaf $\mathscr{F}$ on X, show that there exist locally free sheaves $\mathscr{E}_0$ and $\mathscr{E}_1$ and an exact sequence $0 \to \mathscr{E}_1 \to \mathscr{E}_0 \to \mathscr{F} \to 0$. Let $r_0 = \text{rank } \mathscr{E}_0$ and $r_1 = \text{rank } \mathscr{E}_1$, and define

$\det \mathscr{F} = (\bigwedge^{r_0} \mathscr{E}_0) \otimes (\bigwedge^{r_1} \mathscr{E}_1)^{-1} \in \operatorname{Pic}(X)$. Show that $\det \mathscr{F}$ is independent of the resolution chosen, and that it gives a homomorphism $\det : K(X) \to \operatorname{Pic}(X)$. Finally show that if D is a divisor, then $\det(\psi(D)) = \mathscr{L}(D)$.

(c) If $\mathscr{F}$ is any coherent sheaf of rank r, then show there is a divisor D on X and an exact sequence $0 \to \mathscr{L}(D)^{\oplus r} \to \mathscr{F} \to \mathscr{J} \to 0$, where $\mathscr{J}$ is a torsion sheaf, conclude that if $\mathscr{F}$ is a sheaf of rank r, then $\gamma(\mathscr{F}) - r\gamma(\mathcal{O}_X) \in \operatorname{Im} \psi$.

(d) Using the maps $\psi, \det, \operatorname{rank}$, and $1 \mapsto \gamma(\mathcal{O}_X)$ from $\mathbb{Z} \to K(X)$, show that $K(X) \cong \operatorname{Pic} X \oplus \mathbb{Z}$.

Proof. (a) First, the skyscraper sheaf $k(P)$ for any P is coherent since on a neighbourhood of P it's $\widetilde{k(P)}$ and zero sheaf otherwise. so ψ make sense. Let D as a Cartier divisor be $\{(U_i, f_i)\}$ then $\mathcal{O}_D \cong \mathcal{O}_X/I$ where I is the ideal sheaf generated by f_i, so $\mathcal{O}_D|_{U_i}$ is the coherent O_X-module $\mathscr{F}_{P_i}$ defined by $= A_i/(f_i) = A_i/(t_i^{n_i})$, so $\gamma(\mathcal{O}_D) = \oplus \mathscr{F}_{P_i}$. Then consider $0 \to m_P^{i-1}/m_P^i \to A/m_P^i \to A/m_P^{i-1} \to 0$ we have $\gamma(\mathscr{F}_P) = n_i \gamma(k(P_i))$, so finally we have $\gamma(\mathcal{O}_D) = \sum n_i \gamma(k(P_i)) = \psi(D)$. Let $D' \sim D$, so we have $\mathscr{L}(-D) \cong \mathscr{L}(-D')$, then we have exact sequence $0 \to \mathscr{L}(-D) \to \mathcal{O}_X \to \mathcal{O}_D$ and $0 \to \mathscr{L}(-D') \to \mathcal{O}_X \to \mathcal{O}_{D'}$ we have $\psi(D) = \gamma(\mathcal{O}_D) = \gamma(O_X) - \gamma(\mathscr{L}(-D)) = \gamma(O_X) - \gamma(\mathscr{L}(-D')) = \gamma(\mathcal{O}_{D'}) = \psi(D')$, so $\psi : \operatorname{Cl}(X) \to K(X)$.

(b) Let $\mathscr{F}$ be any coherent sheaf, then we have $0 \to \mathscr{G} \to \oplus_n \mathcal{O}_X \to \mathscr{F} \to 0$ where $\mathscr{G}$, check on every stalk we have the exact sequence on PID $\mathcal{O}_x, X$, so $\mathscr{G}_x$ is free, so $\mathscr{G}$ is locally free.

Let $0 \to \mathscr{E}_1' \to \mathscr{E}_0' \to \mathscr{F} \to 0$, Let $\mathscr{G} = \ker(\mathscr{E}_0 \oplus \mathscr{E}_1 \to \mathscr{F})$ where we take the difference. By nine lemma, we have $0 \to \mathscr{E}_1 \to \mathscr{G} \to \mathscr{E}_0' \to 0$ and $0 \to \mathscr{E}_1' \to \mathscr{G} \to \mathscr{E}_0 \to 0$, so $(\bigwedge \mathscr{E}_0) \otimes (\bigwedge \mathscr{E}_1)^{-1} \cong (\bigwedge \mathscr{E}_0) \otimes (\bigwedge \mathscr{E}_1)^{-1} \otimes (\bigwedge \mathscr{E}_0')^{-1} \otimes (\bigwedge \mathscr{E}_1') \cong (\bigwedge \mathscr{E}_0) \otimes (\bigwedge \mathscr{G})^{-1} \otimes (\bigwedge \mathscr{E}_0') \cong (\bigwedge \mathscr{E}_0) \otimes (\bigwedge \mathscr{E}_1')^{-1} \otimes (\bigwedge \mathscr{E}_0)^{-1} \otimes (\bigwedge \mathscr{E}_0') \cong (\bigwedge \mathscr{E}_0') \otimes (\bigwedge \mathscr{E}_1')^{-1}$. Hence, it does not depend on the choice of free resolution. In order to prove det is well-defined we have to prove any exact sequence in $K(X)$, say $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$, we have $\det \mathscr{F} \cong \det \mathscr{F}' \otimes \det \mathscr{F}''$, we use the method of Horseshoe lemma, we can use commutative diagram to prove it.

Now we prove $\det(\psi(D)) = \mathscr{L}(D)$. if D is effective, consider exact sequence $0 \to \mathscr{I}_D \to \mathcal{O}_X \to \mathcal{O}_D \to 0$, it's easy to see $\mathscr{I}_D$ is also locally free, so this is a free resolution, then $\det(\mathcal{O}_D) = \det(\mathcal{O}_X) \otimes \det(\mathscr{I}_D)^{-1} = \mathcal{O}_X \otimes \mathscr{I}_D^{-1} = \mathscr{L}(-D)^{-1} = \mathscr{L}(D)$. For any divisor $D = D_+ - D_-$ where D_+, D_- is both effective, we have $\det(\psi(D_+ - D_-)) = \det(\psi(D_+) - \psi(D_-)) = \det(\psi(D_+)) \otimes \det(\psi(D_-))^{-1} = L(D_+) \otimes L(D_-)^{-1} = L(D)$.

(c) The idea is to take a basis of $\mathscr{F}_\eta$ to find a $\mathscr{L}(D)$ such that this basis gives global section of $\mathscr{L}(D) \otimes \mathscr{F}$ then we have $\mathcal{O}_X^{\oplus n} \to \mathscr{L}(D) \otimes \mathscr{F}$ which we show to be injective, and then tensoring $\mathscr{L}(-D)$. Covering X with finitely many open affines U_i, and on each $\mathscr{F}|_{U_i} \cong \widetilde{M_i}$. Now consider the stalk $\mathscr{F}_\eta$ at the generic point. Since X is integral and so the generic point appears as (0) in each U_i, we have $\mathscr{F}_e ta \cong \operatorname{Frac}(A_i) \otimes_{A_i} M_i$ for each i. Let $e_1, ..., e_n$ be a basis, and we have $e_j = \frac{m_i j}{a_i}$ for each i. Now we try to let $\{(U_i, a_i)\}$ define a Cartier divisor. First shrink $U' = U/V(a_i)$, and if U_i' can't cover X again, the point set can't be covered will be a finitely many points, so pick such a point $x \in V(a_i)$ for some i and add x to U_i' again. Then for any U_i' and U_j', we have any $x \in V(a_j) \bigcup V(a_i)$ can lie in both of U_i' and U_j', so such $\{(U_i, a_i)\}$ can actually define a Cartier divisor. Now $\frac{1}{a_i} \otimes m_{ij}$ can be glued up to a global section in $\Gamma(X, \mathscr{L}(D) \otimes_{\mathcal{O}_X} \mathscr{F})$. Then we can define $\mathcal{O}_X(U') \to \Gamma(U', \mathscr{L}(D') \otimes_{\mathcal{O}_X} \mathscr{F})$ by an obvious way, and it compatible in $U_i' \bigcap U_j'$ so glue up to a morphism of $\mathcal{O}_X^n \to \mathscr{L}(D) \otimes \mathscr{F}$, by property of localization we have this map is injective. Taking tensoring we have $0 \to \mathscr{L}(-D)^n \to \mathscr{F} \to \mathscr{J} \to 0$ where $\mathscr{J}$ is the cokernel, taking stalk on η we have $\mathscr{J}_\eta = 0$, so it's a torsion sheaf.

still have to prove $\gamma(\mathscr{F}) - r\gamma(\mathcal{O}_X) \in \operatorname{Im} \psi$. By previous exact sequence we have $\gamma(\mathscr{F}) - r\gamma(\mathcal{O}_X) = \gamma(\mathscr{L}(D)^{\oplus r}) + \gamma(\mathscr{J}) - r\gamma(\mathcal{O}_X) = r(\gamma(\mathscr{L}(D)) - \gamma(\mathcal{O}_X)) + \gamma(\mathscr{J})$. By previous part, we have $0 \to \mathscr{L}(-D) \to \mathcal{O}_X \to \mathcal{O}_D \to 0$ for D is effective and every such $\mathcal{O}_D \in \operatorname{Im} \psi$, so $\gamma(\mathscr{L}(D)) - \gamma(\mathcal{O}_X) \in \operatorname{Im} \psi$. For any divisor D, let $D = D_+ - D_-$ where D_+ and D_- are both effective. So left to show $\gamma(\mathscr{J}) \in \operatorname{Im} \psi$. Since $\mathscr{J}$ is coherent and torsion, for every affine subset U, the associated prime ideals of $\mathscr{J}|_U$ is finite and not include generic point, so $\operatorname{Supp}(\mathscr{J})$ is a finite point (closed) subset i.e. $\mathscr{J}$ is a skyscraper sheaf, and any skyscraper sheaf is in the image of ψ. so it's done.

(d) Previous answer make this exact sequence exact and split:

$$0 \overset{1\mapsto\gamma(\mathcal{O}_X)}{\to} K(X) \to \mathrm{Cl}(X) \overset{\det}{\to} \mathrm{Pic}(X) \to 0,$$

, and the splitting map is given by ψ, so $K(X) \cong \mathrm{Pic}(X) \oplus \mathbb{Z}$. □

Exercise 10 (2.7.1)**.** Let $(X, \mathcal{O}_X)$ be a locally ringed space, and $f : \mathscr{L} \to \mathscr{M}$ be a surjective map of invertible sheaves on X. Show that f is an isomorphism.

Proof. Sufficient to prove for any $p \in X$ we have $f_p : \mathscr{L}_p \to \mathscr{M}_p$ is an isomorphism. Sufficient to prove that $f_p : \mathcal{O}_p \to \mathcal{O}_p$ is surjective as $\mathcal{O}_p$-module, then it's injective. Since f_p is surjective then 1 has a preimage denote a, and f_p determined by $b = f_p(1)$, so $1 = f_p(a) = af_p(1) = ab$, so we have b is a unit and f_p is injective. □

Exercise 11 (2.7.2)**.** Let X be a scheme over a field k, Let $\mathscr{L}$ be an invertible sheaf on X, and let $\{s_0, \ldots, s_n\}$ and $\{t_0, \ldots, t_m\}$ be two sets of sections of $\mathscr{L}$, which generate the same subspace $V \subset \Gamma(X, \mathscr{L})$, and which generate the sheaf $\mathscr{L}$ at every point. Suppose $n \leq m$. Show that the corresponding morphism $\varphi : X \to \mathbb{P}^n_k$ and $\psi : X \to \mathbb{P}^m_k$ differ by a suitable linear projection $\mathbb{P}^m - L \to \mathbb{P}^n$ and an automorphism of $\mathbb{P}^n$, where L is a linear subspace of $\mathbb{P}^m_k$ of dimension $m - n - 1$.

Proof. Since $\{s_0, \ldots, s_n\}$ and $\{t_0, \ldots, t_m\}$ generates the same vector space, we may assume $s_i = \sum a_{ij} t_j$ where $a_{ij} \in k$. Let the coordinate ring of $\mathbb{P}^n_k, \mathbb{P}^m_k$ be $k[Y_0, \cdots, Y_n]$ and $k[X_0, \cdots, X_m]$. On $\mathcal{O}_{\mathbb{P}^m_k}(1)$ and $\mathcal{O}_{\mathbb{P}^\ltimes_\urcorner}(1)$ we have $\varphi^*(Y_i) = s_i$ and $\psi^*(X_j) = t_j$. We may define rational map $(x_0, \ldots, x_m) \to (\sum a_{0j} x_j, \ldots, \sum a_{nj} x_j)$, which is well-defined on $X - L$ where

$$L = \{P \in X | (u_i)_P \in m_P \mathscr{L}_P, i = 1, \ldots, n\}, u_i = \sum a_{ij} X_j.$$

It's easy to see that L is a linear subspace of dimension $m-n-1$. So we gain a unique $\rho : \mathbb{P}^m - L \to \mathbb{P}^n$ such that $u_i = \rho^*(x_i)$. By the definition of $\psi : X \to \mathbb{P}^m_k$, we have $\psi^*(u_i) = \psi^*(\sum a_{ij} X_j) = \sum a_{ij} \psi^*(X_j) = s_i$. By proposition 7.1 we have $\rho \circ \psi = \varphi$. □

Exercise 12 (7.3)**.** Let $\varphi : \mathbb{P}^n_k \to \mathbb{P}^m_k$. Then:

(a) either $\varphi(\mathbb{P}^n) = pt$ or $m \geq n$ and $\dim \varphi(\mathbb{P}^n)$;

(b) in the second case, φ can be obtained as the composition of (1) a d-uple embedding $\mathbb{P}^n \to \mathbb{P}^N$ for a unique determined $d \geq 1$, (2) a linear projection $\mathbb{P}^N - L \to \mathbb{P}^m$, and (3) an automorphism of $\mathbb{P}^m_k$. Also, φ has finite fibres.

Proof. (a) Given a morphism of $\varphi : \mathbb{P}^n \to \mathbb{P}^m$ is equivalent to give a invertible sheaf $\mathscr{L} \cong \mathcal{O}_{\mathbb{P}^n}(d)$ generated by global sections $s_0, \ldots, s_m$, and we know it's a proper morphism. If $m < n$ and $d > 0$ then consider $V(s_0) \bigcap \ldots \bigcap V(s_m)$ is a closed subset of dimension > 0, so this contradicts with $s_0, \ldots, s_m$ generates at every stalk. When $d < 0$ there is no global sections other than 0, so $d = 0$, so we have s_i are constants in k, and by the way we define the morphism $\varphi(\mathbb{P}^n) = pt$.

Now we assume $m \geq n$, If φ is surjective then we implies $\dim \varphi(\mathbb{P}^n) = \dim \mathbb{P}^m = m$, so we have $m = n$. If not surjective then we let $\varphi' : \mathbb{P}^n \to \mathbb{P}^m - \{P\} \to \mathbb{P}^{m-1}$, then by induction we have either φ' maps to a point or $\dim \varphi'(\mathbb{P}^n) = n$ which implies $\dim \varphi(\mathbb{P}^n) = n$. If φ' maps to a point, then $\varphi(\mathbb{P}^n)$ is a subset of preimage of P for map $\mathbb{P}^m \to \mathbb{P}^{m-1}$, which is isomorphic to $\mathbb{A}^1$, so this implies $\varphi : \mathbb{P}^n \to \mathbb{A}^1$. Since φ is porper and $\mathbb{P}^n$ is intergral, so $\varphi(\mathbb{P}^n) = pt$.

(b) We may assume that φ is determined by $s_i \in k[x_0, ..., x_n]_{(d)}$ for $i = 0, ..., m$. Let $\varphi_1 : \mathbb{P}^n \to \mathbb{P}^N$ be d-uple embedding, and we have $\varphi_1^*(t_i) = s_i$ and let U be the open set such that $\{t_j\}$ define a morphism $\varphi_2 : U \to \mathbb{P}^m$ such that $\varphi_2^*(x_i) = t_i$ where $x_i \in \Gamma(\mathbb{P}^m, \mathcal{O}_{\mathbb{P}^m}(1))$. Then we have $\varphi_1^* \varphi_2^*(x_i) = s_i$, by theorem 7.1 we have $\varphi = \varphi_2 \circ \varphi_1$. As for automorphism of $\mathbb{P}^m$, we may have $\varphi_2^*(x_i)$ spans the same sub linear space spanned by t_i, this is when we need a automorphism of $\mathbb{P}^m$. □

Exercise 13 (7.4)**.** (a) Use (7.6) to show that if X is a scheme of finite type over a noetherian ring A, and if X admits an ample invertible sheaf, then X is separated.

(b) Let X be the affine line over a field k with the origin doubled. Calculate Pic X, determine which invertible sheaves are generated by global sections, and then show directly that there is no ample invertible sheaf on X.

Proof. (a) Let $\mathscr{L}$ be the ample sheaf over X then we have $\mathscr{L}^{\otimes n}$ for some n is very ample, which determined a immersion $X \to \mathbb{P}^n$ which is a separated morphism, then $\mathbb{P}^n$ is proper so X is separated.

(b) For any invertible sheaf $\mathscr{L}$ on X, let U_1, U_2 be two copies of affine line in X, we have $L|_{U_1} \cong \mathcal{O}_X|_{U_1}$ and $L|_{U_2} \cong \mathcal{O}_X|_{U_2}$, so on $U_1 \bigcap U_2$, we have a good isomorphism to glue up two copy, where $U_1 \bigcap U_2 \cong \operatorname{Spec} k[x, x^{-1}]$. The isomorphism is determined by an automorphism of $k[x, x^{-1}]$ as modules, so such an automorphism is determined by the image of 1, possibly be ax^n where $a \in k$ and $n \in \mathbb{Z}$. Any two invertible sheaf determined by ax^n and bx^m, they are isomorphic unless $n = m$, since use the language of Cartier divisor it's obvious. so Pic $X \cong \mathbb{Z}$. In order to prove there's no ample sheaf on X, enough to claim that any invertible sheaf $\mathscr{L}$ can't be generated by global sections. So just consider Cartier divisor $\{(U_1, 1), (U_2, x^n)\}$, which determines a invertible subsheaf of $\mathscr{K}$, generated by $(U_1, 1)$ and (U_2, x^{-n}), so obviously it can't be generated by global section since global sections are $k[x]$. □

Exercise 14 (2.7.5)**.** Establish the following properties of ample and very ample invertible sheaves on a neotherian scheme X. $\mathscr{L}, \mathscr{M}$ will denote invertible sheaves, and for (d), (e) we assume furthermore that X is of finite type over a neotherian ring A.

(a) If $\mathscr{L}$ is ample and $\mathscr{M}$ is generated by global section, then $\mathscr{L} \otimes \mathscr{M}$ is ample.
(b) If $\mathscr{L}$ is ample and $\mathscr{M}$ is arbitrary, then $\mathscr{M} \otimes \mathscr{L}^n$ is ample for sufficiently large n.
(c) If $\mathscr{L}, \mathscr{M}$ are both ample, so is $\mathscr{L} \otimes \mathscr{M}$.
(d) If $\mathscr{L}$ is very ample and $\mathscr{M}$ is generated by global sections, then $\mathscr{L} \otimes \mathscr{M}$ is very ample.
(e) If $\mathscr{L}$ is ample, then there is an $n_0 > 0$ such that $\mathscr{L}^n$ is very ample for all $n \geq n_0$.

Proof. (a) For any given coherent sheaf $\mathscr{F}$, by definition for n large enough we have $\mathscr{F} \otimes \mathscr{L}^n$ generated by global section, and since $\mathscr{M}^{\otimes n}$ is generated by global section we have $\mathscr{F} \otimes (\mathscr{L} \otimes \mathscr{M})^n$ is generated by global section.

(b) By definition for any $n > n_0$ we have $\mathscr{M} \otimes \mathscr{L}^n$ is generated by global section, then for any coherent $\mathscr{F}$ and large enough m we have $\mathscr{F} \otimes (\mathscr{M} \otimes \mathscr{L}^{n_0+1})^m$ and by (a) we have $\mathscr{M} \otimes \mathscr{L}^{n_0+1}$ ample.

(c) Just let $n_0 = \max\{n_1, n_2\}$ where n_1, n_2 is the number making $\mathscr{L}, \mathscr{M}$ ample.

(d) By definition we have an immersion $i : X \to Y_1 = \mathbb{P}^n_A$ such that $i_* \mathcal{O}_{Y_1}(1) \cong \mathscr{L}$ and a morphism $\varphi : X \to Y_2 = \mathbb{P}^m_A$ such that $\mathscr{M} \cong \varphi_* \mathcal{O}_{Y_2}(1)$, this induce that $f : X \to \mathbb{P}^n_A \times_A \mathbb{P}^m_A \to \mathbb{P}^N_A$ where the last map is Segre embedding. By an exercise in section 5, we have $\mathscr{L} \otimes \mathscr{M} \cong f_* \mathcal{O}_{\mathbb{P}^N_A}(1)$.

(e) We know that for some $m > 0$ $\mathscr{L}^m$ is very ample, and by definition there exists $d_0 > 0$ such that for any $d > d_0$ we have $\mathscr{L}^d$ is generated by global section, let $n_0 = m + d_0$ and by (d) we have the property. □

Exercise 15 (7.6(The Riemann-Roch Problem))**.** Let X be a nonsingular projective variety over an algebraic closed field, and let $|D|$ be a divisor on X. For any $n > 0$ we consider the complete linear system $|nD|$. Then the Riemann-Roch problem is to determine $\dim |nD|$ as a function of n, and, in particular, its behaviour for large n. If $\mathscr{L}$ is the corresponding invertible sheaf, then $\dim |nD| = \dim \Gamma(X, \mathscr{L}^n) - 1$, so an equivalent problem is to determine $\dim \Gamma(X, \mathscr{L}^n)$ as a function of n.

(a) Show that if D is very ample, and if $X \hookrightarrow \mathbb{P}^n$ is the corresponding embedding in projective space, then for all n sufficient large, $\dim |nD| = P_X(n) - 1$, where P_X is the Hilbert polynomial of X. Thus in this case $\dim |nD|$ is a polynomial function of n for n large.
(b) If D corresponds to a torsion element of Pic X, of order r, then $\dim |nD| = 0$ if $r|n$, -1 otherwise. In this case the function is periodic of period r.

Proof. (a) Since D is very ample, then $X \cong \operatorname{Proj} S$ for some graded k-algebra S which is quotient of $k[x_0, ..., x_n]$, then $\dim |nD| = \dim \Gamma(X, \mathcal{O}_X(n)) - 1 = \dim_k S_n - 1 = P_X(n) - 1$ for sufficient large n. (b) By previous (a), we have $\dim |krD| = \dim \Gamma(X, \mathcal{O}_X) - 1 = 0$ for any $k \in \mathbb{Z}$. if $r \nmid n$ then let

$n_0 \equiv n \mod r$, $\dim |nD| = \dim |n_0 D|$, let $\mathscr{L}$ be the invertible sheaf determined by D and $s \in \Gamma(X, \mathscr{L})$ be a global section. For any affine subset U of X, then $\mathscr{L} \cong \mathcal{O}_X$, let Z be the locus of $S|_U$ by the isomorphism, since $s^{\otimes r}$ is a global section of $\mathcal{O}_X$, it must be a constant in k, so this implies Z is either empty of U, so we have $\mathrm{Supp}(s)$ is either empty of X. If Z is empty, notice that D is of degree 0, since $rD \sim (0)$, and we have $k \subset \Gamma(X, \mathscr{L}^n)$ so $D = 0$, so $r = 1$. If $Z = X$ then $s = 0$, so there's no global section of $\mathscr{L}$ i.e. $\dim |D| = -1$. □

Exercise 16 (7.7(Some Rational Surfaces))**.** Let $X = \mathbb{P}^2_k$, and let $|D|$ be the complete linear system of all divisors of degree 2 on X. D corresponds to the invertible sheaf $\mathcal{O}(2)$, whose space of global sections has a basis $x^2, y^2, z^2, xy, xz, yz$, where x, y, z are the homogeneous coordinates of X.

(a) The complete linear system $|D|$ gives an embedding of $\mathbb{P}^2$ in $\mathbb{P}^5$, whose image is the Veronese surface.
(b) Show that the subsystem defined by $x^2, y^2, z^2, y(x-z), (x-y)z$, gives a closed immersion of X into $\mathbb{P}^4$. The image is called the Veronese surface in $\mathbb{P}^4$.
(c) Let $\mathfrak{d} \subset |D|$ be the linear system of all conics passing through a fixed point P. Then $\mathfrak{d}$ gives an immersion of $U = X - P$ into $\mathbb{P}^4$. Furthermore, if we blow up P, to get a surface $\tilde{X}$, then this map extends to give a closed immersion of $\tilde{X}$ in $\mathbb{P}^4$. Show that $\tilde{X}$ is a surface of degree 3 in $\mathbb{P}^4$, and that the lines in X through P are transformed into straight lines in $\tilde{X}$ which do not meet. $\tilde{X}$ is the union of all these lines, so we say $\tilde{X}$ is a ruled surface.

Proof. (a) Since $|D|$ has a corresponding to $\Gamma(X, \mathcal{O}_X(2))$ so $|D|$ gives the morphism just is the 2-uple embedding with a automorphism of $\mathbb{P}^5$.

(b) We use criterion of proposition 7.3. First the linear system separates point:

$$D(x^2) \cup D(y^2) \cup D(z^2) = D(x) \cup D(y) \cup D(z) = \mathbb{P}^2$$

, so it's equivalent to say x^2, y^2, z^2 separated points. Then we prove the linear system separated tangent vectors: Since y and Z is symmetry so enough to prove for $D(x)$ and point $P = [1:0:0]$. On $D(x)$ we have $\mathcal{O}_X(2)|_{D(x)} \xrightarrow{\cdot \frac{1}{x^2}} \mathcal{O}_X|_{D(x)}$ is an isomorphism, $m_P/m_P^2 = (Z, Y)/(Z^2, Y^2, ZY)$ where $Y = \frac{y}{x}$ and $Z = \frac{z}{x}$, so the image of the linear system is the submodule generated by $1, Y^2, Z^2, Y(1_Z), Z(1_Y)$. it's clear that $Y(1-Z)$ and $Z(1-Y)$ generates the tangent vector space. On the $D(z)$ we have similar situation but more easy: V is locally generated by $1, X^2, Y^2, Y(X-1), X_Y$, for any $P \in D(z)$ let $m_P/m_P^2 = (X-a, Y-b)/((X-a)^2, (Y-b)^2, (X-a)(Y-b)$, so by some trivial algebra calculation we have $X-a$ and $Y-b$ can be generated by the linear system so for any point $P \in D(z)$ we have V separated tangent vector. Similarly, z amd y is symmetry, so we are done.

(c) We use coordinates y_0, y_1, y_2, y_3, y_4 of $\mathbb{P}^4$ and x_0, x_1, x_2 of $\mathbb{P}^2$ and let the point be $P = [0:0:1]$ so the linear system will be $x_0^2, x_1^2, x_0x_1, x_0x_2, x_1x_2$, and the morphism maps $U = X - P$ to an open set of $V = V(y_2y_3 - y_1y_4, y_0y_3 - y_2y_4)$, and we also have the image of $\tilde{X}$ is an irreducible closed subset, so $\tilde{X}$ maps to V, in order to describe the degree of V, let $y_0 \in \Gamma(X, \mathcal{O}(1))$, we have three irreducible component, so it's degree 3. By the property of blow up, we notice that different tangent vector blow up to different points, so such two line $\mathscr{L}, \mathscr{L}'$ intersect only at P with different tangent direction, so in $\tilde{X}$ we have $\tilde{\mathscr{L}}$ has no intersection with $\tilde{\mathscr{L}}'$. □

Exercise 17 (7.8)**.** Let X be a neotherian scheme, let $\mathscr{E}$ be a coherent locally free sheaf on X, and let $\pi : \mathbb{P}(\mathscr{E}) \to X$ be the corresponding projective space bundle. Show that there is a natural 1-1 correspondence between sections of π and quotient invertible sheaves $\mathscr{E} \to \mathscr{L} \to 0$ of $\mathscr{E}$.

Proof. It's directly from proposition 7.12. □

Exercise 18 (7.9)**.** Let X be a regular neotherian scheme, and $\mathscr{E}$ a locally free coherent sheaf of rank ≥ 2 on X.

(a) Show that $\mathrm{Pic}\ \mathbb{P}(\mathscr{E}) \cong \mathrm{Pic}\ X \times \mathbb{Z}$.
(b) If $\mathscr{E}'$ is another locally free coherent sheaf on X, show that $\mathbb{P}(\mathscr{E}) \cong \mathbb{P}(\mathscr{E}')$(over X). if and only if there is an invertible sheaf $\mathscr{L}$ on X such that $\mathscr{E}' \cong \mathscr{E} \otimes \mathscr{L}$.

Proof. (a) Define a morphism $\alpha : \text{Pic } X \times \mathbb{Z} \to \text{Pic } \mathbb{P}(\mathscr{E}), (\mathscr{L}, n) \mapsto (\pi^*\mathscr{L}) \otimes \mathcal{O}(n)$, and clearly this is a group morphism.

Injective: Suppose that $\pi\mathscr{L} \otimes \mathcal{O}(n) \cong \mathcal{O}_{\mathbb{P}(\mathscr{E})}$, then we have $\pi_*(\pi^*\mathscr{L} \otimes \mathcal{O}(n)) \cong \mathcal{O}_X$, by Projection Formula we have $\mathscr{L} \otimes \pi_*\mathcal{O}(n)$ where $\pi_*(\mathscr{E})$ is the degree n part of a symmetry algebra rank ≥ 2, so $n = 0$, and we have $\mathscr{L} \cong \mathcal{O}_X$. so α is injective.

Surjective: Let U_i be open covers of X such that $\mathscr{E}|_{I_i}$ is trivial, so we have $V_i = \mathbb{P}(\mathscr{E}|_{U_i}) \cong \mathbb{P}^n_{U_i}$ is a open cover of $\mathbb{P}(\mathscr{E})$. Since X is regular, in particular regular of codimension one, V_i satisfies the (*) condition of Weil divisor, and by previous exercise, we have $\text{Pic}(V_i) \cong \mathrm{U_i} \times \mathbb{Z}$. Now let $\mathscr{L} \in \text{Pic } \mathbb{P}(\mathscr{E})$, we restrict $\mathscr{L}$ to each V_i we have $\mathscr{L}_i \in U_i$ and $n \in \mathbb{Z}$, such that $\mathcal{O}_i(n_i) \otimes \pi^*(\mathscr{L}_i) \cong \mathscr{L}|_{V_i}$, and check the rank locally we have $n_i = n_j$, so by the way we define $\mathbb{P}(\mathscr{E})$ we have $\mathcal{O}_i(n)|_{V_{ij}} \cong \mathcal{O}_{ij}(n)$ so we have isomorphism $\mathcal{O}_{ij}(n) \otimes \pi^*\mathscr{L}_i|_{V_{ij}} \cong \mathcal{O}_{ij}(n) \otimes \pi^*\mathscr{L}_j|_{V_{ij}}$ so tensoring $\mathcal{O}_{ij}(-n)$ we have $\pi^*\mathscr{L}_i|_{V_{ij}} \cong \pi^*\mathscr{L}_j|_{V_{ij}}$ and by projection formula we have $\mathscr{L}_i|_{U_{ij}} \cong \mathscr{L}_j|_{U_{ij}}$ hence we have cocycle condition to glue up $\mathcal{O}_i(n) \otimes \pi^*(\mathscr{L}_i)$ via this isomorphism. Now we have $\mathscr{M}$ glue up by $\mathscr{L}_i$ and we have $\pi^*\mathscr{M} \otimes \mathcal{O}(n)$.

(b) If $\mathscr{E}' \cong \mathscr{E} \otimes \mathscr{L}$, let $\mathscr{S}, \mathscr{S}'$ be the symmetry $\mathcal{O}_X$-algebra of $\mathscr{E}, \mathscr{E}'$, then we have $\mathscr{S}' \cong \mathscr{S} * \mathscr{L}$, so by Proposition 7.9 we have $\mathbb{P}(\mathscr{E}') \cong \mathbb{P}(\mathscr{E})$. (And we also have a very beautiful proof using Yoneda Lemma, by some abstract nonsense)

If $\mathbb{P}(\mathscr{E}) \cong \mathbb{P}(\mathscr{E}')$, let $f : \mathbb{P}(\mathscr{E}') \to \mathbb{P}(\mathscr{E})$ be the isomorphism, we know that by (a) $f^*\mathcal{O}'(1) \cong \pi^*\mathscr{L} \otimes \mathcal{O}(1)$, then push forward by π, we have $\pi_*(f^*\mathcal{O}'(1)) \cong \pi_*(\pi^*\mathscr{L} \otimes \mathcal{O}(1)) \cong \mathscr{L} \otimes \pi_*\mathcal{O}(1)$ where the second isomorphism is given by Projective Formula. Since $\pi_* = \pi'_* f_*$, we have $\pi_*(f^*(\mathcal{O}'(1))) = \pi'_*(f_* f^*\mathcal{O}'(1))$, and $f_* f^*\mathcal{O}'(1) \cong \mathcal{O}'(1) \otimes \pi^*\mathscr{M}$ for some $\mathscr{L} \in \text{Pic } X$. so we conclude $\pi_*(f^*(\mathcal{O}'(1))) = \pi_*(\mathcal{O}'(1) \otimes \pi^*\mathscr{L}) = \mathscr{E}' \otimes \mathscr{M}$, so we implies $\mathscr{E}' \cong \mathscr{E} \otimes \mathscr{L} \otimes \mathscr{M}^{-1}$. □

Exercise 19 (7.10.($\mathscr{P}^n$-Bundle Over a Scheme))**.** Let X be a neotherian scheme.

(a) By analogy with the definition of a vector bundle, define the notion of a *projective n-space bundle* over X, as a scheme P with a morphism $\pi : P \to X$ such that P is locally isomorphic to $U \times \mathscr{P}^n$, $U \subset X$ open, and the transition automorphism on Spec $A \times \mathbb{P}^n$ are given by A-linear automorphisms of the homogeneous coordinate ring $A[x_0, ..., x_n]$.
(b) If $\mathscr{E}$ is a locally free sheaf of rank $n + 1$ on X, then $\mathbb{P}(\mathscr{E})$ is a $\mathscr{P}^n$-bundle over X.
(c) Assume X is regular, and show that every $\mathbb{P}^n$-bundle P over X is isomorphic to $\mathbb{P}(\mathscr{E})$ for some locally free sheaf $\mathscr{E}$ on X.
(d) Conclude(in the case X regular) that we have a 1-1 correspondence between $\mathbb{P}^n$-bundle over X, and equivalence classes of locally free sheaves $\mathscr{E}$ of rank $n+1$ under the equivalence relation $\mathscr{E}' \sim \mathscr{E}$ if and only if $\mathscr{E}' \cong \mathscr{E} \otimes \mathscr{M}$ for some invertible sheaf $\mathscr{M}$ on X.

Proof. (a) A $\mathbb{P}^n$-bundle of rank n over X is a scheme P and a morphism $f : P \to X$, together with additional data consisting of an open covering $\{U_j\}$ of X, and isomorphisms $\psi_j : f^{-1}(U_j) \to \mathbb{P}^n_{U_i}$, such that for any i, j and let $V = \text{Spec } A \subset U_i \cap U_j$ we have $\psi_{ij} = \psi_j \circ \psi_i^{-1}$ of $\mathbb{P}^n_V$ is defined via an automorphism, an A-linear automorphism of $A[x_0, ..., x_n]$.

(b) For any open affine $U \subset X$ we have $\mathbb{P}(\mathscr{E})|_U \cong \mathbb{P}^n_U$ so only to check if $\mathbb{P}(\mathscr{E})$ is glue up by A-linear automorphism. Let V be in the intersection of two local trivialization, clearly the transition mao is defined by $\mathcal{O}^{\oplus n}_{U_i}|_V \xrightarrow{\sim} \mathcal{O}^{\oplus n}_{U_j}|_V$, satisfying the condition.

(c) I can't...

(d) Given a locally free sheaf of rank $n + 1$ we obtain a projective bundle $\mathbb{P}(\mathscr{E})$ by part (b), so we have $\mathbb{P}(\cdot) : \mathscr{L}\mathscr{F}_{n+1}(X) \to \mathscr{P}\mathscr{B}_n(X)$ which is from locally free sheaves of rank $n + 1$ over X modulo the equivalent relation to $\mathbb{P}^n$-budnle over X, Conversely, if we admits (c), we have $(\cdot) : \mathscr{P}\mathscr{B}_n(X) \to \mathscr{L}\mathscr{F}_{n+1}(X)$ with $\mathbb{P} \circ \mathscr{E} = \text{id}$. To check $\mathscr{E} \circ \mathbb{P} = \text{id}$, we let $\mathscr{F}' = \mathscr{E} \circ \mathbb{P}(\mathscr{F})$ and we have $\mathbb{P}(\mathscr{F}') = P(\mathscr{F})$, by exercise 7.9(b) $\mathscr{F}' \cong \mathscr{F} \otimes \mathscr{L}$ where $\mathscr{L}$ is an invertible sheaf over X. we're done. □

Exercise 20 (7.11)**.** On a noetherian scheme X, different sheaves of ideals can give rise to isomorphic blown up schemes.

(a) If $\mathscr{I}$ is any coherent sheaf of ideals on X, show that blowing up $\mathscr{I}^d$ for any $D \geq 1$ gives a scheme isomorphic to the blowing up of $\mathscr{I}$.

(b) If $\mathscr{I}$ is any coherent sheaf of ideals, and if $\mathscr{J}$ is an invertible sheaf of ideals, then $\mathscr{I}$ and $\mathscr{I}\cdot\mathscr{J}$ give isomorphic blowings-ups.

(c) If X regular, show that (7.17) can be strengthened as follows. Let $U \subset X$ be the largest open set such that $f : f^{-1}(U) \to U$ is an isomorphism. Then $\mathscr{I}$ can be chosen such that the corresponding closed subscheme Y has support equal to $X - U$.

Proof. (a) By definition we have $\mathrm{Blow}_{\mathscr{I}} X = \mathrm{Proj}\ \bigoplus_{n\geq 0} \mathscr{I}^n$ and $\mathrm{Blow}_{\mathscr{I}^d} X = \mathrm{Proj}\ \bigoplus_{n\geq 0} \mathscr{I}^{dn}$, by previous exercise we have such two schemes is isomorphic.

(b) It directly implies from Lemma 7.9, since if we let $\mathscr{S},\mathscr{S}'$ be the symmetry $\mathcal{O}_X$-algebra of $\mathscr{I}$ and $\mathscr{I}\cdot\mathscr{J}$, we have $\mathscr{S}' \cong \mathscr{S} * \mathscr{J}$.

(c) We just to prove that □

Exercise 21 (7.12)**.** Let X be a noetherian scheme, and let Y, Z be two closed subschemes, neither one containing the other. Let $\tilde{X}$ be obtained by blowing up $Y \cap Z$. Show that the strict transform $\tilde{Y}$ and $\tilde{Z}$ of Y and Z in $\tilde{X}$ do not meet.

Proof. The question is local, so we may assume that $X = \mathrm{Spec}\ A$, and let I_Y, I_Z be the ideal of Y, Z, $I = I_Z + I_Y$, so we have $\widetilde{X} = \mathrm{Proj}(\bigoplus_{n\geq 0} I^n)$, $\widetilde{Z} = \mathrm{Proj}(A/I_Z \oplus I/I_Z \oplus I^2/I_Z^2 \oplus \cdots)$ and $\widetilde{Y} = \mathrm{Proj}(A/I_Y \oplus I/I_Y \oplus I^2/I_Y^2 \oplus \cdots)$. The closed immersion of $\psi_1 : \widetilde{Y} \hookrightarrow \widetilde{X}$ and $\psi_2 : \widetilde{Z} \hookrightarrow \widetilde{X}$ is clear by quotients of ring. Let P fall into the image of $\widetilde{Y} \bigcap \widetilde{Z}$ we have the ideal P contains $(I_Y + I_Z)S$ where S is the graded ring of $\widetilde{X}$. So we have ψ_1^{-1} contains $I_Y S_Z$ where S_Z is the graded ring of $\widetilde{Z}$ i.e. $\psi_1^{-1} \in \mathrm{Proj}(A/(I_Z + I_Y) \oplus I/(I_Y + I_Z) \oplus \cdots) = \mathrm{Proj}(A/I_Z \oplus 0 \oplus \cdots) = \varnothing$, contradiction! □

Exercise 22 (7.14)**.** .

(a) Give an example of a noetherian scheme X and a locally free coherent sheaf $\mathscr{E}$, such that the invertible sheaf $\mathcal{O}(1)$ on $\mathbb{P}(\mathscr{E})$ is not very ample relative to X.

(b) Let $f : X \to Y$ be a morphism of finite type, let $\mathscr{L}$ be an ample invertible sheaf on X, and let $\mathscr{S}$ be a sheaf of graded $\mathcal{O}_X$-algebras satisfying (†). Let $P = \mathrm{Proj}\ \mathscr{S}$, let $\pi : P \to X$ be the projection, and let $\mathcal{O}_P(1)$ be the associated invertible sheaf. Show that for all $n >> 0$, the sheaf $\mathcal{O}_P(1) \otimes \pi^*(\mathscr{L})$ is very ample on P relative to Y.

Proof. (a) Take $X = \mathbb{P}^1$, and let $\mathscr{E} = \mathcal{O}(-1)$. Since $\mathscr{E}$ is an invertible sheaf, we have $\mathbb{P}(\mathscr{E}) \cong \mathbb{P}^1$, by this isomorphism $\mathcal{O}_{\mathbb{P}(\mathscr{E})}(1) \cong \mathcal{O}_{\mathbb{P}^1}(-1)$, which can't be very ample.

(b) By proposition 7.10 we have $\mathcal{O}_P(1) \otimes \pi^*\mathscr{L}^n$ is very ample on P relative over X, by Ex5.12 we have $\mathscr{L}^m$ is very ample over Y(note that we are using a theorem stronger than the version in textbook, which doesn't require Y to be affine, the detail is in Stack Project), so by exercise 5.12, $\mathcal{O}_P(1) \otimes \pi^*\mathscr{L}^{m+n}$ is very ample relative to Y. □

Exercise 23 (8.1)**.** Here we strengthen the results of the text.

(a) Let B be a local ring containing a field k, and assume that the residue field $k(B) = B/\mathfrak{m}$ of B is a separable extension of k. Then the exact sequence of (8.4A),

$$0 \to \mathfrak{m}/\mathfrak{m}^2 \xrightarrow{\delta} \Omega_{B/k} \otimes k(B) \to \Omega_{k(B)/k} \to 0$$

is exact on the left also.

(b) With B, k as above, assume furthermore that k is perfect, and that B is a localization of an algebra of finite type over k. Then show that B is a regular local ring if and only if $\Omega_{B/k}$ is free of rank $= \dim B + \mathrm{tr.d.}k(B)/k$.

(c) Let X be an irreducible scheme of finite type over a perfect field k, and let $\dim X = n$. For any point $X \in X$, not necessarily closed, show that the local ring $\mathcal{O}_{x,X}$ is a regular local ring if and only if the stalk $(\Omega_{X/k})_x$ of the sheaf of differentials at x is free of rank n.

(d) If X is a variety over an algebraically closed field k, then $U = \{x \in X | \mathcal{O}_x \mathrm{isaregularlocalring}\}$ is an open dense subset of X.

Proof. (a) In order to prove the injective, equivalent to prove $\delta^* : \mathrm{Der}_k(B,k) \to \mathrm{Hom}_k(\mathfrak{m}/\mathfrak{m}^2, k)$ is surjective. For any $h \in \mathrm{Hom}_k(\mathfrak{m}/\mathfrak{m}^2, k)$, we try to define a derivation d with $d \circ \delta = h$, by theorem 8.25A we can consider $k(B)$ as as subfield of $B/\mathfrak{m}^2$, every $b \in B/\mathfrak{m}^2$ can be uniquely written as $b = \lambda + c$ where $\lambda \in k(B), c \in \mathfrak{m}/\mathfrak{m}^2$, we put $db = h(c)$, we have $d(bb') = h(\lambda c' + \lambda' c) = bdb' + b'db$, and by definition $d \circ \delta = h$, so surjective.

(b) $\Leftarrow$: By the exact sequence of (a), we have $\mathrm{rank}(\mathfrak{m}/\mathfrak{m}^2) = \dim A$, so regular.

$\Rightarrow$: By the previous (a) and k is perfect, we have $\dim \Omega_{B/k} \otimes k(B) = \dim B + \mathrm{rank}(\Omega_{k)(B)/k}) = \dim B + \mathrm{tr.d.}(k(B)/k)$, only to prove that $\dim \Omega_{B/k} \otimes K$ have the same. we have $\dim \Omega_{B/k} \otimes K = \dim \Omega_{K/k} = \mathrm{tr.d.}(K/k)$. Now use the condition B is a localization of A, where A is a finite generated k-algebra. Put $B = A_{\mathfrak{p}}$, we have the fraction field of A equals to the fraction field of A, now we have $\mathrm{tr.d.}(K) = \dim A = \dim A/\mathfrak{p} + \mathrm{ht}(\mathfrak{p}) = \mathrm{tr.d.}(k(B)/k) + \dim B$.

(c) By the result in (b), we have that the local ring $\mathcal{O}_{x,X}$ is a regular local ring if and only if $\Omega_{X/k}$ is free on an open affine neighborhood Spec B of x of rank $\dim A_{\mathfrak{p}} + \mathrm{tr.d.}(k(B)/k) = \mathrm{ht}(\mathfrak{p}) + \mathrm{tr.d.}(k(B)/k) = \dim X$ where $A_{\mathfrak{p}}$ is the local ring of x.

(d) We already know that U is dense, only to say it's open. for any $x \in U$, we have $(\Omega_{X/k})_x$ is free of rank n, so there's an open neighborhood V of x, such that $\Omega_{X/k}|_V$ is free of rank n, V is nonsingular, so $V \subset U$, implies U is open. $\square$

Exercise 24 (8.2)**.** Let X be a variety of dimension n over k. Let $\mathscr{E}$ be a locally free sheaf of rank $> n$ on X, and let $V \subset \Gamma(X, \mathscr{E})$ be a vector space of global sections which generate $\mathscr{E}$. Then show that there is an element $s \in V$, such that for each $x \in X$, we have $s_x \notin \mathfrak{m}_x \mathscr{E}_x$. Conclude that there is a morphism $\mathcal{O}_X \to \mathscr{E}$ giving rise to an exact sequence

$$0 \to \mathcal{O}_X \to \mathscr{E} \to \mathscr{E}' \to 0$$

where $\mathscr{E}'$ is also locally free.

Proof. Let $B := \{(x,s) \in X \times V | s_x \in m_x \mathscr{E}_x\}$, now we have two projections $p_1 : B \to X$ and $p_2 : B \to V$. Since V generates $\mathscr{E}$, we have p_1 is surjective, and for any point $x \in X$, the fibre on x is the kernel of $V \otimes k(x) \to \mathscr{E} \otimes k(x)$, notice that it's surjective, so the dimension of fibre is $\dim V - \mathrm{rank}\mathscr{E}$, so $\dim B = \dim X + \dim V - \mathrm{rank}\mathscr{E} < \dim V$, so p_2 is never a surjective, so we can pick a $s \in V$ satisfying the condition.

Define $\mathcal{O}_X \to \mathscr{E}$ by $1 \to s$, we have it's injective, and for any $x \in X$, the quotient of stalk $\mathscr{E}_x/\mathcal{O}_{x,X}$ is again a free $\mathcal{O}_{x,X}$-module. $\square$

Exercise 25 (8.3)**.** Product Schemes.

(a) Let X and Y be schemes over another scheme S. Use (8.10) and (8.11) to show that $\Omega_{X\times Y/S} \cong p_1^*\Omega_{X/S} \oplus p_2^*\Omega_{Y/S}$.
(b) If X and Y are nonsingular varieties over a field k, show that $\omega_{X\times Y} \cong p_1^*\omega_X \otimes p_2^*\omega_Y$.
(c) Let Y be a nonsingular plane cubic curve, and let X be the surface $Y \times Y$. Show that $p_g(X) = 1$ but $p_a(X) = -1$. This shows that the arithmetic genus and the geometric genus of a nonsingular projective variety may be different.

Proof. (a) By (8.10) we have $p_1^*\Omega_{X/S} \cong \Omega_{X\times Y}/Y$ and $p_2^*\Omega_{Y/S} \cong \Omega_{X\times Y/X}$, and by (8.11) we have two exact sequence

$$p_1^*\Omega_{X/S} \to \Omega_{X\times Y/S} \xrightarrow{\varphi} p_2^*\Omega_{Y/S} \to 0 p_2^*\Omega_{Y/S} \xrightarrow{\psi} \Omega_{X\times Y/S} \to p_1^*\Omega_{X/S} \to 0$$

and we write $\varphi \circ \psi$ in local situation we have

$$d(1\otimes b) \mapsto db \otimes (1 \otimes 1) \mapsto d(1 \otimes b),$$

so ψ is injective and the exact sequence split, we have $\Omega_{X\times Y/S} \cong p_1^*\Omega_{X/S} \oplus p_2^*\Omega_{Y/S}$.

(b) Since X and Y are both nonsingular, so $\Omega_{X/k}$ and $\Omega_{Y/k}$ are locally free, by definition we have

$$\begin{aligned}
\omega_{X\times Y} &= \bigwedge^{mn} \Omega_{X\times Y} \\
&= \bigwedge^{mn} (p_1^*\Omega_{X/k} \oplus p_2^*\Omega_{Y/k}) \\
&= (\bigwedge^{m} p_1^*\Omega_{X/k}) \otimes (\bigwedge^{n} p_2^*\Omega_{Y/k}) \\
&= (p_1^* \bigwedge^{m} \Omega_{X/k}) \otimes (p_2^* \bigwedge^{n} \Omega_{Y/k}) \\
&= p_1^*\omega_{X/k} \otimes p_2^*\omega_{Y/k}.
\end{aligned}$$

(c) For such Y, We have $d = 3$ and $n = 2$, so $\omega_Y = \mathcal{O}_X(-n + d - 1) = \mathcal{O}_{\mathcal{X}}$ so $\omega_X = \mathcal{O}_X$, by definition $p_g = \dim_k \Gamma(X, \omega_X) = 1$. By previous exercise we have $p_a(X) = p_a(X)^2 - 2p_a(X) = -1$. □

Exercise 26 (8.5, Blowing up a Nonsingular Subvariety)**.** Let X be a nonsingular variety, let Y be a nonsingular subvariety of codimension $r \geq 2$, let $\pi : \tilde{X} \to X$ be the blowing up of X along Y, and $Y' = \pi^{-1}(Y)$.

(a) Show that the maps $\pi^* : \text{Pic}X \to \text{Pic}\tilde{X}$, and $\mathbb{Z} \to \text{Pic}\tilde{X}$ define by $n \mapsto$ classof nY', give rise to an isomorphism $\text{Pic}\tilde{X} \cong \text{Pic}X \oplus \mathbb{Z}$

(b) Show that $\omega_{\tilde{X}} \cong f^*\omega_X \otimes \mathscr{L}((r-1)Y')$.

Proof. (a) By proposition 6.5,we have an exact sequence

$$\mathbb{Z} \to \text{Pic}\tilde{X} \to \text{Pic}(\tilde{X} - Y') \to 0,$$

and we have $\tilde{X} - Y' \cong X - Y$ and Y is of codimension ≥ 2, we have $\text{Pic}(\tilde{X} - Y') \cong \text{Pic}X$. And by theorem 8.24 we have $\mathcal{O}_{\tilde{X}}(nY')|_{Y'} \cong \mathcal{O}_{Y'}(n)$, so the first map is injective. Recall that we have a splitting map given by $\pi^* : \text{Pic}(X) \to \text{Pic}(\tilde{X})$, so $\text{Pic}\tilde{X} \cong \text{Pic}X \oplus \mathbb{Z}$.

(b) By (a) we may assume that $\omega_{\tilde{X}} \cong f^*\mathscr{M} \otimes \mathscr{L}(mY')$. Let $U = X - Y$, we have $\omega_{\tilde{X}}|_U = \omega_U$, by (a) again we have $\mathscr{M} = \omega_X$. Now we try to determine m: we have $\omega_{Y'} = \omega_{\tilde{X}} \otimes (Y') \otimes \mathcal{O}_{Y'} = f^*\omega_X \otimes \mathcal{O}_{Y'}(-m-1)$. Check on the fibre of $y \in Y$ we have $\omega_Z = \pi_2^*\omega_{Y'} = \omega_Z(-q-1)$. Since the fibre is a projective space of dimension $r - 1$, so we implies $m = r - 1$. □

Exercise 27 (8.6, The Infinitesimal Property)**.** Let k be an algebraically closed field, let A be a finitely generated k-algebra such that Spec A is nonsingular variety over k. Let $0 \to I \to B' \to B \to)$ be an exact sequence, where B' is a k-algebra, and I is an ideal with $I^2 = 0$. Finally suppose given a k-algebra homomorphism $f : A \to B$. Then there exists a k-algebra homomorphism $g : A \to B'$ making a commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] \\
 & I \arrow[d] \\
 & B \arrow[d] \\
A \arrow[ur, dashed, "g"] \arrow[r, "f"] & B' \arrow[d] \\
 & 0
\end{tikzcd}$$

Proof. We prove this proposition by following steps(after the guidance on Hartshorne):

Step 1: By the construction of Kahler differentials, we have the canonical isomorphism:

$$\mathrm{Der_k(A, \cdot) \cong Hom(\Omega_{A/k}, \cdot)}.$$

Categorically we have tuple (Ω, d) represent the functor $M \mapsto \mathrm{Der}_k(A, M)$. Given two lifting of f, namely g, g', let $\theta = g - g'$, we have $\theta|_k = 0$, and

$$\begin{aligned}\theta(aa') &= g(aa') - g'(aa') \\ &= g(a)(g(a') - g'(a')) + g'(a')(g(a) - g'(a)) \\ &= g(a)\theta(a') + g'(a')\theta(a),\end{aligned}$$

which says $\theta \in \mathrm{Der}_k(A, I)$, or equivalently we say $\theta \in \mathrm{Hom}_A(\Omega_{A/k}, I)$. Conversely, if we fix an element $\theta \in \mathrm{Hom}_A(\Omega_{A/k}, I)$, denote the same θ for $\theta \circ d$,,which is the corresponding element of $\mathrm{Der}_k(A, I)$ and denote $g' = g + \theta$, we have

$$\begin{aligned}g'(a)g'(a') &= (g(a) + \theta(a))(g(a') + \theta(a')) \\ &= g(aa') + g(a)\theta(a') + g(a')\theta(a) + \theta(a)\theta(a') \\ &= g(aa') + g(a)\theta(a') + g(a')\theta(a) \\ &= g(aa') + \theta(aa'),\end{aligned}$$

and $g'|_k = g|_k = \mathrm{id}$, so it's a morphism of k-algebra. Now we conclude that if we fix a lifting of f, then we have a one-to-one correspondence between all the lifting maps of f and $\mathrm{Hom}_A(\Omega_{A/k}, I)$ as sets. Later on, we use this correspondence to do some adjustment in order to find the right lifting.

Step 2: Since A is finitely generated over k, we have the following exact sequence $0 \to J \to P \to A \to 0$, where $P = k[x_1, \ldots, x_n]$ and J is the kernel. We want to construct the following map $h : P \to B'$ making a commutative diagram,

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
J \arrow[d] & I \arrow[d] \\
P \arrow[r, "\mathrm{h}"] \arrow[d, "\mathrm{p}"'] & B \arrow[d] \\
A \arrow[r, "\mathrm{f}"] \arrow[d] & B' \arrow[d] \\
0 & 0
\end{tikzcd}$$

Actually, there should be a lots of choice of such h by previous result, but we only need decide one and we obtain all of them up to a derivation. Let $h(x_i)$ be the an (arbitrary) element of $f(p(x_i)) + I$, and make it multiplicative. Note $h(J) \subset I$ and $h(J^2) \subset I^2 = 0$, so h induces $\bar{h} : J/J^2 \to I$.

Step 3: Since Spec A is nonsingular, we have the following exact sequence

$$0 \to J/J^2 \to \Omega_{P/k} \otimes A \to \Omega_{A/k} \to 0.$$

Applying the functor $\mathrm{Hom}(\cdot, I)$ we have

$$0 \to \mathrm{Hom}_A(\Omega_{A/k}, I) \to \mathrm{Hom}_P(\Omega_{P/k}, I) \to \mathrm{Hom}_A(J/J^2, I) \to 0,$$

We explain that in the middle we have $\mathrm{Hom}_A(\Omega_{P/k} \otimes A, I) = \mathrm{Hom}_P(\Omega_{P/k}, I)$, this is clear because $A = P/J$. And the last term of this sequence should be $\mathrm{Ext}^1(\Omega_{A/k}, I)$, since $\Omega_{A/k}$ is a free module of

A, we have $\operatorname{Ext}^1(\Omega_{A/k}, I) = 0$. Let $\theta \in \operatorname{Hom}_P(\Omega_{P/k}, I)$ be a preimage of $\bar{h}$, and let $h' = h - \theta$ be a homomorphism from P to B, we have $h'(J) = 0$, so it induces a lifting $h' : A \to B$. $\square$

Exercise 28 (8.7)**.** If X is affine and nonsingular, then any infinitesimal extension of X by a coherent sheaf $\mathscr{F}$ is isomorphic to the trivial one.

Proof. Since everything is affine, we are able to translate it into a pure algebraic description: There is ring isomorphism $A \cong B/I$ where $I^2 = 0$ and note that I is a B/I-module and by this isomorphism I is also a A-module. We need to prove that $B \cong A \oplus I$ as ring and the multiplication of $A \oplus I$ is given by $(a \oplus f) \cdot (a' \oplus f') = aa' \oplus (af' + a'f)$. The given information determined a split exact sequence of abelian groups:

$$0 \to I \to B \to A \to 0,$$

where the split map is given by the infinitesimal lifting, so $B \cong A \oplus I$ as abelian groups, to complete the proof only to figure out the multiplication:

$$(a, i)(a', i') = aa' + ai' + a'i + ii' = (aa', ai' + a'i).$$

This proves the result we want. $\square$

Exercise 29 (8.8)**.** Let X be projective nonsingular variety over k. For any $n > 0$ we define the nth plurigenus of X to be $P_n = \dim_k \Gamma(X, \omega_X^{\otimes n})$. Also, for any q, $0 \le q \le \dim X$ we define an integer $h^{q,0} = \dim_k \Gamma(X, \Omega^q_{X/k})$. The integer $h^{q,0}$ are called Hodge numbers. Show that P_n and $h^{q,0}$ are birational invariants of X.

Proof. Let X, X' be birational projective nonsingular variety over k, since X, X' is birational, so we may find the largest open subset V of X, representing the birational map $f : V \to X'$. Note that $X - V$ is of codimension ≥ 2, since X is nonsingular and X' is proper(this is direct result from valuation criterion of proper). Such a f induces $f^*\Omega_{X'/k} \to \Omega_{V/k}$, and these are both locally free of dimension n, so further induces $f^*\omega_{X'/k} \to \omega_{V/k}$ and $(f^*\omega_{X'/k})^{\otimes n} \to \omega_{V/k}^{\otimes n}$, and also $(f^*\Omega_{X'/k})^q \to \Omega^q_{V/k}$, which implies $\Gamma(X', \omega^{\otimes n}_{X'/k}) \to \Gamma(V, \omega^{\otimes n}_{V/k})$ and $\Gamma(X', \Omega^q_{X'/k}) \to \Gamma(V, \Omega^q_{V/k})$. These maps restrict on a open dense set U of X becomes isomorphism, since X, X' is birational. Plus the fact that on an open dense set global sections don't vanish, we implies the maps above are both injective. Now we use algebraic Hartog to prove that $\Gamma(X, \omega^{\otimes n}_{X/k}) \to \Gamma(V, \omega^{\otimes n}_{V/k})$ is bijective, so is the other type. This implies that $P_n(X') \le P_n(X)$ and $h^{0,q}(X') \le h^{0,q}(X)$. Switch the place of X and X', we are done. $\square$

Exercise 30 (3.6.1)**.** Let $(X, \mathcal{O}_X)$ be a ringed space, and $\mathscr{F}', \mathscr{F}'' \in \mathfrak{Mod}(X)$. An extension of $\mathscr{F}''$ by $\mathscr{F}'$ is a short exact sequence

$$0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$$

in $\mathfrak{Mod}(X)$. Two extension are isomorphic if there is an isomorphism of the short exact sequences, inducing the identity maps on $\mathscr{F}$ and $\mathscr{F}''$. Given an extension as above consider the long exact sequence arising from $\operatorname{Hom}(\mathscr{F}, \cdot)$, in particular the map

$$\delta : \operatorname{Hom}(\mathscr{F}'', \mathscr{F}'') \to \operatorname{Ext}^1(\mathscr{F}'', \mathscr{F}'),$$

and let $\zeta \in \operatorname{Ext}^1(\mathscr{F}'', \mathscr{F})$ be $\delta(1_{\mathscr{F}''})$. Show that this process gives a one-to-one correspondence between isomorphism classes of extension of $\mathscr{F}''$ by $\mathscr{F}'$, and elements of the group $\operatorname{Ext}^1(\mathscr{F}'', \mathscr{F}')$.

Proof. We construct two functor being inverse to each other. One is given in the above description, we just let $\zeta = \delta(1_{\mathscr{F}''})$. Now we construct its inverse. Let $\eta \in \operatorname{Ext}^1(\mathscr{F}'', \mathscr{F}')$, we embed $\mathscr{F}'$ into an injective object $\mathscr{I}$ and let $\mathscr{R}$ be its cokernel i.e. we have $0 \to \mathscr{F}' \to \mathscr{I} \to \mathscr{R} \to 0$. Applying $\operatorname{Hom}(\mathscr{F}'', \cdot)$, we have such long exact sequence

$$0 \to \operatorname{Hom}(\mathscr{F}'', \mathscr{F}') \to \operatorname{Hom}(\mathscr{F}'', \mathscr{I}) \to \operatorname{Hom}(\mathscr{F}'', \mathscr{R}) \to \operatorname{Ext}^1(\mathscr{F}'', \mathscr{F}') \to 0.$$

By the last map is surjective, we can lift η to $\operatorname{Hom}(\mathscr{F}'', \mathscr{R})$, and now we define $\mathscr{F}$ to be the pullback of h and η:

$$\begin{tikzcd} 0 \arrow[r] & \mathscr{F}' \arrow[r, "\mathrm{f}"] & \mathscr{I} \arrow[r, "\mathrm{h}"] & \mathscr{R} \arrow[r] & 0 \\ 0 \arrow[r] & \mathscr{F}' \arrow[r, "\mathrm{g}"] \arrow[u] & \mathscr{F} \arrow[r] \arrow[u] & \mathscr{F}'' \arrow[r] \arrow[u] & 0 \end{tikzcd}$$

To be concrete, define $\mathscr{F}$ to be the sheaf associated to the kernel of $\mathscr{F}'' \oplus \mathscr{I} \to \mathscr{R}, (m, i) \mapsto \eta(m) - h(i)$ and g is given by $g(m) = (0, f(m))$. We define the image of the inverse functor to be the lower exact sequence. One can check such diagram is commutative, and taking Ext we have the lower line gives $\delta(1_{\mathscr{F}''}) = \eta$, so one side inverse. As for the other, just a consequence of 5-lemma. □

Exercise 31 (3.6.3)**.** Let X be a neotherian scheme, and let $\mathscr{F}, \mathscr{G} \in \mathfrak{Mod}(X)$.

(a) If $\mathscr{F}, \mathscr{G}$ are both coherent, then $\mathscr{E}xt^i(\mathscr{F}, \mathscr{G})$ is coherent, for all $i \geq 0$.
(b) If $\mathscr{F}$ is coherent and $\mathscr{G}$ is quasi-coherent, then $\mathscr{E}xt^i(\mathscr{F}, \mathscr{G})$ is quasi-coherent, for all $i \geq 0$.

Proof. Since the question is local by previous proposition, we may assume $X = \operatorname{Spec} A$, and $\mathscr{F} \cong \widetilde{M}, \mathscr{G} \cong \widetilde{N}$. We first prove (b): assume that M is finite generated, we may make a finite free resolution $F^\bullet \to M \to 0$, or equivalent we have $\widetilde{F^\bullet} \to \widetilde{M} \to 0$. The above Ext can be compute by $h(\operatorname{Hom}(\widetilde{F^\bullet}, \widetilde{N}))$, which is generated by $\operatorname{Ext}^i(M, N)$, so it's quasi-coherent. If plus N is a finite generated A-module, we have $h(\operatorname{Hom}(\widetilde{F^\bullet}, \widetilde{N})) = h(\operatorname{Hom}(A^{n_i}, N)) = h(N^{n_i})$, since N is finite generated, so is $h(N^{n_i})$ for all $i \geq 0$, so the $\mathscr{E}xt^i(\mathscr{F}, \mathscr{G})$ is coherent, for all $i \geq 0$. □

Exercise 32 (3.6.6)**.** Let A be a regular local ring, and M be a finite generated A-module. In this case, strengthen the result (6.10A) as follows.

(a) M is projective if and only if $\operatorname{Ext}^i(M, A) = 0$ for all $i > 0$.
(b) Use (a) to show for any n, pd $M \leq n$ if and only if $\operatorname{Ext}^i(M, A) = 0$ for all $i > n$.

Proof. (a) Only if part: Since M is projective, we have $\operatorname{Hom}_A(M, \bullet)$ is an exact functor, and Ext is defined to be its derived functor, so all of them are 0.

If part: For $n > \dim A$, we have Ext^n is zero, use the method of shifting dimension we can use finite free resolution to prove $\mathrm{Ext}^{\mathrm{i}}(M, N) = 0$ for all $i > 0$. Let $A^n \to M \to 0$ and let K be its kernel then taking Ext we have

$$0 \to \operatorname{Hom}(M, K) \to \operatorname{Hom}(M, A^n) \to \operatorname{Hom}(M, M) \to 0,$$

then $\mathrm{id}_M \in \operatorname{Hom}(M, M)$ lifted to $\operatorname{Hom}(M, A^n)$, we see M is an direct summand of A^n.

(b) The only if part is obvious. For the if part, we take the truncated projective resolution of M as follows:

$$0 \to M \to P_0 \to \cdots \to P_{n-1} \to R \to 0$$

where R is cokernel of previous map. Applying $\operatorname{Hom}(\bullet, A)$ to this long exact sequence we have $\mathrm{Ext}^{\mathrm{i}}(R, A) = \mathrm{Ext}^{\mathrm{n+i}}(M, A) = 0$ for all $i > 0$. By (a) we have R is projective, so pd $M \leq n$. □

Exercise 33 (3.7.1)**.** Let X be an integral projective scheme of dimension ≥ 1 over a field k, and let $\mathscr{L}$ be an ample invertible sheaf on X. Then $H^0(X, \mathscr{L}^{-1}) = 0$.

Proof. If $\dim H^0(X, \mathscr{L}^{-1}) \neq 0$, we may assume that $0 \neq s \in H^0(X, \mathscr{L}^{-n})$, and $\mathscr{L}^n$ is a very ample sheaf over X related with $i : X \to \mathbb{P}^n_k$ and $i^*\mathcal{O}(1) \cong \mathscr{L}^n$. We define an morphism $\mathscr{L}^n \to \mathcal{O}_X$ by $m \mapsto s(m) \in \mathcal{O}_X$. And $\dim_k \Gamma(X, \mathscr{L}^n) = \dim_k \Gamma(X, i^*\mathcal{O}(1)) \geq \dim X + 1 \geq 2$, the first inequality is because X is defined by a homogeneous ideal in $k[x_0, \ldots, x_n]$. So we have contradiction since $\dim \Gamma(X, \mathcal{O}_X) = 1$. □

Exercise 34 (3.7.3)**.** Let $X = \mathbb{P}^n_k$. Show that $H^q(X, \Omega^p_X) = 0$ for $p \neq q$, k for $p = q$, $0 \leq p, g \leq n$.

Proof. We use the consequence of an exercise from 2.5. Consider $\bigwedge^r(\oplus^{n+1}\mathcal{O}_X(-1))$, we have a filtration

$$\bigwedge^r(\oplus^{n+1}\mathcal{O}_X(-1)) = F^0 \supset F^1 \supset \cdots \supset F^{r+1} = 0,$$

with each $F^i/F^{i+1} = \Omega_X^p \otimes \bigwedge^{r-p}\mathcal{O}_X$. Since we have $\bigwedge^p \mathcal{O}_X = 0$ if $p \neq 0, 1$, or $\mathcal{O}_X$ if $p = 0, 1$. So we have the following exact sequence:

$$0 \to \Omega_X^r \to \bigwedge^r(\mathcal{O}_X(-1)^{\oplus n+1}) \to \Omega_X^{r-1} \to 0.$$

And the middle term is many $\mathcal{O}_X(-r)$ direct sum, by taking cohomology turning into a long exact sequence we have $H^i(X, \Omega_X^r) = H^{i-1}(X, \Omega_X^{r-1})$ for $i > 0$. So $H^p(X, \Omega_X^p) = H^1(X, \Omega_X) = k$(from Euler sequence), also $H^{p>q}(X, \Omega_X^q) = H^{p-q}(X, \mathcal{O}_X) = 0$ and $H^{p<q}(X, \Omega_X^q) = H^{p+n-q}(X, \Omega_X^n) = H^{p+n-q}(X, \mathcal{O}_X(-n-1)) = 0$. By Serre duality, we're done. □

Exercise 35 (3.4.5)**.** For any ringed space $(X, \mathcal{O}_X)$, let Pic X be the group of isomorphism classes of invertible sheaves. Show that Pic $X \cong H^1(X, \mathcal{O}_X^*)$, where $\mathcal{O}_X^*$ denotes the sheaf whose sections over an open set U are the units in the ring $\Gamma(U, \mathcal{O}_X^*)$, with multiplication as the group operation.

Proof. We try to construct a map Φ : Pic $X \to H^1(X, \mathcal{O}_X^*)$ and check it's an isomorphism. Let $\mathscr{L} \in$ Pic X, let U_i be the local trivialization of $\mathscr{L}$ such that $\varphi_i : \mathcal{O}_X|_{U_i} \to \mathscr{L}|_{U_i}$ is isomorphism and denote $\varphi_{ij} = \varphi_j^{-1} \circ \varphi_i$. Let $\alpha \in H^1(X, \mathcal{O}_X^*)$ be $\alpha_{ij} = \varphi_{ij}(1) \in \Gamma(X, \mathcal{O}_X^*)$, since these trivialization map has cocycle condition we have $\alpha \in \ker d_2$, we define $\Phi(\mathscr{L}) = \alpha$. For the same reason, any element $\beta \in H^1(X, \mathcal{O}_X^*)$ satisfy cocycle condition, so we can glue up to an invertible sheaf $\mathscr{M}$ such that $\Phi(\mathscr{M}) = \beta$ i.e. Φ is surjective. For injective, if we have any $\Phi(\mathscr{L}) \in \mathrm{Im}\ d_1$, denote $\Phi(\mathscr{L})_{ij} = a_{ij}$, then there is a tuple $\{a_i \in \Gamma(U_i, \mathcal{O}_X^*)\}$ such that $a_i a_j^{-1} = a_{ij}$. Then we may define $\psi_i : \mathcal{O}_X|_{U_i} \to \mathscr{L}|_{U_i}, 1 \mapsto a_i^{-1}\varphi_i(1)$, this is a set of local isomorphism and compatible on the intersection $U_i \cap U_j$, so we glue up to an isomorphism $\psi : \mathcal{O}_X \to \mathscr{L}$, i.e. $\mathscr{L} \in$ Pic X is identity. And forget to say, obviously, it's a group morphism. □

Exercise 36 (4.1.1)**.** Let X be a curve, and let $P \in X$ be a point. Then there exists a nonconstant rational function $f \in K(X)$, which is regular everywhere except at P.

Proof. Let g be the genus of X, just take $D = nP$ with n large enough, then applying RR formula we have $l(nP) = n + 1 - g > 2$, then we have a nonconstant $s \in H^0(X, \mathcal{O}_X(nP))$, then we have $(s)_0 \sim D$, we have a $f \in K(X)$ such that $(f) = D' - nP$ satisfying the condition. □

Exercise 37 (4.1.2)**.** Again let X be a curve, and let $P_1, ..., P_r \in X$ be points. Then there is a rational function $f \in K(X)$ having poles at each of the P_i and regular elsewhere.

Proof. By previous exercise we have $f_i \in \Gamma(X, \mathcal{O}_X)$ such that f_i only have a pole at P_i, and regular elsewhere, let $f = \sum f_i$ and f have poles at P_i and regular elsewhere. □

Exercise 38 (4.1.3)**.** Let X be an integral, separated, regular. one-dimensional scheme of finite type over k, which is not proper over k. Then X is affine.

Proof. We can embed any X into an complete $\bar{X}$, and let $\bar{X} - X = \{P_1, ..., P_r\}$. Use exercise 4.1.2, we have f such that f have some poles at $P_1, ..., P_r$ and regular anywhere else. In fact, we require f must have poles(which is stronger than we did in previous exercise), but only to let n large enough $l(nP) = n+1-g$, when n turning into $n+1$ the dimension also plus 1, so we have $s \in H^0(X, \mathcal{O}_X((n+1)P)) - H^0(X, \mathcal{O}_X(nP))$, so we can find such f_i and $f = \sum f_i$. Such f determines a finite morphism $f : \bar{X} \to \mathbb{P}_k^1$, which only maps P_i to ∞, so $f^{-1}(A^1) = X$, which implies X is affine. □

Exercise 39 (4.1.4)**.** Show that a separated, one-dimensional scheme of finite type over k, none of whose irreducible components is proper over k, is affine.

Proof. By Ex 3.3.1 and 3.3.2 we may assume X is integral. Then since X is not proper, so is the normalization $\bar{X}$. By Ex 4.1.3 we have that $\bar{X}$ is affine, so by Ex 3.4.2 since $\bar{X} \to X$ is finite, s o X is also affine. □

Exercise 40 (4.1.5)**.** For an effective divisor D on curve X of genus g, show that $\dim |D| \leq \deg D$. Furthermore, equality holds if and only if $D = 0$ or $g = 0$.

Proof. We have $\dim |D| = l(D) - 1 = l(K - D) + \deg D + 1 - g \leq \deg D$, and the equality holds if and only if $l(K - D) = l(K) = g$. Obviously when $g = 0$ or $D = 0$, this happened. Conversely, if $l(K - D) = l(K) = g$, we have $D \sim 0$, plus D effective so $D = 0$. □

Exercise 41 (4.1.6)**.** Let X be a curve of genus g. Show that there is a finite morphism $f : X \to \mathbb{P}^1$ of degree $\leq g + 1$.

Proof. Let $D = (g+1)P$ for some point $P \in X$, we have $l(D) \geq g+1+1-g = 2$, so we have $f \in K(X)$ such that $v_P(f) \leq g + 1$, i.e. $f : X \to \mathbb{P}^1_k$ with $f^{-1}(\infty)$ is a single point set with multiplicity $\leq g + 1$, so f is a morphism of degree $\leq g + 1$. □

Exercise 42 (4.1.7)**.** A curve X is called hyperelliptic if $g \geq 2$ and there exists a finite morphism $f : X \to \mathbb{P}^1$ of degree 2.

(a) If X is a curve of genus $g = 2$, show that the canonical divisor defines a complete linear system $|K|$ of degree 2 and dimension 1, without base points. Use (II,7.8.1) to conclude that X is hyperelliptic.
(b) Show that the curves constructed in (1.1.1) all admit a morphism of degree 2 to $\mathbb{P}^1$. Thus there exist hyperelliptic curves of any genus $g \geq 2$.

Proof. (a) If $g = 2$, we have $\deg K = 2$ and $\dim |K| = \dim H^0(X, \mathcal{O}_X(K)) - 1 = 1$. Then we use proposition 4.3.1(a) to prove that there is no base point for $|K|$, equivalently, for any point $P \in X$ we have $\dim |K-P| = \dim |K|-1$. Put $D = K-P$ and use RR formula we have $l(P) = l(K-P)$, and we know that $l(P) = 1$ or 2, and if $l(P) = 2$ we have a morphism $f : X \to \mathbb{P}^1$ of degree 1, contradiction! so $\dim |K - P| = l(P)-1 = 0$, we're done. At last by the proposition of projective morphism, we have X is hyperelliptic.

(b) Let $X \subset Q \cong \mathbb{P}^1 \times \mathbb{P}^1$ be the type of curve $(g+1, 2)$, which has genus g, and compose the second projection we have a morphism $f : X \to \mathbb{P}^1$ of degree 2. □

Exercise 43 (4.1.9, RR for Singular Curves)**.** Let X be an integral projective scheme of dimension 1 over k. Let X_{reg} be the set of regular points of X.

(a) Let $D = \sum n_i P_i$ be a divisor with support in X_{reg}, i.e. all $P_i \in X_{reg}$. Then define $\deg D = \sum n_i$. Let $\mathscr{L}(D)$ be the associated invertible sheaf on X, and show that $\chi(\mathscr{L}(D)) = \deg D + 1 - p_a$.
(b) Show that any Cartier divisor on X is the difference of two ample Cartier divisors.
(c) Conclude that every invertible sheaf on X is isomorphic to $\mathscr{L}(D)$ for some divisor D with support in X_{reg}.
(d) Assume furthermore that X is a locally complete intersection in some projective space. Then by (III, 7.11) the dualizing sheaf ω° is an invertible sheaf on X, so we can define the canonical divisor K to be a divisor with support in X_{reg} corresponding to ω°. Then the formula of (a) becomes
$$l(D) - l(K - D) = \deg D + 1 - p_g.$$

Proof. (a) Follow the idea of proof in text, first we have $D = 0$ is right: Since $p_g = \dim_k H^1(X, \mathcal{O}_X)$ and $\chi(\mathcal{O}_X) = \dim_k H^0(X, \mathcal{O}_X) - \dim_k H^1(X, \mathcal{O}_X)$, by $\dim_k H^0(X, \mathcal{O}_X) = 1$, the equality holds. Then we prove This equality holds for D if and only if it holds for $D + P$,and this is proved in the same way in text, we don't repeat again.

(b) Let D be a Cartier divisor, $\mathscr{M} = \mathscr{L}(D)$, and let $\mathscr{L}$ be very ample. Choose n large enough such that $\mathscr{M} \otimes \mathscr{L}^{\otimes n}$ is generated by global sections. Then by Ex 2.7.5(d) both of $\mathscr{M} \otimes \mathscr{L}^{n+1}$ and $\mathscr{L}^{n+1}$ are very ample. So we may have $\mathscr{M} \otimes \mathscr{L}^{n+1} \cong \mathscr{L}(D')$ and $\mathscr{L}^{n+1} = \mathscr{L}(D'')$, then $D' - D'' \sim D$.

(c) By (b) we reduce to D is very ample. Let $f : X \to \mathbb{P}^n$ be the morphism determined by D, we have $\mathscr{L}(D) \cong f^*\mathcal{O}_{\mathbb{P}^n}(1)$. By Bertini's theorem we can find a hyperplane H such that $H \cap X \subset X_{reg}$, let $D' = X \cap H$, we have $\mathscr{L}(D') \cong \mathscr{L}(D)$.

(d) Locally complete intersection is contained in Cohen-Macaulay, so Serre duality holds. And the dualizing sheaf ω° is locally free, so by (c) we may define the canonical divisor K to be a divisor with support in X_{reg}. Then (a) turns

$$l(D) - l(K - D) = \deg D + 1 - p_g.$$

□

Exercise 44 (4.2.1)**.** Use (2.5.3) to show that $\mathbb{P}^n$ is simply connected.

Proof. We use induction on n, assume that $\mathbb{P}^i$ is simply connected if $i < n$, then we prove $\mathbb{P}^n$ is also simply connected. Let $f : X \to \mathbb{P}^n$ be an étale covering, then we consider $\bar{f} := f|_{f^{-1}(H)} : f^{-1}(H) \to H$ where H is a hyperplane, then by the previous assumption we have $\bar{f}$ is identity map, this implies f is of degree 1, hence an isomorphism. □