# Étale cohomology and Weil conjecture

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**Abstract**

These are notes from the online lecture given by Daniel Litt. Here is the link of the course page and the videos are reported on Bilibili. The writer takes all the responsibility for mistakes.

## Contents

# 1 Basic theory

## 1.1 Étale morphisms

**Slogan 1.1.** The structure on cohomology implies Riemann Hypothesis. This is the central idea to define étale cohomology and use it to prove the 'Riemann Hypothesis' of a finite field.

**Definition 1.2** (étale morphism)**.** Let $f : X \to Y$ be a morphism of schemes, then $f$ is étale if it is locally of finite presentation, flat and unramified.

Recall that $f$ is unramified if the relative differentials $\Omega_{X/Y} = 0$, or equivalently for any $x \in X$, let $y = f(x)$, then $\mathfrak{m}_y \cdot \mathcal{O}_x = \mathfrak{m}_x$ and the residue extensions are separable. To see this, please check the [1] and [2].

**Definition 1.3.** Another definition of étale morphism is locally finite presentation plus formally étale. and $f$ is formally étale if the following lifting uniquely exists:

$$\begin{tikzcd} \mathrm{Spec}(A/I) \arrow[r] \arrow[d] & X \arrow[d] \\ \mathrm{Spec}(A) \arrow[r] \arrow[ur, dashed, "\exists\,!"] & Y \end{tikzcd}$$

where $I^n = 0$ for some integer $n$.

*Remark* 1.4. étale morphisms behave like a covering space or local diffeomorphism. we define a morphism $f : X \to Y$ to be "standard étale" as following: For any $x \in X, y \in f(x)$, there exists $y \in V = \text{Spec } R, x \in U = \text{Spec } R[x]_h/g$ such that $f(U) \subset V$, and $g'$ is a unit in $R[x]_h$ and $g$ is monic. The geometric interpretation is that in fact, you take an affine line over $Y$ and then take a locally closed subset as the locus of $g$, and $g'$ is a unit that means $g$ has no double roots in each fiber.

**Exercise 1.5.** Check that standard étale is étale.

**Example 1.6.** multiple $[n]$ on an elliptic curve is étale if $n$ is invertible in the base.

**Example 1.7.** Let $\mathbb{G}_m = \text{Spec } k[t, t^{-1}]$ and the morphism defined by $t \mapsto t^n$ is étale if $n$ prime to char$k$. To see why, a hint is that $\frac{\partial}{\partial t}(t^n) = nt^{n-1}$ is a unit.

**Example 1.8.** Let $i : \mathbb{G}_m \to \mathbb{A}^1$ is the natural inclusion, it's étale.

**Proposition 1.9.** *There are several properties of étale morphisms.*

1. *Any open immersion is étale.*
2. *composition of étale morphisms is étale.*
3. *base change of étale morphism is étale.*
4. *Let $\phi : Y \to Z$ and $\psi : X \to Y$, if $\phi \circ \psi$ is étale, then $\psi$ is étale.*

*Proof.* In fact, another definition of étale is smooth of dimension 0, see [2]. □

**Example 1.10** (An étale morphism which is not quasi-finite)**.** Consider $X = \mathbb{G}_m - \{1\}$ and $f : X \to \mathbb{G}_m$ defined by $t \mapsto t^2$ and assume the characteristic of base field is not 2. then $f$ is not finite because it's not proper and every finite morphism is proper.

**Example 1.11.** finite separable field extension is étale.

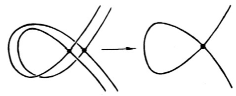

Figure 1: An finite étale covering of nodal curve

**Example 1.12.** There is a good example in Hartshorne [2], which gives us a finite étale covering of nodal curve $X = \text{Spec } k[x, y]/(y^2 - x^3 - x^2)$. We take $Y = \bar{X}_1 \coprod \bar{X}_2$ to be two copies of its normalization, on each copy $X_i$ there are two points $x_{i1}, x_{i2}$ in the preimage of the original point, then we glue a point $x_{11}$ with $x_{21}$ in the other copy, $x_{12}$ with $x_{22}$ so that every point in $X$ has two points in its preimage. One can check $Y \to X$ is étale.

**Example 1.13** (What is not étale)**.** Let $X = \text{Spec } k[x, y]/(xy)$ and $\bar{X}$ be its normalization. then $\bar{X} \to X$ is not flat, hence not étale.

Let $\mathbb{A}^1 \to \mathbb{A}^1$ defined by $t \mapsto t^2$, it is ramified at the origin, and we can compute that the Kahler differentials $\Omega_f = k[t]\mathrm{d}t/\mathrm{d}(t^2) = k[t]\mathrm{d}t/2t\mathrm{d}t$, so its support is the origin point if char $\neq 2$. If charis 2, then $\Omega_f$ is not torsion.

Let $L/K$ be a Galois extension of a number field, then $\mathcal{O}_K \hookrightarrow \mathcal{O}_L$ is étale if and only if every prime is unramified.

Let $f : \mathbb{A}^m \to \mathbb{A}^m$ determined by polynomials $f_1, ..., f_m$, one can ask where $f$ is étale. Actually, it is just the open subscheme where we invert $\det(\frac{\partial f_i}{\partial t_j})$.

**Proposition 1.14.** *Étale morphisms on varieties over an algebraically closed field k induce isomorphisms on complete local rings at closed points.*

*Proof.* Use criterion for formal étaleness. □

## 1.2 Sites-generalization of topological spaces/sheaf

**Question 1.15.** *What parts of the definition of topological space do you need to define a sheaf?*

- Open sets, inclusions *i.e.* the 'category of open set'. A presheaf on $X$ is also defined to be a contravariant functor from the category of open sets on $X$.
- Sheaf condition.
- collections of morphisms which are 'covers'.
- existence of certain fiber products (intersection).

**Definition 1.16** (Grothdndieck topology/site)**.** A Grothendieck topology on a category $\mathcal{C}$ is the assignment to each object $U$ of $\mathcal{C}$ of a collection of sets of arrows $\{U_i \to U\}$, called coverings of $U$, denoted by $\text{Cov}(U)$ such that

(i) (intersections exists) If $X_i \to X$ appear in a covering families then so does $X_i \times_X Y$.

(ii) If $\{X_i \to X\}$ is a covering family, then so is $\{Y \times_X X_i \to Y\}$.

(iii) (composition of covers is cover)If $\{X_i \to X\}$ and $X_{ij} \to X_i$ are covers, then the composition $\{X_{ij} \to X\}$ is a covering family.

(iv) If $f : X \to Y$ is an isomorphism, then $X \to Y$ is a covering family.

**Example 1.17.** Let $X$ be any topological space and $\mathcal{C}$ be the category of open sets of $X$ with $\{U_i \to U\} \in \text{Cov}(X)$ if and only if $U = \bigcup U_i$, is the motivating example.

**Example 1.18.** Let $M$ be a manifold, let $\mathcal{C}$ be the category of all $M' \to M$ locally isomorphic, and $\{M_i \to M'\} \in \text{Cov}(M)$if and only if $\bigcup \text{Im}(f_i) = M'$.

**Example 1.19.** Let $X$ be a scheme, and let $X_{ét}$ be the category whose objects are all étale morphisms $Y \to X$ and morphism are $X$-morphisms, note that it is étale automatically by proposition 1.9. For any $Y \in \mathrm{Ob}(X_{ét})$ let $\{Y_i \to Y\} \in \mathrm{Cov}(Y)$ if and only if the union of the image is surjective. Note that the union means the underlying topology space union. $X_{ét}$ is called the small étale site.

Let $X_{Ét}$ be a category, whose objects are all $X$-schemes, and whose morphisms are $X$-morphism. For $Y \in \mathrm{Ob}(X)$, $\{Y_\alpha \to Y\} \in \mathrm{Cov}(Y)$ if and only if each $Y_\alpha \to Y$ is étale and the union of image is $Y$. This is called the big Étale site, which has much more objects than the small étale site.

**Example 1.20.** Let $X$ be a complex analytic space. Let $X_{an\cdot ét}$ be the analytic étale site, whose objects are $f : Y \to X$ such that locally $f$ is an analytic isomorphism, and morphisms are $X$ morphisms. Covers are just covers.

*Remark* 1.21. Note that $\mathrm{Sh}(X_{an\cdot ét}) \cong \mathrm{Sh}(X^{top})$, the magical point is that the categories are totally different but the sheaves on them are isomorphic.

**Example 1.22.** Let $X$ be a scheme and denote $X_{zar} = \mathrm{Open}(X^{top})$. Let $X_{Zar}$ be a category, whose objects are all $X$-schemes, and morhpisms are $X$-morphisms, covers are $f_\alpha : U_\alpha \to U$ and $f_\alpha$ open embeddings and $\bigcup \mathrm{Im}(f_\alpha) = U$.

**Example 1.23** (fppf topology)**.** fppf stands for flat and locally of finite presentation, which comes from French. Let X be a scheme, and $X_{fppf}$ be the small fppf site, whose objects are flat morphisms $Y \to X$ locally of finite presentation and morphism are $X$-morphisms, a cover of $Y \to X$ is fppf morphisms that cover $Y$.

**Definition 1.24** (Presheaf on $\mathcal{C}$)**.** A $\mathcal{D}$-valued presheaf is a contravariant functor $\mathscr{F} : \mathcal{C} \to \mathcal{D}$. note that presheaf doesn't need Grothendieck topology.

**Exercise 1.25.** Let $X$ be a topological space, a $\mathcal{D}$-valued presheaf on $X$ is the same as a presheaf on $\mathrm{Open}(X)$.

**Definition 1.26** (Sheaf on a site $\mathcal{C}$)**.** A sheaf $\mathscr{F}$ is a presheaf such that the following diagram

$$\mathscr{F} \to \prod_{\alpha} \mathscr{F}(U_\alpha) \rightrightarrows \prod_{\alpha,\alpha'} \mathscr{F}(U_\alpha \times_U U_{\alpha'})$$

to be an equalizer for all covering families $\{U_\alpha \to U\} \in \mathrm{Cov}(U)$.

**Example 1.27.** Let $\mathscr{F}$ be a sheaf on a site, then if exist, there is an isomorphism $\mathscr{F}(\coprod U) \to \prod F(U)$ for any object $U$ and $\{\coprod U \to U\} \in \mathrm{Cov}(U)$.

**Definition 1.28** (Morphisms of sheaves/presheaves)**.** A morphism $\mathscr{F}_1 \to \mathscr{F}_2$ is the natural transformation.

**Example 1.29.** (of sheaves on $X_{Ét}$) We will prove any representable functor is a sheaf on $X_{Ét}/X_{Fppf}$.

**Example 1.30.** Let $\mu_n$ be the functor represented by Spec $k[t]/(t^n - 1)$, then we have

$$\mu_n(U) = \{f \in \mathcal{O}_U(U) |\ f^n = 1\}.$$

Let $\mathcal{O}_X^{ét}(U) := \mathcal{O}_U(U)$, and it is represented by $\mathbb{A}^1_X$, so it's also a sheaf. The constant sheaf $\underline{\mathbb{Z}/l^n\mathbb{Z}}$ is represented by $\mathbb{Z}/l^n\mathbb{Z} \times X$. By the definition of constant sheaf, if giving $\mathbb{Z}/l^n\mathbb{Z}$ discrete topology one can see

$$\underline{\mathbb{Z}/l^n\mathbb{Z}}(U) = \mathrm{Hom}_{cont}(U^{top}, \mathbb{Z}/l^n\mathbb{Z}).$$

Let $\mathbb{G}_m(U) = \mathcal{O}_U(U)^\times$, represented by $\mathbb{G}_{m,X} = \mathrm{Spec}\ \mathbb{Z}[t, t^{-1}] \times_{\mathbb{Z}} X$. Another important example is $\mathbb{P}^n : U \mapsto \mathrm{Hom}_X(U, \mathbb{P}^n)$

*Remark* 1.31. We now can see something of étale cohomology: Next, we will prove that $\underline{\mathbb{Z}/l^n\mathbb{Z}}$ is a sheaf on $X_{\acute{E}t}$, then prove that the category of sheaves on $X_{\acute{E}t}$ of abelian groups is an abelian category with enough injectives, which is nontrivial. Then we can define

$$H^i(X_{\acute{e}t}, \underline{\mathbb{Z}/l^n\mathbb{Z}}) = \mathrm{R}^i\Gamma_X(\underline{\mathbb{Z}/l^n\mathbb{Z}}).$$

**Example 1.32** (A tricky difference)**.** The epimorphisms on the zariski site are different from those on étale site. There is an example. Consider $f : \mathbb{G}_m \to \mathbb{G}_m$ giving by sending $t$ to $t^n$ where $n$ is invertible on the base and $f$ induces a morphism of sheaves on both zariski site and étale site, which is on zariski site $\mathcal{O}^\times \to \mathcal{O}^\times$ and on étale site $\mathcal{O}^\times_{\acute{e}t} \to \mathcal{O}^\times_{\acute{e}t}$ and both sends $f$ to $f^n$. In general, this is not a surjective on the zariski site, for example let $X = \operatorname{Spec} \mathbb{R}$ and $n = 2$, then we have $\mathbb{R}^\times \to \mathbb{R}^\times$ giving by $t \to t^2$ is clearly not surjective. But in étale site, it's surjective in general. In fact, for any $U \in \mathrm{Ob}(X)$ and $s \in \mathcal{O}^\times_{\acute{e}t}(U)$, we look for an étale morphism $\psi : U' \to U$ such that $s$ lifts in $\mathcal{O}^\times_{\acute{e}t}(U')$ has a $n$-root. Since $n$ is invertible in the base field, by the standard étale criterion $f$ is an étale morphism, we can look at the following pullback square,

$$\begin{array}{ccc} U \times \mathbb{G}_m & \longrightarrow & \mathbb{G}_m \\ \downarrow f' & & \downarrow f \\ U & \longrightarrow & \mathbb{G}_m \end{array}$$

so $f'$ is also étale. And one can tell $s$ really has a $n$-root on $U \times \mathbb{G}_m$ by definition.

## 1.3 Descent

**Definition 1.33** (continuous map of site)**.** Let $(\mathcal{C}, \mathcal{T}_1), (\mathcal{D}, \mathcal{T}_2)$ be two sites, A continuous map of $\mathcal{T}_1 \to \mathcal{T}_2$ is a functor $\mathcal{D} \to \mathcal{C}$ preserves fibre product and sends covering families to covering families.

**Example 1.34.** Let $f : X \to Y$ be a continuous map of topological spaces, one can define a continuous map of sites $\mathrm{Open}(X) \to \mathrm{Open}(Y)$, namely $U \mapsto f^{-1}(U)$ where $U$ any open subset of $Y$.

**Example 1.35.** Let $X$ be a scheme. Let $X_{Fppf}$ be the category, whose objects are all schemes over $X$ and morphisms are $X$-morphisms. For any $Y \to X$, Let $\{Y_\alpha \to Y\} \in \mathrm{Cov}(Y)$ if and only if each $Y_\alpha \to Y$ is flat and locally of finite presentation and union of the images are $Y$. We have the following (inclusion) maps of sites,

$$X_{Fppf} \to X_{\acute{E}t} \to X_{\acute{e}t} \to X_{zar}.$$

*Remark* 1.36 (terminology). Now what we define is in fact called Grothendieck pre-topology. The notion of topos is defined to be a category that is isomorphic to a sheaves category on some category.

**Question 1.37.** *How to check if a presheaf is a sheaf and construct sheaves on $X_{\acute{e}t}/X_{fppf}$?*

**Theorem 1.38.** *Let $X$ be a scheme and use the notation $X_{Fppf}/X_{\acute{E}t}/X_{\acute{e}t}$ defined as above.*

1. *If $Y$ is an $X$-scheme, the functor $Z \mapsto \mathrm{Hom}_X(Z, Y)$ is a sheaf on $X_{Fppf}/\ X_{\acute{E}t}/X_{\acute{e}t}$.*
2. *Let $\mathscr{F}$ be a quasi-coherent sheaf on $X$, the functor sends $f : Z \to X$ to $\Gamma(Z, f^*\mathscr{F})$ is a sheaf on $X_{Fppf}/X_{\acute{E}t}/X_{\acute{e}t}$. Denote the associated sheaf on $X_{\acute{e}t}$ by $\mathscr{F}^{\acute{e}t}$.*

We start to prove (2). The question is that, Let $U = \coprod U_i \to X$ is an fppf cover of $X$ and $\mathscr{F}$ is a quasi-coherent sheaf on $X$, when does $\mathscr{F}$ come from a quasi-coherent sheaf on $X$? To be more precious, What extra structure do you need to "descend it" to a quasi-coherent sheaf on $X$? And the same when does an arbitrary morphism $f : \mathscr{F}_1|_U \to \mathscr{F}_2|_U$ come from a morphism $\mathscr{F}_1 \to \mathscr{F}_2$ pulling back? To discover it, we consider the situation in zariski site, which is quite simple.

**Example 1.39.** Let $U = \coprod U_i \to X$ be the zariski cover, which means that every $U_i$ is an open subset of $X$ and $\bigcup U_i = X$. The data we need to get a sheaf on $X$ is the "gluing data" *i.e.* isomorphisms $\phi_{ij} : \mathscr{F}_i|_{U_i \cap U_j} \to \mathscr{F}_j|_{U_i \cap U_j}$ satisfying cocycle condition. A morphism $\mathscr{F} \to \mathscr{G}$ on $X$ is the same as morphisms $\mathscr{F}_i \to \mathscr{G}_i$ commuting with gluing data. This is the motivating example for the following definition.

**Definition 1.40** (Descent data for quasi-coherent sheaves)**.** Let $f : U \to X$ be a morphism, and let $\pi_i : U \times_X U \to U$ be two projections to correspondent factors and $\pi_{ij} : U \times_X U \times_X U \to U \times_X U$ be projections to correspondent factors for $i, j = 1, 2$. The descent data for quasi-coherent sheaf on $U$ over $X$ is the following:

1) A quasi-coherent sheaf $\mathscr{F}$ on $U$;

2) Isomorphism $\varphi : \pi_1^* \mathscr{F} \to \pi_2^* \mathscr{F}$;

3) These morphisms satisfy $\pi_{23}^* \varphi \circ \pi_{12}^* \varphi = \pi_{13}^* \varphi$.

**Exercise 1.41.** Check this agrees with what we had before if $U \to X$ is a Zariski cover.

**Definition 1.42** (Morphism of descent data)**.** Given descent data $(\mathscr{F}, \varphi)$ and $(\mathscr{G}, \psi)$, a morphism $(\mathscr{F}, \varphi) \to (\mathscr{G}, \psi)$ is a map $h : \mathscr{F} \to \mathscr{G}$ such that the following diagram commutes.

$$\begin{tikzcd} \pi_1^* \mathscr{F} \arrow[r, "\pi_1^* h"] \arrow[d, "\varphi"] & \pi_1^* \mathscr{G} \arrow[d, "\psi"] \\ \pi_2^* \mathscr{F} \arrow[r, "\pi_2^* h"] & \pi_2^* \mathscr{G} \end{tikzcd}$$

**Example 1.43** (Descent data of a vector bundle on zariski site)**.** Let $U = \coprod U_i \to X$ be zariski covers and let $\mathcal{O}_{U_i}^{\oplus n}$ be $n$ copy of structure sheaf for an integer $n$. To glue up to a vector bundle on $X$, we need isomorphisms $\varphi : \mathcal{O}_{U_i \cap U_j}^{\oplus n} \to \mathcal{O}_{U_i \cap U_j}^{\oplus n}$ satisfying cocycle condition, by Exercise 3.10, it's just descent data, so there's an equivalence between all vector bundles and all descent data of vector bundle on zariski site.

**Example 1.44.** Let $L/K$ be a finite Galois extension with Galois group $G$, Consider the descent data on Spec $L$ over Spec $K$, *i.e.* a $L$-vector space $V$ with gluing data $\varphi : \pi_1^* V \to \pi_2^* V$ on Spec $L \otimes_K L$ satisfying cocycle condition. Since $L$ is finite separable over $K$, we can write $L = K(\alpha)$, where $\alpha$ is algebraic and $f$ denotes its minimal polynomial, then

$$L \otimes_K L = L[x]/f(x) = \bigoplus_{g \in G} L[x]/(x - g\alpha) = \bigoplus_{g \in G} L.$$

Hence Spec $L \otimes_K L = \coprod_G \text{Spec } L$, which is copies of Spec $L$ indexed by $G$. One who is familiar with Galois descent will find out the descent we defined here is the same as Galois descent data, but I'm not.

Our strategy to prove Theorem 3.6 is first to make some preparation for faithfully flat, and then prove the theorem of descent data, which eventually implies Theorem 3.6.

**Lemma 1.45.** *Let $R \to S$ be a faithfully flat ring morphism, and let $N$ be a $R$-module, then*

$$N \to N \otimes_R S \rightrightarrows N \otimes_R S \otimes_R S$$

*is an equalizer diagram, where the left one is $f : n \mapsto n \otimes 1$ and the right ones are $\mathrm{id}_N \otimes \mathrm{id}_S \otimes 1$ and $\mathrm{id}_N \otimes 1 \otimes \mathrm{id}_S$, denoted by $p_1, p_2$.*

*Proof.* The exactness is equivalent to the exactness of

$$0 \to N \xrightarrow{f} N \otimes S \xrightarrow{p_1 - p_2} N \otimes S \otimes S.$$

Since $R$ is faithfully flat over $S$, it is enough to prove that

$$0 \to N \otimes S \xrightarrow{f \otimes \mathrm{id}} N \otimes S \otimes S \xrightarrow{(p_1 - p_2) \otimes \mathrm{id}} N \otimes S \otimes S \otimes S.$$

The key fact is that we can define a section by multiplication. More preciously, we have

$$N \otimes S \xleftarrow{\gamma} N \otimes S \otimes S \xleftarrow{\tau} N \otimes S \otimes S \otimes S$$

where $\tau(n \otimes s \otimes s' \otimes s'') = n \otimes s \otimes s's''$ and $\gamma(n \otimes s \otimes s') = n \otimes ss'$. One can check $\gamma(f \otimes \mathrm{id})$ is identity, so $f \otimes \mathrm{id}$ is injective. Also, easy to check $(p_2 - p_1) \circ f$ is zero map. Let $\alpha \in N \otimes R \otimes R$ with $(p_1 \otimes \mathrm{id})(\alpha) = (p_2 \otimes \mathrm{id})(\alpha)$. then one can check

$$\alpha = \tau(p_1 \otimes \mathrm{id})(\alpha) = \tau(p_2 \otimes \mathrm{id})(\alpha) = (f \otimes \mathrm{id})\gamma(\alpha),$$

which implies the exactness in the middle. □

*Remark* 1.46. Given $R \to S$ is faithfully flat, then the complex

$$N \to N \otimes S \to N \otimes S \otimes S \to \cdots \to N \otimes S^{\otimes r} \to \cdots$$

is exact. The proof is more or less the same.

**Theorem 1.47** (Grothendieck, descent for quasi-coherent sheaf)**.** *Suppose $U \to X$ is fppf. Then $f^*$ induces an equivalence of categories between the category of quasi-coherent sheaves* $\mathrm{QCoh}(X)$ *and the category of descent data on $U$ over $X$.*

*Proof.* Explicitly, given $\mathscr{F} \in \mathrm{QCoh}(X)$, we have $f^*\mathscr{F} \in \mathrm{QCoh}(U)$, since $\pi_1 \circ f = \pi_2 \circ f$, we have an isomorphism $(\pi_1 \circ f)^*\mathscr{F} = (\pi_2 \circ f)^*\mathscr{F}$ on $U \times_X U$, identified as the gluing data. This maps a quasi-coherent sheaf to descent data. There are 2 steps to prove this theorem: (1)prove that the above functor $f^*$ is fully faithful; (2)prove that $f^*$ is essentially surjective.

Step 1: To prove fully faithfulness, it is saying that given $\mathscr{F}_1, \mathscr{F}_2 \in \mathrm{QCoh}(X)$, claim that the following diagram being equalizer is equivalent to fully faithfulness.

$$\begin{tikzcd} & & \mathrm{Hom}_{U \times_X U}(\pi_1^* f^* \mathscr{F}_1, \pi_1^* f^* \mathscr{F}_2) \arrow[d, "\wr"] \\ \mathrm{Hom}_X(\mathscr{F}_1, \mathscr{F}_2) \arrow[r, "f^*"] & \mathrm{Hom}_U(f^*\mathscr{F}_1, f^*\mathscr{F}_2) \arrow[ur, "\pi_1^*"] \arrow[dr, "\pi_2^*"'] & \\ & & \mathrm{Hom}_{U \times_X U}(\pi_2^* f^* \mathscr{F}_1, \pi_2^* f^* \mathscr{F}_2) \end{tikzcd}$$

First, let $g \in \mathrm{Hom}_U(f^*\mathscr{F}_1, f^*\mathscr{F}_2)$, claim that $g$ is a morphism of descent data if and only if $\pi_1^*(g) = \pi_2^*(g)$. It's straightforward from the definition. The image of $f^*$ maps to the same thing by $\pi_1^*$ and $\pi_2^*$ means pulling back by $f$ gives descent data, in the other way it says all descent data is given by a morphism $\mathscr{F}_1 \to \mathscr{F}_2$ pulling back.

To prove the diagram is an equalizer, first reduce to the affine situation. First, assume $X$ is affine. Since faithfully flat morphism implies openness and $X$ is quasi-compact, we may assume $U$ is covered by finitely many affine open subschemes $U_i \subset U, i \in I$. Consider the disjoint $\bar{U} := \coprod_{i \in I} U_i$ of those schemes. Let $u : \bar{U} \to U$ be the canonical morphism, $\bar{f} = f \circ u$. Then we obtain a diagram that the above diagram can be embedded into, which means there are injectives from each term above to each term below.

$$\begin{tikzcd} & & \mathrm{Hom}_{\bar{U} \times_X \bar{U}}(\pi_1^* \bar{f}^* \mathscr{F}_1, \pi_1^* \bar{f}^* \mathscr{F}_2) \arrow[d, "\wr"] \\ \mathrm{Hom}_X(\mathscr{F}_1, \mathscr{F}_2) \arrow[r, "\bar{f}^*"] & \mathrm{Hom}_{\bar{U}}(\bar{f}^*\mathscr{F}_1, \bar{f}^*\mathscr{F}_2) \arrow[ur, "\pi_1^*"] \arrow[dr, "\pi_2^*"'] & \\ & & \mathrm{Hom}_{\bar{U} \times_X \bar{U}}(\pi_2^* \bar{f}^* \mathscr{F}_1, \pi_2^* \bar{f}^* \mathscr{F}_2) \end{tikzcd}$$

The first term on the left doesn't change, the injective between two terms in the middle is given by pullback along $u$, and the injective between the right term is pullback along $u \times u$. The reason why it is injective is that $u$ is faithfully flat. So the exactness of the first diagram can be checked on the second diagram and for the second one, everything is affine. Now let $X$ be arbitrary schemes, we need to argue carefully, please see theorem 4.1.10 of [5]. In it, we first assume $\mathscr{F}$ to be a big Zariski sheaf, then reduce it to the affine case.

Now we translate it into commutative algebra. Let $R \to S$ be a faithfully flat morphism and let $N, M$ be $R$-modules. We want to prove

$$\mathrm{Hom}_R(M, N) \to \mathrm{Hom}_S(M \otimes S, N \otimes S) \rightrightarrows \mathrm{Hom}_{S \otimes S}(M \otimes S \otimes S, N \otimes S \otimes S)$$

to be an equalizer diagram. The map on the left side is induced by $N \to N \otimes S$, Hence injective. The two maps on the right side can also be induced by the maps in Lemma 3.13, Hence exactness in the middle.

Step 2: Given descent data $(\mathscr{F}, \varphi)$ on $U$ over $X$, we want to prove there exists $\mathscr{G} \in \mathrm{QCoh}(X)$ such that $f^*\mathscr{G} \cong \mathscr{F}$ as descent data. First, one can reduce to the affine morphism.

Now let $f : R \to S$ be fppf morphism and $M$ be a $S$-module with an isomorphism $\varphi : S \otimes_R M \to M \otimes_R S$ of $S \otimes_R S$-modules satisfying cocycle condition. We have two maps

$$M \rightrightarrows M \otimes_R S$$

define by $m \to m \otimes 1$ and $m \to \varphi(1 \otimes m)$, set $K := \mathrm{eq}(M \rightrightarrows M \otimes S)$, *i.e.* the subset of $M$ which map to the same elements in $M \otimes S$ through two maps. Claim that the natural map $K \otimes_R S \to M$ is an isomorphism compatible with descent data, which finishes our proof. After tensoring $S$ over $R$, the equalizer we define becomes

$$K \otimes_R S \to M \otimes_R S \rightrightarrows M \otimes_R S \otimes_R S$$

. Since $R \to S$ is faithfully flat, it's also an equalizer diagram. Then we can easily define two $R$-isomorphisms $M \otimes_R S \otimes_R S \to M \otimes_R S \otimes_R S$ such that the following diagram commutes:

$$\begin{tikzcd}
K \otimes_R S \arrow[r] \arrow[d, dashed] & M \otimes_R S \arrow[r, shift left] \arrow[r, shift right] \arrow[d, equal] & M \otimes_R S \otimes_R S \arrow[d, shift right] \arrow[d, shift left, "\wr"] \\
M \arrow[r] & M \otimes_R S \arrow[r, shift left] \arrow[r, shift right] & M \otimes_R S \otimes_R S
\end{tikzcd}$$

By 5-Lemma, we obtain a unique isomorphism $K \otimes_R S \xrightarrow{\sim} M$, also by definition we have

$$\begin{tikzcd}
(K \otimes_R S) \otimes_R S \arrow[r] \arrow[d] & S \otimes_R (K \otimes_R S) \arrow[d] \\
M \otimes_R S \arrow[r] & S \otimes_R M
\end{tikzcd}$$

Hence, it's an isomorphism as descent data. □

*Remark* 1.48. What a beautiful theorem! It's enough to prove on the fpqc site, see Stack project.

**Corollary 1.49.** *The associated presheaf on sites $\mathscr{F}^{\text{ét}}/\mathscr{F}^{fppf}$ is a sheaf.*

*Proof.* Apply the above theorem to $\mathscr{F}_1 = \mathcal{O}_X$ and $\mathscr{F}_2 = \mathscr{F}$. □

**Theorem 1.50.** *Let $p : U \to X$ is an fppf cover, then induced functor $p^* : \mathrm{Sch}/X \to \mathcal{D}$, where $\mathcal{D}$ is the category of descent data is fully faithful.*

*Proof.* Using the same strategy in proving Theorem 3.15, we can reduce to the case of affine morphism. It's sufficient to consider quasi-coherent $\mathcal{O}_X$-algebras since any affine morphism can be written as the relative spectrum $\mathcal{S}pec\ \mathscr{A}$ for a quasi-coherent $\mathcal{O}_X$-algebras $\mathscr{A}$.

Again, translate what we want into a result of algebra. Let $Y, Z$ be two $X$-schemes, it's sufficient to prove the following diagram is an equalizer.

$$\mathrm{Hom}_X(Y, Z) \to \mathrm{Hom}_U(p^*Y, p^*Z) \rightrightarrows \mathrm{Hom}_{U\times_X U}(\pi^*p^*Y, \pi^*p^*Z).$$

By the discussion above, for $Y, Z$ affine over $X$, suffice to prove

$$\mathrm{Hom}_{\mathrm{QCohAlg}_X}(\mathcal{O}_Z, \mathcal{O}_Y) \to \mathrm{Hom}_U(p^*\mathcal{O}_Z, p^*\mathcal{O}_Y) \rightrightarrows \mathrm{Hom}_{U\times_X U}(\pi^*p^*\mathcal{O}_Z, \pi^*p^*\mathcal{O}_Y)$$

is an equalizer diagram. If one forget the structure of algebra, This follows from Theorem 3.13 straightforward. □

**Corollary 1.51.** *If $Z$ is a scheme over $X$, then* $\mathrm{Hom}_X(-, Z)$ *is a sheaf on* $X_{fppf}, X_{\acute{E}t}, X_{\acute{e}t}$.

*Remark* 1.52. But $p^*$ is not essentially surjective in general for schemes, it is for affine schemes and polarized schemes *i.e.* projective schemes with an very ample line bundle.

**Example 1.53.** Hence we have a lot of sheaves listed before. But we never mention before that Hilbert functor/Quot functor are sheaves since representable.

# 2 Cohomology

## 2.1 Construction

Now we have prepared something for developing a new cohomology theory on schemes and recall what do we need to define a cohomology. Here are the ingredients:

- The category of abelian sheaves on $X_{ét}$ to be an abelian category.
- Enough injectives.

In fact, both facts are true for the category of abelian sheaves on any site. We are going to prove it in general. We are going to prove the following theorem.

**Theorem 2.1.** *Let $\tau$ be a site. The forgetful functor* $\mathrm{Sh}(\tau) \to \mathrm{PreSh}(\tau)$ *has a left adjoint, which is called sheafification.*

*Proof.* We say that a presheaf is separated if it satisfies the injective of the equalizer diagram of being a sheaf. Then, we have two inclusions

$$\text{separated presheaves on } C \hookrightarrow \text{presheaves on } C$$

and

$$\text{sheaves on } C \hookrightarrow \text{separated presheaves on } C.$$

We want to show that each of them has a left adjoint functor. First, we construct the first one. Let $\mathscr{F}$ be a presheaf on $\tau$, we define $\mathscr{F}^s$ to be the quotient of $\mathscr{F}$ which to any object $U$ on $\tau$ let $\mathscr{F}^s(U) = \mathscr{F}(U)/\sim$, where any two sections $a, b \in \mathscr{F}(U)$ are equivalent if there exists a covering $\{U_i \to U\}_{i \in I} \in \mathrm{Cov}(U)$ such that $a$ and $b$ have the same image under $\mathscr{F}(U) \to \prod_{i \in I} \mathscr{F}(U_i)$. To check it is a presheaf, if $V \to U$ is a morphism in $\tau$ and any cover $\{U_i \to U\}_{i \in I}$, we have $\{U_i \times_U V \to V\}$ and a commutative diagram:

$$\begin{tikzcd}
\mathscr{F}(U) \arrow[r] \arrow[d] & \prod_{i \in I} \mathscr{F}(U_i) \arrow[d] \\
\mathscr{F}(V) \arrow[r] & \prod_{i \in I} \mathscr{F}(U_i \times_U V).
\end{tikzcd}$$

Hence, it follows that the composition

$$\mathscr{F}(U) \to \mathscr{F}(V) \to \mathscr{F}(V)/\sim$$

factors through $\mathscr{F}(U)/\sim$, so $\mathscr{F}^s$ is a presheaf. It is also clear from the construction that if $\mathscr{G}$ is a separated presheaf then any map $\mathscr{F} \to \mathscr{G}$ factor through $\mathscr{F}^s$. Then this is really a left adjoint $\mathscr{F} \to \mathscr{F}^s$.

For the second, let $\mathscr{F}$ be a separated presheaf on $\tau$, and define $\mathscr{F}^a$ to be the presheaf that associates to any $U \in \tau$ the set of pairs

$$(\{U_i \to U\}_{i \in I}, \{a_i\}),$$

consisting of a covering $\{U_i \to U\}_{i \in I}$ of $U$ and an element

$$\{a_i\} \in \mathrm{Eq}(\prod_{i \in I} \mathscr{F}(U_i) \rightrightarrows \prod_{i,j \in I} \mathscr{F}(U_i \times_U U_j)),$$

of the equalizer, modulo the equivalence relation(which we may consider as refinement)

$$(\{U_i \to U\}_{i \in I}, \{a_i\}) \sim (\{V_j \to U\}_{j \in J}, \{b_j\})$$

if $a_i$ and $b_j$ have the same image in $\mathscr{F}(U_i \times_U V_j)$ for all $i \in I, j \in J$. For the same reason as above $\mathscr{F}^a$ is a presheaf and $\mathscr{F}^a$ is separated because $\mathscr{F}$ is separated. To see the exactness in the middle, it's just given by definition. So it's a sheaf. Also, there's a natural map $\mathscr{F} \to \mathscr{F}^a$ which sends $a \in \mathscr{F}(U)$ to $(\{\mathrm{id} : U \to U\}, \{a\}) \in \mathscr{F}^a(U)$, through which any morphism $\mathscr{F} \to \mathscr{G}$ factors where $\mathscr{G}$ is a sheaf. This gives the left adjoint functor of the second inclusion. By composing two functors, we obtain the left adjoint functor of the forgetful functor. $\square$

Now we do some preliminaries.

**Definition 2.2** (pushforward)**.** Let $f : \tau_1 \to \tau_2$ is a continuous morphism of sites. Given a sheaf $\mathscr{G} \in \mathrm{Sh}(\tau_1)$, we define the pushforward $f_*\mathscr{G}$ associated each $U$ with $f_*\mathscr{G}(U) := \mathscr{G}(f^{-1}(U))$.

**Exercise 2.3.** An exercise is that above $f_*\mathscr{G}$ is a sheaf.

**Example 2.4.** Let $f : X \to Y$ be a morphism of schemes, we get an induced morphism of sites $f : X_{\acute{e}t} \to Y_{\acute{e}t}$ which send $U \to Y$ to $U \times_Y X \to X$. So we can push forward a sheaf $\mathscr{F}$ on $X_{\acute{e}t}$ to $Y_{\acute{e}t}$.

If let $k$ be a algebraically closed field, then one can easily see that $\mathrm{Sh}((\mathrm{Spec}\ k)_{\acute{e}t}) = \mathrm{Sets}$. To give a geometric point in $X$ *i.e.* $\iota_{\bar{x}} : \mathrm{Spec}\ k \to X$, one have

$$(\iota_{\bar{x}})_*\mathscr{F}(U \to X) = \mathscr{F}(U \times_X \{\bar{x}\}) = \mathscr{F}(\coprod \mathrm{Spec}\ k) = \prod \mathscr{F}(\mathrm{Spec}\ k).$$

where the coproduct and product are indexed by the preimage of $\bar{x}$.

**Definition 2.5** (skyscarper sheaf)**.** The sheaf $(\iota_{\bar{x}})_*\mathscr{F}$ is called a skyscraper sheaf.

**Theorem 2.6.** *Let $f : \tau_1 \to \tau_2$ be a continuous map between sites. we assume the underlying categories are small categories. then the functor $f_* : \mathrm{Sh}(\tau_1) \to \mathrm{Sh}(\tau_2)$ has a left adjoint functor $f^*$, called pullback long $f$.*

*Scratch of the proof.* First, if we admit that $f_*$ has a left adjoint $\hat{f}^*$ in presheaves, then just compose $\hat{f}^*$ with sheafification functor then we have the $f^*$ in sheaves. To construct it in presheaves, it's quite similar with what we do in classical topology: for any object $U \in \tau_1$ we define a category $I_U$, its objects are pairs $(U', \rho)$ where $U' \in \tau_2$ and $\rho : U \to f^{-1}(U')$ is a morphism in $\tau_1$, a morphism $(U', \rho) \to (V', \varepsilon)$ is a morphism $g : U' \to V'$ such that $f(g) \circ \rho = \varepsilon$, then define

$$(\hat{f}^*\mathscr{F})(U) := \varinjlim_{(U',\rho) \in I_U^{op}} \mathscr{F}(U').$$

At last, try to find a natural morphism of functors $\hat{f}^* f_* \to \mathrm{id}_{\mathrm{PreSh}(\tau_1)}$ and $\mathrm{id}_{\mathrm{PreSh}(\tau_2)} \to f_*\hat{f}^*$. To see the details, please see proposition 2.2.26 in [5]. $\square$

**Example 2.7.** Let $k$ be an algebraically closed field and let $\iota_{\bar{x}} : \mathrm{Spec}\ k \to X$ define the stalk $\mathscr{F}_{\bar{x}} := \iota_{\bar{x}}^*\mathscr{F} = \varinjlim_{(U,\bar{u})} \mathscr{F}(U)$.

**Example 2.8** (Strictly Henselization)**.** The local ring at a point on a differential manifold depends only on the dimension and whether the manifold is real or complex. In the étale sense, it is more or less for schemes. In fact, let $\varphi : X \to Y$ is étale at a geometric point $\bar{y} \in Y$, then the map $\mathcal{O}_{X,\varphi(\bar{y})} \to \mathcal{O}_{Y,\bar{y}}$ induced by $\varphi$ is an isomorphism. Another fact is that: Let $P \in X$ be a nonsingular point and $d = \dim X$, then one can define a regular map $\varphi : U \to \mathbb{A}^d$ étale at $P$ where $U$ is a Zariski neighborhood of $P$. Another striking fact is that: every étale stalk of $\mathcal{O}_X$ at a geometric point $\bar{x}$, denoted by $\mathcal{O}_{X,\bar{x}}^{\acute{e}t}$, is a strictly Henselization of $\mathcal{O}_{X,\bar{x}}$(the Zariski stalk).

**Example 2.9.** Let $f : Y \to X$ be a morphism of schemes, $f^*\underline{\mathbb{Z}/l\mathbb{Z}} = \underline{\mathbb{Z}/l\mathbb{Z}}$. Let $Z$ be a $X$-scheme and $\mathscr{F} = \mathscr{H}om_X(-, Z)$, then $f^*\mathscr{F} = \mathscr{H}om_Y(-, \underline{Y \times_X Z})$.

**Lemma 2.10.** *Let $\mathscr{F}, \mathscr{G}$ are sheaves of abelian groups on $X_{ét}$. The following are equivalent:*

*(i) $\mathscr{F} \to \mathscr{G}$ is an epimorphism.*

*(ii) $\mathscr{F} \to \mathscr{G}$ is locally subjective* i.e. *given any $s \in \mathscr{G}(U)$ for any $U$, there exists $U' \to U$ such that $s|_{U'}$ is the image of some $t' \in \mathscr{F}(U')$.*

*(iii) $\mathscr{F}_{\bar{x}} \to \mathscr{G}_{\bar{x}}$ is surjective for all geometric points $\bar{x} \to X$. Let $\bar{x}$ vary, the*

*Proof.* (ii)⇒(i): If let $a, b$ be two maps such that the composition

$$\mathscr{F} \to \mathscr{G} \rightrightarrows \mathscr{H}$$

is equal, then we want $a = b$. This is the same as the situation in usual topological spaces.

(i) ⇒ (iii): Assume that $\mathscr{F}_{\bar{x}} \to \mathscr{G}_{\bar{x}}$ is not surjective for some $\bar{x}$ with the cokernel $\Lambda$. Then we have the following diagram

$$\mathscr{F} \to \mathscr{G} \rightrightarrows (\iota_{\bar{x}})_* \Lambda$$

where two maps on the right are zero maps and the cokernel map. this contradicts (i).

(iii)⇒ (ii): Given $s \in \mathscr{G}(U)$, we want to find a morphism $U' \to U$ such that $s|_{U'}$ comes from $\mathscr{F}(U')$. Now choose $\bar{x} \in U$. Since $\mathscr{F}_{\bar{x}} \to \mathscr{G}_{\bar{x}}$ is surjective, there exists some étale neighborhood $(V, \bar{v})$ of $\bar{x}$ such that $s_V$ is in the image of $\mathscr{F}$. We are not done because $V \to U$ is not a cover, but let $\bar{x}$ vary and we obtain a cover eventually. □

**Lemma 2.11.** *Let the following diagram be a sequence of abelian sheaves on $X_{ét}$*

$$0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{H}.$$

*The following are equivalent:*

*(i) the sequence is exact* i.e. *$\mathscr{F} = eq(\mathscr{G} \rightrightarrows H)$, where the two maps are zero map and original map.*

*(ii) the following sequence is exact for all $U$*

$$0 \to \mathscr{F}(U) \to \mathscr{G}(U) \to \mathscr{H}(U).$$

*(iii) the following sequence is exact for all geometric point $\bar{x} \in X$*

$$0 \to \mathscr{F}_{\bar{x}} \to \mathscr{G}_{\bar{x}} \to \mathscr{H}_{\bar{x}}.$$

*Proof.* An exercise. □

*Remark* 2.12. In the course, we construct the existence of sheafification by "espace étale", which only works in étale topology $X_{ét}$. But the method at the beginning of this section is a general one. The "espace étale" means that given $\mathscr{F} \in \mathrm{PreSh}(X_{ét})$ we define

$$\mathrm{Esp}(\mathscr{F}) := \prod_{\bar{x}} (\iota_{\bar{x}})_* \mathscr{F}_{\bar{x}},$$

then the sheafification of $\mathscr{F}$ exists as a subsheaf of $\mathrm{Esp}(\mathscr{F})$.

**Corollary 2.13.** *Colimits exists in* $\mathrm{Sh}(X_{ét})$*.*

*Scratch of proof.* (i)Colimits exist in the category of presheaves; (ii)Left adjoint sends colimits to colimits. □

**Corollary 2.14.** *The category* $\mathrm{Sh}(X_{ét})$ *is abelian.*

*Scratch of the proof.* The colimits exist as above. The limits exist because we can define them pointwise. The cokernel exists because it is a special type of colimits. To check image is a coimage, just check on the stalk. Note that this is quite similar to usual topological space, but if one wants to prove this on arbitrary sites, this will be much more complicated, see [5]. ☐

The next step is to show the category $\mathrm{Sh}(X_{ét})$ has enough injectives.

**Theorem 2.15.** *Let $X$ be a scheme, then $\mathrm{Sh}(X_{ét})$ has enough injectives,* i.e. *for any sheaf $\mathscr{F} \in \mathrm{Sh}(X_{ét})$, there exists a sheaf $\mathscr{I} \in \mathrm{Sh}(X_{ét})$ with $\mathscr{F} \hookrightarrow \mathscr{I}$.*

*Proof.* For each $x \in X$, choose a geometric point $\bar{x} \to x \to X$ and let $I(\bar{x})$ be an injective abelian group with $\mathscr{F}_{\bar{x}} \hookrightarrow I(\bar{x})$. I claim that $\mathscr{I} = \prod_{\bar{x}} (\iota_{\bar{x}})_* I(x)$ will work. First, the map $\mathscr{F} \to \mathscr{I}$ is induced by $\mathscr{F}_{\bar{x}} \to \mathscr{I}$ so obviously injective. To check it is an injective morphism, it's enough to check on the stalk. ☐

*Remark* 2.16. For any site, $\tau$, the category of sheaves of abelian groups $\mathrm{Sh}(\tau)$ is abelian with enough injectives.

Now we are prepared to "compute" étale cohomology.

**Definition 2.17.** Given $\mathscr{F} \in \mathrm{Sh}(X_{ét})$, then define the i-th étale cohomology to be

$$H^i(X_{ét}, \mathscr{F}) = R^i\Gamma(X, \mathscr{F}).$$

To compute it, we can choose an injective resolution(or cyclic resolution)

$$\mathscr{F} \to \mathscr{I}^0 \to \mathscr{I}^1 \to \mathscr{I}^2 \to \cdots.$$

we have $H^i(X_{ét}, \mathscr{F}) = H^i(\Gamma(X, \mathscr{I}^\bullet))$. Also analogue with the usual cohomology, we can also define high direct image as a relative version, that is, let $\pi : X_{ét} \to Y_{ét}$ and $\mathscr{F}$ be a sheaf on $X_{ét}$, then $R^i\pi_*\mathscr{F} = H^i(\pi_*\mathscr{I}^\bullet) \in \mathrm{Sh}(Y_{ét})$.

**Exercise 2.18.** Note that $L^i\pi^*\mathscr{G} = 0$ for any $\mathscr{G} \in \mathrm{Sh}(Y_{ét})$ and $i > 0$, which means pullback is exact.

Now it's time to rewind some basic properties of cohomology:

(I) $H^0(X_{ét}, \mathscr{F}) = \mathscr{F}(X) = \Gamma(X, \mathscr{F})$.

(II) $H^i(X_{ét}, \mathscr{I}) = 0$ for $i > 0$ if $\mathscr{I}$ is injective.

(III) Short exact sequence implies long exact sequence *i.e.* if there is an exact sequence

$$0 \to \mathscr{F}_1 \to \mathscr{F}_2 \to \mathscr{F}_3 \to 0,$$

then we have

$$\cdots \to H^i(X_{ét}, \mathscr{F}_2) \to H^i(X_{ét}, \mathscr{F}_3) \to H^{i+1}(X_{ét}, \mathscr{F}_1) \to \cdots.$$

**Example 2.19.** Let $k$ be a field, then consider $\mathrm{Sh}((\mathrm{Spec}\ k)_{ét})$ and let $G = Gal(k^s/k)$. Claim that there is an equivalent of categories between the above one and the category of discrete $G$-modules, given by

$$\iota : \mathscr{F} \mapsto \varinjlim_{k \subset L \subset k^s} \mathscr{F}(\mathrm{Spec}\ L).$$

From this, one immediately has the following corollary.

**Corollary 2.20.** $H^i((\mathrm{Spec}\ k)_{ét}, \mathscr{F}) = H^i(G, \iota(\mathscr{F}))$

*Proof of the claim.* Note that any étale morphism $V \to \text{Spec } k$ is of form $V = \coprod_{k'/k} \text{Spec } k'$ where all $k'$ is finite separable over $k$. We construct the inverse functor: given an étale morphism $V \to \text{Spec } k$ and a discrete $G$-module $M$, define

$$\eta : M \mapsto (V \mapsto \prod M^{Gal(k^s/k')}).$$

Note that the above equivalence also implies that $H^0 \xleftrightarrow{\iota} (-)^G$, which implies the fact that étale cohomology = group cohomology. Galois information is recorded into a sheaf of étale topology. □

**Example 2.21** (elliptic curves)**.** Let $E$ be an elliptic curve and the functor $\text{Hom}(-, E)$ is equivalent to $E(k^s)$ by the above equivalence.

## 2.2 Čech cohomology

An important tool to compute is Čech cohomology. First, we have some warnings

1. Čech cohomology does not always compute étale cohomology.
2. Čech cohomology is not actually computable, because in general cyclic covers do not exists.

Now we define Čech cohomology. Let $U = \coprod_i U_i \to X$ be an étale cover. we have the following complex

$$X \longleftarrow U \leftleftarrows U \times_X U \Lleftarrow U \times_X U \times_X U \cdots$$

then apply $\mathscr{F}$ on this complex we have

$$\mathscr{F}(U) \rightrightarrows \mathscr{F}(U \times_X U) \Rrightarrow \mathscr{F}(U \times_X U \times_X U) \cdots$$

then take the alternating sum, as usual, it gives us the complex for any $U \to X$

$$\check{C}^\bullet(U/X, \mathscr{F}) : 0 \to \mathscr{F}(U) \to \mathscr{F}(U \times_X U) \to \mathscr{F}(U \times_X U \times_X U) \to \cdots,$$

then define the Čech complex

$$\check{C}^\bullet(X_{ét}, \mathscr{F}) := \varinjlim \check{C}^\bullet(U/X, \mathscr{F})$$

where the colimit is for the covering families.

**Definition 2.22.** Define the Čech cohomology to be

$$\check{H}^i(U/X, \mathscr{F}) := H^i(\check{C}^\bullet(U/X, \mathscr{F}))$$

and

$$\check{H}^i(X_{ét}, \mathscr{F}) := H^i(\check{C}^\bullet(X_{ét}, \mathscr{F})).$$

**Proposition 2.23.** $\check{H}^0(U/X, \mathscr{F}) = H^0(X_{ét}, \mathscr{F}) = H^0(X, \mathscr{F})$.

*Proof.* By sheaf condition. □

**Proposition 2.24.** $\check{H}^i(U/X, \mathscr{I}) = \check{H}^i(X_{ét}, \mathscr{I}) = 0$ *for any injective object* $\mathscr{I}$.

*Proof.* Enough to show that the complex $\check{C}^\bullet(U/X, \mathscr{I})$ is exact everywhere away from 0. Denote that $\mathbb{Z}_U = \mathbb{Z}[\mathrm{Hom}_X(-, U)]$. Then by the Yoneda Lemma and $\mathscr{I}$ is injective, the complex $\check{C}^\bullet(U/X, \mathscr{F})$ is isomorphic to

$$0 \to \mathrm{Hom}(\mathbb{Z}_U, \mathscr{I}) \to \mathrm{Hom}(\mathbb{Z}_{U\times_X U}, \mathscr{I}) \to \mathrm{Hom}(\mathbb{Z}_{U\times_X U\times_X U}, \mathscr{I}) \to \cdots.$$

The reason is that any morphism $\phi : \mathbb{Z}_U \to \mathscr{I}$ is determined by the image $\phi_U(\mathrm{id}) \in \mathscr{I}$ where $\mathrm{id} \in \mathbb{Z}[\mathrm{Hom}(U, U)]$. Since $\mathscr{I}$ is injective, then it is enough to prove the following sequence is exact

$$\mathbb{Z}_U \leftarrow \mathbb{Z}_{U\times_X U} \leftarrow \mathbb{Z}_{U\times_X U\times_X U} \leftarrow \cdots.$$

In fact, this is exact from a more general statement, let $S$ be a set then the sequence

$$\mathbb{Z}[S] \leftarrow \mathbb{Z}[S \times S] \leftarrow \mathbb{Z}[S \times S \times S] \leftarrow \cdots$$

is exact. □

**Theorem 2.25.** *For any short exact sequence in* $\mathrm{Sh}(X_{ét})$

$$0 \to \mathscr{F}_1 \to \mathscr{F}_2 \to \mathscr{F}_3 \to 0,$$

*if the induced Čech complex*

$$0 \to \check{C}^\bullet(U/X, \mathscr{F}_1) \to \check{C}^\bullet(U/X, \mathscr{F}_2) \to \check{C}^\bullet(U/X, \mathscr{F}_3) \to 0$$

*is exact, then* $\check{H}^i(X_{ét}, \mathscr{F}) = H^i(X_{ét}, \mathscr{F})$ *for all* $i > 0$ *and* $\mathscr{F}$.

*Proof.* This is the generalization of theorem 4.5 in [2]. The proof follows more or less the same, just to prove both two functors are the universal $\delta$-functor. □

**Theorem 2.26.** *The condition will be satisfied when $X$ is quasi-compact and any finite subset of $X$ is contained in some affine. (e.g. quasi-projective).*

*Proof.* See III in [6]. □

Note that on any site, the first cohomology group equals the first derived functor group. We write some crucial propositions without proof, see more on [6].

**Proposition 2.27.** *For a sheaf $\mathscr{F}$ on $X_{ét}$, let $\mathcal{H}^r(\mathscr{F})$ be the presheaf $U \mapsto H^r(U, \mathscr{F}|_U)$. For all $r > 0$, the sheaf associated with $\mathcal{H}^r(\mathscr{F})$ is 0.*

**Corollary 2.28.** *Let $s \in H^r(X, \mathscr{F})$ for some $r > 0$. Then there exists a covering $(U_i \to X)$ such that the image of $s$ in each group $H^r(U_i, \mathscr{F})$ is zero.*

Now we can define the isomorphism $H^1(X, \mathscr{F}) \to \check{H}^1(X, \mathscr{F})$. First, choose an injective embedding $\mathscr{F} \hookrightarrow \mathscr{I}$, and take $\mathscr{G}$ to be the cokernel. We have an exact sequence

$$0 \to \mathscr{F} \to \mathscr{I} \to \mathscr{G} \to 0,$$

which induces a long exact sequence

$$0 \to \mathscr{F}(X) \to \mathscr{I}(X) \to \mathscr{G}(X) \to H^1(X, \mathscr{F}) \to 0.$$

Let $s \in H^1(X, \mathscr{F})$, let $t \in \mathscr{G}(X)$ maps to $s$. According to the above corollary, there is a covering $U_i \to X$ such that $s$ restricts to zero on each $U_i$, and so $t|_{U_i}$ lifts to an element $\tilde{t}_i \in \mathscr{I}(U_i)$. Let $s_{ij} = \tilde{t}_j|_{U_{ij}} - \tilde{t}_i|_{U_{ij}}$ regarded as an element of $\mathscr{F}(U_{ij})$(just because $\mathscr{G}$ is a sheaf). One can check easily that $s_{ij}$ is a one-cocycle.

**Proposition 2.29.** *The map $s \mapsto (s_{ij})$ defines an isomorphism $H^1(X, \mathscr{F}) \to \check{H}^1(X.\mathscr{F})$.*

*Proof.* Chase diagrams. □

Now we start to construct the so-called Čech-to-derived spectral sequence to give an alternative proof of the last proposition. First, we state Grothendieck's theorem on the existence of spectral sequence.

**Theorem 2.30.** *Let $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ be abelian categories and assume that $\mathbf{A}$ and $\mathbf{B}$ have enough injectives. Let $F : \mathbf{A} \to \mathbf{B}$ and $G : \mathbf{B} \to \mathbf{C}$ be left exact functors, and assume that $(R^r G)(FI) = 0$ for $r > 0$ if $I$ is injective(if $F$ takes injectives to injectives). Then there is a spectral sequence*

$$E_2^{rs} = (R^r G)(R^s F)(A) \Rightarrow R^{r+s}(FG)(A).$$

*Proof.* In the sense of derived category, it is almost trivial. But one can also prove this using spectral sequence. □

When we compute the derived cohomology of $\mathscr{F}$, we take an injective resolution of $\mathscr{F}$,

$$\mathscr{F} \to \mathscr{I}^0 \to \mathscr{I}^1 \to \mathscr{I}^2 \to \cdots .$$

Also, a cover $U \to X$ gives us

$$\check{C}^\bullet(X_{\acute{e}t}, \mathscr{I}^0) \to \check{C}^\bullet(X_{\acute{e}t}, \mathscr{I}^1) \to \check{C}^\bullet(X_{\acute{e}t}, \mathscr{I}^2) \to \cdots$$

as a spectral sequence, then we have

$$E_2 = \check{H}^i(X_{\acute{e}t}, H^j(\mathscr{F})) \Rightarrow H^{i+j}(X_{\acute{e}t}, \mathscr{F}).$$

Why? Because the derived functors of the forgetful functor $i : \mathrm{Sh}(X_{\acute{e}t}) \to \mathrm{PreSh}(X_{\acute{e}t})$ are

$$\mathcal{H}^r(-) : \mathscr{F} \mapsto \{U \mapsto H^r(U, \mathscr{F})\}.$$

According to Corollary 2.28, $\check{H}^r(X_{\acute{e}t}, \mathcal{H}^s(\mathscr{F})) = 0$ for $s > 0$. Thus, for a sheaf $\mathscr{F}$,

$$\check{H}^r(X, \mathscr{F}) \cong H^r(X, \mathscr{F}) \quad \text{for } r = 0, 1.$$

and there is an exact sequence

$$0 \to \check{H}^2(X, \mathscr{F}) \to H^2(X, \mathscr{F}) \to \check{H}^1(X, \mathcal{H}^1(\mathscr{F})) \to \check{H}^3(X, \mathscr{F}) \to H^3(X, \mathscr{F}).$$

Similarly, for any étale covering $\mathcal{U} = (U_i \to X)$ of $X$, there exists a spectral sequence

$$\check{H}^r(\mathcal{U}, \mathcal{H}^s(\mathscr{F})) \Rightarrow H^{r+s}(X_{\acute{e}t}, \mathscr{F}).$$

Note that spectral sequence is a generalization of the Mayer-Victoris sequence. Let $U = U_0 \bigcup U_1$ where $U_i \subseteq U$ is a Zariski open subset. We have the following proposition.

**Proposition 2.31.** *There exists a functorial long exact sequence.*

$$\cdots \to H^s(U, \mathscr{F}) \to H^s(U_0, \mathscr{F}) \oplus H^s(U_1, \mathscr{F}) \to H^s(U_0 \cap U_1, \mathscr{F}) \to H^{s+1}(U, \mathscr{F}) \to \cdots$$

*Proof.* For the presheaf $\mathcal{H}^s(\mathscr{F})$, one can easily check that we have the following exact sequence

$$0 \to \check{H}^0(\mathcal{U}, \mathcal{H}^s(\mathscr{F})) \to H^s(U_0, \mathscr{F}) \oplus H^s(U_1, \mathscr{F}) \to H^s(U_0 \cap U_1, \mathscr{F}) \to \check{H}^1(\mathcal{U}, \mathcal{H}^s(\mathscr{F})) \to 0.$$

In the spectral sequence

$$\check{H}^r(\mathcal{U}, \mathcal{H}^s(\mathscr{F})) \Rightarrow H^{r+s}(X, \mathscr{F}).$$

Since the cover $\mathcal{U}$ only has two parts, we have $\check{H}(\mathcal{U}, \mathcal{H}^s(\mathscr{F})) = 0$ unless $r = 0, 1$. Hence, the filtration given by the spectral sequence is

$$0 \to \check{H}^1(\mathcal{U}, \mathcal{H}^s(\mathscr{F})) \to H^{s+1}(X, \mathscr{F}) \to \check{H}^0(\mathcal{U}, \mathcal{H}^{s+1}(\mathscr{F})) \to 0,$$

for all $s > 0$. Then we can splice the sequences together. □

**Theorem 2.32.** *Let $X$ be a scheme and $\mathscr{F} \in \mathrm{QCoh}(X)$. Then $H^i(X, \mathscr{F}) = H^i(X_{\acute{e}t}, \mathscr{F}^{\acute{e}t}) = H^i(X_{fppf}, \mathscr{F}^{fppf})$.*

*Proof.* We only prove it with a stronger assumption. Assume $X$ is quasi-compact and separated, also assume that Čech cohomology computes derived functor. These imply that every cover can be refined to a finite affine cover.

Suppose $X$ is affine, let $U \to X$ is an fppf affine cover. Claim that $\check{C}^\bullet(U/X, \mathscr{F})$ is exact if $\mathscr{F} = \widetilde{M}$ if a quasi-coherent sheaf. Equivalently, let $U = \mathrm{Spec}\ B$ and $X = \mathrm{Spec}\ A$, and $M$ is an fppf $A$-module, we have the following diagram

$$M \to M \otimes B \to M \otimes B \otimes B \to \cdots$$

is exact, which we have already mentioned in the descent section. This claim implies that the higher cohomology is vanishing and the zero cohomologies are the global sections for the affine situations. Again for $X$ is affine, the above discussion shows that

$$\check{H}^r(X_{\acute{e}t}, \mathscr{F}^{\acute{e}t}) = \begin{cases} \mathscr{F}(X) & r = 0 \\ 0 & r > 0. \end{cases}$$

The reason is that affine covers are cofinal in the inverse system taken by Čech cohomology, so every non-zero element must be killed by refining to an affine cover.

Now for an arbitrary scheme $X$, satisfying quasi-compact, separated and the Čech cohomology agrees with the derived functor cohomology, take an affine cover $\mathcal{U} \to X$, use Čech-to-derived spectral sequence. First, compute the Čech cohomology vertically on $E_0$, then the only non-zero row in $E_1$ is

$$0 \to \mathscr{F}(X) \to \mathscr{I}^0(X) \to \mathscr{I}^1(X) \to \cdots,$$

which degenerate in $E_2$, computing $H^\bullet(X, \mathscr{F})$. Notice that this spectral sequence converges to $H^\bullet(X_{\acute{e}t}, \mathscr{F}_{\acute{e}t})$ or $H^\bullet(X_{fppf}, \mathscr{F}_{fppf})$, it's done. □

*Remark* 2.33. One should notice that an important condition is that $\mathscr{F}$ is a quasi-coherent sheaf instead of a sheaf of abelian groups. For such $\mathscr{F}$ and $X$ is qcqs(*i.e.* quasi-separated and quasi-compact), we also have an informal relation

$$H^i(X_{zar}, \mathscr{F}) = \mathrm{Ext}^i_{\mathrm{Sh}(X_{zar})}(\underline{\mathbb{Z}}, \mathscr{F}) = \mathrm{Ext}^i_{\mathrm{QCoh(X)}}(\mathcal{O}_X, \mathscr{F}) = H^i(\mathrm{QCoh}(X), \mathscr{F}).$$

**Example 2.34.** Finally, we can compute some examples of étale cohomology. Let $X = \mathbb{P}^n_k$ and $\mathscr{F} = \mathcal{O}_X$, we have

$$H^i(X_{\acute{e}t}, \mathcal{O}_X^{\acute{e}t}) = \begin{cases} k & r = 0 \\ 0 & r > 0. \end{cases}$$

Now let $X$ be a quasi-projective variety over $\mathbb{F}_p$. Let's try to compute $H^i(X_{\acute{e}t}, \underline{\mathbb{F}_p})$. Our idea is to construct an exact sequence in which $\underline{\mathbb{F}_p}$ appears somewhere. In fact, we have

$$0 \to \underline{\mathbb{F}_p} \to \mathbb{G}_a \xrightarrow{x^p - x} \mathbb{G}_a \to 0.$$

where $\mathbb{G}_a = \mathrm{Hom}(-, \mathbb{A}^1) = \mathcal{O}_X^{\acute{e}t}$ and the third arrow means sending $f \in \mathcal{O}_U(U)$ to $f^p - f \in \mathcal{O}_U(U)$. Obviously, the sequence is left exact. Only need to prove that $\varphi : \mathbb{A}^1 \to \mathbb{A}^1, f \mapsto f^p - f$ is an epimorphism, *i.e.* given any $f \in \mathcal{O}_U(U)$, only need to solve $x^p - x = f$ étale-locally on $U$. Consider the following diagram,

$$\begin{tikzcd} U \times_{\mathbb{A}^1} \mathbb{A}^1 \arrow[r] \arrow[d] & \mathbb{A}^1 \arrow[d, "x^p - x"] \\ U \arrow[r] & \mathbb{A}^1, \end{tikzcd}$$

We have $\frac{\partial}{\partial x}(x^p - x) = px^{p-1} - 1 = -1$ is a unit, since it is over $\mathbb{F}_p$. Hence $\varphi$ is étale, and so is its base change. In the étale-neighborhood $U \times_{\mathbb{A}^1} \mathbb{A}^1$, $x^p - x = f$ has a solution.

So far, we have proved it's a short exact sequence, known as the Artin-Schreier exact sequence, so it gives us a long exact sequence of their cohomologies. Combining with Theorem 2.32, we have

$$\cdots \to H^i(X, \mathcal{O}_X) \xrightarrow{x^p - x} H^i(X, \mathcal{O}_X) \to H^{i+1}(X_{ét}, \underline{\mathbb{F}_p}) \to \cdots .$$

Once we know how to compute the usual derived cohomology, we can get the result we want. For example, let $X = \mathbb{A}^1 = \text{Spec } \mathbb{F}_p[t]$, we have

$$0 \to H^0(\mathbb{A}^1_{ét}, \mathbb{F}_p) \to \mathbb{F}_p[t] \xrightarrow{t \mapsto t^p - t} \mathbb{F}_p[t] \to H^1(\mathbb{A}^1_{ét}, \mathbb{F}_p) \to 0.$$

It turns out that étale cohomology over $\mathbb{F}_p$ with $\mathbb{F}_p$ coefficients behaves bad, because the $H^1(\mathbb{A}^1_{ét}, \mathbb{F}_p)$ is really huge. It is not finite generated.

*Remark* 2.35. Note that if $X$ is proper over $\mathbb{F}_p$, then $H^i(X_{ét}, \mathbb{F}_p)$ is finitely dimensional because of proper pushforward for coherent cohomology.

*Remark* 2.36. If we take $X = (\text{Spec } k)_{ét}$, then compute Čech cohomology is exactly the Galois cohomology.

## 2.3 Compute étale cohomology

Now fix a connected curve $C$ over an algebraically closed field of characteristic $p$, we are going to compute the $H^i(C_{ét}, \mathbb{Z}/l^n\mathbb{Z})$ for $l \neq p$. We know that $H^0$ agrees with global sections, so let us turn to $H^1$.

**Definition 2.37** ($\mathcal{G}$-torsor)**.** Let $\mathcal{G}$ be a sheaf of groups on $X_{ét}$. A $\mathcal{G}$-torsor is a sheaf $T \in \text{Sh}^{Sets}(X)$ with an action $\mathcal{G} \times T \to T$ such that

$$\mathcal{G} \times T \xrightarrow{(a, \pi_2)} T \times T$$

is an isomorphism. It also has another name called principal homogeneous space.

**Example 2.38.** In fact, $\mathcal{G}$ is a $\mathcal{G}$-torsor by multiplication(trivial torsor).

**Example 2.39.** Let $G$ be a finite group and $\mathcal{G} = \underline{G}$ is the constant sheaf. We claim that every $\mathcal{G}$-torsor is a Galois covering of $X$ with Galois group $G$. When $X$ is connected, the Galois covering of $X$ with $G$ is classified by the continuous homomorphisms $\pi_1(X, \bar{x}) \to G$ where $\bar{x}$ is a geometric point of $X$. Thus, we have

$$H^1(X_{ét}, \mathcal{G}) = \text{Hom}_{cont}(\pi_1(X, \bar{x}), G).$$

**Example 2.40.** Let $\mathbb{G}_m = \text{Hom}(-, \text{Spec } k[t, t^{-1}])$ which sends $U$ to $\mathcal{O}_U^\times(U)$. Note that $\mathbb{G}_m$ also has a group structure by multiplication. Let $\mathcal{L}$ be any line bundle, consider

$$\mathcal{L} \mapsto \mathcal{S}pec \bigoplus_{n \in \mathbb{Z}} \mathcal{L}^{\otimes n}.$$

Locally the relative spec is isomorphic to $\mathcal{O}_U \times_k \text{Spec } k[t, t^{-1}]$, so obviously it's a $\mathbb{G}_m$-torsor.

Why do we turn to define such a $\mathcal{G}$-torsor, or equivalently a principal homogeneous space for $\mathcal{G}$? In fact, we can translate $\check{H}^1(X_{ét}, \mathcal{G})$ into the language of $\mathcal{G}$-torsor.

**Theorem 2.41.** *Let $\mathcal{G}$ be a sheaf of groups on $X_{ét}$ and $\mathcal{U} = \{U_i\}_{i \in I}$ be a étale cover of $X$. We say that the cover $\mathcal{U}$ splits a $\mathcal{G}$-torsor $\mathcal{S}$ if $\mathcal{S}(U_i) \neq \varnothing$. There is a bijection from the isomorphism classes of $\mathcal{G}$-torsor split by $\mathcal{U}$ to $\check{H}^1(X, \mathcal{G})$.*

*Proof.* Choose $(s_i) \in \mathcal{S}(U_i)$, consider $s_i|_{U_{ij}}, s_j|_{U_{ij}} \in \mathcal{S}(U_{ij})$, since $\mathcal{S}$ is a $\mathcal{G}$-torsor, there exists a $g_{ij} \in \mathcal{G}(U_{ij})$ such that

$$s_i|_{U_{ij}} g_{ij} = s_j|_{U_{ij}}$$

for any $i, j \in I$. Moreover, such $(g_{ij})_{I\times I}$ satisfies the cocycle condition, *i.e.*

$$s_i|_{U_{ijk}} \cdot g_{ij}|_{U_{ijk}} \cdot g_{jk}|_{U_{ijk}} = s_k|_{U_{ijk}} = s_i|_{U_{ijk}} \cdot g_{ik}|_{U_{ijk}},$$

which implies the cocycle condition. Now we check that different choices of $(s_i)$ give the same cohomological class. Let $s_i' = s_i h_i$ where $h_i \in \mathcal{G}(U_i)$, then the corresponding $g_{ij}' = h_i g_{ij} h_j^{-1}$, so by definition it falls in the same cohomological class as $(g_{ij})_{I\times I}$. Denote this cohomological class by $c(S)$ where $c : S \mapsto c(S)$.

Let $\alpha : \mathcal{S} \to \mathcal{S}'$ be an isomorphism of $\mathcal{G}$-torsors. Choose $s_i \in \mathcal{S}(U_i)$, we have(omitting the restriction maps)

$$s_i g_{ij} = s_j \Rightarrow \alpha(s_i) g_{ij} = \alpha(s_j).$$

This shows that the map $c$ is well-defined on the isomorphism class of $\mathcal{G}$-torsors.

Now check $c$ is injective. Let $c(\mathcal{S}) = c(\mathcal{S}')$, fix $(s_i) \in \mathcal{S}(U_i)$. Let $t \in \mathcal{S}(X)$ be an arbitrary element. We have unique $g_i \in \mathcal{G}(U_i)$ for any $i \in I$ such that

$$t|_{U_i} = s_i g_i.$$

Since $t|_{U_i}|_{U_{ij}} = t|_{U_j}|_{U_{ij}}$, we have the cocycle $(g_{ij})$ given by $(s_i)$

$$g_{ij} = g_i g_j^{-1}.$$

Because $\mathcal{S}$ is a sheaf $t \mapsto (g_i)_I$ is bijective. The same statement is also true for $\mathcal{S}'$, by such two bijections we have a canonical map $\mathcal{S}(X) \to \mathcal{S}'(X)$. For $S(V)$ where $V \in \mathrm{Ob}(X_{\acute{e}t})$, only need to check the same thing for the étale cover $\{U_i \times_X V\}_I$.

The last thing is surjective. Let $(g_{ij})_{I\times I}$ be a representing element of an element in $\check{H}^1(\mathcal{U}/X, \mathcal{G})$ where $\mathcal{U} = \{U_i\}_I$ is an étale cover. Define the following sheaf $\mathcal{S}$: for any $V \to X$ étale, consider $V_i \to V$ is an étale cover of $V$ where $V_i = U_i \times_X V$, let $S(V)$ be the set of families $(g_i)_I, g_i \in \mathcal{G}(V_i)$ such that

$$g_i|_{V_{ij}} = g_{ij} g_j|_{V_{ij}}.$$

By definition $\mathcal{S}$ is a sheaf and a $\mathcal{G}$-torsor with $c(\mathcal{S}) = (g_{ij})_{I\times I}$. □

*Remark* 2.42. Note that Čech cohomology also makes sense for any sheaf of groups $\mathcal{G}$.

We have shown the interpretation of the first étale cohomology of a sheaf of groups in terms of $\mathcal{G}$-torsors. Now let's turn to a more specific example. Let $L_n(X_{zar})$ be the set of locally free sheaves of rank $n$ in regular sense.

**Theorem 2.43** (Hilbert 90)**.** *There are natural bijections*

$$L_n(X_{zar}) \leftrightarrow \check{H}^1(X_{zar}, \mathrm{GL}_n) \leftrightarrow H^1(X_{\acute{e}t}, \mathrm{GL}_n) \leftrightarrow H^1(X_{fppf}, \mathrm{GL}_n).$$

*Proof.* It's more or less the same as what we did in the previous theorem. We construct the bijection between $L(X_*)$ and $\check{H}^1(X_*, \mathrm{GL}_n)$ where $* = zar, et, fppf$. Now let $\mathcal{M}$ be a locally free sheaf of $\mathcal{O}_X$-module. No matter what $*$ is we can always find an open affine covering $\mathcal{U} = \{U_i\}_I$ of $X$ such that $\mathcal{M}|_{U_i} = \mathcal{O}_{U_i}^n$ for all $i \in I$. Then we can choose the "local trivialization" of $\mathcal{M}$ *i.e.*

$$\theta_i = \mathcal{M}|_{U_i} \to \mathcal{O}_{U_i}^n,\ \theta_{ij} := (\theta_i|_{U_{ij}}) \circ (\theta_j|_{U_{ij}})^{-1} \in \mathrm{GL}_n(U_{ij}).$$

One can check in the same method as the previous theorem that $(\theta_{ij})_{I\times I}$ is a one-cocycle and different choice of "local trivialization" gives the same cohomological class. Hence, we have bijection $L_n(X_*) \leftrightarrow \check{H}^1(X_*, \mathrm{GL}_n)$.

Now left to check that $L_n(X_*)$ is the same for different $*$. We need to show the maps $\mathcal{M} \mapsto M^{\acute{e}t}$ and $\mathcal{M} \mapsto M^{fppf}$ give bijections $L_n(X_{zar}) \to L_n(X_{\acute{e}t})$ and $L_n(X_{zar}) \to L_n(X_{fppf})$. Need to check the following things:

(a) every locally free $\mathcal{O}_{X_{fppf}}$-module is of the form $\mathcal{M}^{fppf}$;

(b) let $\mathcal{M}$ be an $\mathcal{O}_{X_{zar}}$-module, if $\mathcal{M}^{fppf}$ is locally free, then so is $\mathcal{M}$.

(c) for $\mathcal{M}$ and $\mathcal{N}$ in $L^n(X_{zar})$, $\mathcal{M} \cong \mathcal{N}$ if and only if $\mathcal{M}^{fppf} \cong \mathcal{N}^{fppf}$.

We start at (c). If $\mathcal{M} \cong \mathcal{N}$, then $\mathcal{M}^{fppf} \cong \mathcal{N}^{fppf}$ by definition. Conversely, it is a consequence of Theorem 1.46, because the isomorphism on the fppf site gives an isomorphism of descent data.

To prove (b) and (a) enough to prove the affine case, now we assume that $X = \text{Spec } A$. In fact, any finite generated $A$-module $M$ to be locally free is equivalent to say $M$ is projective. Now let $M$ be a finite generated $A$-module and $A \to B$ be a faithfully flat homomorphism. Note that M is projective $\iff$ $M \otimes_A B$ is projective, because of

$$B \otimes_A \text{Hom}_A(M, N) = \text{Hom}_B(B \otimes_A M, B \otimes_A N).$$

Now we need to translate a flat covering of $X$ into a faithfully flat algebra. Suppose that $\mathcal{U} = \{U_i\}_I$ is a flat covering where $M$ becomes free. One may refine it to make it to be an affine covering. Notice that $\varphi_i : U_i \to X$ is open, hence $\varphi_i(U_i)$ is open in $X$. Since $X$ is quasi-compact, we may assume that $I$ is finite. Then $Y = \coprod U_i$ is affine and we reduce everything to affine cases, which proves (b).

To prove (c), let $\mathcal{M}$ be a locally free sheaf on $X_{fppf}$, we can also assume a flat covering is replaced by a homomorphism $A \to B$ and assume on the flat neighborhood $B$, $\mathcal{M}$ becomes a sheaf associated with a module, say $N$. Recall the proof of Theorem 1.46, we only need to construct the descent data related to $\mathcal{M}$, just pull back the identity of $\mathcal{M}$ along different projections, which gives a descent data. Note that we prove more than what we need, we just prove that the categories of coherent sheaves on $X_*$ for $* = zar, et, fppf$ are equivalent. □

**Corollary 2.44.** *There is a canonical isomorphism* $H^1(X_{\acute{e}t}, \mathbb{G}_m) \cong \text{Pic}(X)$.

*Proof.* The Picard group is defined to be $L_1(X_{zar})$, Hence we have $\text{Pic}(X) \cong \check{H}^1(X_{\acute{e}t}, \mathbb{G}_m)$. Since $\mathbb{G}_m$ is abelian, we have $\check{H}^1(X_{\acute{e}t}, \mathbb{G}_m) = H^1(X_{\acute{e}t}, \mathbb{G}_m)$ by Proposition 2.29. □

*Remark* 2.45. Let $k$ be a field, the corollary reduces to the classical Hilbert 90, *i.e.*

$$H^1(Gal(k^s/k), \bar{k}^\times) = H^1((\text{Spec } k)_{\acute{e}t}, \mathbb{G}_m) = 0.$$

**Example 2.46.** Let $X$ be a scheme over an algebraically closed field $k$ where a prime number $l$ is invertible. Consider the Kummer sequence, which is

$$1 \to \mu_l \to \mathbb{G}_m \xrightarrow{z \mapsto z^l} \mathbb{G}_m \to 1.$$

Note that the morphism $\mathbb{G}_m \to \mathbb{G}_m, z \mapsto z^l$ is surjective in étale site. The long exact sequence derived from this gives us

$$1 \to H^1(X_{\acute{e}t}, \mu_l) \to \text{Pic}(X) \xrightarrow{[l]} \text{Pic}(X) \to \cdots.$$

Hence, we have $H^1(X_{\acute{e}t}, \mu_l) = \text{Pic}(X)[l]$, which is the $l$-torsion subgroup of $\text{Pic}(X)$. Note that $\underline{\mathbb{Z}/l\mathbb{Z}} \cong \mu_l$(depends on a choice of primitive $l$-th root of unity). Hence, we have

$$H^i(X_{\acute{e}t}, \mathbb{Z}/l\mathbb{Z}) \cong H^i(X_{\acute{e}t}, \mu_l).$$

**Example 2.47.** In the case of abelian groups, Čech cohomology agrees with derived cohomology in the first cohomology, also Theorem 2.41 shows the geometric interpretation of the first cohomology group, which in fact classifies all the torsor associated with the given group sheaf. One may ask how to write the corresponding torsor explicitly. For $\mu_l$, it is quite easy. For $[Y] = H^1(X_{\acute{e}t}, \mathbb{F}_p) = \text{coker}(\mathcal{O}_X \xrightarrow{x^p - x} \mathcal{O}_X)$, we can write $Y(U) := \{y^p - y = a\}$ where $a \in \mathcal{O}_X(U)$.

# 3 The Cohomology of Curves

The goal of this section is to compute the cohomology of curves. We assume all the curves are smooth over an algebraically closed field.

**Theorem 3.1.** *Let $X$ be a smooth curve over an algebraically closed field $k$. Then*

$$H^i(X_{\acute{e}t}, \mathbb{G}_m) = \begin{cases} \mathcal{O}_X(X)^\times & i = 0 \\ \mathrm{Pic}(X) & i = 1 \\ 0 & i > 1 \end{cases}$$

Only need to compute when $i > 1$. Due to a Grothendieck's theorem, the derived cohomology in the small Zariski site vanishes when $i > \dim(X)$, so for a curve the cohomology groups for $i > 1$ vanishes. However, the conclusion is no longer trivial for étale cohomology. Assuming this theorem, we have the following corollary.

**Corollary 3.2.** *Let $X$ be a smooth proper connected curve of genus $g$ over an algebraically closed field and fix a prime number $l \neq \mathrm{char}\ k$. Then*

$$H^i(X_{\acute{e}t}, \underline{\mathbb{Z}/l^n\mathbb{Z}}) = \begin{cases} \mathbb{Z}/l^n\mathbb{Z} & r = 0 \\ \mathrm{Pic}(X)[l^n] = (\mathbb{Z}/l^n\mathbb{Z})^{2g} & r = 1 \\ \mathbb{Z}/l^n\mathbb{Z} & i = 2 \\ 0 & i > 2 \end{cases}$$

*Proof.* We have to use a black box from the theory of abelian variety, we have the following exact sequence

$$0 \to \mathrm{Jac}(X) \to \mathrm{Pic}(X) \xrightarrow{\deg} \mathbb{Z} \to 0$$

where $\mathrm{Jac}(X) = \mathrm{Pic}^0(X)$ is the Jacobian of $X$ and $\mathrm{Jac}(X)$ itself is a $g$-dimensional abelian variety where $g$ is the genus of $X$ and for the $l^n$-torsion subgroup(as an abstract group), we have

$$\mathrm{Jac}(X)[l^n] \cong (\mathbb{Z}/l^n\mathbb{Z})^{2g}.$$

Consider the Kummer sequence, we have the long exact sequence

$$0 \to H^1(X_{\acute{e}t}, \underline{\mathbb{Z}/l^n\mathbb{Z}}) \to \mathrm{Pic}(X) \xrightarrow{[l]} \mathrm{Pic}(X) \to H^2(X_{\acute{e}t}, \underline{\mathbb{Z}/l^n\mathbb{Z}}) \to 0.$$

Hence,

$$H^1(X_{\acute{e}t}, \underline{Z/l^n\mathbb{Z}}) = \mathrm{Pic}(X)[l^n] = \mathrm{Jac}(X)[l^n] = (\mathbb{Z}/l^n\mathbb{Z})^{2g}.$$

and

$$H^2(X_{\acute{e}t}, \underline{\mathbb{Z}/l^n\mathbb{Z}}) = \mathrm{coker}(\mathrm{Pic}(X) \xrightarrow{[l^n]} \mathrm{Pic}(X)) = \mathrm{coker}(\mathrm{Jac}(X) \xrightarrow{[l^n]} \mathrm{Jac}(X)) = \mathbb{Z}/l^n\mathbb{Z}$$

where the second equality is implied by the Jacobian is divisible. □

*Remark* 3.3. These isomorphisms are not Galois equivariant.

We have three main ingredients to help us to prove Theorem 3.1.

(1) Leray spectral sequence;

(2) Divisor exact sequence;

(3) Brauer groups.

## 3.1 Pushforward and Leray spectral sequence

Recall that let $f : X \to Y$ be a morphism of schemes. This induces the pushforward

$$f_* : \mathrm{Sh}(X_{\acute{e}t}) \to \mathrm{Sh}(Y_{\acute{e}t}), \mathscr{F} \mapsto \{f_*\mathscr{F} : (U \to Y) \mapsto \mathscr{F}(U \times_Y X)\}.$$

This is a left exact functor because of the adjunction with $f^*$, so we can define the right derived functor

$$R^i f_* : \mathrm{Sh}^{ab}(X_{\acute{e}t}) \to \mathrm{Sh}^{ab}(Y_{\acute{e}t}).$$

In fact, it is the relative version of taking the global sections $\Gamma(X, -)$, so hopefully, it is expected to compute the cohomology along each fiber, which is unfortunately not true in general. One can prove that

$$R^i f_* \mathscr{F} = \text{sheafification of presheaf } V \mapsto H^i(f^{-1}(V), \mathscr{F}).$$

**Proposition 3.4.** *Assume that $f$ is finite(or closed immersion), then $R^i f_* = 0$ for all $i > 0$.*

*Proof.* The proposition is equivalent to saying that $f_*$ is also right exact, so enough to check on stalks. Since $f$ is finite, so for any sheaf $\mathscr{F}$ on $X$ and any geometric point $\bar{y} \in Y$, we have

$$(f_*\mathscr{F})_{\bar{y}} = \bigoplus_{\bar{x} \in f^{-1}(\bar{y})} \mathscr{F}_{\bar{x}},$$

so it is obvious. □

**Proposition 3.5.** *Let $X \to Y$ be a morphism of schemes. Then $f_*$ preserves injectives.*

*Proof.* This is true for any functor with an exact left adjoint, which is in this case $f^*$. □

**Corollary 3.6** (Leray spectral sequence)**.** *Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of schemes and $\mathscr{F}$ be a sheaf on $X$. Then one has the spectral sequence*

$$R^i g_* R^j f_* \mathscr{F} \Rightarrow R^{i+j}(g \circ f)_* \mathscr{F}.$$

*Particularly, for $Z = \mathrm{Spec}\ k$, we have*

$$H^i(Y, R^j f_* \mathscr{F}) \Rightarrow H^{i+j}(X, \mathscr{F}).$$

*Proof.* Proposition 3.5 plus Theorem 2.30. □

**Example 3.7.** Let $k$ be a field that is not algebraically closed and $X$ be a scheme over $k$. Leray spectral sequence gives us

$$H^i(k, R^j \pi_* \mathscr{F}) \Rightarrow H^{i+j}(X_{\acute{e}t}, \mathscr{F})$$

where $\pi$ is the projection map and $\mathscr{F}$ is any sheaf on $X$. We know that the left hand side can also be understood as Galois cohomology, so by Corollary 2.20 we have the left hand side is in fact the Galois cohomology

$$H^i(k, H^j((X_{k^s})_{\acute{e}t}, \mathscr{F})).$$

According to Galois cohomology, if $k$ is a finite field we have

$$H^i(k, V) = \begin{cases} V^G & i = 0 \\ V_G & i = 1 \end{cases}$$

and vanishes for higher dimension *i.e.* the cohomology dimension is 1. This will help us to count the cohomological dimension for any $X$ smooth projective over $k$.

*Remark* 3.8. Let $\pi : X \to Y$ be a smooth proper morphism of varieties over $\mathbb{C}$. In this case, one has

$$H^i(Y, R^j\pi_*\underline{\mathbb{Q}}) \Rightarrow H^{i+j}(X, \underline{\mathbb{Q}})$$

which is the singular cohomology. There is a fact that Deligne proved in Weil II, which says that this spectral sequence degenerated on page two.

**Proposition 3.9.** *Let $\pi : X \to Y$ be a morphism of schemes, then*

$$R^i f_*\mathscr{F} = \text{sheafification of presheaf } V \mapsto H^i(f^{-1}(V), \mathscr{F}).$$

*Proof.* Let $0 \to \mathscr{F} \to \mathscr{I}^\bullet$ be an injective resolution, then

$$R^i f_*\mathscr{F} = H^i(\pi_* I^\bullet) = H^i(a \circ \hat{\pi}_* \circ Forg\acute{e}t(I^\bullet)) = a \circ \pi_* H^i(Forg\acute{e}t(I^\bullet))$$

where $a$ is sheafification and $\hat{\pi}_*$ is pushforward as presheaf(both are exact). $\square$

**Example 3.10.** Let $X$ be an integral scheme, $\iota : \eta \to X$ be the generic point and $\mathscr{F}$ be a sheaf on $\eta_{\acute{e}t}$. Our goal is to understand $(R^i\iota_*\mathscr{F})_\eta = H^i(\text{Spec } K_\eta, \mathscr{F})$, where $K_\eta$ is the field of fractions of $\mathcal{O}_{X,\eta}$. Since the sheafification preserves stalks, then

$$(R^i\iota_*\mathscr{F})_\eta = \varinjlim (R^i\iota_*\mathscr{F})(U) = \varinjlim H^i(U_\eta, \mathscr{F}|_{U_\eta}).$$

We can set $\eta = \text{Spec } K$. The strictly local ring $\mathcal{O}_{X,\eta}$ is $K^{sep}$ because the normalization of $X$ in any finite extension $L$ over $K$ will be étale over $X$ on some nonempty open subset. Hence,

$$(R^0 g_*\mathscr{F})_\eta = M.$$

and vanishes for higher degrees, where $M$ is the corresponding discrete $G$-module with respect to $\mathscr{F}$.

## 3.2 Divisor exact sequence

Again, our goal is to understand $H^i(X, \mathbb{G}_m)$ for a curve $X$ over a separably closed field $k$ and $i > 1$. An idea is to reduce this question to questions in Galois cohomology.

**Proposition 3.11.** *Let $X$ be a regular variety over $k$ and $\eta \hookrightarrow X$ be a generic point. Then there is a short exact sequence in* $\text{Sh}(X_{\acute{e}t})$.

$$0 \to \mathbb{G}_m \to \eta_*\mathbb{G}_m \to \bigoplus_{\text{ht}(Z)=1} \iota_{Z*}\underline{\mathbb{Z}} \to 0.$$

*Proof.* Nothing but a fancy way to say, for a ring $A$ which allows Weil divisor, then one has

$$0 \to A^\times \to K^\times \to \bigoplus_{\text{ht}(p)=1} \mathbb{Z} \to 0,$$

where $K$ is the fraction field of $A$. Plus the fact that Cartier divisor agrees with Weil divisor if regular. This sequence of abelian groups is not exact, but as sheaves, the sequence is exact. $\square$

**Corollary 3.12.** *The above short exact sequence gives a long exact sequence.*

$$\cdots \to H^{i-1}(X_{\acute{e}t}, \oplus_{\text{codim}(Z)=1}\iota_{Z*}\underline{\mathbb{Z}}) \to H^i(X_{\acute{e}t}, \mathbb{G}_m) \to H^i(X_{\acute{e}t}, \eta_*\mathbb{G}_m) \to \cdots .$$

*Remark* 3.13. We don't need regular to make Weil agree with Cartier, just integral separated noetherian and locally factorial *i.e.* the local ring is UFD. See [2].

**Proposition 3.14.** *Let $X$ be a curve over a field $k$ such that $k = k^s$, then for $i > 0$,*

$$H^i(X_{\acute{e}t}, \oplus_{ht(Z)=1}\iota_{Z*}\underline{\mathbb{Z}}) = 0.$$

*Proof.* Equivalent to prove for any $Z$ be a closed subscheme of codimension 1, then $H^i(X_{\acute{e}t}, \iota_{Z*}\underline{\mathbb{Z}}) = 0$ for any $i > 0$. Consider Leray spectral sequence

$$H^i(X_{\acute{e}t}, R^j\iota_{Z*}\underline{\mathbb{Z}}) \Rightarrow H^{i+j}(Z_{\acute{e}t}, \iota_{Z*}\underline{\mathbb{Z}}).$$

By Proposition 3.5, we have $f_*$ is exact, *i.e.*

$$R^j\iota_{Z*}\underline{\mathbb{Z}} = \begin{cases} \iota_{Z*}\underline{\mathbb{Z}} & j = 0 \\ 0 & j > 0 \end{cases}$$

Hence, the spectral sequence degenerates on page one. This implies

$$H^i(X_{\acute{e}t}, \iota_{Z*}\underline{\mathbb{Z}}) = H^i(Z_{\acute{e}t}, \mathbb{Z}).$$

Notice that $Z$ is a single point and we are working on a separably closed field, so $H^0(Z_{\acute{e}t}, \mathbb{Z}) = \mathbb{Z}$ and vanishes for higher degrees. $\square$

**Corollary 3.15.** *Let $X$ be a smooth curve over $k$ with $k^s = k$. Then for $i > 1$,*

$$H^i(X_{\acute{e}t}, \mathbb{G}_m) \cong H^i(X_{\acute{e}t}, \eta_*\mathbb{G}_m).$$

*Proof.* Short exact sequence induces long exact sequence. $\square$

Our new goal is to compute the right hand side then, use Leray spectral sequence again,

$$H^i(X_{\acute{e}t}, R^j\eta_*\mathbb{G}_m) \Rightarrow H^{i+j}(\eta, \mathbb{G}_m).$$

We have

$$(R^j\eta_*\mathbb{G}_m)_{\bar{\eta}} = \varinjlim H^j(U_\eta, \mathbb{G}_m|_{U_\eta}) = \varinjlim H^j(\mathrm{Gal}(K^{sep}/L), K^{sep})$$

by Example 3.10. More general, if we replace the generic point by any closed point, we have the Galois cohomology of a fraction field of strictly henselization of a DVR. If the higher degree vanishes, then the Leray spectral sequence degenerates and we have

$$H^i(X_{\acute{e}t}, \eta_*\mathbb{G}_m) = H^i(\eta, \mathbb{G}_m) = 0.$$

for $i > 1$. So all we need is the following lemma(more general).

**Lemma 3.16.** *Let $K$ be the function field of a curve over an algebraically closed field or $K$ be equal to its strictly henselization associated with a geometric point of a curve over an algebraically closed field. Then for $i > 0$,*

$$H^i((\mathrm{Spec}\ K)_{\acute{e}t}, \mathbb{G}_m) = 0.$$

Note that Hilbert 90 shows the situation when $i = 1$. For $i = 2$, let us define the famous Brauer group.

## 3.3 Brauer group

**Definition 3.17** (Cohomological Brauer group)**.** Let $X$ be a scheme, then define the cohomological Brauer group to be

$$\mathrm{Br}^{coh}(X) = H^2(X_{\acute{e}t}, \mathbb{G}_m)_{tors}.$$

We can understand this geometrically in terms of $\mathrm{PGL}_n$-torsors. We have the following short exact sequence in $\mathrm{Sh}^{gp}(X_{\acute{e}t})$ by definition

$$1 \to \mathbb{G}_m \to \mathrm{GL}_n \to \mathrm{PGL}_n \to 1.$$

This gives us a long exact Čech cohomology sequence, particularly

$$\{\text{etale locally trivial } \mathrm{PGL}_n\text{−torsors}\} \cong \check{H}^1(X_{\acute{e}t}, \mathrm{PGL}_n) \to H^2(X_{\acute{e}t}, \mathbb{G}_m).$$

Let $L_n(X)$ denote the set on the left-hand side and $L := \bigcup L_n(X)$, we have a map

$$\delta : L \to H^2(X_{\acute{e}t}, \mathbb{G}_m).$$

**Definition 3.18** (Brauer group)**.** Define the Brauer group $\mathrm{Br}(X)$ to be the image of $\delta$.

To write $\delta$ explicitly in terms of Čech cohomology. Let $[T] \in L_n(X) = \check{H}^i(X_{\acute{e}t}, \mathrm{PGL}_n)$, assume that $T$ is split by an étale cover $\mathcal{U} \to X$. $T$ gives a section $s \in \Gamma(\mathcal{U} \times_X \mathcal{U}, \mathrm{PGL}_n)$ satisfying the cocycle condition. After refining $\mathcal{U}$, $s$ can be lifted to $\Gamma(\mathcal{U} \times \mathcal{U}, \mathrm{GL}_n)$, but may not satisfy the cocycle condition, but we can measure the obstruction

$$\pi_{23}^* s \cdot \pi_{12}^* s \cdot (\pi_{13}^* s)^{-1} \in \Gamma(\mathcal{U} \times_X \mathcal{U} \times_X \mathcal{U}, \mathbb{G}_m),$$

and this defines an element $\delta([T]) \in H^2(X_{\acute{e}t}, \mathbb{G}_m)$. Hence, we have a slogan: "$\delta([T])$, which is called Brauer class, measures the obstruction to lifting $T$ to a $\mathrm{GL}_n$-torsor".

On the geometric side, there is a general principle: one can construct a natural bijection on any site

$$\{\text{Locally split } G\text{−torsors}\} \longleftrightarrow \{\text{Sheaves on } X_{\acute{e}t} \text{ locally isomorphic to T}\}.$$

Give an element $F$ on right hand side, define $F \mapsto \underline{\mathrm{Isom}}(F, T)$, conversely given a $G$-torsor $\tau$ that is locally split, then define $\tau \mapsto a((\tau \times T)/G)$, where $a$ is the sheafification. A concrete example is that we stated in Theorem 2.43,

$$\{\mathrm{GL}_n\text{−torsors}\} \longleftrightarrow \{\text{Vector bundles}\}.$$

**Exercise 3.19.** For any scheme $X$, one can prove

$$\mathrm{Aut}_X(\mathbb{P}_X^{n-1}) = \mathrm{PGL}_n.$$

A hint is to use the formal function theorem.

**Corollary 3.20.** *On étale site, one has*

$$\{\mathrm{PGL}_n\text{−torsors}\} \longleftrightarrow \{\text{Schemes over X locally isomorphic to } \mathbb{P}_{\mathrm{X}}^{\mathrm{n}-1}\}.$$

In fact, we can describe what we did above in another definition.

**Definition 3.21** (twisted sheaf)**.** Let $[\alpha] \in \check{H}^2(X_{\acute{e}t}, \mathbb{G}_m)$. An $\alpha$-twisted sheaf on an étale cover $\mathcal{U}$ is a quasi-coherent sheaf $\mathscr{F}$, such that there is an isomorphism $\varphi : \pi_1^*\mathscr{F} \xrightarrow{\sim} \pi_2^*\mathscr{F}$, which satisfies the cocycle condition up to $\alpha$, *i.e.*

$$\pi_{23}^*\varphi \circ \pi_{12}^*\varphi \circ \pi_{13}^* = \alpha.$$

We define a category $\mathrm{QCoh}(X, \alpha)$. The objects are all $\alpha$-twisted sheaves and the morphisms are morphisms commuting with $\varphi$.

**Example 3.22.** $\mathrm{QCoh}(X, 1) = \mathrm{QCoh}(X)$ by étale descent.

**Proposition 3.23.** *Let $\alpha, \alpha'$ are 2-cocycles for $\mathbb{G}_m$,*

*(1)* $\mathrm{QCoh}(X, 1) = \mathrm{QCoh}(X)$

*(2)* $[\alpha] \in \mathrm{Br}(X) \Leftrightarrow$ *there exists $\alpha$-twisted sheaf.*

*(3)* $\mathrm{QCoh}(X, \alpha)$ *is an Abelian category with enough injectives(if $X$ is good enough).*

*(4)* $\otimes : \mathrm{QCoh}(X, \alpha) \times \mathrm{QCoh}(X, \alpha') \to \mathrm{QCoh}(X, \alpha + \alpha')$

*(5)* $\mathrm{Hom} : \mathrm{QCoh}(X, \alpha) \times \mathrm{QCoh}(X, \alpha') \to \mathrm{QCoh}(X, \alpha' - \alpha)$.

*(6)* $\mathrm{Sym}^n, \bigwedge : \mathrm{QCoh}(X, \alpha) \to \mathrm{QCoh}(X, n\alpha)$.

*Proof.* Exercise. □

**Corollary 3.24.** $\mathrm{Br}(X)$ *is a subgroup of* $H^2(X_{\acute{e}t}, \mathbb{G}_m)$.

*Proof.* Let $\alpha, \alpha' \in \mathrm{Br}(X)$, then by the previous proposition we have $\mathscr{E}, \mathscr{E}'$ to be the $\alpha, \alpha'$-twisted sheaf. Then consider $\mathscr{E} \otimes_X \mathscr{E}' \in \mathrm{QCoh}(X, \alpha+\alpha')$ is an $\alpha+\alpha'$-twisted sheaf, hence $[\alpha+\alpha'] \in \mathrm{Br}(X)$. Similarly, to check $-\alpha \in \mathrm{Br}(X)$, equivalent to check $\check{\mathscr{E}} \in \mathrm{QCoh}(X, -\alpha)$. □

**Proposition 3.25.** *Let $\alpha$ is a 2-cocycle for $\mathbb{G}_m$. then $[\alpha]$ is trivial is equivalent to saying there exists an $\alpha$-twisted line bundle.*

*Proof.* If $[\alpha]$ is trivial, then any line bundle can be an $\alpha$-twisted line bundle, saying $\mathcal{O}_X$. Conversely, if there exists an $\alpha$-twisted line bundle $L$, then it gives you a descent data, which satisfies the cocycle condition, implying $[\alpha]$ is trivial. □

**Corollary 3.26.** *Suppose $\mathscr{E}$ is an $\alpha$-twisted vector bundle of rank $n$, Then $[\alpha] \in H^2(X_{\acute{e}t}, \mathbb{G}_m)$ is $n$-torsion. Moreover, $\mathrm{Br}(X) \subseteq \mathrm{Br}^{coh}(X) := H^2(X_{\acute{e}t}, \mathbb{G}_m)_{tors}$.*

*Proof.* Use Proposition 3.23(6), we have $\bigwedge^n \mathscr{E}$ to be a $n\alpha$-twisted line bundle, Hence $[n\alpha] = n[\alpha]$ is trivial. □

**Example 3.27.** Let $X$ be the zero loci of $x^2+y^2+z^2 = 0$ in $\mathbb{P}^2_{\mathbb{R}}$. we know that $X$ is a smooth projective curve with no rational points. If $X$ has a rational point then it's isomorphic to $\mathbb{P}^1_{\mathbb{R}}$, hence $X_{\mathbb{C}} \cong \mathbb{P}^1_{\mathbb{C}}$, which implies that $X$ is a scheme over $\mathbb{R}$ and étale-locally isomorphic to $\mathbb{P}^1_{\mathbb{R}}$. In fact, $\delta([X])$ generates $\mathrm{Br}(\mathbb{R}) = \mathbb{Z}/2\mathbb{Z}$.

**Example 3.28** (TBA). $\mathrm{Br}(\mathbb{Q}_p) = \mathbb{Q}/\mathbb{Z}$.

**Example 3.29** (TBA).

$$0 \to \mathrm{Br}(\mathbb{Q}) \to \bigoplus_{p \text{ place}} \mathrm{Br}(\mathbb{Q}_p) \to \mathbb{Q}/\mathbb{Z} \to 0.$$

## 3.4 Finish the computation

To finish the proof of Theorem 3.1, the last thing we need is to give a proof of Lemma 3.16. In fact, we have the following theorem

**Definition 3.30.** a field $k$ is said to be quasi-algebraically closed if every nonconstant homogeneous polynomial $f(T_1, \ldots, T_n) \in k[T_1, \ldots, T_n]$ of degree $d < n$ has a nontrivial zero in $k^n$.

**Theorem 3.31** (Tsen). *Suppose $k$ is a quasi-algebraically close field, then $\mathrm{Br}(k) = 0$.*

**Example 3.32.** the following fields are quasi-algebraically close fields

(a) a function field of a curve over an algebraically closed field;

(b) the field of fractions $K$ of a Henselian discrete valuation ring $R$ with algebraically closed residue field provided that the completion of $K$ is separable over $K$.

**Definition 3.33** (reduced norm). Let $\mathscr{E}$ be an $\alpha$-twisted sheaf, define the reduced norm

$$\mathrm{Nm} : \mathcal{E}nd(\mathscr{E}) \to \mathcal{E}nd(\bigwedge^{top} \mathscr{E}) = \mathcal{O}_X$$

be the map induced by $\bigwedge$.

**Proposition 3.34.** *Given $f \in \mathcal{E}nd(\mathscr{E})$, then $f$ is invertible if and only if $\mathrm{Nm}(f)$ is a unit.*

*Proof.* Check locally, it becomes a matrix. □

*Proof of Tsen's thoerem.* Our goal is that given any $[\alpha] \in \mathrm{Br}(k)$, we can find an $\alpha$-twisted sheaf that is a line bundle. By the previous discussion, we can realize $\alpha$ as an $\alpha$-twisted vector bundle $\mathscr{E}$. Trivial if $\mathrm{rank}(\mathscr{E}) = 1$ by Proposition 3.25. We want to find some nontrivial subbundle of $\mathscr{E}$ if $\mathrm{rank}(\mathscr{E}) > 1$, *i.e.* find $f \in \mathcal{E}nd(\mathscr{E})$ which $\mathrm{Nm}(f) = 0$.

Étale locally, $\mathcal{E}nd(\mathscr{E})$ is an affine $\mathrm{rank}(\mathscr{E})^2$-dimensional affine space after base change to a larger field. Then Nm becomes a polynomial function in $\mathrm{rank}(\mathscr{E})^2$ variables of degree $\mathrm{rank}(\mathscr{E})$. Since $k$ is a quasi-algebraically closed field and $\mathrm{rank}(\mathscr{E}) > 1$, Nm has a nontrivial zero, *i.e.* there exists a nontrivial $f \in \mathcal{E}nd(\mathscr{E})$ such that $\mathrm{Nm}(f) = 0$. Let $\mathscr{E}' = \ker(f)$, which is an $\alpha$-twisted vector bundle of rank $< \mathrm{rank}(\mathscr{E})$, Use induction. □

**Corollary 3.35.** *Let $k$ quasi-algebraically closed, then $H^2(k, \mathbb{G}_m) = 0$.*

*Proof.* For a field, one have $\mathrm{Br}(k) = H^2(k, \mathbb{G}_m)$. We do not give it proof here and this is true in a lot of situations. □

**Proposition 3.36.** *Let $k$ be a function field of a curve $C$ over an algebraically closed field, then $k$ is quasi-algebraically closed.*

*Proof.* Given a $f \in k[x_1, \dots, x_n]$ homogeneous such that $\deg(f) < n$, we want to find a nontrivial solution of $f$ in $k^n$. Choose an ample divisor $D$ on $C$ and $m \in \mathbb{Z}$, then $f$ induces a map

$$\Gamma(C, \mathcal{O}(mD))^n \xrightarrow{f} \Gamma(C, \mathcal{O}((\deg f)mD) + D')$$

where $D'$ is contributed by the coefficient of $f$. Denote this map to be $\hat{f} : X \to Y$ and we know that because of the Riemann-Roch theorem if we take $m$ large enough, then the dimension of $X$ and $Y$ are linear about $m$, to be clear, $X$ is an affine $mn$-space and $Y$ is more or less affine $(\deg f)m$-space. Hence, for $m$ large enough, we have $\dim X > \dim Y$, this implies that any non-empty fiber has a positive dimension. Plus $f^{-1}(0)$ is non-empty since $f$ is homogeneous. □

In the case of Example 3.32(c), it is a pity that we cannot prove it here, we admit it. Please see Lang's thesis.

**Theorem 3.37.** *Let $K$ be a field in Example 3.32, then $H^i(K, \mathbb{G}_m) = 0$ for all $i > 0$.*

*Scratch of the proof.* This is again by virtue of étale cohomology is Galois cohomology, our goal is to prove

$$H^i(L/K, \mathbb{G}_m) = H^i(\mathrm{Gal}(L/K), L^\times) = 0$$

for all finite separable extensions of $K$. For $i = 1, 2$ it's immediately from Hilbert 90 and Tsen's theorem and inflation-restriction(Group cohomology).

First, assume that $L/K$ is cyclic *i.e.* $\mathrm{Gal}(L/K)$ is cyclic, then the cohomology is 2-periodic, it's done. Then if $L/K$ is solvable, use inflation and restriction exact sequence. For general $L/K$, since all $p$-groups are solvable, for all $G_p \subseteq \mathrm{Gal}(L/K)$ $p$-sylow group, use corestriction and restriction. □

By virtue of this theorem, we can go back to the discussion at the end of Section 3.2, since $(R^i\eta_*\mathbb{G}_m)_{\bar{x}} = 0$ for any point and $i > 0$, one can imply that all the higher image vanishes. Eventually, we prove that for $i > 1$

$$H^i(X_{\acute{e}t}, \eta_*\mathbb{G}_m) = H^i(\eta, \mathbb{G}_m) = H^i(k(C), \mathbb{G}_m) = 0,$$

which completes the proof of Theorem 3.1 and Corollary 3.2.

*Remark* 3.38. If you look at the Riemann surfaces of genus $g$, then the singular cohomology with coefficients in $\mathbb{Z}/l^n\mathbb{Z}$ looks exactly the same as what we have computed. And this becomes the first evidence that étale cohomology is a good one and a good replacement for singular cohomology in algebraic geometry. Another example would be $\mathbb{A}^2 - \{0\}$ and $\mathbb{C}^2 - \{0\}$, they have the same cohomology étale wise and singular wise.

# 4 TBA

## 4.1 Cohomology with compact support

Let $j : U \to X$ be an open embedding, for $\mathscr{F} \in \mathrm{Sh}(U_{ét})$ the support of $j_*\mathscr{F}$ may not be $U$ (this is true when $j$ is a closed immersion), hence we define the following thing.

**Definition 4.1** (Extension by zero)**.** Let $j : U \to X$ be an open embedding, we define $j_! : \mathrm{Sh}^{ab}(U) \to \mathrm{Sh}^{ab}(X)$ to be the sheafification of

$$V \to \begin{cases} \mathscr{F}(V \times_X U) & \text{if } \mathrm{Im}(V) \subseteq U \\ 0 & \text{otherwise.} \end{cases}$$

*Remark* 4.2. $j_!$ exists for more general morphisms but non-trivial.

**Proposition 4.3.** *$j_!$ is left adjoint to $j^*$.*

*Proof.* Check this on presheaves and use the adjoint property of sheafification. □

**Proposition 4.4.** *Let $\bar{x} \in X$ be a geometric point of $X$, then*

$$(j_!\mathscr{F})_{\bar{x}} = \begin{cases} \mathscr{F}_{\bar{x}} & \bar{x} \in \mathrm{Im}(j) \\ 0 & \bar{x} \notin \mathrm{Im}(j) \end{cases}$$

*Proof.* By definition. □

**Corollary 4.5.** *$j_!$ is exact.*

*Proof.* Check on stalks. □

**Proposition 4.6.** *Let $\mathscr{F} \in \mathrm{Sh}^{ab}(X_{ét})$, let $j : U \to X$ be an open embedding, and let $Z$ be the complement of $U$ in $X$ and $\iota : Z \to X$ be the inclusion. Then one has the following exact sequence*

$$0 \to j_!j^*\mathscr{F} \to \mathscr{F} \to \iota_*\iota^*\mathscr{F} \to 0.$$

*Proof.* Check on stalks again. Choose a geometric point $\bar{x} \in X$, then either $\bar{x} \in U$ or $\bar{x} \in Z$. If $\bar{x} \in U$, then the sequence becomes

$$0 \to \mathscr{F}_{\bar{x}} \to \mathscr{F}_{\bar{x}} \to 0 \to 0.$$

Otherwise, the sequence becomes

$$0 \to 0 \to \mathscr{F}_{\bar{x}} \to \mathscr{F}_{\bar{x}} \to 0.$$

Exactness on every stalk implies exactness of sheaves. □

**Definition 4.7** (Cohomology with compact support)**.** Let $\mathscr{F}$ be an abelian sheaf on étale site of $U$, $j : U \to X$ is an open embedding and $X$ is proper.

$$H^i_c(U_{ét}, \mathscr{F}) := H^i(X_{ét}, j_!\mathscr{F}).$$

There are two obvious questions in this definition: Why does $X$ exist (*i.e.* a compactification)? and Why is this definition independent from $j_!X$?

**Theorem 4.8** (Nagata)**.** *Let $j : U \to S$ be a separated morphism, then there exists an open embedding $f : U \to X$ with $\bar{j} : X \to S$ proper and $j = \bar{j} \circ f$.*

Hence, we don't need to worry about the existence of compactification. Or if we work on varieties, we can simply take the closure of quasi-projective varieties to be its compactification. As for the second question, the independence for torsion sheaves will require proper base change.

**Proposition 4.9.** *Let $U$ be a connected regular curve over an algebraically closed field $k$ with its characteristic not dividing $n$. Then there is a canonical isomorphism*

$$H^2_c(U_{\acute{e}t}, \mu_n) \cong \mathbb{Z}/n\mathbb{Z}.$$

*Proof.* Let $j : U \to X$ be the canonical regular compactification, we want to compute $H^i(X, j_!\mu_n)$. Using the short exact sequence in this case, we have

$$0 \to j_!j^*\mu_n \to \mu_n \to \iota_*\iota^*\mu_n \to 0.$$

In the case, $j^*\mu_n$ is just restricting $\mu_n$ on $X$ to $U$, which $\mu_n$ on $U$. And $\iota_*\iota^*\mu_n$ is just the direct sum of skyscraper sheaves on $X$. This gives us the long exact sequence

$$\cdots \to H^i(X, \mu_n) \to H^i(X, \iota_*\iota^*\mu_n) \to H^{i+1}_c(U, \mu_n) \to \cdots .$$

The cohomology with coefficient in $\iota_*\iota^*\mu_n$ is easy to compute. By Leray spectral sequence, one has

$$H^i(X, R^j\iota_*\iota^*\mu_n) \Rightarrow H^{i+j}(Z, \mu_n).$$

where $Z$ is the complement of $U$ in $X$, which is finitely many discrete points. We know that $H^i(Z, \mu_n) = 0$ for $i > 0$. By the previous result, we know that for closed embedding (or any finite morphism) pushforward $\iota_*$ is exact, Hence $H^i(\iota_*\iota^*\mu_n) = 0$ for $i > 0$. And obviously

$$H^0(X, \iota_*\iota^*\mu_n) = \bigoplus \mu_n(k).$$

By definition, $H^0_c(U, \mu_n) := H^0(X, j_!\mu_n) = \Gamma(X, j_!\mu_n)$, which have to be 0. So we gain the following exact sequence

$$0 \to \mu_n(k) \to \bigoplus \mu_n(k) \to H^1_c(U, \mu_n) \to (\mathrm{Pic}\ X)[n] \to 0$$

and

$$0 \to H^2_c(U, \mu_n) \to H^2(X, \mu_n) \to 0.$$

We obtain more than we ask. □

**Definition 4.10.** Let $\mathscr{F} \in \mathrm{Sh}(X_{\acute{e}t})$. We say $\mathscr{F}$ is constructible if

(a) For every closed embedding $i : Z \to X$, there exists an non-empty open $U \subset Z$ such that $(i^*\mathscr{F})|_U$ is locally constant, *i.e.* there exists a cover $V \to U$ such that $(i^*\mathscr{F})|_V$ is constant.

(b) stalks of $\mathscr{F}$ is finite.

The reason we introduce constructible sheaves is that the class of locally constant sheaves is not stable under the formation of direct images, even by proper maps (not even closed immersions), so we enlarge this class. See [7]. In [7], the proper base change theorem in topology is introduced, which I think is very helpful. If you don't know what the proper base change theorem is talking about, I strongly recommend you to have a look.

**Example 4.11.** Let $j : U \to X$ be an open embedding, then $j_!\mu_n$ is constructible.

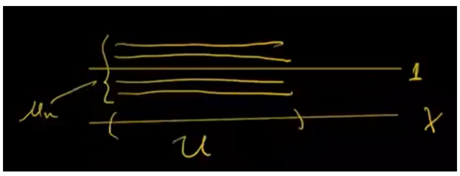


Figure 2: Picture of $\mu_n$

**Example 4.12.** Let $\mathscr{F}$ be represented by a quasi-finite $X$-scheme.

**Theorem 4.13.** *Let $f : X_{\acute{E}t} \to X_{\acute{e}t}$ be the natural map of sites and $\mathscr{F} \in \mathrm{Sh}(X_{\acute{E}t})$, then*

*(1) $f^* f_* \mathscr{F} \to \mathscr{F}$ is an isomorphism.*

*(2) $f_* \mathscr{F}$ constructible.*

This is hard to prove, and we will not touch the proof. But it tells us some interesting facts. First, any sheaf of big étale site can be the pullback of some constructible sheaf on small étale site. Since constructible sheaves $\mathscr{F}$ on small étale site are represented by a quasi-finite scheme, so is $f^*\mathscr{F}$, which means that every sheaf on big étale site is representable. A warning here is that a quasi-finite scheme is generally not étale.

**Theorem 4.14.** *Let $\pi : X \to S$ be a proper morphism, and $\mathscr{F} \in \mathrm{Sh}^{ab}(X_{\acute{e}t})$. Then $R^i\pi_*\mathscr{F}$ are constructible for $i \geq 0$ and*

$$(R^i\pi_*\mathscr{F})_{\bar{s}} = H^i(X_{\bar{s}}, \mathscr{F}_{X_{\bar{s}}})$$

*for all geometric points $\bar{s} \in S$.*

The key idea of the proof is this: First reduce the case to the relative curve, and use Devissage to reduce to the case where $\mathscr{F} = \mu_n$. Now assume $\pi : X \to S$ to be a relative curve, after applying derived functor to the Kummer sequence, we have

$$0 \to \pi_*\mu \to \pi_*\mathbb{G}_m \to \pi_*\mathbb{G}_m \to R^1\pi_*\mu_n \to R^1\pi_*\mathbb{G}_m \to R^1\pi_*\mathbb{G}_m \to R^2\pi_*\mu_m \to 0.$$

Our goal is to show that $\pi_*\mu_n, R^1\pi_*\mu_n, R^2\pi_*\mu_m \to 0$ is represented by quasi-finite $S$-schemes. In fact, $R^1\pi_*\mathbb{G}_m = \mathrm{Pic}(X/S)$ is represented by a $S$-scheme locally of finite type, which is called Picard scheme due to Grothendieck. Precisely, the Picard functor is defined as follows

$$\mathrm{Pic}(X/S)(T) = \text{sheafification of}(\{\text{line bundles on } \mathrm{X_T}/\}/\pi_T^*\{\text{line bundle on T}\}).$$

Now we have

$$R^1\pi_*\mu_n = \ker(\mathrm{Pic}(X/S) \xrightarrow{[n]} \mathrm{Pic}(X/S))$$

and

$$R^2\pi_*\mu_n = \mathrm{coker}(\mathrm{Pic}(X/S) \xrightarrow{[n]} \mathrm{Pic}(X/S)),$$

this will result in the fact that they are quasi-finite.

**Corollary 4.15.** *Let $X$ be a proper scheme over a separably closed field, then*

*(a) $H^i(X_{\acute{e}t}, \mathscr{F})$ is finite.*

*(b) Let L be an extension of separably closed fields, then*

$$H^i(X_{\acute{e}t}, \mathscr{F}) \cong H^i(X_{L,et}, \mathscr{F}|_{X_{L,\acute{e}t}})$$

*Proof.* (a) is directly implied by the fact that $R^i\pi_*\mathscr{F}$ is constructible on $(\mathrm{Spec}\ k)_{\acute{e}t}$. (b) Consider $(R^i\pi_*\mathscr{F})_{\mathrm{Spec}\ L \to \mathrm{Spec}\ k}$, and $(R^i\pi_*\mathscr{F})_{\mathrm{Spec}\ k}$ is already a constant sheaf, so is the latter one. □

**Example 4.16** (Non-example)**.** Let $X = \mathbb{A}^1_{\mathbb{F}_p}$ and $\mathscr{F} = \mathbb{Z}/p\mathbb{Z}$, we have computed the $H^1(X_{\acute{e}t}, \mathbb{Z}/p\mathbb{Z})$ is infinite.

**Corollary 4.17.** *Let $\pi : X \to S$ and $f : S' \to S$, then we have the following Cartesian square.*

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "\pi'"] & X \arrow[d, "\pi"] \\ S' \arrow[r, "f"'] & S \end{tikzcd}$$

*For any $\mathscr{F} \in \mathrm{Sh}^{ab}(X_{ét})$, there is a natutal map*

$$f^*(R^i\pi_*\mathscr{F}) \to R^i\pi'_*f'^*\mathscr{F},$$

*If $\pi$ is proper and $\mathscr{F}$ is torsion, this map is an isomorphism.*

*Proof.* We first define this map, then check on stalks. To define a morphism between $f^*(R^i\pi_*\mathscr{F}) \to R^i\pi'^*f'^*\mathscr{F}$, suffice to define $R^i\pi_*\mathscr{F} \to f_*R^i\pi'^*f'^*\mathscr{F}$, since $f^*$ and $f_*$ are adjoint. By id $\to f'_* \circ f'^*$, we have $R^i\pi_* \to R^i\pi_* \circ f'_* \circ f'^*$. The Leray spectral sequence gives us an injection $R^i\pi_* \circ f'_* \circ f'^* \to R^i(\pi \circ f')_*f'^* = R^i(f \circ \pi)_* \circ f'^*$ and a surjection $R^r(f \circ \pi')_* \circ f'^* \to f_* \circ R^r\pi'_* \circ f'^*$. By composing all three maps, we have the map we want. One can see the map agrees with the one in theorem 4.14 on stalks, hence isomorphism on stalks. □

*Remark* 4.18. Note that a torsion sheaf is the union (filtered colimit) of its constructible subsheaves.

*Remark* 4.19. Corollary 4.17 is theorem 4.14 if takes $T$ to be a geometric point of $S$.

**Example 4.20.** Let $\pi : X \to S$ be a smooth proper morphism. By the Leray spectral sequence, we have

$$H^i(S_{ét}, R^j\pi_*\mathbb{Z}/n\mathbb{Z}) \Rightarrow H^{i+j}(X_{ét}, \mathbb{Z}/n\mathbb{Z}).$$

On the left-hand side, we know that $R^j\pi_*\mathbb{Z}/n\mathbb{Z}$ is constructible and its stalks are $H^j(X_{\bar{s}}, \mathbb{Z}/n\mathbb{Z})$, of which the order does not depend on $\bar{s}$. This shows that it is a constant sheaf. Even though we don't require $\pi$ to be smooth, we know that smoothness is an open condition, so it is a locally constant sheaf on the locus of smoothness.

**Theorem 4.21.** *Let $U$ be a separated scheme and $\mathscr{F}$ constructible sheaf on $U$. Then the cohomology with compact support*

$$H_c(U, \mathscr{F}) := H^i(X_{ét}, j_!\mathscr{F}),$$

*where $j : U \to X$ is an open embedding and $X$ is proper, doesn't depend on the choice of $X$.*

*Proof.* Let $j_1 : U \to X_1$ and $j_2 : U \to X_2$ be such two maps. Let $X$ be the closure of the image of $U \xrightarrow{(j_1,j_2)} X_1 \times X_2$. Now we have the following

$$\begin{tikzcd} U \arrow[r, "j"] \arrow[dr, "j_1"'] & X \arrow[d, "\pi"] \\ & X_1 \end{tikzcd}$$

where $X, X_1$ is proper and $j, j_1$ is open embedding. Note that $\pi$ is proper because it is a map between proper schemes. Again by the Leray sequence, we have

$$H^r(X_1, R^s\pi_*j_!\mathscr{F}) \Rightarrow H^r(X, j_!\mathscr{F}).$$

By proper base change, we have

$$(R^s\pi_* j_!\mathscr{F})_{\bar{x}} = H^s(\pi^{-1}(\bar{x})_{ét}, j_!\mathscr{F}),$$

If $\bar{x} \in \mathrm{Im}(j)$, it's the cohomology of a geometric point, if not the stalk will be zero. Hence, the stalk is zero for $s > 0$. Our spectral sequence degenerates immediately, so

$$H^r(X_1, \pi_* j_!\mathscr{F}) = H^r(X, j_!\mathscr{F}).$$

Then only need to check there is a canonical isomorphism $\pi_* j_!\mathscr{F} \cong (j_1)_!\mathscr{F}$. □

**Proposition 4.22.** *Given a short exact sequence of constructible abelian sheaves on $U_{ét}$*

$$0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{H} \to 0,$$

*one can get a long exact sequence*

$$\cdots \to H_c^{i-1}(X_{ét}, \mathscr{H}) \to H_c^i(X_{ét}, \mathscr{F}) \to H_c^i(X_{ét}, \mathscr{G}) \to \cdots.$$

*Proof.* Just because $j_!$ is exact. □

**Proposition 4.23.** *Let $\mathscr{F}$ be a constructible sheaf on $U_{ét}$, then $H_c^i(X_{ét}, \mathscr{F})$ is finite.*

*Proof.* Only to prove $j_!\mathscr{F}$ is constructible: First, $j_!\mathscr{F}$ has finite stalks since $\mathscr{F}$ does. Second, for any closed subset $T \subset X$, just consider $T \cap U$. □

## 4.2 Purity and Gysin sequence

Let $\Lambda = \underline{\mathbb{Z}/n\mathbb{Z}}$ with $n$ invertible on the base. Denote the sheaves of $\Lambda$-module by $\mathrm{Sh}^\Lambda$. For example, $\mu_n$ is a sheaf of $\Lambda$-module. Given $\mathscr{F} \in \mathrm{Sh}^\Lambda$, we put

$$\mathscr{F}(r) := \mathscr{F} \otimes_\Lambda \mu_n^{\otimes r}.$$

Note that in an algebraically closed field, $\mathscr{F}(r)$ is isomorphic to $\mathscr{F}$, but it will have different Galois cohomology if not in one.

The general way of relating cohomology on an open to cohomology on the complement is called cohomology with supports. Let $Z$ be a closed subscheme, we define the section which has support on $Z$ to be

$$\Gamma_Z(X, -) := \ker(\Gamma(X, -) \to \Gamma(U, -)).$$

This is a left exact functor (exercise), so we can define the cohomology with supports on $Z$.

**Definition 4.24** (cohomology with supports)**.** $H_Z^*(X, -)$ is the right derived functors of $\Gamma_Z$.

**Theorem 4.25.** *The cohomology with supports is functorial and short exact sequence induces long exact sequence of cohomology with supports. And there is a long exact sequence*

$$\cdots \to H_Z^i(X_{ét}, -) \to H^i(X_{ét}, -) \to H^*(U_{ét}, -) \to \cdots.$$

*Proof.* The only non-trivial part is the last statement. Let $U = X - Z$ and denote $j : U \to X$ and $i : Z \to X$ be the inclusions. We have proved there is an exact sequence

$$0 \to j_! j^* \underline{\mathbb{Z}} \to \underline{\mathbb{Z}} \to i_* i^* \underline{\mathbb{Z}} \to 0.$$

I claim that

$$\mathrm{Hom}(i_* i^* \underline{\mathbb{Z}}, \mathscr{F}) \cong \Gamma_Z(X_{ét}, \mathscr{F}).$$

We apply $\mathrm{Hom}(-, \mathscr{F})$ to the above exact sequence

$$0 \to \mathrm{Hom}(i_* i^* \underline{\mathbb{Z}}, \mathscr{F}) \to \mathrm{Hom}(\underline{\mathbb{Z}}, \mathscr{F}) \to \mathrm{Hom}(j_! j^* \underline{\mathbb{Z}}, \mathscr{F})$$

The middle term is $\Gamma(X, \mathscr{F})$ The last term is in fact

$$\mathrm{Hom}(j_! j^* \underline{\mathbb{Z}}, \mathscr{F}) = \mathrm{Hom}(j^* \underline{\mathbb{Z}}, j^* \mathscr{F}) = \mathrm{Hom}(\underline{\mathbb{Z}}, j^* \mathscr{F}) = \Gamma(U, \mathscr{F}|_U).$$

Hence, the above exact sequence turns to

$$0 \to \mathrm{Hom}(i_* i^* \underline{\mathbb{Z}}, \mathscr{F}) \to \Gamma(X_{\acute{e}t}, \mathscr{F}) \to \Gamma(U, \mathscr{F}|_U).$$

So we prove the claim. This isomorphism of functors $\mathrm{Hom}(i_* i^* \underline{\mathbb{Z}}, -) \cong \Gamma_Z(X_{\acute{e}t}, -)$ induces isomorphism of derived functors

$$H^i_Z(X_{\acute{e}t}, \mathscr{F}) \cong \mathrm{Ext}^i_{\mathrm{Sh}^{ab}(X_{\acute{e}t})}(i_* i^* \underline{\mathbb{Z}}, \mathscr{F}).$$

Since Ext can induce long exact sequences, we are done. □

**Theorem 4.26** (Purity)**.** *Let $Z \subseteq X$ be a closed subset of a scheme over a field $k$. Assume that $Z$ and $X$ are smooth and $Z$ is of pure codimensional $c$ on $X$. Then for any $\mathscr{F} \in \mathrm{Sh}^{ab}(X_{\acute{e}t})$ which is locally constant constructible (l.c.c) and the order of which is prime to the characteristic of the base field, there is a canonical isomorphism*

$$H^{r-2c}(Z, \mathscr{F}(-c)) \cong H^r_Z(X, \mathscr{F}).$$

**Example 4.27.** Let $Z$ be a closed point of $\mathbb{A}^1_k$ where $k$ is an algebraically closed field of characteristic not divide $n$, then $c = 1$. Hence, we have

$$H^r((\mathbb{A}^1_k)_{\acute{e}t}, \mathbb{Z}/n\mathbb{Z}) \cong H^{r-2}(Z, \mathbb{Z}/n\mathbb{Z}) = \begin{cases} \mathbb{Z}/n\mathbb{Z} & r = 2 \\ 0 & \text{otherwise} \end{cases}$$

In fact, we can check this theorem in this situation. By using the Kummer sequence, we can compute the $H^i((\mathbb{A}^1)_{\acute{e}t}, \mathbb{Z}/n\mathbb{Z})$ and $H^i((\mathbb{G}_m)_{\acute{e}t}, \mathbb{Z}/n\mathbb{Z})$, because we know their Picard groups are both 0. We are done by the exact sequence in the last part of theorem 4.25.

**Corollary 4.28** (Gysin sequence)**.** *Let $X, Z$ be as above and $U = X - Z$. Then for $0 \leq r < 2c - 1$, the restriction map*

$$H^r(X_{\acute{e}t}, \mathscr{F}) \to H^r(U_{\acute{e}t}, \mathscr{F})$$

*is an isomorphism and the short exact sequence*

$$0 \to H^{2c-1}(X, \mathscr{F}) \to H^{2c-1}(U, \mathscr{F}|_U) \to H^0(Z, \mathscr{F}(-c)) \to H^{2c}(X, \mathscr{F}) \to H^{2c}(U, \mathscr{F}) \to H^1(Z, \mathscr{F}(-c)) \to \cdots.$$

*Proof from Theorem to Corollary.* Just replace the $H^r_Z(X, \mathscr{F})$ by $H^{r-2c}(Z, \mathscr{F}(-c))$. □

**Example 4.29** (Cohomology of projective space)**.** Let $n$ be an integer prime to the characteristic of the base field. We computed that $H^i(\mathbb{A}^1, \mu_n) = 0$ if $i > 0$ and $\mu_n$ if $i = 0$. By Kunneth theorem, we know that it does not change if we replace $\mathbb{A}^1$ by $\mathbb{A}^n$. Then by Gysin sequence for $\mathbb{P}^k, \mathbb{P}^{k-1}$ with codimension $c = 1$, we have

$$H^0(\mathbb{P}^k, \mu_n) = H^0(\mathbb{A}^k, \mu_n).$$

and

$$0 \to H^1(\mathbb{P}^k, \mu_n) \to H^1(\mathbb{A}^k, \mu_n) \to H^0(\mathbb{P}^{k-1}, \mathbb{Z}/n\mathbb{Z}) \to H^2(\mathbb{P}^k, \mu_n) \to H^2(\mathbb{A}^k, \mu_n) \to H^1(\mathbb{P}^{k-1}, \mathbb{Z}/n\mathbb{Z}) \to \cdots.$$

This gives us $H^1(\mathbb{P}^k, \mu_n) = 0$ and $H^i(\mathbb{P}^k, \mathbb{Z}/n\mathbb{Z}) \cong H^{i-2}(\mathbb{P}^{k-1}, \mathbb{Z}/n\mathbb{Z})$ for $i$. By induction,

$$H^r(\mathbb{P}^k, \mathbb{Z}/n\mathbb{Z}) = \begin{cases} (\mathbb{Z}/n\mathbb{Z})(-\frac{r}{2}) & r \text{ even} \\ 0 & \text{otherwise} \end{cases}$$

It will take some effort to prove the purity theorem, but worth it. Let $Z, X$ be as above and the inclusion map $i : Z \to X$. We roughly see that $i^*$ and $i_*$ define an equivalence between the category of étale sheaves on $X$ with support on $Z$ and the category of étale sheaves on $Z$. The exactness and preserving injectives allow us to abuse $i_*\mathscr{F}$ on $X$ with $\mathscr{F}$ on $Z$. When it comes to an open subset $U \subseteq X$ with inclusion $j : U \to X$, we define the upper shrink of a sheaf $\mathscr{F}$ to be

$$\mathscr{F}^! := \ker(\mathscr{F} \to j_*j^*\mathscr{F}),$$

which is the largest subsheaf of $\mathscr{F}$ with support on $Z$. For any étale $\varphi : V \to X$,

$$\mathscr{F}^!(V) = \Gamma_{\varphi^{-1}(Z)}(V, \mathscr{F}) = \ker(\mathscr{F}(V) \to \mathscr{F}(\varphi^{-1}(U))).$$

Let $\mathscr{G}$ be any sheaf with support on $Z$, it's easy to see

$$\mathrm{Hom}_X(\mathscr{G}, \mathscr{F}^!) \cong \mathrm{Hom}_X(\mathscr{G}, \mathscr{F}).$$

Hence, if put $i^!\mathscr{F}$ to be $i^*\mathscr{F}^!$, $i^!$ is a functor from $\mathrm{Sh}(X_{\acute{e}t})$ to $\mathrm{Sh}(Z_{\acute{e}t})$ with

$$\mathrm{Hom}_Z(\mathscr{G}, i^!\mathscr{F}) \cong \mathrm{Hom}_X(i_*\mathscr{G}, \mathscr{F}).$$

Hence $i^!$ has a left adjoint, which implies that it is left exact. It preserves injectives as well. With all this notion, we can reorganize the purity theorem as follows.

**Theorem 4.30** (cohomological purity)**.** *Let $Z, X$ be as above with codimension $c$. For any locally constant sheaf of $\Lambda$-modules on $X$,*

$$R^{2c}i^!\mathscr{F} \cong (i^*\mathscr{F})(-c)$$

*and $R^r i^!\mathscr{F} = 0$ for $r \neq 2c$.*

*Proof from Theorem 4.30 to 4.26.* Consider

$$\mathrm{Sh}(X_{\acute{e}t}) \xrightarrow{i^!} \mathrm{Sh}(Z_{\acute{e}t}) \xrightarrow{\Gamma(Z,-)} \mathrm{Ab},$$

the composition is the functor $\Gamma_Z(X, -)$, so we have a spectral sequence

$$H^r(Z, R^s i^!\mathscr{F}) \Rightarrow H^{r+s}_Z(X, \mathscr{F}).$$

By the theorem 4.30, the spectral sequence degenerates at page two, because the only non-zero line is $s = 2c$, and

$$H^r(X, \mathscr{F}) = H^{r-2c}(Z, R^s i^!\mathscr{F}) = H^{r-2c}(Z, \mathscr{F}(-c)).$$

$\square$

The question is local in étale topology. The following lemma reduces the case to the $X = \mathbb{A}^m$ and $Z = \mathbb{A}^{m-c}$.

**Lemma 4.31.** *Let $Z, X$ be as usual with codimension $c$. Let $P \in Z$. There exists an open neighbourhood $V$ of $P$ and an étale morphism $\alpha : V \to \mathbb{A}^m$, where $m = \dim X$,with $\alpha|_{V\cap Z}$ is an étale morphism to $\mathbb{A}^{m-c}$.*

*Proof.* Since we require $Z, X$ to be smooth, the tangent space $T_P(Z)$ is a subspace of $T_P(X)$ of codimension $c$. Choose regular functions $f_1, \ldots, f_m$ on an open neighbourhood $V$ of $P$ such that $df_1, \ldots df_m$ form a basis for the cotangent space $T_P(X)^*$ and $df_1, \ldots, df_{m-c}$ is a basis of the cotangent space $T_P(Z)^*$. Consider the morphism

$$\alpha : V \to \mathbb{A}^m, Q \mapsto (f_1(Q), \ldots, f_m(Q)).$$

Then we know that $d\alpha$ is an isomorphism of tangent spaces. Therefore, $\alpha$ is étale at (a neighborhood of) $P$. Similarly, $\alpha_{Z\cap V} : Z \cap V \to \mathbb{A}^{m-c} \subseteq \mathbb{A}^m$ is étale at $P$. $\square$

Then we use induction on $m, c$, we did the case $m = c = 1$ and $\mathscr{F} = \mathbb{Z}/n\mathbb{Z}$. We are not going to talk about it in detail.

## 4.3 Comparison Theorems

**Definition 4.32** (elementary fibration)**.** Consider the following diagram

$$\begin{tikzcd} U \arrow[r, "j"] \arrow[dr, "f"'] & X \arrow[d, "h"] & Z \arrow[l] \arrow[dl, "g"] \\ & S & \end{tikzcd}$$

with the condition:

(1) $j$ is a Zariski open, and $j(U)$ is fibrewise dense in $Y$, $Z = Y - U$ and $i : Z \to Y$ is closed embedding.

(2) $h$ is smooth projective with geometrically irreducible fibers, relative dimension 1.

(3) $g$ is finite étale.

The key to this definition is to say "the topology of each fiber of $f$ is constant". When over $\mathbb{C}$, you can put $U$ be a bunch of Riemann surfaces of the same genus minus some points.

**Proposition 4.33** (Artin)**.** *Let $X$ be a smooth variety over an algebraically closed field $k$. For each $x \in X$, there exists a Zariski open $x \in U$ such that $U$ fits into an elementary fibration.*

*Proof.* The proof is basically picking a $U$ and embedding it into $\mathbb{P}^n$, then projecting it down until the base has codimension 1. □

**Theorem 4.34.** *Let $X$ be a variety over $\mathbb{C}$. Let $\mathscr{F} \in \mathrm{Sh}^{ab}(X_{ét})$ be constructible. There are three sites*

$$\begin{tikzcd} & X_{an\cdot ét} \arrow[dl, "\pi"'] \arrow[dr, "\mathrm{an}"] & \\ X^{an} & & X_{ét} \end{tikzcd}$$

*(1) $\pi_* : \mathrm{Sh}(X_{an\cdot ét}) \to \mathrm{Sh}(X^{an})$ is a equivalence between category, hence $\pi$ induces isomorphism on cohomology of all abelian sheaves.*

*(2) $\mathrm{an}^* : H^i(X_{ét}, \mathscr{F}) \to H^i(X_{an\cdot ét}, \mathrm{an}^*\mathscr{F})$ is an isomorphism.*

*Let $X^{an}$ be the same $X$ with Euclidean topology and we denote the site associated to the open sets of $X^{an}$ by the same notation. The $X_{an\cdot ét}$ is the category of complex-analytic spaces mapping to $X^{an}$ via local analytic isomorphisms (covers are covers). Here $\pi$ is just the inclusion map and the functor* an *maps any étale morphism $f : X \to Y$ to $f^{an} : X^{an} \to Y^{an}$. This is well-defined because the analytification of an étale morphism is locally isomorphism.*

**Corollary 4.35.** *For $\mathscr{F}$ as above, there is a canonical isomorphism*

$$H^i(X_{ét}, \mathscr{F}) \xrightarrow{\sim} H^i(X^{an}, \mathscr{F}^{an})$$

*where $\mathscr{F}^{an}$ is restricting $\mathrm{an}^*\mathscr{F}$ to $X^{an}$* i.e. *$\pi_*\mathrm{an}^*\mathscr{F}$.*

Part one of Theorem 4.34 is a purely topological statement (exercise), and we will only prove (2) under the assumption that $X$ is smooth and $\mathscr{F}$ is locally constant constructible.

*Sketch Proof of (2).* I claim that it is enough to prove

(a) the canonical map $\mathscr{F} \to \mathrm{an}_*\mathrm{an}^*\mathscr{F}$ is an isomorphism.

(b) for any $i > 0$, $R^i\mathrm{an}_*\mathrm{an}^*\mathscr{F} = 0$.

If assuming this, the spectral Leray sequence gives what we want.

First to prove $\mathscr{F} \to \mathrm{an}_*\mathrm{an}^*\mathscr{F}$ is an isomorphism when $\mathscr{F}$ is locally constant constructible (l.c.c), *i.e.* locally constant with finite stalks. This is a local question, so we may assume that $\mathscr{F}$ is a constant sheaf $\underline{\Lambda}$ on $U$. And we have

$$\Gamma(U, \underline{\Lambda}) \to \Gamma(U^{an}, \mathrm{an}^*\underline{\Lambda}) = \Gamma(U^{an}, \underline{\Lambda}).$$

To prove this is an isomorphism, enough to prove that $U$ and $U^{an}$ have the same connected component, or equivalently

$$\pi_0(U^{an}) \to \pi_0(U)$$

is a bijection. With the assumption that $X$ is smooth, we may deal with it more easily. By a point-set topology argument, one can prove that if $U$ has an open dense connected subset, then $U$ is connected. So we can replace $U$ by its open dense subset and use elementary vibration to it (to lower the dimension and use induction).

$$\begin{tikzcd} U \arrow[r, "j"] \arrow[dr, "f"'] & Y \arrow[d, "h"] & Z \arrow[l] \arrow[dl, "g"] \\ & S & \end{tikzcd}$$

We can assume that $U$ is connected without loss of generality, (hence $Y$ is connected) and $U^{an}$ is not connected (hence $Y^{an}$ is not connected). Hence, $Y = Y_1 \bigcup Y_2$ where $Y_i$ is both open and closed in complex topology. Since each fiber of $h$ is connected and admitting the case of Riemann surfaces, $Y_i$ must be the union of fibers. By the fact that $h$ is proper, that means $h(Y_1)$ and $h(Y_2)$ is two connected components of $S$, now we are done by induction.

Secondly, prove $R^i\mathrm{an}_*\mathrm{an}^*\mathscr{F} = 0$ for $i > 0$. This is also a local question. Without loss of generality, we assume $\mathscr{F}$ is constant on $U$ and $U$ fits into an elementary fibration. Let $\mathrm{an} : U_{an\cdot ét} \to U_{ét}$ and $\underline{\Lambda}$ is a constant sheaf on $U_{ét}$. We want to prove that $R^i\mathrm{an}_*\underline{\Lambda} = 0$ for $i > 0$. This is implied by the following lemma. $\square$

**Lemma 4.36.** *Let $U$ be a connected smooth variety over $\mathbb{C}$ and $\mathscr{F}$ locally constant with finite stalks on $U_{an\cdot ét}$. Then for any $r > 0$ and any $s \in H^r(U_{an\cdot ét}, \mathscr{F})$, there exists an étale cover $U' \to U$ such that $s$ maps to zero in $H^r(U'_{an\cdot ét}, \mathscr{F})$.*

This means that any $\mathscr{F}$ as above can be killed after changing to an étale cover. So we have $(R^r\mathrm{an}_*\mathscr{F})_{\bar{x}} = 0$ for any $\bar{x}$, and therefore $R^r\mathrm{an}_*\mathscr{F} = 0$.

*Sketch proof of the Lemma.* Again, $U$ fits in an elementary fibration.

$$\begin{tikzcd} U \arrow[r, "j"] \arrow[dr, "f"'] & Y \arrow[d, "h"] & Z \arrow[l, "i"'] \arrow[dl, "g"] \\ & S & \end{tikzcd}$$

Let $s \in H^r(U_{an\cdot ét}, \mathscr{F})$. Again, by the Leray spectral sequence, we have

$$H^i(S_{an\cdot ét}, R^jf_*\mathscr{F}) \Rightarrow H^{i+j}(U_{an\cdot ét}, \mathscr{F}).$$

Recall that $R^jf_*\mathscr{F}$ is the sheafification of

$$V \mapsto H^j(f^{-1}(V), \mathscr{F}).$$

By induction, one can kill the contribution coming from $R^jf_*\mathscr{F}$ when $j > 0$. So the only contribution is from $H^i(S_{an\cdot ét}, f_*\mathscr{F})$. In fact, $f_*\mathscr{F}$ is a locally constant sheaf with finite stalks, because $f$ is good enough, and each fiber of $f : U \to S$ has the same topology (in complex sense), which is a punctured curve. This is not rigorous, but helpful to understand. Now $S$ is lower dimensional. So we may assume $\dim U = 1$.

Now we assume that $U$ is one dimensional and $\underline{\Lambda}$ is locally constant sheaf with finite stalks on $U_{an\cdot ét}$. Then $U^{an}$ is compact Riemann surfaces. To kill $H^2(U_{an\cdot ét}, \underline{\Lambda})$, pass to any affine cover. For $s \in H^1(U_{an\cdot ét}, \underline{\Lambda})$, it correspondent to a $\underline{\Lambda}$-torsor. To trivialize it in the algebraic sense, we need the Riemann existence theorem. □

**Theorem 4.37** (Riemann existence theorem)**.** *There is an equivalence of categories*

$$U_{f\cdot ét} \xrightarrow{\sim} U_{f\cdot an\cdot ét},$$

*where f stands for finite.*

**Example 4.38.** Let $X$ be a K3 surface over $\mathbb{C}$, we can analytify it as a complex manifold and compute its étale cohomology with coefficients in $\mathbb{Z}/n\mathbb{Z}$.

## 4.4 Étale fundamental group

In this section, we will assume $X$ to be normal connected variety (scheme).

**Definition 4.39** (finite étale site)**.** Let $X$ be a locally Noetherian scheme. Define the site $X_{fét}$ to be a category, whose objects are finite étale morphisms $Y \to X$ and morphisms are morphisms over $X$, with the covers which are just covers.

*Remark* 4.40. This is a site.

**Definition 4.41.** Let $\bar{x} \in X$ be a geometric point of $X$. Then define a functor $F_{\bar{x}}$ : $X_{fét} \to \text{FinSets}$ by $(Y \to X) \mapsto Y_{\bar{x}}$. Define $\pi_1^{ét}(X, \bar{x}) := \text{Aut}(F_{\bar{x}})$ as the automorphism groups of the functor. Give it the weakest topology such that for all $Y \to X$ finite étale, $\pi_1^{ét}(X, \bar{x}) \to \text{Aut}(F_{\bar{x}}(Y))$ is continuous ($\text{Aut}(F_{\bar{x}}(Y))$ is discrete).

**Exercise 4.42.** Let $X = \text{Spec } k$ where $k$ is a field with $\bar{x} : \text{Spec } \bar{k} \to \text{Spec } k$ be its geometric point. We have $X_{fét}$ as the opposite category of all finite étale extensions $L/k$, which is finite product of finite separable extensions over $k$. Then

$$\pi_1^{ét}(\text{Spec } k, \bar{x}) = \text{Gal}(k^s/k).$$

**Example 4.43.** Let $E$ elliptic curve over an algebraically closed field $k$ with characteristic $p > 0$. Then for any geometric point $\bar{x} \in E$,

$$\pi_1^{ét}(E, \bar{x}) = \varprojlim_n E[n] = \begin{cases} \mathbb{Z}_p \times \prod_{l \neq p} \mathbb{Z}_l^2 & E \text{ ordinary} \\ \prod_{l \neq p} \mathbb{Z}_l^2 & E \text{ super singular} \end{cases}$$

**Example 4.44.** Let $X$ normal over $\mathbb{C}$ and connected. Then

$$\pi_1^{ét}(X, \bar{x}) \xrightarrow{\sim} \pi_1\widehat{(X^{an}}, x)$$

where the right-hand side means the profinite completion of the classical fundamental group.

**Theorem 4.45** (SGA1)**.** *There is an equivalence of category between finite étale site and finite continuous $\pi_1^{ét}$-sets for normal connected variety $X$.*

**Example 4.46.** Let $k$ be a field of positive characteristic and $\bar{x}$ be a geometric point of $X = \mathbb{A}^1_k$. Then $\pi_1^*(\mathbb{A}^1_k, \bar{x})$ is not topologically finite generated.

Indeed, we computed that

$$H^1(X, \mathbb{F}_p) = \text{coker}(k[t] \xrightarrow{x \mapsto x^p - x} k[t]).$$

Hence not finite generated. We claim that

$$\mathrm{Hom}_{cont}(\pi_1^{\acute{e}t}, \mathbb{F}_p) \xrightarrow{\sim} H^1(\mathbb{A}_k^1, \mathbb{F}_p) \cong \{\mathbb{F}_p\text{−torsor}\}.$$

In fact, we mentioned in Example 2.39 that for any finite group constant sheaf $\mathcal{G}$ on $X_{\acute{e}t}$, the $\mathcal{G}$-torsors are one-to-one correspondent to the Galois coverings of $X$. Now we try to prove the left-hand side is also Galois coverings.

First, $\mathbb{F}_p$ has an action on itself by addition. So one can see that left-hand side is the set of finite continuous $\pi_1^{\acute{e}t}$-sets such that the action factor through a map $\pi_1^{\acute{e}t} \to \mathbb{F}_p$, and this sets is correspondent to some finite étale morphisms by Theorem 4.45. I claim that these are just Galois coverings without proof.

**Corollary 4.47.** *For any $\bar{x}_1, \bar{x}_1$ geometric points of $X$, then*

$$\pi_1^{\acute{e}t}(X, \bar{x}_1) \cong \pi_1^{\acute{e}t}(X, \bar{x}_2).$$

*Proof.* This is just because both $\pi_1^{\acute{e}t}(X, \bar{x}_i)$-sets are finite étale sites on $X$. And the category determines abstract groups (exercise). □

*Remark* 4.48. In fact, this isomorphism is well-defined up to inner conjugation, just like the classical situation.

**Theorem 4.49** (Comparison theorem)**.** *Let $X$ be a normal connected scheme over $\mathbb{C}$. then there is a natural isomorphism*

$$\pi_1(\widehat{X^{an}, \bar{x}^{an}}) \to \pi_1^{\acute{e}t}(X, \bar{x}).$$

*Sketch of proof.* We claim that an : $\mathrm{FEt}(X) \to \mathrm{FinCov}(X)$ is an equivalence of categories by Riemann existence theorem. We realize the $\pi_1(X^{an}, \bar{x}^{an})$ as the automorphism of the functor $F_{\bar{x}}^{an}$ : Cov $\to$ Sets. And we claim that the diagram commutes:

$$\begin{tikzcd} \mathrm{FEt}(X) \arrow[r, "\mathrm{an}"] \arrow[d, "F_{\bar{x}}"] & \mathrm{Cov}(X^{an}) \arrow[dl, "F_{\bar{x}}^{\mathrm{an}}"] \\ \mathrm{Sets} & \end{tikzcd}$$

The completion comes from taking limits of finite covering spaces. □

**Corollary 4.50.** *Let $X$ be a smooth proper curve of genus $g$ over $\mathbb{C}$, then*

$$\pi_1^{\acute{e}t}(X) = \widehat{< a_1, b_1, \ldots, a_g, b_g |} \prod [a_i, b_i] = 1 >.$$

**Theorem 4.51.** *Let $X$ be a smooth proper scheme over a field $k$. Let $\bar{k}$ be its algebraic closure and $X_{\bar{k}}$ be the base extension. Then one has the following short exact sequence*

$$1 \to \pi_1^{\acute{e}t}(X_{\bar{k}}, \bar{x}) \to \pi_1^{\acute{e}t}(X, \bar{x}) \to \mathrm{Gal}(\bar{k}/k) \to 1.$$

*Idea of proof.* The surjectivity follows from the geometric connectedness of $X$. To prove the exactness in the middle, any finite étale cover of $X$ which maps to identity "splits" into a disjoint union of copies of $X_{\bar{k}}$ after base changing to $\bar{k}$. □

To compute the cohomology of varieties over positive characteristics, we discuss specialization maps. Let $X$ be a proper flat over a complete DVR $R$ with geometrically connected fibers. Put $K = \mathrm{Frac}(R)$ and $m$ its maximal ideal with $k = R/m$.

**Theorem 4.52.** *Given a geometric point on special fiber $\bar{x} \to X_k$, the natural map*

$$\pi_1^{\acute{e}t}(X_k, \bar{x}) \to \pi_1^{\acute{e}t}(X, \bar{x})$$

*is an isomorphism of topological groups.*

*Proof.* Enough to show the finite étale site of $X$ restricting to the finite étale site of $X_k$ is an equivalence of categories. We only try to explain why this is essentially surjective.

Given $Y \to X_k$ finite étale, the first step is to construct $\mathcal{Y} \to \widehat{X}$ where $\widehat{X}$ is the formal scheme, *i.e.* the completion of $X$ along the special fiber. So the question is to fill in the following diagram

$$\begin{tikzcd} Y_n \arrow[r, dashed] \arrow[d] & ? \arrow[d, dashed] \\ X \otimes R/m^n \arrow[r, hook, "i"] & X \otimes R/m^{n+1} \end{tikzcd}$$

To answer it, we need some deformation theory: This existence is determined by the obstruction class $\alpha \in \mathrm{Ext}^2_{\mathrm{X_k}}(\Omega^1_{Y/X_k}, \mathscr{I})$ where $\mathscr{I}$ is the ideal defining $i$. If $\alpha = 0$, then there exists such $Y_{n+1}$ flat over $X \otimes R/m^{n+1}$ making the diagram Cartesian. But in our étale setting $\mathrm{Ext}^2_{\mathrm{X_k}}(\Omega^1_{Y/X_k}, \mathscr{I}) = 0$. And also the sets of such $Y_{n+1}$ is a $\mathrm{Ext}^1_{\mathrm{X_k}}(\Omega^1_{Y/X_k}, \mathscr{I})$-torsor, but this is also a trivial group, hence $Y_{n+1}$ is unique. And one can check $Y_{n+1} \to X \otimes R/m^{n+1}$ is étale by computing $\Omega^1_{Y_{n+1}/(X\otimes R/m^{n+1})} = 0$.

Now admitting unique $\mathcal{Y} \to \widehat{X}$, we want to lift it to $Y \to X$. Since $X$ is proper, this is the formal GAGA. □

*Remark* 4.53. A good geometric interpretation is that the spectrum of a DVR is an open unit disk, the closed point is the origin point and the generic point is the punctured one.

Given $X$ as above $\bar{\eta} \to X_K$ be a geometric point on generic fiber specializing to a geometric point on special fiber $\bar{\xi} \to X_k$, the specialization map is the composition

$$sp : \pi_1^{\acute{e}t}(X_K, \bar{\eta}) \to \pi_1^{\acute{e}t}(X, \bar{\eta}) \xrightarrow{\sim} \pi_1^{\acute{e}t}(X, \bar{\xi}) \xrightarrow{\sim} \pi_1^{\acute{e}t}(X_k, \bar{\xi}).$$

**Theorem 4.54.** *Assume such $X$ is normal, then $sp : \pi_1^{\acute{e}t}(X_K, \bar{\eta}) \to \pi_1^{\acute{e}t}(X_k, \bar{\xi})$ is surjective.*

*Proof.* To be added. □

**Theorem 4.55.** *Let $X$ be a variety over an algebraically closed field $k$ of characteristic* 0*. Assume $L/k$ is an extension of algebraically closed fields, then*

$$\pi_1^{\acute{e}t}(X_L) \to \pi_1^{\acute{e}t}(X)$$

*is an isomorphism*

*Proof.* To be added (Galois descent). □

**Theorem 4.56.** *Let $X$ be a smooth proper curve over an algebraically closed field $k$ of characteristic $p > 0$. Then $\pi_1^{\acute{e}t}(X, \bar{x})$ is topologically generated by at most $2g(X)$ elements, where $g(X)$ is the genus of $X$.*

*Sketch of proof.* (1) Lift to characteristic 0 by deformation theory, since $X$ is a curve the $\mathrm{Ext}^2$ always vanishes by Grothendieck vanishing. Hence the obstruction does not exist. Plus formal GAGA, we're done. (2) By the surjective specialization map $\pi_1^{\acute{e}t}(X_{\bar{K}}) \to \pi_1^{\acute{e}t}(X)$, we can compute the former one in $\mathbb{C}$. □

*Remark* 4.57. Probably this is still the only way we can compute the fundamental groups of a curve of positive characteristics.

Here are some important facts.

**Theorem 4.58** (SGA1)**.** *Let $X$ as above, then*

$$\pi_1^{\acute{e}t}(X_K) \to \pi_1^{\acute{e}t}(X_k)$$

*induces an isomorphism on prime-to-p completions.*

**Proposition 4.59.** *There is an equivalence of categories*

$$\{\text{local constant constructible sheaves on } X_{\acute{e}t}\} \xrightarrow{\sim} \{\text{finite continuous } \pi_1^{\acute{e}t}-\text{modules}\}$$

*Sketch of proof.* Let us construct two maps. Any locally constant constructible sheaf $\mathscr{F}$ is represented by a finite étale $X$-scheme. By Theorem 4.45, finite étale $X$-schemes are just $pi_1^{\acute{e}t}$-sets. Also by the virtue of Theorem 4.45, Given a $\pi_1^{\acute{e}t}$-sets is equivalent to giving a finite étale $X$-scheme $Y$, we take the sheaf to be $h_Y$. $\square$

**Corollary 4.60.** *Let $X$ be connected. There is a canonical map*

$$H^i_{cont}(\pi_1^{\acute{e}t}(X,\bar{x}),M) \to H^i(X_{\acute{e}t},\mathscr{F}_M)$$

*induces an isomorphism on $i = 0, 1$, where $\mathscr{F}_M$ is the correspondent l.c.c sheaf of $M$.*

*Sketch of proof.* We have the inclusion $i : X_{\acute{e}t} \to \mathrm{FEt}(X)$. We claim that

$$\mathrm{FEt}(X) = \pi_1^{\acute{e}t}-\text{sets}$$

and $\mathscr{F}_M = f^*M$ in this sense. We're done if $R^1 f_*\mathscr{F}_m = 0$. We have done it before (a slogan is "Torsors kill themselves"). $\square$

## 4.5 Finiteness and $\ell$-adic sheaves

**Theorem 4.61.** *Let $X$ be a variety over a separably closed field $k$ and $\mathscr{F}$ constructible on $X_{\acute{e}t}$. If*

*(1) $X$ is proper, or*

*(2) the stalks of $\mathscr{F}$ have the order prime to* $\mathrm{char}(k)$,

*then $H^r(X_{\acute{e}t},\mathscr{F})$ is finite for $r \geq 0$.*

*Sketch of the proof.* (1) is part of the proper base change. To prove (2) if assuming smoothness, use induction on the dimension by elementary fibration (SGA4). $\square$

*Remark* 4.62. The counterexample will always be $\mathbb{A}^1_{\mathbb{F}_p}$.

I have to copy an important paragraph from Milne's notes to continue:

So far, we have talked only of torsion sheaves. However, it will be important for us to have cohomology groups that are vector spaces over a field of characteristic zero in order, for example, to have a good Lefschetz fixed-point formula. However, the étale cohomology groups with coefficients in nontorsion sheaves are anomalous. For example, when X is normal,

$$H^1(X_{\acute{e}t},\mathbb{Z}) = \mathrm{Hom}_{cont}(\pi_1^{\acute{e}t}(X,\bar{x}),\mathbb{Z}),$$

where $\mathbb{Z}$ equip with the discrete topology. Since $\pi_1^{\acute{e}t}(X,\bar{x})$ is profinite, hence compact, so the image is finite, which implies the right-hand side is 0. It will change nothing if we replace $\mathbb{Z}$ by $\mathbb{Z}_\ell$ because we still give discrete topology to $\mathbb{Z}_\ell$ as a $\pi_1^{\acute{e}t}$-module.

The solution to this is to define

$$H^r(X_{\acute{e}t},\mathbb{Z}_\ell) = \varprojlim_n H^r(X_{\acute{e}t},\mathbb{Z}/\ell^n\mathbb{Z}).$$

This cohomology does not commute with inverse limits of sheaves. With this definition,

$$H^1(X_{\acute{e}t},\mathbb{Z}_\ell) := \varprojlim_n H^1(X_{\acute{e}t},\mathbb{Z}/\ell^n\mathbb{Z}) \cong \varprojlim_n \mathrm{Hom}_{cont}(\pi_1^{\acute{e}t}(X,\bar{x}),\mathbb{Z}/\ell^n\mathbb{Z}) \cong \mathrm{Hom}_{cont}(\pi_1^{\acute{e}t}(X,\bar{x}),\mathbb{Z}_\ell)$$

where $\mathbb{Z}/\ell^n\mathbb{Z}$ has discrete topology and the last isomorphism is a result of continuous group cohomology.

To give a finite generated $\mathbb{Z}_\ell$-module $M$ is the same as to give a family $(M_n, f_{n+1} : M_{n+1} \to M_n)_{n\in\mathbb{N}}$ such that

(a) for all $n$, $M_n$ is a finite $\mathbb{Z}/\ell^n\mathbb{Z}$-module;

(b) for all $n$, the map $f_{n+1} : M_{n+1} \to M_n$ induces an isomorphism $M_{n+1}/\ell^n M_{n+1} \to M_n$.

Let $(M_n, f_{n+1} : M_{n+1} \to M_n)_{n\in\mathbb{N}}$ satisfying such conditions. By induction, we obtain a canonical isomorphism $M_{n+s}/\ell^n M_{n+s} \cong M_n$. On tensoring

$$0 \to \mathbb{Z}/\ell^s\mathbb{Z} \xrightarrow{\ell^n}\to \mathbb{Z}/\ell^{s+n}\mathbb{Z} \to \mathbb{Z}/\ell^n\mathbb{Z} \to 0$$

with $M$, we obtain a sequence

$$0 \to M_s \to M_{n+s} \to M_n \to 0,$$

which is exact if $M$ is flat (or equivalently torsion-free).

**Definition 4.63.** $(\mathcal{M}_n, f_{n+1} : \mathcal{M}_{n+1} \to \mathcal{M}_n)$ is a sheaf of $\mathbb{Z}_\ell$-modules if

(a) each $\mathcal{M}_n$ is a constructible sheaf of $\mathbb{Z}/\ell^n\mathbb{Z}$-module;

(b) $f_{n+1}$ induces an isomorphism $\mathcal{M}/\ell^n\mathcal{M}_{n+1} \xrightarrow{\sim} \mathcal{M}_n$.

**Definition 4.64** (flat sheaf of $\mathbb{Z}_\ell$-module)**.** A sheaf of $\mathbb{Z}_\ell$-module is flat if the following short sequence

$$0 \to \mathcal{M}_s \xrightarrow{\ell^n} \mathcal{M}_{n+s} \to \mathcal{M}_n \to 0$$

is exact.

**Definition 4.65.** Let $\mathcal{M} = (\mathcal{M}_n, f_{n+1})$ be a sheaf of $\mathbb{Z}_\ell$-module, we define its cohomology to be

$$H^r(X_{\acute{e}t}, \mathcal{M}) = \varprojlim_n H^r(X_{\acute{e}t}, \mathcal{M}_n).$$

Its compact support cohomology is

$$H^r_c(X_{\acute{e}t}, \mathcal{M}) = \varprojlim_n H^r_c(X_{\acute{e}t}, \mathcal{M}_n).$$

**Example 4.66.** Let $X$ be a smooth proper curve of genus $g$ over a separably closed field $k$ and has char not equal to $\ell$. Then

$$H^i(X_{\acute{e}t}, \mathbb{Z}_\ell) = \varprojlim_n H^i(X_{\acute{e}t}, \mathbb{Z}/\ell^n\mathbb{Z}) = \begin{cases} \mathbb{Z}_\ell & i = 0 \\ T_\ell(\mathrm{Jac}(X))(-1) & i = 1 \\ \mathbb{Z}_\ell(-1) & i = 2 \\ 0 & i > 2 \end{cases}$$

When $i = 1$, it's the $\ell$-adic Tate module.

**Theorem 4.67.** *Let $\mathcal{M}$ be a flat sheaf of $\mathbb{Z}_\ell$-module on a variety $X$ over a separably closed field $k$. If $X$ proper and $\ell \neq \mathrm{char}(k)$, then*

*(a) $H^r(X_{\acute{e}t}, \mathcal{M})$ is a finite generated $\mathbb{Z}_\ell$-module for any $r \geq 0$;*

*(b) there is a long exact sequence*

$$\cdots \to H^{r-1}(X_{\acute{e}t}, \mathcal{M}_n) \to H^r(X_{\acute{e}t}, \mathcal{M}) \xrightarrow{\ell^n} H^r(X_{\acute{e}t}, \mathcal{M}) \to H^r(X_{\acute{e}t}, \mathcal{M}_n) \to \dots.$$

*Proof.* (1) Reduce to the previous finiteness theorem, (2) Build the long exact sequence above out of the long exact sequence arising from

$$0 \to \mathcal{M}_s \xrightarrow{\ell^n} \mathcal{M}_{n+s} \to \mathcal{M}_n \to 0$$

by taking inverse limits. □

**Definition 4.68.** A $\mathbb{Z}_\ell$-sheaf $\mathcal{M} = (\mathcal{M}_n, f_{n+1})$ is locally constant if each $\mathcal{M}_n$ is locally constant. We call $\mathcal{M}$ lisse if it is flat and locally constant.

*Warning* 4.69. A $\mathbb{Z}_\ell$-sheaf $\mathcal{M}$ is locally constant does not imply there is an étale cover of $X$ such that it is constant on it. Simply because there are infinitely many $\mathcal{M}_n$.

Let $\mathcal{M} = (\mathcal{M}_n, f_{n+1})$ be a locally constant $\mathbb{Z}_\ell$-sheaf, then there are a natural $\pi_1^{ét}$-representation associated to it. Denote the fiber of $\mathcal{M}_n$ on a geometric point $\bar{x} \in X$ by $\mathcal{M}_{n,\bar{x}}$ (not stalk), then there are natural continuous maps

$$\begin{tikzcd}
 & \cdots \arrow[d] \\
\pi_1^{ét}(X, \bar{x}) \arrow[ur] \arrow[r, "\rho_{n+1}"] \arrow[dr, "\rho_n"'] \arrow[ddr] & \mathrm{Aut}(\mathcal{M}_{n+1}, \bar{x}) \arrow[d] \\
 & \mathrm{Aut}(\mathcal{M}_n, \bar{x}) \arrow[d] \\
 & \cdots
\end{tikzcd}$$

This induces a one-to-one correspondence

$$\{\text{locally constant } \mathbb{Z}_\ell-\text{sheaves}\} \leftrightarrow \{\text{continuous } \pi_1^{ét}-\text{reps on f.g. } \mathbb{Z}_\ell-\text{modules}\},$$

and

$$\{\text{lisse } \mathbb{Z}_\ell-\text{sheaves}\} \leftrightarrow \{\text{continuous } \pi_1^{ét}-\text{reps on f.g. free } \mathbb{Z}_\ell-\text{modules}\}.$$

It's a little bit complicated to give a definition to $\mathbb{Q}_\ell$-sheaves. The objects are just $\mathbb{Z}_\ell$-sheaves but the morphisms are "localization' categorically. We simply denote that for a $\mathbb{Q}_\ell$-sheaf $\mathcal{M}$

$$H^i(X_{ét}, \mathcal{M}) = (\varprojlim_n H^i(X_{ét}, \mathcal{M}_n)) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell,$$

and

$$H_c^i(X_{ét}, \mathcal{M}) = (\varprojlim_n H_c^i(X_{ét}, \mathcal{M}_n)) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell.$$

Note that given a $\mathbb{Q}_\ell$-sheaf whose underlying $\mathbb{Z}_\ell$-sheaf is locally constant gives a continuous representation

$$\rho : \pi_1^{ét}(X, \bar{x}) \to GL_n(\mathbb{Q}_\ell).$$

In fact, this induces an equivalence of category.

## 4.6 Smooth Base Change

We can only summarize it here. Put

$$\begin{tikzcd}
X_T \arrow[r, "f'"] \arrow[d, "\pi'"'] & X \arrow[d, "\pi"] \\
T \arrow[r, "f"] & S
\end{tikzcd}$$

**Theorem 4.70** (Smooth base change)**.** *Let $\mathscr{F}$ be a constructible sheaf on $X$ with $\mathscr{F}_{\bar{x}}$ has order invertible on the base, assuming $f$ to be smooth. Then*

$$f^*(R^r \pi_* \mathscr{F}) \to R^r \pi'_* f'^* \mathscr{F}$$

*is an isomorphism.*

*Remark* 4.71. This still applies when $f$ is an inverse limit of smooth morphisms:

$f$ is the inclusion of a generic point;

fraction field of strictly Henselization.

**Theorem 4.72** (smooth-proper base change theorem)**.** *Assume the $\pi : X \to S$ is smooth and proper, $\mathscr{F}$ is an l.c.c. sheaf on $X$ and has the order invertible on $S$, then $R^r\pi_*\mathscr{F}$ are l.c.c.*

**Exercise 4.73** (TBA)**.** Find a counterexample if drop the hypothesis on the orders of the fibers. Hint: Take a family of elliptic curves of char $p$ such that the generic fiber is ordinary but at least one fiber is supersingular.

Let $X$ be a scheme, and $x_0, x_1$ be points of $X$ schematically (points that are not closed are allowed). We say that $x_0$ is a specialization of $x_1$ if it is contained in the closure of $x_1$. Assume that $x_0$ is a specialization of $x_1$. Choose geometric points $\bar{x}_0 \to x_0 \hookrightarrow X$ and $\bar{x}_1 \to x_1 \hookrightarrow X$. Any étale neighborhood $(U, u)$ of $\bar{x}_0$ can be considered as an étale neighborhood of $\bar{x}_1$ because the image of $U \to X$ is open then we can choose a morphism $\bar{x}_1 \to U$. Hence for any $\mathscr{F}$ constructible on the étale site of $X$, we have a non-canonical map (we call it cospecialization)

$$\mathscr{F}_{\bar{x}_0} \to \mathscr{F}_{\bar{x}_1}.$$

For example, Let $R$ be a DVR with residue field $k$ and fraction field $K$. Let $\pi : X \to \operatorname{Spec} R$ be a proper morphism. Since proper, $\mathscr{F} = R^i\pi_*(\underline{\mathbb{Z}/\ell})$ is constructible and $\mathscr{F}_k = H^i(X_k, \mathbb{Z}/\ell)$ and $\mathscr{F}_K = H^i(X_K, \mathbb{Z}/\ell)$. By the above argument, we have a non-canonical map

$$H^i(X_k, \mathbb{Z}/\ell) \to H^i(X_k, \mathbb{Z}/\ell).$$

**Proposition 4.74.** *Let $\mathscr{F}$ be a constructible sheaf. Then $\mathscr{F}$ being locally constant is equivalent to all cospecialization maps are isomorphisms.*

*Proof.* Exercise. □

**Corollary 4.75.** *Let $\pi : X \to S$ be a smooth proper and $\mathscr{F}$ l.c.c. sheaf on $X_{\acute{e}t}$ such that the order of its stalks is invertible on $S$. Given $\bar{\eta}, \bar{\xi}$ geometric points of $S$ and $\bar{\eta}$ is a specialization of $\bar{\xi}$, then the cospecialization map*

$$H^r(X_{\bar{\eta}}, \mathscr{F}) \to H^r(X_{\bar{\xi}}, \mathscr{F})$$

*is an isomorphism.*

**Slogan 4.76.** The fiber of smooth proper morphism does not change.

**Corollary 4.77.** *Let $k$ be a field of char $p > 0$, and $X$ be a smooth proper variety over $k$. Then if $X$ lifts to char 0, we can compute its cohomology with coefficients $\mathbb{F}_\ell$ or $\mathbb{Z}_\ell$, where $\ell \neq \operatorname{char} k$.*

"Lifts to char 0" means there exists a smooth proper $R$-scheme $\mathcal{X}$ where $R$ is a DVR with residue field $k$ such that $X_k \cong X$. We discussed previously that if $H^2(X, \mathcal{T}_X) = 0$, then we have a formal lift. This is not enough. But if you can also formally lift ample line bundle, then one can apply formal GAGA.

"Can compute cohomology" means by smooth proper base change theorem, it is enough to compute $X_{\bar{K}}$ where $K$ is the fraction field of a DVR to which $X$ lifts. Then one can choose an embedding $K$ into $\mathbb{C}$, by the Artin comparison theorem, which is singular cohomology.

## 4.7 Kunneth Formula and cycle class map

Kunneth formula and Poincare duality are best suitable under the setting of derived category. In this section and the next one, I will only give a summary (without proof) in the language of derived category.

**Theorem 4.78.** *Let $X, Y$ be proper $k$-schemes where $k$ is algebraically closed. Let $\mathscr{F}$ be a constructible sheaf on $X_{\text{ét}}$ and $\mathscr{G}$ be a constructible sheaf on $Y_{\text{ét}}$. Then*

$$R\Gamma(X_{\text{ét}}, \mathscr{F}) \otimes^L R\Gamma(Y_{\text{ét}}, \mathscr{G}) \to R\Gamma((X \times Y)_{\text{ét}}, \mathscr{F} \boxtimes G)$$

*is an isomorphism in $D^\flat(\mathrm{Ab})$. The $\mathscr{F} \boxtimes \mathscr{G}$ is the tensor product of pullback $\mathscr{F}$ and $\mathscr{G}$ over $X \times Y$.*

**Corollary 4.79.** *There is a spectral sequence*

$$\bigoplus_{i+j=s} \mathrm{Tor}^{\Lambda}_{-r}(H^i(X_{\text{ét}}, \Lambda), H^j(Y_{\text{ét}}, \Lambda)) \Rightarrow H^{r+s}(X \times Y, \Lambda)$$

*Proof.* This is a standard result in homological algebra induced from the above theorem. □

Passing to $\mathbb{Z}_\ell$-modules, one has

**Theorem 4.80.** *For varieties $X, Y$, there is a canonical exact sequene*

$$0 \to \bigoplus_{r+s=m} H^r(X, \mathbb{Z}_\ell) \otimes H^s(Y, \mathbb{Z}_\ell) \to H^m(X \times Y, \mathbb{Z}_\ell) \to \bigoplus_{r+s=m+1} \mathrm{Tor}_1^{\mathbb{Z}_\ell}(H^r(X, \mathbb{Z}_\ell), H^s(Y, \mathbb{Z}_\ell)) \to 0.$$

*Remark* 4.81. Note that these maps can be obtained by the cup product on their Čech cohomology. Cup product also exists in derived functor sheaf cohomology.

We have the following consequence: let $\Lambda = \mathbb{Z}/\ell^n\mathbb{Z}$ or $\mathbb{Z}_\ell$, if $H^*(X, \Lambda)$ or $H^*(Y, \Lambda)$ is free $\mathbb{Z}_\ell$-module, then

$$H^*(X, \Lambda) \otimes H^*(Y, \Lambda) \to H^*(X \times Y, \Lambda)$$

is an isomorphism (in fact an isomorphism of graded rings). After, tensoring $\mathbb{Q}_\ell$ one has an isomorphism

$$H^*(X, \mathbb{Q}_\ell) \otimes H^*(Y, \mathbb{Q}_\ell) \to H^*(X \times Y, \mathbb{Q}_\ell).$$

**Example 4.82.** Let $C$ be a smooth proper curve. Consider the cohomology of $C \times P^1$ with $\mathbb{Q}_\ell$ coefficients.

$$H^i(C, \mathbb{Q}_\ell) = \begin{cases} \mathbb{Q}_\ell & i = 0 \\ T_\ell(\mathrm{Jac}(X))(-1) \otimes \mathbb{Q}_\ell & i = 1 \\ \mathbb{Q}_\ell(-1) & i = 2 \\ 0 & i > 2 \end{cases}$$

and

$$H^i(\mathbb{P}^1, \mathbb{Q}_\ell) = \begin{cases} \mathbb{Q}_\ell & i = 0 \\ 0 & i = 1 \\ \mathbb{Q}_\ell(-1) & i = 2 \\ 0 & i > 2 \end{cases}$$

so

$$H^i(C \times \mathbb{P}^1, \mathbb{Q}_\ell) = \begin{cases} \mathbb{Q}_\ell & i = 0 \\ T_\ell(\mathrm{Jac}(X))(-1) \otimes \mathbb{Q}_\ell & i = 1 \\ \mathbb{Q}_\ell(-1) \bigoplus \mathbb{Q}_\ell(-1) & i = 2 \\ T_\ell(\mathrm{Jac}(X))(-2) \otimes \mathbb{Q}_\ell & i = 3 \\ \mathbb{Q}_\ell(-2) & i = 4 \end{cases}$$

Let us at least talk a little bit about the proof. There are two main ingredients: the projection formula and the base change theorem. Assume this Cartesian diagram

$$\begin{tikzcd} & X \times_S Y \arrow[dl, "p"'] \arrow[dr, "q"] \arrow[dd, "h"] & \\ X \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\ & S & \end{tikzcd},$$

In the language of derived category, one has

$$\begin{tikzcd} Rf_*\mathscr{F} \otimes^L Rg_*\mathscr{G} \arrow[d, "\sim"] & \text{(projection formula)} \\ Rf_*(\mathscr{F} \otimes Lf^* Rg_*\mathscr{G}) \arrow[d, "\sim"] & \text{(base change)} \\ Rf_*(\mathscr{F} \otimes Rp_*(Lq^*\mathscr{G})) \arrow[d, "\sim"] & \text{(projection formula)} \\ Rf_*(Rp_*(p^*\mathscr{F} \otimes q^*\mathscr{G})) \arrow[d, "\sim"] & (Rf_* \circ Rp_* = R(f \circ p)_*) \\ Rh_*(p^*\mathscr{F} \otimes q^*\mathscr{G}). & \end{tikzcd},$$

Now we introduce cycle class maps. Let $X$ be nonsingular variety over $k$ with char $\neq \ell$. Put $C^r(X)$ be the free abelian group on prime cycles of codimension $r$. Let $\Lambda = \mathbb{Z}/\ell^n\mathbb{Z}$, $\mathbb{Z}_\ell$ or $\mathbb{Q}_\ell$. What we want is to define

$$cl^r : C^r(X) \to H^{2r}(X, \Lambda(r)).$$

which is functorial and agrees with

$$C^1(X) \to \mathrm{Pic}(X) = H^1(X, \mathbb{G}_m) \to H^2(X, \Lambda(1)).$$

The first map comes the isomorphism between the divisor class group and the Picard group and the last one is induced by the Kummer sequence. Now we define for any $Z \subseteq X$, $cl^r(Z)$ to be the image of $1 \in H^0(Z, \Lambda) \xrightarrow{\sim} H_Z^{2r}(X, \Lambda(r)) \to H^{2r}(X, \Lambda(r))$ where the first isomorphism is by the purity. For $Z \subseteq X$ singular, we have the following lemma.

**Lemma 4.83.** *Let $Z \subseteq X$ of codimension $r$. Then for $s < 2r$,*

$$H_Z^s(X, \Lambda) = 0$$

*and*

$$H_Z^{2r}(X, \Lambda) = H_{Z-Z^{sing}}^{2r}(X - Z^{sing}, \Lambda)$$

We omit the proof. This gives us the way to define $cl^r(Z)$ when $Z$ is singular. Note that $cl : C^r(X_{\bar{k}}) \to H^{2r}(X_{\bar{k}}, \Lambda(r))$ is Galois equivariant.

## 4.8 Chern class and Poincare duality

Let $X$ be a nonsingular projective variety and $\mathcal{E}$ be a locally free sheaf of $\mathcal{O}_X$-module of rank $m+1$, one can construct the relative projective space $\mathbb{P}(\mathcal{E})$. If $\mathcal{E}$ is free, $\mathbb{P}(\mathcal{E})$ is $X \times \mathbb{P}^m$. By previous computation,

$$H^*(\mathbb{P}^m, \Lambda) \cong \Lambda[T]/[T^{m+1}], \xi \mapsto T$$

where $\xi$ is the class of hyperplane section and $\Lambda = \mathbb{Z}/\ell^n\mathbb{Z}$ or $\mathbb{Z}_\ell$. By the Kunneth formula when $\mathcal{E}$ is free we have

$$H^*(\mathbb{P}(\mathcal{E})) = H^*(X \times \mathbb{P}^m) = H^*(X) \otimes H^*(\mathbb{P}^m).$$

Note that $H^*(X)$ is in fact a ring. Combined with above fact, we know that $H^*(\mathbb{P}(\mathcal{E}))$ is a free module of $H^*(X)$ generated by $1, \xi, \dots, \xi^m$. For any locally free $\mathcal{E}$, this is still true induced by Mayer-Vietoris sequence.

Regarding $H^*(\mathbb{P}(\mathcal{E}))$ as an $H^*(X)$-module, there is a linear relation between $1, \xi, \dots, \xi^m, \xi^{m+1}$, and unique if normalizing the coefficient of $\xi^{m+1}$. We define the $r$-th Chern class $ch_r(\mathcal{E}) \in H^{2r}(X, \Lambda(r))$ to be those coefficients (take $\Lambda(r)$ for having the same Galois action on different $r$) *i.e.*

$$\begin{cases} \displaystyle\sum_{r=0}^{m+1} ch_r(\mathcal{E}) \cdot \xi^{m+1-r} = 0 \\ \qquad\qquad ch_0(\mathcal{E}) = 1. \end{cases}$$

and the total Chern class of $\mathcal{E}$ to be

$$ch(\mathcal{E}) = \sum ch_r(\mathcal{E}) \in H^{2i}(X, \Lambda(i)).$$

In fact, the Chern class has another equivalent definition. Let $\kappa : \mathrm{Pic}(X) \to H^2(X, \Lambda(1))$, then $ch_1(\mathcal{E}) := \kappa(\det(\mathcal{E})$. For the rest $r > 1$, we have the following theorem.

**Theorem 4.84.** *There exists a unique assignment $\mathcal{E} \to c_i(\mathcal{E})$ such that*

*(1) (Functoriality) If $f : X \to Y$ is a morphism of smooth varieties, then $c_i(f^*\mathcal{E}) = f^*(c_i(\mathcal{E}))$*

*(2) (Nomalization) $c_1(\mathcal{E}) = \kappa(\det(\mathcal{E}))$.*

*(3) (Additivity) If*

$$0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$$

*is an exact sequence of vector bundles on $X$, then*

$$ch_t(\mathcal{E}) = ch_t(\mathcal{E}') \cdot ch_t(\mathcal{E}'').$$

*Remark* 4.85. From the additivity, we know the Chern class facter through

$$K(X) \to \bigoplus_i H^{2i}(X, \mathbb{Q}_\ell(i))$$

where $K(X)$ is the Grothendieck group. On the left-hand side, the tensor product makes it a ring; On the right-hand side, the cup product makes it a ring. But $K(X)$ has no graded structure, but one can define $K^r(X)$ to be the subgroup of $K(X)$ generated by coherent sheaf with support in codimension $\geq r$, then $K^\bullet(X)$ define a filtration of $K(X)$, put

$$GK^*(X) := gr(K(X)) = \bigoplus K^r(X)/K^{r+1}(X).$$

The map $ch : K^*(X) \to H^*(X)$ induces a ring homomorphism $GK^*(X) \to H^*(X)'$ where $H^*(X)' = H^*(X)$ but with multiplication alternating the sign (see Milne's notes).

Let $CH^*(X)$ be the Chow group, one can define a graded ring morphism

$$CH^*(X) \to GK^*(X), Z \mapsto (\text{Alternating sum of finite free resolution of } \mathcal{O}_Z).$$

Most surprisingly, the composition

$$CH^*(X) \to GK^*(X) \to H^*(X)$$

is the cycle class map we define in the last section, which is a highly non-trivial fact. And one aslo has

$$CH^*(X)_{\mathbb{Q}} \xrightarrow{\sim} GK^*(X)_{\mathbb{Q}}.$$

sdflwgk 4.84.

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