Height Filtration on Flag Bundles over a Curve

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Height filtration and successive minima

Let K be either a number field or $K = k(C)$ where C is a projective smooth curve over a field k. Let X/K be a projective variety of dimension d and L be an adelic line bundle on X. These data induce an Arakelov height function h_L on X ([**Zhang1994SMALLPA**], see also [**Yuan2011AlgebraicD**] §9 for a survey). A typical case is the geometric height, which is the one we concern in this project. Let C be a smooth projective curve over a field k and let $K = k(C)$ be its function field. Let $\mathcal{X} \longrightarrow C$ be a projective flat morphism with $\mathcal{X}$ integral, and let $\mathcal{L}$ be a line bundle on $\mathcal{X}$. Consider the following diagram

$$\begin{array}{ccc} X & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ \operatorname{Spec} K & \longrightarrow & C \end{array}$$

and define a height function $h_{\mathcal{L}} : X(\overline{K}) \to \mathbb{R}$ associated to $\mathcal{L}$ by

$$x \mapsto \frac{\overline{\{x\}} \cdot \mathcal{L}}{\deg(x)},$$

where $\overline{\{x\}}$ is the cloure of x in $\mathcal{X}$ and $(- \cdot -)$ means taking the intersection number. If K is a number field, the height function can be defined similarly by arithmetic intersection theory. Assume L is ample. Let $Z_t(X, h_{\overline{L}})$ be the Zariski closure of the set $\left\{x \in X(\overline{K}) : h_{\overline{L}}(x) < t\right\}$. Note that

- $t \longmapsto Z_t(X, h_{\overline{L}})$ is an increasing filtration of Zariski closed subsets.
- $Z_t(X, h_{\overline{L}}) = X$ when $t \gg 0$ and $Z_t(X, h_{\overline{L}}) = \varnothing$ when $t \ll 0$.

Since Zariski topology on X is Noetherian, the filtration $Z_t(X, h_{\overline{L}})$ gives a finite filtration $X_0 = X \supsetneq X_1 \supsetneq X_2 \supsetneq \cdots \supsetneq X_r = \varnothing$, the *height filtration*. Its jumping points $\zeta_i(X, h_{\overline{L}}) = \inf\left\{t : Z_t(X, h_{\overline{L}}) = X_{i-1}\right\}$ are called the *successive minima.*
The goal of this project is to compute the height filtration and successive minima on flag varieties over function fields $K = k(C)$ when $\operatorname{char}(k) = p$.

On flag bundle

For any linear algebraic group Γ/k, a principal Γ-bundle on C is a variety F equipped with a right action of Γ and a Γ-equivariant smooth morphism $F \longrightarrow C$ such that the map

$$F \times_C (C \times \Gamma) \longrightarrow F \times_C F, \quad (f, (x, g)) \longmapsto (f, fg)$$

is an isomorphism. Attached to any principal Γ-bundle F, one has the degree map

$$\deg(F) : X(\Gamma) \longrightarrow \mathbb{Z}, \quad \lambda \longmapsto \langle \deg(F), \lambda\rangle = \deg\left(F \times_\Gamma k_\lambda\right).$$

Here $\deg\left(F \times_\Gamma k_\lambda\right)$ is the degree of the line bundle $F \times_\Gamma k_\lambda$ on the curve C.
Let H be a closed subgroup of a linear algebraic group Γ/k. A reduction of structure group of F to H is a pair (F_H, φ) where F_H is a principal H-bundle and $\varphi : F_H \times_H \Gamma \cong F$ is an isomorphism. By the universal property of the quotient F/H, the assignment to any section $\sigma : C \longrightarrow F/H$ the reduction $\sigma^* F$ of F to H is a one-one correspondence between reductions of structure group of F to H and sections of $F/H \longrightarrow C$.
Let F be a principal G-bundle on C for the connected reductive group G. Note that for any parabolic subgroup $P \subseteq G$ with Levi subgroup L_P, the natural inclusion

$$X(P) \longrightarrow X\left(L_P\right) \longrightarrow X\left(Z\left(L_P\right)\right)$$

becomes an isomorphism after tensoring with $\mathbb{Q}$. Thus we have

$$X(T)_{\mathbb{Q}} \longrightarrow X\left(Z\left(L_P\right)\right)_{\mathbb{Q}} = X(P)_{\mathbb{Q}}$$

and by taking duals, we get the so-called slope map $X(P)_{\mathbb{Q}}^{\vee} \longrightarrow X(T)_{\mathbb{Q}}^{\vee}$ introduced in [23, §2.1.3]. Let F_P be a reduction of F to P. By applying the slope map to $\deg\left(F_P\right)$ we can define $\left\langle \deg\left(F_P\right), \lambda\right\rangle$ for any $\lambda \in X(T)$.

Semistablity and canonical reduction

Definition 1 *A principal G-bundle F on C is called semistable if for any parabolic subgroup P, any reduction F_P of F to P and any dominant character λ of P which is trivial on $Z(G)$, we have $\left\langle \deg\left(F_P\right), \lambda\right\rangle \leq 0$.*
We say F is strongly semistable if for any non-constant finite morphism $f : C' \to C$, the pullback $f^* F$ is semistable.

Definition 2 *Let F be a principal G-bundle on C. A reduction F_Q of F to a parabolic subgroup Q is called canonical if the following two conditions hold:*

- *The principal L_Q bundle $F_Q \times_Q L_Q$ is semistable, where L_Q is the Levi subgroup of Q.*
- *For any non-trivial character λ of Q which is non-negative linear combination of simple roots, $\left\langle \deg\left(F_Q\right), \lambda\right\rangle > 0$.*

We say F_Q is strongly canonical if for any non-constant finite morphism $f : C' \to C$, the pullback $f^*(F_Q)$ is canonical.

Remark 1 *When $G = \mathrm{GL}_n$, this semistability recovers the one for vector bundles; The canonical reduction is equivalent to the Harder-Narasimhan filtration of vector bundles.*

Remark 2 *If the base field is of charateristic 0, then there is no difference between semistability (resp. canonical reduction) and strongly semistability (resp. strongly canonical reduction).*

Computation of HN filtration

Let V be any Q-representation of highest weight $\lambda \in X^*(T)$and let $V = \bigoplus_\nu V[\nu]$ be its weight decomposition. Furthermore, let F_Q be the canonical reduction of F. We define a filtration $V_\bullet$ on the vector space V:
For any rational number $q \in \mathbb{Q}$, we define the subspace V_q as the sum of weight spaces

$$V_q := \bigoplus_{\langle \deg F_Q, \nu\rangle \geq q} V[\nu]$$

Clearly, $V_{q'} \subseteq V_q$ whenever $q' \geq q$. We will consider the subspaces V_q only for the finitely many $q \in \mathbb{Q}$ where a jump occurs, i.e., only for those q such that $V_{q'} \subsetneq V_q$ for all $q' > q$. Let q_0 be the smallest and q_1 the largest rational number occurring among such q. Then V_{q_1} is the smallest nonzero filtration step, and V_{q_0} equals V.
Then, by twisting the Q-subrepresentations V_q above by F_P, we obtain a filtration $V_{\bullet, F_Q}$ of the vector bundle $V_{F_G} = F_G \times^G V$ by subbundles

$$0 \neq V\left[\lambda + \mathbb{Z}R_M\right]_{F_M} = \bigoplus_{\nu \in \lambda + \mathbb{Z}R_M} V[\nu] = V_{q_1, F_Q} \subsetneq \cdots \subsetneq V_{q, F_Q} \subsetneq \cdots \subsetneq V_{q_0, F_Q} = V_{F_G}.$$

Improving the result of Schieder, we proved that:

Proposition 1 *Assume the redution F_Q is the strongly canonical reduction, then the filtration $V_{\bullet, F_Q}$ of the vector bundle V_{F_G} is the Harder-Narasimhan filtration of V_{F_G}.*

Here, we used following facts: [1] If F is a strongly semistable M-bundle and $\rho : M \to H$ is a homomorphism that maps the connected component of of the center of M into that of H, then $F \times^M H$ is a semistable H-bundle. & E is a vector bundle and $0 = E_0 \subsetneq E_1 \subsetneq \cdots \subsetneq E_r = E$ is filtration of subbundles. If E_i/E_{i-1} is semistable of slope $\mu(E)$ for all i, then E is semistable.

Height lower bound on Schubert cells

Let F be a principal G-bundle with strong canonical reduction F_Q to some parabolic subgroup $Q \subseteq G$. Let $P \subseteq G$ be a parabolic subgroup. Set $\mathcal{X} = F/P$ and $X = \mathcal{X}_K$.
A character $\lambda : P \longrightarrow \mathbb{G}_m$ is called strictly anti-dominant if the natural pairing $\langle \alpha^\vee, \lambda\rangle < 0$ for any $\alpha \in \Delta \backslash \Delta_P$. Let $\lambda : P \longrightarrow \mathbb{G}_m$ be a stricly anti-dominant character. Then the line bundle $M_\lambda = G \times_P k_\lambda$ on G/P is ample. Therefore $\mathcal{L}_\lambda = F \times_G M_\lambda$ is a relatively ample line bundle on $\mathcal{X} = F \times_G G/P$ and induces a height function $h_{\mathcal{L}_\lambda}$.
For $w \in W_Q \backslash W / W_P$, write $\mathcal{C}_w = F_Q \times_Q QwP/P, \mathcal{X}_w = F_Q \times_Q \overline{QwP}/P, C_w = \mathcal{C}_{w,K}$ and $X_w = \mathcal{X}_{w,K}$.

Proposition 2 *For any $x \in C_w(\bar{K})$, $h_{\mathcal{L}_\lambda}(x) \geq \langle \deg F_Q, w\lambda\rangle$.*

Main theorem A

Theorem 1 *Assume the strongly canonical reduction of F exists. The height filtration of $h_{\mathcal{L}_\lambda}$ on F/P is given by successively deleting Schubert cells $C_w = \left(F_Q \times_Q QwP/P\right)_K$ for $w \in W_Q \backslash W / W_P$, i.e.*

$$Z_t = X \setminus \coprod_{\langle \deg(F_Q), w\lambda\rangle \geq t} C_w = \coprod_{\langle \deg(F_Q), w\lambda\rangle < t} C_w.$$

In particular, successive minima are $\zeta_w = \left\langle \deg\left(F_Q\right), w\lambda\right\rangle$ and Zhang's successive minima are $e_i = \min\left\{\zeta_w : \ell(w) = \dim G/P - i + 1\right\}$ where

$$\ell(w) = \max_{\sigma \in W_Q w W_P} \min_{\tau \in \sigma W_P} \ell(\tau).$$

Main theorem B

In general, a theorem by Langer [2] shows for a principal G-bundle F (might not admits a strongly canonical reduction), the strongly canonical reduction exists for $(\mathrm{Fr}^n)^* F$, when n is sufficiently large.
We have a cartesian diagram

$$\begin{array}{ccc} (\mathrm{Fr}^n)^* F/P & \xrightarrow{\phi} & F/P \\ \downarrow & & \downarrow \\ C & \xrightarrow{\mathrm{Fr}^n} & C \end{array}$$

Here Fr is the absolute Frobenius on C.
Suppose n is large enough such that $(\mathrm{Fr}^n)^* F$ has strongly canonical reduction, then we have:

Theorem 2 *The height filtration of $h_{\mathcal{L}_\lambda}$ on F/P is given by the image of the height filtration of the height filtration of $(\mathrm{Fr}^{\mathrm{n}})^* F/P$, the successive minima are $\frac{1}{p^n}$ of the succesive minima of $(\mathrm{Fr}^{\mathrm{n}})^* F/P$.*

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