Notes on Analytic stack

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1 Introduction

1.1 Motivation

Classically, there are several different theories from geometry and arithmetic.

1. complex analytic spaces (in the smooth case we have complex manifolds)

2. Locally analytic manifolds: Let $K, |\bullet|$ be a complete valued field, one can glue up K^d along local analytic maps. However, in the case of $K = \mathbb{Q}_p$ the topology is totally disconnected.

3. Rigid analytic geometry due to J. Tate: -focus on the local rings of functions instead of local topology. This is generalized to the adic space by Huber, which relieves the "finite type" condition.

4. Berkovich space: glue up local models given by some Banach ring (this construction, in particular, works for archimedean case).

- We want to unify all the cases in a single theory. The only theory allowing all the cases is Berkovich's, however, the gluing is not well worked out unless we take the "finite type" assumption.

- Even individually, these theories are less flexible than, for example, the theory of schemes.

– Issues of descent in $\mathrm{QCoh}(-)$.

- Reasons from Langlands: Fargues-Scholze gives the "geometrized" local Langlands

$$\mathbb{Q}_p \rightsquigarrow X_{FF},$$

where X_{FF} is the Fargues-Fontaine curve, an adic space (or scheme) of infinite type. Speculatively, we could also hope for a geometrization of global Langlands.

$$\mathbb{Z} \rightsquigarrow \text{Exotic analytic spaces over } \mathbb{Z}$$

1.2 Condensed math

all the local models above are in fact topological rings. For example, if one tensor two Tate algebras together, it is not complete, hence we need the complete tensor product. However, the topological rings and topological modules over them are not suitable for a general theory. The category of topological modules is **not** abelian.

To solve this issue, we go back to the basis of these topologies and define a replacement of the category of topological modules. The basic idea is this: We single out a collection of nice "test spaces" S, and instead of encoding topological spaces X traditionally, we just record the data of continuous maps $S \to X$. Formally our test sets are profinite sets, i.e. totally disconnected compact Hausdorff spaces.

Definition 1.1. A *light profinite set* is a countable inverse limit of finite sets. A *light condensed set* is a sheaf of sets on the category of light profinite sets with respect to the Grothendieck topology, where the covers are finite collections of jointly surjective maps. More explicitly, a light condensed set is a functor

$$\mathrm{LightProf}^{op} \to \mathrm{Sets}$$

such that

- $X(\varnothing) = *$.
- $X(S \coprod T) = X(S) \times X(T)$.
- If $T \twoheadrightarrow S$, then $X(S) \simeq \mathrm{eq}(X(T) \rightrightarrows X(T \times_S T))$.

Remark 1.2. The morphism of profinite sets are the continuous maps, or equivalently the maps come from the inverse systems.

Example 1.3. Any topological space X, the functor

$$X \mapsto X(S) = \mathrm{Cont}(S, X)$$

is a condensed set.

Remark 1.4. **Why this precise definition?** If $X = \{*\}$, then the correspondent condensed set $\underline{X}$ is the "underlying set". If $X = \mathbb{N} \bigcup \{\infty\}$ which is the one-point compatification of $\mathbb{N}$, $\underline{X}$ is the "convergent sequences".

Remark 1.5.
- Allowing all the surjective maps into our cases really simplified the structure of light condensed sets, hence giving good homological properties.
- Reunion the Grothendieck topology to be the finity gives good categorical compactness properties for

$$\mathrm{LightProf} \hookrightarrow \mathrm{CondSets}^{\mathrm{Light}}.$$

Even for all the metrizable compact Hausdorff spaces. For example, consider

$$\{0,1\}^{\mathbb{N}} \twoheadrightarrow [0,1].$$

1.3 (Light) Analytic Ring

$$\text{Condsets} \rightsquigarrow \text{CondRing}/\text{CondMod},$$

is not enough to give good geometry. Note that in the category of Condensed rings, one can formulate the relative tensor product, with

$$(\underline{A} \otimes_{\underline{k}} \underline{B})(*) = \underline{A}(*) \otimes_{\underline{k}(*)} \underline{B}(*) = A \otimes_k B.$$

But this ring is expected to be the complete tensor product, which corresponds to the fiber product in geometry. In order to fix this, we need extra structure. We record some classes of (Condensed) modules to be considered "complete". The right tensor product here should be a derived tensor product, hence we work with derived rings and there are two options:

- $\mathbb{E}_\infty$-algebras.
- Animated rings (presented by simplicial commutative ring).

Definition 1.6. A *condensed animated rings* R is a hypersheaf of animated rings on the site of profinite sets.

The basic invariant of such a ring R is its derived category $D(R)$. If R is static, i.e. R is an ordinary ring, $D(R)$ is the usual derived category of the abelian category of modules over R.

Definition 1.7. An *analytic ring* is a pair $R = (R^\triangle, D(R))$ where $R^\triangle$ is a condensed (animated) ring and $D(R)$ is a full subcategory of $D(R^\triangle)$, such that:

1. $D(R)$ is closed under all limits and colimits;
2. If $N \in D(R)$ and $M \in D(R^\triangle)$, then
$$RHom_{R^\triangle}(M, N) \in D(R).$$
3. If $\widehat{(-)}_R$ denotes the left adjoint to the inclusion $D(R) \subseteq D(R^\triangle)$, then $\widehat{(-)}_R$ sends $D(R^\triangle)_{\geq 0} \to D(R^\triangle)_{\geq 0}$.
4. $R^\triangle \in D(R)$.

A map of analytic rings $R \to S$ is a map of condensed rings $R^\triangle \to S^\triangle$ such that if $M \in D(S)$, then consider M as an objective in $D(R^\triangle)$, $M \in D(R)$.

Remark 1.8. There exists a t-structure on $D(R)$:

$$D(R)_{\geq 0} = D(R) \bigcap D(R^\triangle)_{\geq 0}, D(R)_{\leq 0} = D(R) \bigcap D(R^\triangle)_{\leq 0}.$$

which implies $D(R)^\heartsuit$ is nothing but the abelian subcategory $D(R) \bigcap D(R^\triangle)^\heartsuit$.

Remark 1.9. For a profinite set S, consider the free R-module

$$R[S] := (\widehat{R^\triangle[S]})_R,$$

and these generate $D(R)_{\geq 0}$ under colimits. The intuition of $R^\triangle[S]$ is "space of $R^\triangle$-linear combinations of Dirac measures on S", and then $R[S]$ is some completion, a bigger space of measures. If $M \in D(R^\triangle)^\heartsuit$ lies in $D(R)^\heartsuit$ if and only if for any $f : R^\triangle[S] \to M$, there exists a unique extension $\tilde{f} : R[S] \to M$. In other words, if there is a continuous map $\varphi : S \to M$ and $\mu \in R[S]$, there's a well-defined $\int_S f d\mu \in M$.

Remark 1.10. **Colimits of analytic rings**: The filtered colimits:

$$(\varinjlim_I R_i)^{\triangle} = \varinjlim_I (R_i^{\triangle}), (\varinjlim_I R_i)[S] = \varinjlim_I (R_i[S])$$

Pushouts: Consider $k \to A$ and $k \to B$, define $D(A \otimes_k A)$ to be the full subcategory of $D(A^{\triangle} \otimes_{k^{\triangle}} B^{\triangle})$ generated by underlying $A^{\triangle}$-modules and $B^{\triangle}$-modules. A warning is that the data

$$(A^{\triangle} \otimes_{k^{\triangle}} B^{\triangle}, D(A \otimes_k B))$$

is **not** an analytic ring, it satisfies 1-3 but not 4. To fix this, we apply the left adjoint to $D(A \otimes_k B)$. Then the category $D(A \otimes_k B)$ stays the same, but $(A^{\triangle} \widehat{\otimes_{k^{\triangle}}} B^{\triangle})$ becomes complete, we denote $(A \otimes_k B)^{\triangle}$.

1.4 Solid analytic rings

Slogan 1.11. Solid recovers adic spaces.

Let R be any ring. It's natural to consider

$$\mathcal{M}_R(\mathbb{N}) = R[\mathbb{N} \cup \{\infty\}]/R[\infty],$$

which classifies nullsequences in R-modules. The addition on $\mathbb{N}$ gives the ring structure on $\mathcal{M}_k(\mathbb{N})$, we have

$$R[T] \to \mathcal{M}_R(\mathbb{N}) \to R[\![T]\!],$$

Definition 1.12. R is solid if $\mathcal{M}_R(\mathbb{N})/(T-1) = 0$.

Theorem 1.13. *There exists a "universal" solid analytic ring $\mathbb{Z}^{\blacksquare} = (\mathbb{Z}, D(\mathbb{Z}^{\blacksquare}))$ such that an analytic ring R is solid if and only if there exists a unique morphism $\mathbb{Z}^{\blacksquare} \to R$. Moreover,*

- $\mathbb{Z}^{\blacksquare}(S) = \varprojlim_n \mathbb{Z}[S_n] = \prod_I \mathbb{Z}$ *(I is countable), where* $S = \varprojlim_n S_n$
- $\prod_I \mathbb{Z}$ *is compact projective generators of* $D(\mathbb{Z})$
- $\prod_I \mathbb{Z}$ *is flat with respect to* $\otimes_{\mathbb{Z}^{\blacksquare}}$ *and*

$$\prod_I \mathbb{Z} \otimes_{\mathbb{Z}^{\blacksquare}} \prod_J \mathbb{Z} = \prod_{I \times J} \mathbb{Z}$$

Remark 1.14. "$\mathbb{Z}^{\blacksquare}$ behaves like a regular ring of dimension 2".

2 Light Condensed Abelian Group

Our goal is to construct global perfectoid spaces over Spec $\mathbb{Z}$. In the theory of perfectoid spaces, we fix a prime number p and form the algebra like

$$\mathbb{C}_p\langle T^{\pm 1/p^{\infty}}\rangle = \mathbb{C}_p\langle T^{\mathbb{Z}[\frac{1}{p}]}\rangle,$$

But to form a similar theory for $\mathbb{R}$, we are asking for some algebra like

$$K\langle T^{\mathbb{Q}}\rangle \subseteq K\langle T^{\mathbb{R}}\rangle.$$

To let this make sense, the basic idea is to replace the spaces/groups/rings with condensed sets/groups/rings. In this framework, some $K\langle T^{\mathbb{Q}}\rangle$ does make sense as an underlying ring. We are supposed to develop the most natural foundation for analytic geometry based on condensed rings and try it out on known formalism.

2.1 Profinite sets

Our starting point is the category of profinite sets.

Proposition 2.1. *The following categories are equivalent:*

1. Pro(FSets), *where objects are* $\varprojlim_{i\in I} S_i$ *and* S_i *are finite sets and* I *is a cofiltered posets. Morphisms are (or continuous maps)*

$$\mathrm{Hom}(\varprojlim_{i\in I} S_i, \varprojlim_{J\in J} T_j) = \varprojlim_{I} \varinjlim_{J} \mathrm{Hom}(S_i, T_j).$$

2. *Totally disconnected separated Hausdorff spaces*
3. (Boolean algebras)$^{\mathrm{op}}$

given by

$$\varprojlim_{I} S_i \mapsto S \textit{ with product topology} \mapsto Cont(S, \mathbb{F}_2) = \varinjlim_{i} \mathbb{F}_2^{S_i}$$

and

$$A \mapsto \mathrm{Spec}\ A = \mathrm{Hom}(A, \mathbb{F}_2) = \varprojlim \mathrm{Hom}(A_i, \mathbb{F}_2),$$

where A_i *runs through all finite subalgebras.*

There are two ways to measure how big $S \in \mathrm{Pro(FSets)}$ are.

Definition 2.2. Let $S = \varprojlim_i S_i$ be a profinite set.

1. The size of S is $\kappa = |S|$;
2. The weight of S is $\lambda = |\mathrm{Cont}(S, \mathbb{F}_2)|$
3. S is light if $\lambda \leq \omega$, i.e. a countable limit of finite sets.

Remark 2.3. λ is equal to $|I|$, where I is smallest possible posets.

Example 2.4.

1. Finite sets.
2. $S = \mathbb{N} \cup \{\infty\} = \varprojlim \{0, 1, 2, \cdots, n, \infty\}$.
3. The Cantor sets $C = \{0, 1\}^{\mathbb{N}} = \varprojlim_n \{0, 1\}^n$.
4. The Stone-Cech compactification of $\mathbb{N}$.

Proposition 2.5. $\lambda \leq \kappa,\ \kappa \leq 2^{\lambda}$. *Also, If* $\kappa = \omega$, *then* $\lambda = \omega$.

Proof. $\lambda = |\mathrm{Cont}(S, \mathbb{F}_2)| \leq |\mathrm{Map(S}, \mathbb{F}_2)| = 2^{\kappa}$ and $\kappa = |\mathrm{Hom}(A, \mathbb{F}_2)| \leq \mathrm{Map}(A, \mathbb{F}_2) = 2^{\lambda}$. Now assume that $|S| = \omega$, hence write $S = \{s_0, s_1, \cdots\}$. For each n, we can choose inductively a series of compatible quotients $S \to S_n$ such that $\{s_1, \cdots, s_n\}$ maps into S_n injectively. This implies that $S \simeq \varprojlim_n S_n$ and consequently $\lambda = \omega$. The same proof shows that if in the infinite situation, $\lambda \leq \kappa$. □

Proposition 2.6. *The following categories are equivalent:*

1. *Countable profinite sets* $\mathrm{Pro}_{\mathbb{Z}}(\mathrm{FSets})$, *where objects are* $\varprojlim_{n\in\mathbb{N}} S_n$.
2. *Metrizable totally disconnected compact Hausdorff spaces.*
3. *(Countable Boolean Algebras)*^{op}.

Proposition 2.7. *The category of light profinite sets has all countable limits and Sequential limits of surjections are surjective.*

Proposition 2.8. *Let S be a light profinite set. Then there exists a surjection*

$$C = \{0,1\}^{\mathbb{N}} \twoheadrightarrow S.$$

Proof. Assume that $S = \varprojlim_i S_i$, then there exists n_i such that $n_i \geq |S_i|$, hence we can define $\{0,1\}^{n_i} \twoheadrightarrow S_i$. Then constructing a map of inverse systems inductively, we get $C \twoheadrightarrow S$. $\square$

Now, two properties are special to **light** profinite sets, which does not hold for general profinite sets.

Proposition 2.9. *Let S be a light profinite set and $U \subseteq S$ an open subset of S. Then U is a countable disjoint union of light profinite sets.*

Remark 2.10. In general, there exists $U \subseteq S$ profinite with

$$\mathrm{H}^i(U, \mathbb{Z}) \neq 0 \text{ for } i > 0.$$

Proof. Let $S = \varprojlim S_i$ and Z be the closed complement of U, which is of form $Z = \varprojlim Z_i$ and Z_i is the image of $Z \to S_i \subseteq S_i$. Hence $U = \bigcup_{i\in\mathbb{Z}} \pi_i^{-1}(S_i \backslash Z_i)$. Each $\pi_i^{-1}(S_i \backslash Z_i)$ is clopen in S. And we have

$$U = \coprod_i (\pi_{i+1}^{-1}(S_{i+1} \backslash Z_{i+1}) \backslash \pi_i^{-1}(S_i \backslash Z_i)).$$

$\square$

Proposition 2.11. *Let S be a light profinite set. Then S is an injective object in* Pro(FSets)*,* i.e. *for any injective map $Z \subseteq X$ of profinite sets, a morphism $Z \to S$ extends to a morphism $X \to S$.*

Proof. First check $S = \{0,1\}$. This is to say

$$\mathrm{Cont}(X, \mathbb{F}_2) \twoheadrightarrow \mathrm{Cont}(Z, \mathbb{F}_2),$$

i.e. each clopen subset of Z extends to a clopen subset in X.

In general, write $S = \varprojlim S_i$ and all $S_{n+1} \to S_n$ are surjective. We inductively extend $Z \to S_n$ to $X \to S_n$ in a compatible way. Without loss of generality, we can assume that S_n is a singleton and replace S_{n+1} by the fibre over that point, now we want to show the diagonal arrow exists.

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & S_n \arrow[d] \\ X \arrow[r] \arrow[ur, dashed, "\exists"] & * \end{tikzcd}$$

We immediately reduce this to the case $S = \{0,1\}$. $\square$

2.2 Condensed sets

Definition 2.12. A **light condensed set** is a sheaf on the category of light profinite sets for Grothendieck topology generated by

- Finite disjoint unions;
- All surjective maps.

Or equivalently, A functor

$$X : \mathrm{Pro(FSets)}^{op} \to \mathrm{Sets}, S \mapsto X(S),$$

such that

1. $X(\varnothing) = *$.

2. $X(S_1 \coprod S_2) \simeq X(S_1) \prod X(S_2)$.

3. For all surjective map $T \twoheadrightarrow S$, we have

$$X(S) \simeq \mathrm{eq}(X(T) \rightrightarrows X(T \times_S T)).$$

Example 2.13. Let A be a topological space. We can form a light condensed set

$$\underline{A} : S \mapsto \mathrm{Cont}(S, A).$$

This is a sheaf by some tautological argument. Now the $X(*)$ is the underlying set of A. And also $\underline{A}(\mathbb{N} \cup \{\infty\})$ is the convergent sequences in A.

Remark 2.14. Denote C to be the Cantor set. A light condensed set X is completely determined by $X(C)$ and the action of $\mathrm{End}(C)$ on it. Here $X(C)$ is an abstract set and $\mathrm{End}(C)$ is an abstract group.

Proposition 2.15. *The functor* $A \mapsto \underline{A}, \mathrm{Top} \to \mathrm{CondSets}^{\mathrm{Light}}$ *has a left adjoint*

$$X \mapsto X(*)_{\mathrm{top}},$$

i.e. *we have*

$$\mathrm{Hom}_{\mathrm{CondSets}}(X, \underline{A}) \simeq \mathrm{Hom}_{\mathrm{Top}}(X(*)_{\mathrm{top}}, A).$$

Proof. We define $X(*)_{\mathrm{top}}$ as $X(*)$ with the quotient topology

$$\coprod_{S, \alpha \in X(S)} S \to X(*),$$

or equivalently

$$\coprod_{\alpha \in X(C)} C \to X(*),$$

where C is the Cantor set. One can prove that this topological space is metrizably compactly generated and for any metrizably compactly generated A, $\underline{A}(*)_{top} \to A$ is an isomorphism. TBA. □

Corollary 2.16.

$$\{\text{Metrizably compactly generated top spaces}\} \hookrightarrow \mathrm{CondSets}^{\mathrm{Light}}$$

is a fully faithful embedding.

Warning 2.17. From a point on, we should get used to considering any profinite set, for example, the Cantor set, as the condensed set using Yoneda embedding.

Definition 2.18. qsqc (for general Topos)

1. An object X is quasi-compact if any cover $\coprod X_i \to X$ admits finite subcover. Here this is equivalent to there exists a surjection form cantor set C to X, or X is empty.

2. An object X is quasi-separated if for all quasi-compact $Y, Z \to X$, the fibre product is quasi-compact. Here for all $f, g : C \to X$, $C \times_X C$ is quasi-compact.

Remark 2.19. Allowing Cantor sets in the test category gives such simplified characterization of qsqc. If only allowed $\mathbb{N} \cup \{\infty\}$, then quasi-compact object would all be countable, therefore Cantor sets will not be qc.

Proposition 2.20. *One has*

1. *{qcqs light condensed sets} $\simeq$ {metrizable compact $Hausdorff$ spaces}*
2. *{qs light condensed sets} $\simeq$ Ind_{inj} {metrizable compact $Hausdorff$ spaces}*

Remark 2.21. Consider on the left, we have $[0,1]$. There is a natural surjection

$$\{0,1\}^{\mathbb{N}} \twoheadrightarrow [0,1], (a_0, a_1, \cdots) \mapsto (0.a_0a_1\cdots) \text{ in binary.}$$

Our major goal is to do homological algebra. Recall that For sheaves on any site, sheaves of abelian groups forms a Grothendieck abelian category with all limits and colimits exist and filtered colimits are exact. So the category

$$\mathrm{CondAb}^{\mathrm{Light}}$$

is a Grothendieck abelian category.

Example 2.22. Consider $\mathbb{R}_{disc} \to \mathbb{R}$, this gives $\underline{\mathbb{R}_{disc}} \to \underline{\mathbb{R}}$, we have

$$(\underline{\mathbb{R}_{disc}}/\underline{\mathbb{R}})(*) = 0$$

and

$$(\underline{\mathbb{R}_{disc}}/\underline{\mathbb{R}})(S) \stackrel{!}{=} \mathrm{Cont}(S, \mathbb{R})/\mathrm{Cont}(S, \mathbb{R}_{disc}),$$

which is nonzero. First we can check $\underline{\mathbb{R}_{disc}}/\underline{\mathbb{R}}$ is already a sheaf, and $\mathrm{Cont}(S, \mathbb{R}_{disc})$ is locally constant.

The category $\mathrm{CondSets}^{\mathrm{Light}}$ also has a tensor product, for $M, N \in \mathrm{CondSets}^{\mathrm{Light}}$ the tensor product is defined to be the sheafification of

$$S \mapsto M(S) \otimes N(S).$$

The unit object is just $\underline{\mathbb{Z}}$.

Theorem 2.23. *The forget functor* $\mathrm{CondAb}^{\mathrm{Light}} \to \mathrm{CondSets}^{\mathrm{Light}}$ *has a left adjoint*

$$X \mapsto \mathbb{Z}[X], X \mapsto \mathbb{Z}[X]$$

where $\mathbb{Z}[X]$ *is the sheafification of* $S \mapsto \mathbb{Z}[X(S)]$.

Remark 2.24. $\mathbb{Z}[X]$ has *some topology* on $\mathbb{Z}[X(*)]$.

Example 2.25.

$$\mathbb{Z}[\mathbb{R}] = \Big\{ \sum_{x \in \mathbb{R}} n_x[x], n_x \in \mathbb{Z} \text{ almost all zero} \Big\} = \varinjlim_{I} \mathbb{Z}[I].$$

and

$$\mathbb{Z}[I] = \bigcup_{n \in \mathbb{N}} \mathbb{Z}[I]_n := \Big\{ \sum_{x \in I} n_x[x] \Big| \sum |n_x| \leq n \Big\},$$

where $\mathbb{Z}[I]_n$ is compact Hausdorff. Hence this is qs. Given by the group structure on $\mathbb{R}$, $\mathbb{Z}[\mathbb{R}]$ is in fact a condensed ring.

Theorem 2.26. *In the category* $\mathrm{CondSets}^{\mathrm{Light}}$*,*

1. *Countable products are exact (and satisfy (AB6) in Tohoko).*
2. *Sequential limits of surjective maps are surjective.*
3. $\mathbb{Z}[\mathbb{N} \cup \{\infty\}]$ *is internally projective,* i.e. $\underline{\mathrm{Ext}}^i(P, -) = 0$ *for all* $i > 0$*, where*

$$\underline{\mathrm{Ext}}^i(M, N)$$

is the sheafification of

$$S \mapsto \mathrm{Ext}^i(M \otimes \mathbb{Z}[S], N).$$

Proof. (2) $\Rightarrow$ (1): Since the limits are always left exact, we only need to show when $f_n : M_n \to N_n$ are surjective, then the product

$$\prod f_n : \prod M_n \to \prod N_n$$

is also surjective. For all m, we have the following surjective map

$$\prod_{n \le m} M_n \times \prod_{n > m} N_n \twoheadrightarrow \prod_n N_n.$$

The map $\prod f_n$ is nothing but

$$\varprojlim_m \Big(\prod_{n \le m} M_n \times \prod_{n > m} N_n \twoheadrightarrow \prod_n N_n\Big),$$

which is surjective by (2).

Now prove (2): To prove a sheaf surjection, enough to prove

$$\begin{tikzcd}
M_\infty = \varprojlim_i (M_i) \arrow[r] & M_0 \\
S' \in \mathrm{Pro}(\mathrm{FSets}) \arrow[u, "\exists ?"] \arrow[r, two heads] & S \in \mathrm{Pro}(\mathrm{FSets}) \arrow[u]
\end{tikzcd}$$

Now we write down the whole limit, for $M_{n+1} \twoheadrightarrow M_n$, we can find S_{n+1}, therefore we can find the following diagram:

$$\begin{tikzcd}
M_\infty \arrow[r, "\sim"] & \varprojlim_i (M_i) & \cdots \arrow[r, two heads] & M_2 \arrow[r, two heads] & M_1 \arrow[rr, two heads] & & M_0 \\
& \cdots \arrow[ur] \arrow[rr, two heads] & & S_2 \arrow[u] \arrow[r, two heads] & S_1 \arrow[u] \arrow[rr, two heads] & & S \in \mathrm{Pro}(\mathrm{FSets}) \arrow[u] \\
S' \in \mathrm{Pro}(\mathrm{FSets}) \arrow[uu, "\exists ?"] \arrow[ur, two heads] \arrow[urrrrrr, two heads] & & & & & &
\end{tikzcd}$$

where we define $S' = \varprojlim_i S_i$. Previously we know that countable limit of surjective are still surjective in $\mathrm{Pro}_{\mathbb{N}}(\mathrm{FSets})$, hence S' is what we want. **Countable limits of covers are still covers**.

Proof of (3): Let $M = \mathbb{Z}[\mathbb{N} \cup \{\infty\}]/\mathbb{Z}[\infty]$. We claim that M is internally projective. First to show projective, we want to do the lifting:

$$\begin{tikzcd}
& \tilde{N} \arrow[d, two heads] \\
M \arrow[r] \arrow[ur, dashed, "\exists ?"] & N
\end{tikzcd}$$

Consider

$$\begin{tikzcd}
& & \tilde{N} \arrow[d, two heads] \\
\mathbb{N} \cup \{\infty\} \arrow[r] & M \arrow[r] & N
\end{tikzcd}$$

where we map ∞ to zero. Since $\tilde{N} \to N$ is surjective, we can find a light profinite set S such that

$$\begin{tikzcd}
S \arrow[rr, "g"] \arrow[d, two heads] & & \tilde{N} \arrow[d, two heads] \\
\mathbb{N} \cup \{\infty\} \arrow[r] & M \arrow[r] & N
\end{tikzcd}$$

We can pick a closed subset $S' \subseteq S$ such that $S \times_{\mathbb{N}\cup\{\infty\}} \mathbb{N} \simeq \mathbb{N}$. WLOG, we assum that $S = S'$. Let S_∞ be $S \times_{\mathbb{N}\cup\infty} \{\infty\}$ and S_∞ is light, therefore S_∞ is an injective object in profinite sets. Therefore there exists a retraction $r : S \to S_\infty$. Now consider

$$\begin{tikzcd}
S \arrow[r, "g-goior"] & \tilde{N} \\
S_\infty \arrow[u, hook] \arrow[r] & 0 \arrow[u, hook]
\end{tikzcd}$$

We claim that there is a pushout square:

$$\begin{tikzcd}
 & & \tilde{N} \\
S \arrow[r, two heads] \arrow[rru, "goior"] & \mathbb{N}\cup\{\infty\} \arrow[ru] & \\
S_\infty \arrow[u] \arrow[r] \arrow[ru, phantom, "\urcorner", very near end] & 0 \arrow[u] \arrow[ruu] &
\end{tikzcd}$$

This gives us $\tilde{f} : M \to \tilde{N}$, which is a lifting of f by definition. For internal projectivity, the argument is essntially the same. $\square$

2.3 Cohomology

Take X is a light condensed set and M is an abelian group.

Definition 2.27.

$$\mathrm{H}^i(X, M) := \mathrm{Ext}^i_{\mathrm{CondSets}^{\mathrm{Light}}}(\mathbb{Z}[X], \underline{M}).$$

Theorem 2.28. *If X is a CW complex, then*

$$\mathrm{H}^i(X, M) = \mathrm{Ext}^i(\mathbb{Z}[X], \underline{M}) \simeq \mathrm{H}^i_{\mathrm{sing}}(X, M).$$

Proof. Such X can be written as an increasing union of metrizable compact Hausdorff X_i,

$$X = \bigcup X_i.$$

On both sides, we are taking the filtered colimits to derived limits. Hence reduce to the case where X is compact. Then the theorem is the consequence of the following theorem. $\square$

Theorem 2.29. *Let X be a metrizable compact Hausdorff CW complex and M be an abelian groups. Then*

$$\mathrm{Ext}^i_{\mathrm{CondAb}^{\mathrm{Light}}}(\mathbb{Z}[\underline{X}], M) \simeq \mathrm{H}^i_{\mathrm{sheaf}}(X, \underline{M}),$$

where the right-hand side refers to the sheaf cohomology of constant sheaf associated with M.

Remark 2.30. For X CW complex, the sheaf cohomology recovers singular cohomology, *i.e.*

$$\mathrm{H}^i_{\mathrm{sheaf}}(X, \underline{M}) \simeq \mathrm{H}^i_{\mathrm{sing}}(X, M).$$

In general, it fails. For example, if X is totally disconnected compact Hausdorff, then the functor $\Gamma(X, -)$ is exact, therefore sheaf cohomology vanishes when $i > 0$ and

$$\mathrm{H}^0_{\mathrm{sheaf}}(X, M)$$

is all the locally constant maps $X \to M$. The 0th singular cohomology is all the maps $X \to M$.

Now we can upgrade it in the sense of topoi. now consider the Grothendieck sites

$$\mathrm{Pro}_{\mathbb{N}}(\mathrm{FSets})_X,$$

which is the profinite sets with a continuous map to X, and the site of open subsets of X

$$\mathrm{Open}(X).$$

Passing to the topoi of sheaves, there is a geometric morphism

$$\lambda : \mathrm{CondSets}^{\mathrm{Light}}{}_{\underline{X}} = \mathrm{Sh}(\mathrm{Pro}_{\mathbb{N}}(\mathrm{FSets})_X) \to \mathrm{Sh}X = \mathrm{Sh}(\mathrm{Open}(X)),$$

given by $\lambda^* U = \underline{U} \in \mathrm{CondSets}^{\mathrm{Light}}{}_{\underline{X}}$ for each $U \subseteq X$. Since $\underline{U} \in \mathrm{Sh}(X)$ are the generators in the $\mathrm{Sh}(X)$, $\lambda^* U$ determines all the $\lambda^* \mathscr{F}$. This defines

$$\lambda^* : D(\mathrm{Sh}(X, \mathrm{Ab})) \to D(\mathrm{Ab}(\mathrm{CondSets}^{\mathrm{Light}}{}_X)).$$

Theorem 2.31. *Assume that X is metrizable compact Hausdorff. On D^+, λ^* is fully faithful. Hence for any sheaf $\mathscr{F}$ on X, we have*

$$H^*(\mathrm{CondSets}^{\mathrm{Light}}{}_X, \lambda^* \mathscr{F}) \simeq H^*_{\mathrm{sheaf}}(X, \mathscr{F}).$$

Sketch. We prove that the adjunction map for $A \in D^+(\mathrm{Sh}(X, \mathrm{Ab}))$ is an ismorphism

$$A \to R\lambda_* \lambda^* A.$$

Check this on stalks at point x of X, *i.e.* to check

$$A_x \to (R\lambda_* \lambda^* A)_x$$

is an isomorphism for each of $x \in X$.

We claim that there is a base change property: taking stalks at x commutes with $R\lambda_*$. This comes from the general fact that "cohomology commutes with filtered colimits" results in general "coherent topoi", which is qcqs in this case (indeed $\underline{X}$ is qcqs). This reduces the case to $X = \{*\}$. And this is obviously true. □

Remark 2.32. Taking colimit along a neighborhood in a profinite topology is taking limits along closed neighborhoods. The key point is that the closed neighborhood is cofinal.

Let's do some concrete computation. Let X be a metrizable compact Hausdorff space, and we want to compute

$$\mathrm{Ext}^{\mathrm{i}}(\mathbb{Z}[\underline{X}], \mathbb{Z}).$$

We try to find some projective resolution or at least acyclic resolution.

First, assume that X is totally disconnected, *i.e.* , X is a profinite set. then X itself

$$\mathrm{Ext}^0(\mathbb{Z}[\underline{S}], \mathbb{Z}) = Cont(S, \mathbb{Z}),$$

and vanishes for all the other $i > 0$.

For any hyper cover of S, *i.e.* ,

$$\begin{tikzcd} \cdots \arrow[r] & S_2 \arrow[r, shift left=4] \arrow[r] \arrow[r, shift right=4] & S_1 \arrow[l, shift right=2] \arrow[l, shift left=2] \arrow[r, shift left=2] \arrow[r, shift right=2] & S_0 \arrow[l] \arrow[r] & S \end{tikzcd}$$

where $S_0 \twoheadrightarrow S$ and $S_1 \twoheadrightarrow S_0$, $\cdots$. Taking free abelian ring, we get

$$\cdots \mathbb{Z}[S_2] \cdots \mathbb{Z}[S_1] \to \mathbb{Z}[S_0] \to \mathbb{Z}[S] \to 0,$$

which is exact. We want to show that

$$0 \to Cont(S, \mathbb{Z}) \to Cont(S_0, \mathbb{Z}) \to Cont(S_1, \mathbb{Z}) \to \cdots.$$

TBA

Remark 2.33. It will follow from the cohomological descent and proper base change theorem.

Now we take X to be a general compact Hausdorff topological space X, and we want to resolve $\mathbb{Z}[\underline{X}]$ by $S \in \mathrm{Pro}_{\mathbb{N}}(\mathrm{FSets})$. we will have

$$\cdots \longrightarrow S_0 \times_{\underline{X}} S_0 \times_{\underline{X}} S_0 \; \substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow} \; S_0 \times_{\underline{X}} S_0 \; \substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow} \; S_0 \longrightarrow\!\!\!\!\to \underline{X}$$

as our resolution. This is a resolution by $\mathrm{Ext}^1(-, \mathbb{Z})$-acyclic. Hence the cohomology of $\mathbb{Z}[\underline{X}]$ is calculated by

$$0 \to Cont(S_0, \mathbb{Z}) \to Cont(S_1, \mathbb{Z}) \to \cdots .$$

We can treat all terms as global sections of sheaves on X, and one can check it resolves the constant sheaf associated with Z on X.

2.4 Locally compact abelian group

Let $\mathrm{LCA}_{\mathrm{m}}$ be the category of metrizable locally compact groups. For example, discrete abelian group, $\mathbb{R}$, $\mathbb{R}/\mathbb{Z}$, $\mathbb{Z}_p$, the adele $\mathbb{A} = \hat{\mathbb{Z}} \otimes \mathbb{Q} \times \mathbb{R}$, and one has

$$0 \to \mathbb{Q} \to \mathbb{A} \to \mathbb{A}/\mathbb{Q} \to 0,$$

where the first term is discrete and the last term is compact.

For the metrizable compact abelian group, we have its structure theorem. it admits a filtration with 3 pieces: discrete, finite-dimensional vector spaces and compact metrizable abelian group.

One can compute the Yoneda-Ext's.

Theorem 2.34. *For $A, B \in \mathrm{LCA}_{\mathrm{m}}$. Then in the category of light condensed abelian group,*

$$\mathrm{Ext}^{\mathrm{i}}(\underline{A}, \underline{B}) = \begin{cases} \mathrm{Hom}_{LCA}(A, B) & i = 0 \\ \mathrm{Ext}^1(A, B) & i = 1 \\ 0 & i \geq 2 \end{cases}$$

Example 2.35.

$$\mathrm{Ext}^{\mathrm{i}}(\underline{A}, \underline{\mathbb{R}/\mathbb{Z}}) = \begin{cases} A^* & i = 0 \\ 0 & i > 0. \end{cases}$$

where A^* is the dual abelian group. This is an analog of Pontryagin duality.

$$\mathrm{Ext}^{\mathrm{i}}(\underline{\mathbb{R}}, \mathbb{Z}) = 0$$

for any $i \geq 0$.

To do some computation we need to find something closed to a projective resolution of $\underline{A}$, which Scholze call it Breen-Deligne resolution.

Theorem 2.36. *There is a resolution of form*

$$\cdots \to \mathbb{Z}[M^n] \to \cdots \to \mathbb{Z}[M^2] \xrightarrow{[(a,b)] \mapsto [a] + [b] - [a+b]} \mathbb{Z}[M] \xrightarrow{[m] \mapsto m} M \to 0$$

functorial in abelian group M.

Remark 2.37. There is no explicit formula for this resolution. By functoriality, this also works for abelian sheaves on any site. Therefore, we reduce the computation of $\mathrm{Ext}^{\mathrm{i}}(\underline{A}, -)$ to computing $\mathrm{Ext}^{\mathrm{i}}(\mathbb{Z}[\underline{A}^i], -)$. And a key fact is

$$\mathrm{Ext}^{\mathrm{i}}(\mathbb{Z}[\underline{X}], \mathbb{R}) = \begin{cases} Cont(X, \mathbb{R}) & i = 0 \\ 0 & i > 0 \end{cases}$$

for X metrizable compact Hausdorff. It also works when one replaces $\mathbb{R}$ with any Banach space V. But it requires a local convexity of V and uses an argument of partition of unity.

Example 2.38.

$$\mathrm{Ext^i}(\underline{\mathbb{R}}, \mathbb{Z}) = 0 \quad \forall i \geq 0.$$

Using Breen-Deligne resolution, we have

$$\cdots \to \mathbb{Z}[\underline{\mathbb{R}}^n] \to \cdots \to \mathbb{Z}[\underline{\mathbb{R}}^2] \to \mathbb{Z}[\underline{\mathbb{R}}] \to \underline{\mathbb{R}} \to 0.$$

By the comparison between singular cohomology, we know

$$\mathrm{Ext^i}(\mathbb{Z}[\underline{\mathbb{R}}^n], \mathbb{Z}) = \mathrm{H}^i_{\text{sing}}(\mathbb{R}^n, \mathbb{Z}),$$

So each term only has one copy of $\mathbb{Z}$. So the Breen-Deligne resolution for $\underline{\mathbb{R}}$ is nothing but

$$\cdots \to \mathbb{Z}[0] \to \cdots \to \mathbb{Z}[0] \to \mathbb{Z}[0] \to 0.$$

Maclane's Q-construction

Corollary 2.39. *For all discrete abelian groups of M,*

$$\mathrm{Ext^i}(\prod_{\mathbb{N}} \mathbb{Z}, M) \begin{cases} \oplus_{\mathbb{N}} M & i = 0 \\ 0 & i > 0 \end{cases}$$

Remark 2.40. $\prod_{\mathbb{N}} \mathbb{Z}$ will become a compact projective generator of $\{\text{Solid}\} \subseteq \mathrm{CondAb}^{\text{Light}}$. Here, for X "compact" means that the corepresentable functor $\mathrm{Hom}(X, -)$ preserves filtered colimits, and "projective generator" means that X can "separate" objects in the category and is a projective object.

Remark 2.41. Here, as a profinite set, we have

$$\prod_{\mathbb{N}} \mathbb{Z} = \bigcup_{f:\mathbb{N}\to\mathbb{N}} \prod_{n\in\mathbb{N}} [-f(n), f(n)].$$

The union is a huge colimit.

Proof. Consider

$$0 \to \prod_{\mathbb{N}} \mathbb{Z} \to \prod_{\mathbb{N}} \underline{\mathbb{R}} \to \prod_{\mathbb{N}} \underline{\mathbb{R}/\mathbb{Z}} \to 0.$$

The last one is a metrizable compact abelian group. Recall that there is an extremely important result that **taking countable product is exact**. We have

$$\mathrm{Ext^i}(\prod_{\mathbb{N}} \mathbb{R}/\mathbb{Z}, M) = \begin{cases} 0 & i = 0 \\ \bigoplus_{\mathbb{N}} M & i = 1 \\ 0 & i > 1 \end{cases}$$

Now only need to check that

$$\mathrm{Ext^i}(\prod_{\mathbb{N}} \mathbb{R}, M) = 0 \quad \forall i \geq 0.$$

Now use the adjunction

$$RHom(\prod_{\mathbb{N}} \mathbb{R}, M) = RHom_{\underline{\mathbb{R}}}(\prod_{\mathbb{N}} \mathbb{R}, RHom(\underline{\mathbb{R}}, M)).$$

Here $RHom(\underline{\mathbb{R}}, M) = 0$ since M is discrete. $\square$

We are wondering in what generality one can take the product outside of Ext to get a coproduct. Consider $(*)$ condition: For all sequential limits

$$\cdots \twoheadrightarrow M_1 \twoheadrightarrow M_0$$

of countable discrete abelian groups and all discrete abelian groups N. Then an Assertion is that

$$\mathrm{Ext}^{\mathrm{i}}(\varprojlim_n M_n, N) \stackrel{?}{=} \varinjlim_n \mathrm{Ext}^{\mathrm{i}}(M_n, N) \quad \forall i$$

which is equivalent to

$$\mathrm{Ext}^{\mathrm{i}}(\prod_{\mathbb{N}} \bigoplus_{\mathbb{N}} \mathbb{Z}, \bigoplus_{\mathbb{N}} \mathbb{Z}) \stackrel{?}{=} 0 \quad \forall i.$$

But it is easy to see that the continuum hypothesis denies it.

Theorem 2.42 (A new progress in set theory). *1. $(*)$ implies that*

$$2^{\aleph_0} > \aleph_\omega.$$

2. It is consistent that $(*)$ holds and

$$2^{\aleph_0} = \aleph_{\omega+1}.$$

2.5 Solod abelian groups

Our goal is to isolate a class of "complete" objects in the category $\mathrm{CondAb}^{\mathrm{Light}}$ and also solve the issue that "the tensor product is not complete", we want to form a tensor product that leads to a notion of symmetry monoidal category with underlying category of "complete abelian groups". For example, in the category of $\mathrm{CondAb}^{\mathrm{Light}}$, we have

$$(\mathbb{Z}[\![T]\!] \otimes_{\mathrm{CondAb}^{\mathrm{Light}}} \mathbb{Z}[\![U]\!])(*) \neq \mathbb{Z}[\![T, U]\!].$$

The LHS is in fact regular tensor product $\mathbb{Z}[\![T]\!] \otimes_{\mathbb{Z}} \mathbb{Z}[\![U]\!]$.

Moreover, we want the condition of "being complete" to define an abelian subcategory of $\mathrm{CondAb}^{\mathrm{Light}}$. For example, consider the SES

$$0 \to \mathbb{Z} \to \underline{\mathbb{Z}_p} \to \underline{\mathbb{Z}_p}/\mathbb{Z} \to 0,$$

where the last term should also be complete in some sense (even though it is highly non-Hausdorff). In fact, it is difficult to find a notion for which $\mathbb{R}$ is complete, but there is a theory that works well in non-archimedean content.

Idea. For any $M \in \mathrm{CondAb}^{\mathrm{Light}}$, it should be "non-archimedean complete" if any null sequence in M is summable.

To formalize this idea, consider the object representing the null sequence

$$P = \mathbb{Z}[\mathbb{N} \cup \{\infty\}]/\mathbb{Z}[\infty] \in \mathrm{CondAb}^{\mathrm{Light}},$$

which is also a free compact projective object. In fact, it is internally projective, *i.e.*

$$\underline{\mathrm{Hom}}(P, -) : \mathrm{CondAb}^{\mathrm{Light}} \to \mathrm{CondAb}^{\mathrm{Light}}, M \mapsto (S \mapsto \mathrm{Hom}(P \otimes \mathbb{Z}[S], M))$$

is exact and preserves all the limits and colimits. Consider the endomorphism

$$f = \mathrm{id} - \mathrm{shift} : P \to P, [n] \mapsto [n] - [n+1].$$

Definition 2.43. Let $M \in \mathrm{CondAb}^{\mathrm{Light}}$, M is *solid* if

$$f^* : \underline{\mathrm{Hom}}(P, M) \xrightarrow{\sim} \underline{\mathrm{Hom}}(P, M)$$

is an isomorphism. To write termwisely, $\underline{\mathrm{Hom}}(P, M)$ is "the space of nullsequences" and

$$f^* : (m_0, m_1, m_2, \cdots) \mapsto (m_0 - m_1, m_1 - m_2, \cdots).$$

Whenever f^* is an isomorphism, the inverse of f^* should be "the map"

$$(m_0, m_1, \cdots) \mapsto (m_0 + m_1 + \cdots, m_1 + m_2 + \cdots, \cdots) = (\sum_{k=i}^{\infty} m_k)_i.$$

Proposition 2.44. *The subcategory* Solid $\subset \mathrm{CondAb}^{\mathrm{Light}}$ *is abelian, stable under kernels, cokernels, extensions, all limits and colimits, internal* $\underline{\mathrm{Hom}}(-,-)$ *and internal* $\underline{\mathrm{Ext}^{\mathrm{i}}(-,-)}$*, and it contains* $\mathbb{Z}$*.*

Proof. All this is for free because P is internally projective. For example, we prove stability under $\underline{\mathrm{Hom}}(-,-)$, *i.e.* for $M \in \mathrm{Solid}, N \in \mathrm{CondAb}^{\mathrm{Light}}$, we want to prove

$$\underline{\mathrm{Hom}}(N, M) \in \mathrm{Solid},$$

which is to check the induced f^* is still an isomorphism on $\underline{\mathrm{Hom}}(P, \underline{\mathrm{Hom}}(N, M))$.

$$\begin{aligned}\underline{\mathrm{Hom}}(P, \underline{\mathrm{Hom}}(N, M)) &= \underline{\mathrm{Hom}}(P \otimes N, M)\\ &= \underline{\mathrm{Hom}}(N, \underline{\mathrm{Hom}}(P, M)),\end{aligned}$$

where the morphism induced by f^* is an isomorphism by the definition of M being solid. Similarly, we have $\underline{\mathrm{Hom}}(P, -)$ is exact, then

$$\underline{\mathrm{Hom}}(P, \underline{\mathrm{Ext}}^{\mathrm{i}}(N, M)) = \underline{\mathrm{Ext}}^{\mathrm{i}}(P \otimes N, M) = \underline{\mathrm{Ext}}^{\mathrm{i}}(N, \underline{\mathrm{Hom}}(P, M)).$$

To show that $\mathbb{Z} \in \mathrm{Solid}$, consider

$$\underline{\mathrm{Hom}}(P, \mathbb{Z}) = \bigoplus_{\mathbb{N}} \mathbb{Z},$$

Naively, since $\mathbb{Z}$ is discrete the null sequence is eventually zero, so it becomes a direct sum. It is easy to compute the f^* on the RHS. □

Corollary 2.45. *There is a left adjoint to the inclusion functor* Solid $\subset \mathrm{CondAb}^{\mathrm{Light}}$*, namely a "solidification"*

$$\mathrm{CondAb}^{\mathrm{Light}} \to \mathrm{Solid}, M \mapsto M^{\blacksquare},$$

for which we have

$$\mathrm{Hom}(M, N) = \mathrm{Hom}(M^{\blacksquare}, N),$$

for $N \in \mathrm{Solid}, N \in \mathrm{CondAb}^{\mathrm{Light}}$*. Moreover,* Solid *acquires unique symmetry monoidal* $\otimes^{\blacksquare}$ *making* $M \mapsto M^{\blacksquare}$ *symmetry monoidal.*

Proof. The existence of $M \mapsto M^{\blacksquare}$ is guaranteed by the adjoint functor theorem. We define in Solid

$$M \otimes^{\blacksquare} N := (M \otimes N)^{\blacksquare}.$$

We want for any $M, N \in \mathrm{CondAb}^{\mathrm{Light}}$,

$$(M \otimes N)^{\blacksquare} \xrightarrow{\sim} (M^{\blacksquare} \otimes N^{\blacksquare})^{\blacksquare}.$$

Only need to check that for all $S \in \mathrm{Solid}$, apply $\mathrm{Hom}(-, S)$, we get the same output, *i.e.*

$$\begin{tikzcd}
\mathrm{Hom}(M^{\blacksquare} \otimes N^{\blacksquare}, S) \arrow[r, "\sim"] \arrow[d, equal] & \mathrm{Hom}(M \otimes N, S) \arrow[d, equal] \\
\mathrm{Hom}(M^{\blacksquare}, \underline{\mathrm{Hom}}(N^{\blacksquare}, S)) \arrow[d, equal] & \mathrm{Hom}(N, \underline{\mathrm{Hom}}(M, S)) \arrow[d, equal] \\
\mathrm{Hom}(M, \underline{\mathrm{Hom}}(N^{\blacksquare}, S)) \arrow[r, equal] & \mathrm{Hom}(N, \underline{\mathrm{Hom}}(M^{\blacksquare}, S))
\end{tikzcd}$$

in which we consequently use the adjunction of solidification and inclusion and Hom-tensor adjunction, also the fact that Solid is stable under Hom functor. $\square$

Proposition 2.46. $\underline{\mathbb{R}}^{\blacksquare} = 0$.

Proof. Since $\underline{\mathbb{R}}^{\blacksquare} = 0$ is a ring, it is enough to prove $1 = 0 \in \underline{\mathbb{R}}^{\blacksquare} = 0$. Consider the map $P \to \underline{\mathbb{R}}$ given the following nullsequence

$$1, \frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \cdots \in \mathbb{R},$$

where 2^{-n} appears 2^n times. Consider the diagram

$$\begin{tikzcd}
& P \arrow[r] \arrow[dr] & \mathbb{R} \arrow[d] \\
\mathbb{Z} \arrow[r, "{[0]}"'] & P \arrow[u, dashed, "f"] \arrow[r, dashed, "\exists !"'] & \underline{\mathbb{R}}^{\blacksquare}
\end{tikzcd}$$

We denote x to be the image in $\underline{\mathbb{R}}^{\blacksquare}$ of $1 \in \mathbb{Z}$ in the lower row. Then intuitively, we know that x satisfies the "formula"

$$\text{“}x = 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \cdots .\text{”}$$

Now we claim that $x = 1 + x$, which implies that $0 = 1$. The proof is the formalization of the following

$$\begin{aligned}
x &= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \cdots \\
&= 1 + (\frac{1}{2} + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \cdots) \\
&= 1 + (1 + \frac{1}{2} + \frac{1}{2} + \cdots) = 1 + x.
\end{aligned}$$

$\square$

Corollary 2.47. *If $M \in \mathrm{CondAb}^{\mathrm{Light}}$ and it admits an $\underline{\mathbb{R}}$-module structure, then $M^{\blacksquare} = 0$. Even more, we have*

$$\mathrm{Ext}^{\mathrm{i}}(M, \mathrm{Solid}) = 0.$$

Proof. Consider the following resolution

$$\cdots \to M \otimes \underline{\mathbb{R}} \otimes \underline{\mathbb{R}} \to M \otimes \underline{\mathbb{R}} \to M \to 0.$$

Taking solidification, then we have

$$0 = M^{\blacksquare} \otimes^{\blacksquare} \underline{\mathbb{R}}^{\blacksquare} \to (M \otimes \underline{\mathbb{R}})^{\blacksquare} \twoheadrightarrow M^{\blacksquare},$$

which we implies that $M^{\blacksquare} = 0$. Similarly, we know that

$$\mathrm{Ext}^{\mathrm{i}}(M, \mathrm{Solid})$$

admits a $\underline{\mathbb{R}}^{\blacksquare}$-modules, which is a 0-module. $\square$

Now we turn to compute what is $P^{\blacksquare}$.

Lemma 2.48. *Let* $\prod_{\mathbb{N}}^{bdd}\mathbb{Z} := \bigcup_{n\in\mathbb{N}}\prod_{\mathbb{N}}(\mathbb{Z}\cap[-n,n])$*, as a subspace of bounded sequence in* $\prod_{\mathbb{N}}\mathbb{Z}$ *(both of them live in* $\mathrm{CondAb}^{\mathrm{Light}}$*). There is a map*

$$P \to \prod_{\mathbb{N}}^{bdd}\mathbb{Z}, [n] \mapsto (0, \cdots, 0, 1_{n-th}, 1, \cdots).$$

Then taking solidification,

$$P^{\blacksquare} \xrightarrow{\simeq} (\prod_{\mathbb{N}}^{bdd}\mathbb{Z})^{\blacksquare}.$$

Moreover, if $M \in \mathrm{Solid}$*, then for* $i \geq 0$*,*

$$\mathrm{Ext}^{\mathrm{i}}(P, M) \simeq \mathrm{Ext}^{\mathrm{i}}(\prod_{\mathbb{N}}^{bdd}\mathbb{Z}, M).$$

Proof. Consider the following diagram

$$\begin{tikzcd}
P\otimes\prod_{\mathbb{N}}^{bdd}\mathbb{Z} \arrow[r] & \prod_{\mathbb{N}}^{bdd}\mathbb{Z} \\
P\otimes\prod_{\mathbb{N}}^{bdd}\mathbb{Z} \arrow[u, "f\otimes\mathrm{id}"] \arrow[ur] \arrow[r, dashed] & P \arrow[u]
\end{tikzcd}$$

To give a map $P\otimes\prod_{\mathbb{N}}^{bdd}\mathbb{Z} \to \prod_{\mathbb{N}}^{bdd}\mathbb{Z}$ by adjunction is the same to give a map

$$P \to \underline{\mathrm{Hom}}(\prod_{\mathbb{N}}^{bdd}\mathbb{Z}, \prod_{\mathbb{N}}^{bdd}\mathbb{Z}),$$

which is the null sequence of endomorphism of $\prod_{\mathbb{N}}^{bdd}\mathbb{Z}$. In this sense, our map is the null sequence of "projection to coordinates $\geq n$". Then the composition

$$P\otimes\prod_{\mathbb{N}}^{bdd}\mathbb{Z} \xrightarrow{f\otimes\mathrm{id}} P\otimes\prod_{\mathbb{N}}^{bdd}\mathbb{Z} \to \prod_{\mathbb{N}}^{bdd}\mathbb{Z}$$

is the null sequence of "projection to nth coordinates, hence the composition factor through P again. Note that this map is split. Now we solidify the diagram:

$$\begin{tikzcd}
(P\otimes\prod_{\mathbb{N}}^{bdd}\mathbb{Z})^{\blacksquare} \arrow[r] & (\prod_{\mathbb{N}}^{bdd}\mathbb{Z})^{\blacksquare} \\
(P\otimes\prod_{\mathbb{N}}^{bdd}\mathbb{Z})^{\blacksquare} \arrow[u, "(f\otimes\mathrm{id})^{\blacksquare}"] \arrow[ur] \arrow[r, dashed] & P^{\blacksquare}, \arrow[u]
\end{tikzcd}$$

where the map $(f\otimes\mathrm{id})^{\blacksquare}$ is an isomorphism. Since the upper row is split and the first column is an isomorphism, we know that $P^{\blacksquare} \to (\prod_{\mathbb{N}}^{bdd}\mathbb{Z})^{\blacksquare}$ is split surjective. To show this is an isomorphism, we only need to starting form P then go around the diagram and check that it is identity on P, indeed, we have

$$[n] \mapsto (0, \cdots, 1_{n-th}, 0, \cdots) \mapsto 1\otimes(0, \cdots, 1_{n-th}, 0, \cdots) \mapsto \text{``}(\sum_{k=0}^{\infty}[k])\text{''}\otimes(0, \cdots, 1_{n-th}, 0, \cdots) \mapsto [n],$$

here "$(\sum_{k=0}^{\infty}[k])$" make sense after taking the solidification. Similar argument also applies to the case $\mathrm{Ext}^{\mathrm{i}}(-, \mathrm{Solid})$. $\square$

Lemma 2.49.
$$(\prod_{\mathbb{N}}^{bdd}\mathbb{Z})^{\blacksquare} \simeq (\prod_{\mathbb{N}}\mathbb{Z})^{\blacksquare} \simeq \prod_{\mathbb{N}}\mathbb{Z}.$$

Moreover,
$$\mathrm{Ext}^{\mathrm{i}}(\prod_{\mathbb{N}}^{bdd}\mathbb{Z}, \mathrm{Solid}) \simeq \mathrm{Ext}^{\mathrm{i}}(\prod_{\mathbb{N}}\mathbb{Z}, \mathrm{Solid})$$

Proof. Consider the SES
$$0 \to \prod_{\mathbb{N}}^{bdd}\mathbb{Z} \to \prod_{\mathbb{N}}\mathbb{Z} \to (\prod_{\mathbb{N}}\mathbb{Z})/(\prod_{\mathbb{N}}^{bdd}\mathbb{Z}) \to 0.$$

We claim that $(\prod_{\mathbb{N}}\mathbb{Z})/(\prod_{\mathbb{N}}^{bdd}\mathbb{Z})$ is a $\underline{\mathbb{R}}$-module, hence solidifies to zero. Then we have the isomorphism. To prove the claim, consider the following isomorphism
$$(\prod_{\mathbb{N}}\mathbb{Z})/(\prod_{\mathbb{N}}^{bdd}\mathbb{Z}) \simeq (\prod_{\mathbb{N}}\mathbb{R})/(\prod_{\mathbb{N}}^{bdd}\mathbb{R}).$$

It is easy to check by hand, hence we are done. Also by a previous result, for $\underline{\mathbb{R}}$-module we have
$$\mathrm{Ext}^{\mathrm{i}}((\prod_{\mathbb{N}}\mathbb{Z})/(\prod_{\mathbb{N}}^{bdd}\mathbb{Z}), \mathrm{Solid}) = 0.$$

□

Corollary 2.50. *$P^{\blacksquare} \simeq \prod_{\mathbb{N}}\mathbb{Z}$ and for all $M \in \mathrm{Solid}$,*
$$\mathrm{Ext}^{\mathrm{i}}(P, M) \simeq \mathrm{Ext}^{\mathrm{i}}(\prod_{\mathbb{N}}\mathbb{Z}, M),$$

which is equal if computing it in the category $\mathrm{CondAb}^{\mathrm{Light}}$, hence when $i = 0$ it is $\oplus_{\mathbb{N}}M$ and vanishes when $i \geq 0$.

Remark 2.51. This is an independent argument of $\prod_{\mathbb{N}}\mathbb{Z}$ is compact projective.

Corollary 2.52.
$$\prod_{\mathbb{N}}\mathbb{Z} \otimes^{\blacksquare} \prod_{\mathbb{N}}\mathbb{Z} \simeq \prod_{\mathbb{N}\times\mathbb{N}}\mathbb{Z},$$

taking consideration of ring structure we have
$$\mathbb{Z}[\![T]\!] \otimes^{\blacksquare} \mathbb{Z}[\![T]\!] \simeq \mathbb{Z}[\![T, U]\!].$$

Proof. We have $P \otimes P \simeq P$ induced by a classical bijection
$$\mathbb{N} \times \mathbb{N} \to \mathbb{N}, (m, n) \mapsto \frac{(m+n)(m+n+1)}{2} + n.$$

Then solidified tensor of them, we obtain our isomorphism. □

Proposition 2.53. *Let S be any infinite light profinite set. Then there exists a map*
$$g : P \to \mathbb{Z}[S]$$

inducing an isomorphism
$$P^{\blacksquare} \simeq \mathbb{Z}[S]^{\blacksquare},$$

and

$$\mathrm{Ext}^{\mathrm{i}}(P, \mathrm{Solid}) \simeq \mathrm{Ext}^{\mathrm{i}}(\mathbb{Z}[S], \mathrm{Solid}).$$

Moreover, we have

$$\mathbb{Z}[S]^{\blacksquare} \simeq \prod_{\mathbb{N}} \mathbb{Z}.$$

Canonically, if $S = \varprojlim_i S_i$, we have

$$\mathbb{Z}[S]^{\blacksquare} \simeq \varprojlim_n \mathbb{Z}[S_n].$$

Sketch. Consider a series of surjection

$$S \twoheadrightarrow \cdots \twoheadrightarrow S_1 \twoheadrightarrow S_0.$$

We construct the following data inductively.

1. section $i_0 : S_0 \to S$.
2. on all elements of S_1 that are not in the image of $i_0(S_0)$, section $i_0 : S_1 \to S$, which gives us a section $i_1 : S_1 \to S$ compatible with i_0.
3. We have a series of sections $i_n : S_n \to S$, with the union of image of all i_n being a dense subset of S.
4. Enumorte all S_i as
$$\mathbb{N} = S_0 \coprod (S_1 \backslash S_0) \coprod (S_2 \backslash S_1) \coprod \cdots$$
5. We define a map
$$g : P \to \mathbb{Z}[S],$$
by on S_0 given by i_0 and on $S_n \backslash S_{n-1}$ given by the difference of maps induced by i_n and $S_n \backslash S_{n-1} \to S_{n-1} \xrightarrow{i_{n-1}} S$.

Now we have the following diagram:

$$\begin{tikzcd}
P \otimes \mathbb{Z}[S] \arrow[r] & \mathbb{Z}[S] \\
P \otimes \mathbb{Z}[S] \arrow[u, "f\otimes\mathrm{id}"] \arrow[r, dashed] & P \arrow[u, "g"']
\end{tikzcd}$$

Using the same understanding, the upper row map is the null sequence of endomorphism of $\mathbb{Z}[S]$ induced by

$$\mathrm{id}, \mathrm{id} - i_0\pi_0, \mathrm{id} - i_1\pi_1, \mathrm{id} - i_2\pi_2, \cdots .$$

Composition with $f \otimes \mathrm{id}$ is the null sequence of endomorphism of $\mathbb{Z}[S]$ induced by

$$i_0\pi_0, i_1\pi_1 - i_0\pi_0, \cdots, i_n\pi_n - i_{n-1}\pi_{n-1}, \cdots .$$

Therefore, the composition factors through g. Then solidify them as before and use the same argument. ☐

Theorem 2.54. *1.* Solid $\subseteq$ CondAb$^{\mathrm{Light}}$ *has a single projective generator* $\prod_{\mathbb{N}} \mathbb{Z}$.

2. For $M \in$ CondAb$^{\mathrm{Light}}$, M is solid if and only if for all $S = \varprojlim_{\mathbb{N}} S_i \in \mathrm{Pro}_{\mathbb{N}}(\mathrm{FSets})$, all $g : S \to M$, there exists a unique extension of g to

$$S \to \mathbb{Z}[S]^{\blacksquare} = \varprojlim_i \mathbb{Z}[S_i] \to M.$$

Proof. Only $\Leftarrow$ is new. We may present M as the cokernel

$$\bigoplus \mathbb{Z}[S_j] \to \bigoplus \mathbb{Z}[S_i] \to M \to 0.$$

in $\mathrm{CondAb}^{\mathrm{Light}}$, and then solidify them

$$\begin{tikzcd}
\bigoplus \mathbb{Z}[S_j] \arrow[r] \arrow[d] & \bigoplus \mathbb{Z}[S_i] \arrow[r] \arrow[d] & M \arrow[r] & 0 \\
\bigoplus \mathbb{Z}[S_j]^{\blacksquare} \arrow[r] & \bigoplus \mathbb{Z}[S_i]^{\blacksquare} \arrow[ur, "\exists!"'] & &
\end{tikzcd}$$

Hence M is also the cokernel of solid abelian groups, which implies M solid. $\square$

Remark 2.55. In fact, we have $\mathbb{Z}[S]^{\blacksquare} = \underline{\mathrm{Hom}}(Cont(S, \mathbb{Z}), \mathbb{Z})$, which is "$\mathbb{Z}$-valued measure on S". In that sense, being solid is nothing but you can always integrate some map $g : S \to M$ to get $\mathbb{Z}[S]^{\blacksquare} \to M$, namely

$$\mu \mapsto \int g\mu \in M.$$

2.6 Complements on solid modules

In this section, we consider the corresponding derived categories.

Definition 2.56. Let $A \in D(\mathrm{CondAb}^{\mathrm{Light}})$ is solid if

$$f^*\underline{\mathrm{RHom}}(P, A) \simeq \underline{\mathrm{RHom}}(P, A),$$

or equivalently

$$f^*\underline{\mathrm{Hom}}(P, \mathrm{H}^i(A)) \simeq \underline{\mathrm{Hom}}(P, \mathrm{H}^i(A)),$$

or equivalently each $\mathrm{H}^(A)$ is solid.

Proposition 2.57.

$$D(\mathrm{Solid}) \to D(\mathrm{CondAb}^{\mathrm{Light}})$$

is a triangulated subcategory stable under infinite $\bigoplus$, $\prod$ *and stable under* $\underline{\mathrm{RHom}}$.

Proposition 2.58.

$$D(\mathrm{Solid}) \to D(\mathrm{CondAb}^{\mathrm{Light}})$$

is fully faithful with essential image A^i solid, and has a left adjoint

$$A^{\mathrm{L}\blacksquare} \hookleftarrow A,$$

and a unique symmetry monoidal $\otimes^{L\blacksquare}$ on $D(\mathrm{Solid})$.

Example 2.59.

$$\mathbb{Z}[S]^{\mathrm{L}\blacksquare} \simeq \mathbb{Z}[S]^{\blacksquare} \simeq \varprojlim \mathbb{Z}[S_i], \;\; P^{\mathrm{L}\blacksquare} \simeq \prod_{\mathbb{N}} \mathbb{Z}.$$

Also,

$$\prod_{\mathbb{N}} \mathbb{Z} \otimes^{\mathrm{L}\blacksquare} \prod_{\mathbb{N}} \mathbb{Z} \simeq \prod_{\mathbb{N}\times\mathbb{N}} \mathbb{Z}.$$

Proposition 2.60. *Let X be CW complex. Then*

$$H_i^{\mathrm{sing}}(X, \mathbb{Z}) \simeq H_i(\mathbb{Z}[\underline{X}]^{\mathrm{L}\blacksquare}).$$

moreover, we have a quasi-isomorphism

$$C_i^{\mathrm{sing}}(X, \mathbb{Z}) \simeq \mathbb{Z}[\underline{X}]^{\mathrm{L}\blacksquare}.$$

Example 2.61. $\mathbb{Z}[\underline{(0,1)}]^{\mathrm{L}\blacksquare} \simeq \mathbb{Z}$, $\mathbb{Z}[\underline{S^1}]^{\mathrm{L}\blacksquare} \simeq \mathbb{Z} \oplus \mathbb{Z}[1]$.

Proof. Formal reduction to the case of finite CW complexes. Assume X is compact. We resolute X by

$$\cdots \to \mathbb{Z}[\underline{S \times_X S}] \to \mathbb{Z}[\underline{S}] \to \mathbb{Z}[\underline{X}] \to 0.$$

So the derived solidification of $\mathbb{Z}[\underline{X}]$ is computed by the complex

$$\cdots \to \mathbb{Z}[\underline{S \times_X S}]^{\blacksquare} \to \mathbb{Z}[\underline{S}]^{\blacksquare} \to 0,$$

since $\mathbb{Z}[S]^{\mathrm{L}\blacksquare}$ is centred at degree 0. And for any profinite set S, we have

$$\mathbb{Z}[S] \simeq \underline{\mathrm{Hom}}(Cont(S,\mathbb{Z}),\mathbb{Z}) \simeq \underline{\mathrm{RHom}}(\underline{\mathrm{RHom}}(\mathbb{Z}[\underline{S}],\mathbb{Z}),\mathbb{Z}).$$

This means that the same formula is true for X.

$$\mathbb{Z}[\underline{X}]^{\mathrm{L}\blacksquare} \simeq \underline{\mathrm{RHom}}(\underline{\mathrm{RHom}}(\mathbb{Z}[X],\mathbb{Z}),\mathbb{Z}).$$

Here $\underline{\mathrm{RHom}}(\mathbb{Z}[X],\mathbb{Z})$ is nothing but $C^{\bullet}_{\mathrm{sing}}(X,\mathbb{Z})$ and $\underline{\mathrm{RHom}}(\underline{\mathrm{RHom}}(\mathbb{Z}[X],\mathbb{Z}),\mathbb{Z})$ is $C^{\mathrm{sing}}_{\bullet}(X,\mathbb{Z})$ as all the H_i^{sing} is finitely generated. □

Now we demonstrate some structure on the category Solid.

- There is a notion of finitely generated objects as quotients of $\prod_{\mathbb{N}} \mathbb{Z}$.
- There is a notion of finitely presented objects, *i.e.* , the cokernel of a map $\prod_{\mathbb{N}} \mathbb{Z} \to \prod_{\mathbb{N}} \mathbb{Z}$.

Theorem 2.62. *The finitely presented objects of* Solid *form an abelian category, stable under kernel, cokernel, and extension. We have*

$$\mathrm{Solid} \simeq \mathrm{Ind}(\mathrm{Solid}^{\mathrm{f.p.}}).$$

Any finitely presented $M \in$ Solid has resolution

$$0 \to \prod_{\mathbb{N}} \mathbb{Z} \to \prod_{\mathbb{N}} \mathbb{Z} \to M \to 0.$$

The key lemma is the following:

Lemma 2.63. *A finitely generated submodule of $\prod_{\mathbb{N}} \mathbb{Z}$ is isomorphic to a product of some copies of $\mathbb{Z}$.*

Proof. We have

$$g : \prod_{\mathbb{N}} \mathbb{Z} \to M \hookrightarrow \prod_{\mathbb{N}} \mathbb{Z}.$$

g is dual to a map $h : \oplus_{\mathbb{N}}\mathbb{Z} \to \oplus_{\mathbb{N}}\mathbb{Z}$. A fact is that a countable abelian group that embeds into a product of copies of $\mathbb{Z}$ is free. This implies the image of h is free. And the kernel K of h splits as a direct summand. Therefore without loss of generality, we assume that h is injective. So we have

$$0 \to \oplus_{\mathbb{N}}\mathbb{Z} \to \oplus_{\mathbb{N}}\mathbb{Z} \to Q \to 0.$$

Let us consider the map

$$Q \to \prod_{\mathrm{Hom}(Q,\mathbb{Z})} \mathbb{Z}$$

and

$$Q \twoheadrightarrow \bar{Q} \hookrightarrow \prod_{\mathrm{Hom}(Q,\mathbb{Z})} \mathbb{Z}.$$

The fact implies that $\bar{Q}$ is free and we have the kernel Q' of $Q \to \bar{Q}$ has no map to $\mathbb{Z}$. Therefore by replacing Q by Q', we assume that Q satisfying $\mathrm{Hom}(Q,\mathbb{Z}) = 0$. Now we dualize h,

$$0 \to \underline{\mathrm{Hom}}(Q,\mathbb{Z}) \to \prod_{\mathbb{N}} \mathbb{Z} \xrightarrow{g} \prod_{\mathbb{N}} \mathbb{Z} \to \underline{\mathrm{Ext}^1}(Q,\mathbb{Z}) \to 0.$$

But $\underline{\mathrm{Hom}}(Q,\mathbb{Z}) = 0$ as the S-valued points are $\mathrm{Hom}(Q \otimes \mathbb{Z}[S], \mathbb{Z}) = \mathrm{Hom}(Q, \mathit{Cont}(S,\mathbb{Z})) = 0$, since $\mathit{Cont}(S,\mathbb{Z})$ is free. Therefore g is injective. □

Corollary 2.64. *Any $M \in \mathrm{Solid}^{f.p.}$ is product of copies of $\mathbb{Z}$ and group of form $\underline{\mathrm{Ext}^{\mathrm{i}}}(Q,\mathbb{Z})$ for some countable abelian group Q with $\mathrm{Hom}(Q,\mathbb{Z}) = 0$.*

Corollary 2.65. *$\prod_{\mathbb{N}} \mathbb{Z}$ is flat with respect to $\otimes^{\blacksquare}$.*

Proof. We need for all the $M \in \mathrm{Solid}$, $M \otimes^{\mathrm{L}\blacksquare} \prod_{\mathbb{N}} \mathbb{Z}$ still sits in degree 0. Without loss of generality, we assume M to be finitely presented, so

$$0 \to \prod_{\mathbb{N}} \mathbb{Z} \to \prod_{\mathbb{N}} \mathbb{Z} \to M \to 0.$$

Now taking the solid tensor with $\prod_{\mathbb{N}} \mathbb{Z}$ the SES stays exact. This shows that

$$M \otimes^{\blacksquare} \prod_{\mathbb{N}} \mathbb{Z} \simeq \prod_{\mathbb{N}} M$$

. □

2.7 Some $\otimes^{\blacksquare}$-computations

Let M be an abelian group and $\widehat{M_p} := \varprojlim_n M/p^n M$. In such a situation, we often use the completed tensor product $\widehat{\otimes}$ which is the p-adic completion of usual $\otimes$. In complete generality, we consider

$$\widehat{M_p} := \mathrm{R}\varprojlim_n M/^{\mathrm{L}} p^n M,$$

where $M/^{\mathrm{L}} p^n M := [M \xrightarrow{p^n} M]$.

Proposition 2.66. *If $N, M \in D_{\geq 0}(\mathrm{Solid})$ are derived p-complete, then $N \otimes^{\mathrm{L}\blacksquare} N$ is derived p-complete.*

Corollary 2.67. $\widehat{(\oplus_{\mathbb{N}}\mathbb{Z})_p} \otimes^{\mathrm{L}\blacksquare} \widehat{(\oplus_{\mathbb{N}}\mathbb{Z})_p} \simeq \widehat{(\oplus_{\mathbb{N}\times\mathbb{N}}\mathbb{Z})_p}$.

Remark 2.68. Nothing special about p for any ring A, $x \in A$, $M, N \in D_{\geq 0}(\mathrm{Solid}_A)$ derived x-complete. Then $M \otimes^{\mathrm{L}\blacksquare} N$ is derived x-complete.

Proof Sketch. First of all, we have $\mathbb{Z}_p \otimes^{\mathrm{L}\blacksquare} \mathbb{Z}_p \simeq \mathbb{Z}_p$, this is because

$$\mathbb{Z}[\![T]\!] \otimes^{\mathrm{L}\blacksquare} \mathbb{Z}[\![U]\!] \simeq \mathbb{Z}[\![T,U]\!],$$

can we take the quotient $T = U = p$, we have

$$\mathbb{Z}_p \otimes^{\mathrm{L}\blacksquare} \mathbb{Z}_p \simeq \mathbb{Z}_p.$$

So everything happens in the subcategory $D(\mathrm{Solid}_{\mathbb{Z}_p})$, which is a full subcategory of Solid. Let us deal with the case where $M = N = \widehat{\bigoplus}_{\mathbb{N}}\mathbb{Z}_p$. Claim that

$$\widehat{\bigoplus_{\mathbb{N}}}\mathbb{Z}_p \simeq \varinjlim_{\substack{f:\mathbb{N}\to\mathbb{N},\\ f(\infty)=\infty}} \prod_{\mathbb{N}} p^{f(n)}\mathbb{Z}_p.$$

There is an obvious map from the right to the left and it is injective. To show it is surjective, and we can pick any $S \in \mathrm{Pro}_{\mathbb{N}}(\mathrm{FSets})$, for a map $g : S \to \widehat{\bigoplus}_{\mathbb{N}} \mathbb{Z}_p$ we want to lift it to a map $S \to \varinjlim_{\substack{f:\mathbb{N}\to\mathbb{N},\\ f(\infty)=\infty}} \prod_{\mathbb{N}} p^{f(n)}\mathbb{Z}_p$. Consider

$$S \xrightarrow{g} \widehat{\bigoplus_{\mathbb{N}}} \mathbb{Z}_p \simeq \varprojlim_n (\bigoplus_{\mathbb{N}} \mathbb{Z}/p^n),$$

Since S is compact, we know that each g_n factors through a finite sub summand $\bigoplus_{n\leq a(n)}(\mathbb{Z}/p^n)$ of $\bigoplus_{\mathbb{N}} \mathbb{Z}/p^n$. Therefore, we can find an increasing function just controlled by $a(n)$.

Therefore we have

$$\begin{aligned} M \otimes^{\mathrm{L}\blacksquare} N &= (\varinjlim_{\substack{f:\mathbb{N}\to\mathbb{N},\\ f(\infty)=\infty}} \prod_{n\in\mathbb{N}} p^{f(n)}\mathbb{Z}_p) \otimes^{\mathrm{L}\blacksquare} (\varinjlim_{\substack{g:\mathbb{N}\to\mathbb{N},\\ g(\infty)=\infty}} \prod_{n\in\mathbb{N}} p^{g(n)}\mathbb{Z}_p) \\ &= \varinjlim_{\substack{f,g:\mathbb{N}\to\mathbb{N},\\ f(\infty)=g(\infty)=\infty}} \prod_{n,m\in\mathbb{N}} p^{f(n)+g(m)}\mathbb{Z}_p. \\ &\simeq (\varinjlim_{\substack{h:\mathbb{N}\times\mathbb{N}\to\mathbb{N},\\ h\to\infty}} \prod_{n\in\mathbb{N}} p^{h(n,m)}\mathbb{Z}_p) \\ &= \widehat{\bigoplus_{\mathbb{N}\times\mathbb{N}}} \mathbb{Z}_p. \end{aligned}$$

□

2.8 Solid functional analysis (TBA)

We work over $\mathbb{Q}_p$ or over all complete nonarchimidean fields. We have

$$D(\mathrm{Solid}_{\mathbb{Q}_p}) \subseteq D(\mathrm{Solid}_{\mathbb{Z}_p}) \subseteq D(\mathrm{Solid}),$$

where the derived category of solid $\mathbb{Q}_p$-modules has compact projective generator $(\prod_{\mathbb{N}} \mathbb{Z}_p)[\frac{1}{p}]$. more usual, we have $(\widehat{\bigoplus_{\mathbb{N}} \mathbb{Z}_p})_p[\frac{1}{p}]$. The first one is an analog of "Smith" spaces and the second is an analog of "Banach" spaces. Usually, we will have a duality between them, which means that the dual of a Smith space is a Banach space and verse visa.

Proposition 2.69. *((light) Smith space)^{op} $\simeq$ ((separable) Banach space).*

Let us consider Frechet spaces now, *i.e.* , the countable limits od Banach spaces under dense transition maps. They have a standard notion of $\widehat{\otimes}$ for them, compatible with such limits.

Proposition 2.70. *If V, W are Frechet $\mathbb{Q}_p$-vector spaces, then*

$$\underline{V} \otimes^{\mathrm{L}\blacksquare} \underline{W} \simeq \underline{V \widehat{\otimes} W}.$$

For example,

$$\prod_{\mathbb{N}} \mathbb{Q}_p \otimes^{\mathrm{L}\blacksquare} \prod_{\mathbb{N}} \mathbb{Q}_p \simeq \prod_{\mathbb{N}\times\mathbb{N}} \mathbb{Q}_p.$$

Proof of Example. Use the interpretation that

$$\prod_{\mathbb{N}} \mathbb{Q}_p \simeq \varinjlim_{\substack{f:\mathbb{N}\to\mathbb{N},\\ f\to\infty,\\ \text{very fast}}} (\prod_{\mathbb{N}} p^{-f(n)}\mathbb{Z}_p)[\frac{1}{p}].$$

□

3 Analytic ring

3.1 The Solid affine line

Recall that a crucial object in our theory is the object of null sequences

$$P = \mathbb{Z}[\mathbb{N} \cup \infty]/\mathbb{Z}[\infty].$$

It is a ring with a ring map $\mathbb{Z}[T] \to P$. We can solidify them and get

$$\mathbb{Z}[T] \to \mathbb{Z}[\![T]\!].$$

Lemma 3.1.

$$\mathbb{Z}[\![T]\!] \otimes^{\mathrm{L}\blacksquare}_{\mathbb{Z}[T]} \mathbb{Z}[\![T]\!] \simeq \mathbb{Z}[\![T]\!].$$

Proof. We know that

$$\mathbb{Z}[\![T_1]\!] \otimes^{\mathrm{L}\blacksquare}_{\mathbb{Z}} \mathbb{Z}[\![T_2]\!] \simeq \mathbb{Z}[\![T_1, T_2]\!],$$

also $\mathbb{Z}[\![T_1]\!] \otimes^{\mathrm{L}\blacksquare}_{\mathbb{Z}} \mathbb{Z}[\![T_2]\!] \to \mathbb{Z}[\![T_1]\!] \otimes^{\mathrm{L}\blacksquare}_{\mathbb{Z}[T]} \mathbb{Z}[\![T_2]\!]$ is nothing but modulo $(T_1 - T_2)$. □

Geometrically, the solidification of P is some subspace of affine line over $\mathbb{Z}$. Note that this is not the formal completion at zero, because if we base change to some non-discrete ring we can distinguish the difference.

Base change to $\mathbb{Q}_p$, we have

$$\mathbb{Q}_p[T] \to \mathbb{Q}_p \otimes^{\mathrm{L}\blacksquare}_{\mathbb{Z}} \mathbb{Z}[\![T]\!],$$

where the latter one is $\mathbb{Q}_p \otimes^{\mathrm{L}\blacksquare}_{\mathbb{Z}} \mathbb{Z}[\![T]\!] = (\mathbb{Z}_p \otimes \mathbb{Z}[\![T]\!])[\frac{1}{p}] = \mathbb{Z}_p[\![T]\!][\frac{1}{p}] \neq \mathbb{Q}_p[\![T]\!]$, in which we use the derived solid tensor commutes with all colimits and limits. But $\mathbb{Z}_p[\![T]\!][\frac{1}{p}]$ is the ring of bounded functions on the open unit disk in $\mathbb{A}^1_{\mathbb{Q}_p}$.

In Tate's rigid geometry, the building blocks are the closed unit disk. So how do we construct it in our content? We can consider it as cutting out an open unit disk centered around the infinity in the affine line. The corresponding map of embedding the unit disk at infinity into the affine line is

$$\mathbb{Z}[T] \to \mathbb{Z}[\![T^{-1}]\!] \otimes \mathbb{Z}[T, T^{-1}] \simeq \mathbb{Z}(\!(T^{-1})\!),$$

so to get the closed unit disk is to kill $\mathbb{Z}(\!(T^{-1})\!)$. Note that we have a resolution

$$\mathbb{Z}[\![U]\!][T] \xrightarrow{UT-1} \mathbb{Z}[\![U]\!][T] \to \mathbb{Z}(\!(T^{-1})\!),$$

using the objects in the category $\mathrm{Mod}_{\mathbb{Z}[T]}(\mathrm{Solid}_{\mathbb{Z}})$. To kill $\mathbb{Z}(\!(T^{-1})\!)$ is the same as to ask $UT - 1$ is an isomorphism. This suggests the following definition.

Definition 3.2. Let M be an object in $\mathrm{Mod}_{\mathbb{Z}[T]}(\mathrm{Solid}_{\mathbb{Z}})$. We call M is $\mathbb{Z}[T]$-solid if

$$\underline{\mathrm{Hom}}(P, M) \xrightarrow[\sim]{\sigma T - 1} \underline{\mathrm{Hom}}(P, M),$$

where σ is the shift map, and equivalently

$$\underline{\mathrm{RHom}}(\mathbb{Z}(\!(T^{-1})\!), M) = 0.$$

Theorem 3.3.

$$\mathrm{Solid}_{\mathbb{Z}[T]} \subseteq \mathrm{Mod}_{\mathbb{Z}[T]}(\mathrm{Solid}_{\mathbb{Z}}) \subseteq \mathrm{Mod}_{\mathbb{Z}[T]}(\mathrm{CondAb}^{\mathrm{Light}})$$

is an abelian subcategory closed under limits, colimits and extensions.

- *If* $M \in \mathrm{Mod}_{\mathbb{Z}[T]}(\mathrm{CondAb}^{\mathrm{Light}})$ *and* $N \in \mathrm{Solid}_{\mathbb{Z}[T]}$, *then* $\underline{\mathrm{Ext}}^{\mathrm{i}}(M, N) \in \mathrm{Solid}_{\mathbb{Z}[T]}$.

- *There exists a left adjoint to the inclusion denoted by* $(-)^{T\blacksquare}$.
- *There exists a symmetry monoidal structure on* $(-)^{T\blacksquare}$.
- *The derived analog holds.*
- $$((\prod_{\mathbb{N}} \mathbb{Z})[T])^{T\blacksquare} = \prod_{\mathbb{N}} \mathbb{Z}[T].$$

Proof. All except the last one is the same as solidification. We claim that

$$M^{LT\blacksquare} = \underline{\mathrm{RHom}}_{\mathbb{Z}[T]}((\mathbb{Z}(\!(T^{-1})\!)/\mathbb{Z}[T])[-1], M)$$

The proof is based on the fact that

$$\mathbb{Z}(\!(T^{-1})\!) \otimes^{\mathrm{L}\blacksquare}_{\mathbb{Z}[T]} \mathbb{Z}(\!(T^{-1})\!) \simeq \mathbb{Z}(\!(T^{-1})\!).$$

Next we claim that the functor

$$D(\mathrm{Solid}_{\mathbb{Z}}) \to D(\mathrm{Solid}_{\mathbb{Z}[T]}), M \mapsto (M \otimes_{\mathbb{Z}} \mathbb{Z}[T])^{LT\blacksquare}$$

is T-exact, preserves limits and colimits and sends $\mathbb{Z}$ to $\mathbb{Z}[T]$. We are done by this claim. The proof of the claim we just refer to episode 7/29 35:44. $\square$

Example 3.4.

$$(\mathbb{Q}_p[T])^{T\blacksquare} = (\mathbb{Z}_p[T])^{T\blacksquare}[\frac{1}{p}] = (\varprojlim_n (\mathbb{Z}/p^n\mathbb{Z}[T])^{T\blacksquare})[\frac{1}{p}] = (\varprojlim_n (\mathbb{Z}/p^n\mathbb{Z}[T]))[\frac{1}{p}] = \widehat{\mathbb{Z}[T]}_p[\frac{1}{p}],$$

which is the "function" on the closed unit disk.

Let us play an open/closed game. Recall that let X be a topological space with an open subset $U \subseteq X$ and Z be the complement. We have the following diagram:

$$D(\mathrm{Sh}(Z)) \underset{\xleftarrow[i^!]{}}{\overset{\xleftarrow{i^*}}{\xrightarrow{i_*}}} D(\mathrm{Sh}(X)) \underset{\xleftarrow[j_*]{}}{\overset{\xleftarrow{j_!}}{\xrightarrow{j^*}}} D(\mathrm{Sh}(U)).$$