GEOMETRIC HEIGHT ON FLAG VARIETIES IN POSITIVE CHARACTERISTIC

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ABSTRACT. Let k be an algebraically closed field of characteristic $p \neq 0$. Let G be a connected reductive group over k, $P \subseteq G$ be a parabolic subgroup and $\lambda : P \longrightarrow G$ be a strictly anti-dominant character. Let C be a projective smooth curve over k with function field $K = k(C)$ and F be a principal G-bundle on C. Then $F/P \longrightarrow C$ is a flag bundle and $\mathcal{L}_\lambda = F \times_P k_\lambda$ on F/P is a relatively ample line bundle. We compute the height filtration and successive minima of the height function $h_{\mathcal{L}_\lambda} : X(\overline{K}) \longrightarrow \mathbb{R}$ over the flag variety $X = (F/P)_K$.

CONTENTS

1. INTRODUCTION

1.1. **Height filtration and successive minima.** Let K be either a number field or $K = k(C)$ where C is a projective smooth curve over a field k. Let X be a projective variety of dimension d over K and $\overline{L}$ be an adelic line bundle on X. These data induce an Arakelov height function $h_{\overline{L}}$ on X (see [9, §9] for a survey). A typical case is the *geometric height*, which is the one we concern in this article. Here we give the definition.

If $K = k(C)$, consider a projective flat morphism $\mathcal{X} \longrightarrow C$ with the generic fiber $X \longrightarrow \mathrm{Spec}(K)$ and a line bundle $\mathcal{L}$ on $\mathcal{X}$ with $\mathcal{L}_K \simeq L$. The data $(\mathcal{X}, \mathcal{L})$ define an adelic line bundle $\overline{L}$ and the height function $h_{\overline{L}}$ is given by

$$h_{\overline{L}} : X(\overline{K}) \longrightarrow \mathbb{Q},\ x \longmapsto \frac{\mathcal{L} \cdot \overline{\{x\}}}{\deg(x)} \text{ where } \overline{\{x\}} \text{ is the closure of } x \text{ in } \mathcal{X}.$$

We also denote this by $h_{\mathcal{L}}$ if there is no ambiguity. If K is a number field, the height function can be defined similarly via arithmetic intersection theory.

For any $t \in \mathbb{R}$, let $Z_t \subseteq X$ be the Zariski closure of the set $\{x \in X(\overline{K}) : h_{\overline{L}}(x) < t\}$. Let's call $\{Z_t : t \in \mathbb{R}\}$ the *height filtration*, and call its jumping points the *successive minima*. Note that our definition of successive minima are slightly different with Zhang [10]. Zhang considers only dimension jumps $e_i = \inf\{t : \dim Z_t \geq d - i + 1\}$, $i = 1, \ldots, d+1$. In this article, e_i will be called the *successive minima of Zhang* to avoid ambiguity.

In [3], Fan-Luo-Qu provides a new case where height filtration can be explicitly computed, which is the geometric height on flag varieties under the assumption $\mathrm{char}(k) = 0$. Following their essence, we examine the case of positive characteristic.

Let k be an algebraically closed field and C be a projective smooth curve over k with function field $K = k(C)$. Let G be a connected reductive group over k, $P \subseteq G$ be a parabolic subgroup and $\lambda : P \longrightarrow \mathbb{G}_m$ be a strictly antidominant character. Let F be a principal G-bundle over C and $\mathcal{X} = F/P$ with generic fiber $X = (F/P)_K$. Let $\mathcal{L}_\lambda = F \times_P k_\lambda$ and $h_{\mathcal{L}_\lambda} : X(\overline{K}) \longrightarrow \mathbb{R}$ be the induced height function by the following diagram.

$$\begin{tikzcd}
X = (F/P)_K \arrow[r] \arrow[d] & \mathcal{X} = F/P \arrow[d] \\
\mathrm{Spec}(K) \arrow[r] & C
\end{tikzcd}$$

Let F_Q be the canonical reduction of F to Q, and $\deg(F_Q) \in X(T)^\vee_{\mathbb{Q}}$ be the induced cocharacter. Let W, W_P and W_Q be the Weyl group of G, L_P and L_Q. For $w \in W_Q \backslash W / W_P$, let $C_w = (F_Q \times_Q QwP/P)_K \subseteq X$ be the corresponding Schubert cell. The numbers $\langle \deg(F_Q), w\lambda \rangle$ are well-defined. One main result in [3] is the following.

Theorem 1.1 (Theorem 2.1, [3])**.** *Assume k has characteristic zero. For any $t \in \mathbb{R}$, let $Z_t \subseteq X$ be the Zariski closure of the set $\{x \in X(\overline{K}) : h_{\mathcal{L}_\lambda}(x) < t\}$. Then*

$$Z_t = X \Bigg\backslash \coprod_{\langle \deg(F_Q), w\lambda \rangle \geq t} C_w = \coprod_{\langle \deg(F_Q), w\lambda \rangle < t} C_w.$$

From now on assume k has positive characteristic unless other description is given. We show that this result does not hold in characteristic p unless one puts the following assumption on F, namely F admits a strongly canonical reduction.

Definition 1.2 (Definition 2.2)**.** *We say a reduction F_Q to a parabolic subgroup Q is strongly canonical if for any non-constant finite morphism $f : C' \to C$, the pullback $f^*(F_Q)$ is the canonical reduction of f^*F.*

Theorem 1.3 (Theorem 3.1)**.** *The result in Theorem 1.1 holds under the assumption that F admit a strongly canonical reduction.*

Canonical reduction is automatically strongly canonical in characteristic 0, whereas strongly canonical reduction may not exist in characteristic $p > 0$. For general G-bundles that may not admit a strongly canonical reduction, we have the following rough form of our another main result.

Theorem 1.4. *The height filtration of X is given by successively deleting some Frobenius twist of Schubert cells.*

More explicitly, a theorem by Langer [5] shows for a principal G-bundle F the strongly canonical reduction exists for $(\mathrm{Fr}^n)^*F$, when n is sufficiently large. Therefore, consider the following Cartesian diagram for n sufficiently large.

$$\begin{array}{ccc} (\mathrm{Fr_C}^n)^*F/P & \xrightarrow{\phi} & F/P \\ \downarrow & & \downarrow \\ C & \xrightarrow{\mathrm{Fr}_C^n} & C \end{array}$$

with its generic fiber

$$\begin{array}{ccc} \tilde{X} = ((\mathrm{Fr_K}^n)^*F/P)_K & \xrightarrow{\phi_K} & X = (F/P)_K \\ \downarrow & & \downarrow \\ \mathrm{Spec}\ K & \xrightarrow{\mathrm{Fr}_C^n} & \mathrm{Spec}\ K \end{array}$$

Here Fr_C is the absolute Frobenius on C. Suppose n is large enough such that $(\mathrm{Fr}_C^n)^*F$ has strongly canonical reduction, then we have:

Theorem 1.5 (Theorem 3.2)**.** *The height filtration of $h_{\mathcal{L}_\lambda}$ on X is given by the image of the height filtration of the height filtration of $\tilde{X}$ along the homeomorphism ϕ_K, the successive minima of X are $\frac{1}{p^n}$ of the succesive minima of $\tilde{X}$.*

Remark 1.6. Here the height filtration of $\tilde{X}$ is computed by Theorem 3.1. Thus, we say the height filtration of X is given by successively deleting some Frobenius twist of Schubert cells as stated in Theorem 1.4.

1.2. **Toy example: projective spaces.** Let k be an algebraically closed field of positive characteristic and C be a curve over k with function field $K = k(C)$. Let E be a vector bundle of rank n on C. We take $\mathcal{X} = \mathbb{P}(E)$, $\mathcal{L}$ to be the relative ample line bundle $\mathcal{O}_{\mathbb{P}(E)}(1)$ and $X = \mathbb{P}(V)$ where V is the generic fiber of E. A special case of a theorem by Ballaÿ [1] show that: Let $\zeta_{ess} := \inf_{t\in\mathbb{R}}\{Z_t = X\}$. Then

$$\zeta_{ess} = \lim_n \frac{\mu_{max}(\pi_*\mathcal{L}^{\otimes n})}{n} (= \lim_n \frac{\mu_{max}(\mathrm{Sym}^n E)}{n}).$$

where $\mu_{\max}(-)$ is the maximal slope appears in the Harder-Narasimhan filtration.

Warning 1.7. *In positive characteristic, the symmetric power of a semistable vector bundle may not remain semistable, making the left-hand side quite difficult to compute. It's important to note that this issue arises only in positive characteristic. However, this problem can be resolved by sufficiently twisting the Frobenius morphism multiple times.*

Proposition 1.8 (Ramanan, Ramanathan, [7])**.** *For above E semistable, there exists N large enough such that for $n \geq N$, any symmetric power of $\mathrm{Fr}_C^{n,*}E$ is semistable, where Fr_C is the absolute Frobenius of C.*

Theorem 1.9 (Baby-version)**.** *Fix a sufficiently large n in the sense of the above proposition. Denote $V' = \mathrm{Fr}_K^{n,*}V$ and $\{V'_i\}$ the generic fiber of Harder-Narasimhan filtration $\{E'_i\}$ of $\mathrm{Fr}_C^{n,*}E$.*

(1) *The height filtration of $\mathbb{P}(V')$ is given by $\mathbb{P}(V') \supsetneq \mathbb{P}(V'/V'_1) \supsetneq \mathbb{P}(V'/V'_2) \cdots \supsetneq \mathbb{P}(V'/V'_r) = \emptyset$ with successive minima $\mu_{\max}(E'_i/E'_{i-1})$.*

(2) *The height filtration of $h_{\mathcal{O}(1)}$ on $\mathbb{P}(V)$ is the image of the height filtration of $h_{\mathcal{O}(1)}$ on $\mathbb{P}(V')$ along the natural map. The successive minima of $\mathbb{P}(V)$ are $\frac{1}{p^n}\mu_{\max}(E'_i/E'_{i-1})$.*

A byproduct would be the following equality:

$$L_{max} = \lim_n \frac{\mu_{max}(\mathrm{Sym}^n E)}{n},$$

where $L_{max} := \max_{f:Y\to C}\{\mu_{max}(f^*E)\}$, where f runs through all finite non-constant morphisms. It would be in particular interesting to have a purely vector bundle theoretical proof of this equality. Note that $L_{max} = \mu_{max}$ in characteristic zero, and $L_{max} = \max_n\{\mu_{max}(Fr_C^{n,*}E)\}$ in positive characteristic.

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2. Strongly canonical reduction

2.1. **Vector bundles.** Let E be a vector bundle on C. The *degree* of E is $\deg(E) := \deg(\det(E))$ and the *slope* of E is $\mu(E) := \deg(E)/\operatorname{rk}(E)$. It is called *slope semistable* if for every subbundle F of E, $\mu(F) \leq \mu(E)$. This is equivalent to $\mu(Q) \geq \mu(E)$ for every quotient bundle Q of E.

There exists uniquely a filtration $0 = E_0 \subseteq E_1 \subseteq \cdots \subseteq E_r = E$ such that

(1) E_i/E_{i-1} is semistable;
(2) $\mu(E_i/E_{i-1}) > \mu(E_{i+1}/E_i)$.

This filtration is called the *Harder-Narasimhan filtration* of E. $\mu(E_1)$ is called *maximal slope* of E and is also denoted by $\mu_{\max}(E)$.

2.2. **Principal bundles.** For any linear algebraic group Γ/k, let $X(\Gamma) = \mathrm{Hom}(\Gamma, \mathbb{G}_m)$ denote the character group of Γ. For a cocharacter $f \in X(\Gamma)^\vee$ and a character $\lambda \in X(\Gamma)$, we shall denote the pairing by $\langle f, \lambda\rangle$. For any $\lambda \in X(\Gamma)$, denote by k_λ the one-dimensional representation on the vector space k with Γ acting by λ.

A *principal* Γ*-bundle* on C is a variety F equipped with a right action of Γ and a Γ-equivariant smooth morphism $F \longrightarrow C$ such that the map

$$F \times_C (C \times \Gamma) \longrightarrow F \times_C F, \quad (f, (x, g)) \longmapsto (f, fg)$$

is an isomorphism.

Attached to a principal Γ-bundle F, one has an associated cocharacter

$$\deg(F) : X(\Gamma) \longrightarrow \mathbb{Z}, \quad \lambda \longmapsto \langle \deg(F), \lambda\rangle = \deg(F \times_\Gamma k_\lambda).$$

Here $\deg(F \times_\Gamma k_\lambda)$ is the degree of the line bundle $F \times_\Gamma k_\lambda$ on the curve C.

Let H be a closed subgroup of a linear algebraic group Γ over k. A *reduction of structure group* of F to H is a pair (F_H, ϕ) where F_H is a principal H-bundle and $\phi : F_H \times_H \Gamma \simeq F$ is an isomorphism.

By the universal property of the quotient F/H, the assignment to a section $\sigma : C \longrightarrow F/H$ the reduction σ^*F of F to H is a one-one correspondence between reductions of structure group of F to H and sections of $F/H \longrightarrow C$.

2.3. **Reductive groups, characters and cocharacters.** Let G be a connected reductive group over k. Fix a Borel subgroup $B \subseteq G$ and a maximal torus $T \subseteq B$. Let W be the Weyl group and Δ be the set of simple roots with respect to (G, B, T). For any $\alpha \in \Delta$, we denote by $\alpha^\vee$ the corresponding simple coroot.

We shall consider only parabolic subgroups containing B. For such a parabolic subgroup P, let $W_P \subseteq W$ be the Weyl group $W(L_P)$ of the Levi factor $L_P \subseteq P$ and $\Delta_P \subseteq \Delta$ be the simple roots of L_P. Note that the natural inclusion

$$X(P) \longrightarrow X(L_P) \longrightarrow X(Z(L_P))$$

becomes an isomorphism after tensoring with $\mathbb{Q}$. Thus we have

$$X(T)_{\mathbb{Q}} \longrightarrow X(Z(L_P))_{\mathbb{Q}} = X(P)_{\mathbb{Q}}$$

and by taking duals, we get the so-called *slope map* $X(P)_{\mathbb{Q}}^\vee \longrightarrow X(T)_{\mathbb{Q}}^\vee$ introduced in [8, §2.1.3]. In other words, a cocharacter on $X(P)_{\mathbb{Q}}$ can be extended canonically to $X(T)_{\mathbb{Q}}$.

2.4. **The strongly canonical reduction to a parabolic subgroup.** Let G be a connected reductive group over k and F be a principal G-bundle over C. Let F_P be a reduction of F to a parabolic subgoup P. Let $\deg(F_P) \in X(T)_{\mathbb{Q}}^\vee$ be the induced (rational) cocharacter.

The G-bundle F is called *semistable* if for any parabolic subgroup P, any reduction F_P of F to P and any dominant character λ of P which is trivial on $Z(G)$, we have $\langle \deg(F_P), \lambda \rangle \leq 0$.

Definition 2.1. *We say F is strongly semistable if for any non-constant finite morphism $f : C' \to C$, the pullback f^*F is semistable.*

Among all filtrations of a vector bundle, there is a canonical one (the Harder-Narasimhan filtration). Similarly, among all reduction to parabolic subgroups, there is a canonical one. A reduction F_P of F to a parabolic subgroup P is called *canonical* if the following two conditions hold:

(1) The the principal L_P bundle $F_P \times_P L_P$ is semistable.
(2) For any non-trivial character λ of P which is non-negative linear combination of simple roots, $\langle \deg(F_P), \lambda \rangle > 0$.

Definition 2.2. *We say F_Q is strongly canonical if for any non-constant finite morphism $f : C' \to C$, the pullback $f^*(F_Q)$ is canonical.*

Remark 2.3. If the base field has characteristic 0, then semistability (resp. canonical reduction) and strongly semistability (resp. strongly canonical reduction) are equivalent and the (strongly) canonical reduction exists uniquely [2]. However, in positive characteristic, the notion of strongly canonical reduction becomes essential for our purposes, despite the fact that it does not generally exist.

3. Height filtrations and successive minima

Let C be a curve over a field k of positive characteristic and K be its function field. Let G be a connected reductive group over k. Let $P \subseteq G$ be a parabolic subgroup and $\lambda : P \longrightarrow \mathbb{G}_m$ be a strictly antidominant character. Let F be a principal G-bundle over C. Let $\mathcal{X} = F/P$ and $X = (F/P)_K$. Let $\mathcal{L}_\lambda = F \times_P k_\lambda$ and $h_{\mathcal{L}_\lambda} : X(\overline{K}) \longrightarrow \mathbb{R}$ be the induced height function.

$$\begin{array}{ccc} X = (F/P)_K & \longrightarrow & \mathcal{X} = F/P \\ \downarrow & & \downarrow \\ \mathrm{Spek}(K) & \longrightarrow & C \end{array}$$

Let F_Q be the canonical reduction of F to Q, and $\deg(F_Q) \in X(T)^{\vee}_{\mathbb{Q}}$ be the degree cocharacter. Let W, W_P and W_Q be the Weyl group of G, L_P and L_Q. For $w \in W_Q \backslash W / W_P$, let $C_w = (F_Q \times_Q QwP/P)_K \subseteq X$ be the corresponding Schubert cell. We have the following theorem.

Theorem 3.1. *Assume F admits strongly canonical reduction. For any $t \in \mathbb{R}$, let $Z_t \subseteq X$ be the Zariski closure of the set $\left\{x \in X(\overline{K}) : h_{\mathcal{L}_\lambda}(x) < t\right\}$. Then*

$$Z_t = X \Bigg\backslash \coprod_{\langle \deg(F_Q), w\lambda\rangle \geq t} C_w = \coprod_{\langle \deg(F_Q), w\lambda\rangle < t} C_w.$$

In particular, successive minima are $\zeta_w = \langle \deg(F_Q), w\lambda\rangle$ and Zhang's successive minima are $e_i = \min\{\zeta_w : \ell(w) = \dim G/P - i + 1\}$ where

$$\ell(w) = \max_{\sigma \in W_Q w W_P} \min_{\tau \in \sigma W_P} \ell(\tau).$$

For general G-bundle F which does not necessarily admits a strongly canonical reduction, consider the following Cartesian diagram for n sufficiently large.

$$\begin{array}{ccc} (\mathrm{Fr_C}^n)^* F/P & \xrightarrow{\phi} & F/P \\ \downarrow & & \downarrow \\ C & \xrightarrow{\mathrm{Fr}_C^n} & C \end{array}$$

with its generic fiber

$$\begin{array}{ccc} \tilde{X} = ((\mathrm{Fr_K}^n)^* F/P)_K & \xrightarrow{\phi_K} & X = (F/P)_K \\ \downarrow & & \downarrow \\ \mathrm{Spec}\ K & \xrightarrow{\mathrm{Fr}_C^n} & \mathrm{Spec}\ K \end{array}$$

Theorem 3.2. *The height filtration of $h_{\mathcal{L}_\lambda}$ on X is given by the image of the height filtration of the height filtration of $\tilde{X}$, the successive minima are $\frac{1}{p^n}$ of the successive minima of $\tilde{X}$.*

3.1. **A height lower bound in Schubert cells.** In subsection 3.1 we follow [3] since there is no difference between characteristic zero and characteristic p. For $w \in W_Q \backslash W / W_P$, write $\mathcal{C}_w = F_Q \times_Q QwP/P, \mathcal{X}_w = F_Q \times_Q \overline{QwP}/P, C_w = \mathcal{C}_{w,K}$ and $X_w = \mathcal{X}_{w,K}$.

Let F_Q be the canonical reduction of F to Q, and $\deg(F_Q) \in X(T)^{\vee}_{\mathbb{Q}}$ be the degree cocharacter. Let W, W_P and W_Q be the Weyl group of G, L_P and L_Q. For $w \in W_Q \backslash W / W_P$, write $\mathcal{C}_w = F_Q \times_Q QwP/P$, $\mathcal{X}_w = F_Q \times_Q \overline{QwP}/P$, $C_w = \mathcal{C}_{w,K}$ and $X_w = \mathcal{X}_{w,K}$.

Note that for any $w' \in W_Q$ and $\lambda \in X(T)$, we have $w'\lambda - \lambda \in \mathbb{Z}[\Delta_Q]$ and consequently $\langle \deg(F_Q), w'\lambda\rangle = \langle \deg(F_Q), \lambda\rangle$. Note also that for any $\lambda \in X(P)$ and $w \in W_P, w\lambda = \lambda$. So the number $\langle \deg(F_Q), w\lambda\rangle$ is well-defined for any $\lambda \in X(P)$ and $w \in W_Q \backslash W / W_P$.

Proposition 3.3. *For any $x \in C_w(\bar{K}), h_{\mathcal{L}_\lambda}(x) \geq \langle \deg F_Q, w\lambda \rangle$.*

Proof. The proof in [3] does not require $\mathrm{char}(k) = 0$, therefore works in our case. $\square$

3.2. **Height filtration and successive minima.** For any $t \in \mathbb{R}$, let $Z_t \subseteq X$ be the Zariski closure of the set $\{x \in X(\overline{K}) : h_{\mathcal{L}_\lambda}(x) < t\}$. Let $\zeta_{\mathrm{ess}}(X) := \inf\{t : Z_t = X\}$ be the *essential minimum* of $h_{\mathcal{L}_\lambda}$ on X.

Note that $\mathcal{X}_w$ is a closed subscheme of $\mathcal{X}$, so $\mathcal{L}_\lambda|_{\mathcal{X}_w}$ induces a height function $h_{\mathcal{L}_\lambda} : X_w(\overline{K}) \longrightarrow \mathbb{R}$, which is nothing but the restriction of $h_{\mathcal{L}_\lambda} : X(\overline{K}) \longrightarrow \mathbb{R}$ to $X_w(\overline{K})$. Let $\zeta_{\mathrm{ess}}(X_w)$ be the essential minimum of $h_{\mathcal{L}_\lambda}$ on X_w. Now we compute the *essential minimum* $\zeta_{\mathrm{ess}}(X_w)$ of X_w.

Lemma 3.4 ([4], Part I, §5.18)**.** *On $\mathcal{X}_w = F_Q \times_Q \overline{QwP}/P \subseteq \mathcal{X}$, we have*

$$\pi_*\left(\mathcal{L}_\lambda|_{\mathcal{X}_w}\right) = F_Q \times_Q \mathrm{H}^0\left(\overline{QwP}/P, M_\lambda\right).$$

Let V be any Q-representation of highest weight $\lambda \in X^*(T)$ and let $V = \bigoplus_\nu V[\nu]$ be its weight decomposition. Furthermore, let F_Q be the strongly canonical reduction of F. We define a filtration $V_\bullet$ on the vector space V: For any rational number $q \in \mathbb{Q}$, we define the subspace V_q as the sum of weight spaces

$$V_q := \bigoplus_{\langle \deg F_Q, \nu \rangle \geq q} V[\nu]$$

Clearly, $V_{q'} \subseteq V_q$ whenever $q' \geq q$. We will consider subspaces V_q only for the finitely many $q \in \mathbb{Q}$ where a jump occurs, that is, only for those q such that $V_{q'} \subsetneq V_q$ for all $q' > q$. Let q_0 be the smallest and q_1 the largest rational number occurring among such q. Then V_{q_1} is the smallest non-zero filtration step and V_{q_0} equals V.

Then, by twisting the Q-subrepresentations V_q above by F_P, we obtain a filtration $V_{\bullet, F_Q}$ of the vector bundle $V_{F_G} = F_G \times^G V$ by subbundles

$$0 \neq V\left[\lambda + \mathbb{Z}\Delta_M\right]_{F_M} = (\bigoplus_{\nu \in \lambda + \mathbb{Z}\Delta_M} V[\nu])_{F_M} = V_{q_1, F_Q} \subsetneq \cdots \subsetneq V_{q, F_Q} \subsetneq \cdots \subsetneq V_{q_0, F_Q} = V_{F_G}.$$

Improving a result of Schieder[1], we proved that:

Proposition 3.5. *Assume the redution F_Q is the strongly*[2] *canonical reduction, then the filtration $V_{\bullet, F_Q}$ of the vector bundle V_{F_G} is the Harder-Narasimhan filtration of V_{F_G}.*

We will need the following well-known theorem.

Theorem 3.6 ([6], Theorem 3.23)**.** *If F is a strongly semistable G-bundle and $\rho : G \to H$ is a homomorphism that maps the connected component of of the center of G into that of H, then $F \times^G H$ is a semistable H-bundle.*

Proof of Proposition 3.5. It will suffice to check two assertions: all qraded pieces are semistable vector bundles and their slopes are decreasing.

As shown in [8] Proposition 5.1, we have

$$\mathrm{gr}_q(V_{\bullet, F_P}) = (\mathrm{gr}_q V_\bullet)_{F_M}.$$

[1]The case of characteristic 0 and a specific case of positive characteristic were proven in [8]

[2]Proposition 3.5 fails to hold if we do not require the notion of strongly canonical reduction.

And its slope is computed as

$$\mu((\mathrm{gr}_q V_\bullet)_{F_M}) = q.$$

Therefore, the slopes of the graded pieces are decreasing by definition.

To prove the semistability, consider the semisimplification W of $(\mathrm{gr}_q V_\bullet)$ as a representation of M. Now apply [8] Proposition 3.2(a), we see that for each simple direct summand W_i of W the associated bundle W_{i,F_M} has slope q. Thus $(\mathrm{gr}_q V_\bullet)_{F_M}$ admits a filtration by subbundles such that each graded piece is a semistable vector bundle of slope q, where the semistability is obtained by applying Theorem 3.6 to $M \to \mathrm{GL}(M_i)$. Then $(\mathrm{gr}_q V_\bullet)_{F_M}$ is also semistable of slope q by the following. $\square$

Lemma 3.7. *If E is a vector bundle on C that admits a filtration of subbundles*

$$0 \subsetneq E_1 \subsetneq \cdots \subsetneq E_r = E,$$

and all graded piece are semistable of slope equals to λ, so is E.

Corollary 3.8. *Assume F_Q is the strongly canonical reduction of F. The Harder-Narasimhan filtration of $F_Q \times_Q H^0\left(\overline{QwP}/P, M_\lambda\right)$ is $F_Q \times_Q H^0\left(\overline{QwP}/P, M_\lambda\right)_{\bullet,\deg(F_Q)}$. Moreover, the maximal slope is $\langle \deg(F_Q), w\lambda\rangle$.*

Proof. The first assertion is obtained by taking V to be $H^0\left(\overline{QwP}/P, M_\lambda\right)$ in Proposition 3.5. For the second assertion, it will suffice to show the highest weights in $H^0\left(\overline{QwP}/P, M_\lambda\right)$ belong to $w\lambda + \mathbb{Z}[\Delta_Q]$. The argument in [3] is also valid without any assumption on the characteristic of base field. Note that one can also apply [4, § 14 Proposition 12] in positive characteristic, since Schubert variety in positive characteristic is normal by [7]. $\square$

Corollary 3.9. *Under the assumption that F has strongly canonical reduction, the essential minimum $\zeta_1(h_{\mathcal{L}_\lambda}, X_w)$ of $h_{\mathcal{L}_\lambda}$ on X_w is $\langle \deg(F_Q), w\lambda\rangle$.*

Proof. By Ballaÿ's theorem [1, 2],

$$\zeta_1(h_{\mathcal{L}_\lambda}, X_w) = \lim_{n\to\infty} \frac{\mu_{\max}(\pi_* \mathcal{L}_{n\lambda}|_{X_w})}{n} = \lim_{n\to\infty} \frac{n\langle \deg(F_Q), w\lambda\rangle}{n} = \langle \deg(F_Q), w\lambda\rangle.$$

$\square$

Proof of Theorem 3.1. The arguments follows from [3] once the lower bound and essential minimum is obtained. See [3, Proof of Theorem 2.1] $\square$

Now we treat the case F may not have a strongly canonical reduction.

Proof of Theorem 3.2. Theorem 5.1 of [5] shows for a principal G-bundle F (might not admits a strongly canonical reduction), the strongly canonical reduction exists for $(\mathrm{Fr}^n)^* F$, when n is sufficiently large. We have the following cartesian diagram

$$\begin{array}{ccc} (\mathrm{Fr_C}^n)^* F/P & \xrightarrow{\phi} & F/P \\ \downarrow & & \downarrow \\ C & \xrightarrow{\mathrm{Fr}^n} & C \end{array}$$

with its generic fibre

$$\begin{array}{ccc} \tilde{X} = ((\mathrm{Fr_K}^n)^*F/P)_K & \xrightarrow{\phi_K} & X = (F/P)_K \\ \big\downarrow & & \big\downarrow \\ \mathrm{Spec}\ K & \xrightarrow{\mathrm{Fr}^n} & \mathrm{Spec}\ K \end{array}$$

Here $\mathrm{Fr_C}$ is the absolute Frobenius on C. Suppose n is large enough such that $(\mathrm{Fr}^n)^*F$ has strongly canonical reduction. Note that ϕ is purely inseparable and we have the equality

$$\frac{1}{p^n} h_{\phi^*\mathcal{L}_\lambda}(x) = h_{\mathcal{L}_\lambda}(\phi_K(x))$$

for all $x \in \tilde{X}$. Our theorem follows. □

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