BEGINNING OF MODULI SPACE AND DEFORMATION

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Abstract. We try to get the first glimpse of moduli spaces and deformation theory in algebraic geometry. Here we rewind the construction of the Hilbert scheme and more generally Quot scheme. Once a scheme represents a functor, then the first-order deformation gives us a surprising description of the kernel and the cokernel of tangent maps. After giving an explanation of the notion of differential geometry about why first-order deformation (i.e. classification of extension over dual number) came into our sight, at last, we state and prove the result in a tangent map.

1. The construction of Quot scheme

Our strategy to prove that the Hilbert scheme exists is that: first, we define and state basic facts about the Quot scheme, then state and prove some very basic properties of Grassmannian as a Quot scheme in special cases, at last, we try to embed our Quot functor into the functor which Grassmannian represent and show that it's relatively represented by a scheme.

1.1. Quot functor.

Notations 1. We always denote the base scheme to be a noetherian scheme S and Sch_S to be the category of locally noetherian schemes over S.

Definition 2 (representable functor)**.** A functor $F : \mathcal{C}^{op} \to \mathrm{Sets}$, is said to be representable, if there exists $X \in \mathrm{Ob}(\mathcal{C})$ such that F is isomorphic to $h_X := \mathrm{Hom}_{\mathcal{C}}(-, X)$ as functors.

Definition 3 (family quotients)**.** Let S be a noetherian and let X be a scheme of finite type over S. Let E be a coherent sheaf on X. For any T in Sch_S, a family of quotients of E parameterised by T means such a pair $(\mathcal{F}, q)$ consisting of:

(1) a coherent sheaf $\mathcal{F}$ on $X_T = X \times_S T$ such that the schematic support of $\mathcal{F}$ is proper over T and $\mathcal{F}$ is flat over T, together with
(2) a surjective $\mathcal{O}_{X_T}$-homomorphism of sheaves $q : E_T \to \mathcal{F}$ where E_T is the pull-back of E under the projection $X_T \to X$.

Two such families $(\mathcal{F}, q)$ and $(\mathcal{F}', q')$ parameterized by T is considered as isomorphic if $\ker(q) = \ker(q')$, obviously this define a equivalent class.

Definition 4 ($\mathfrak{Quot}_{E/X/S}$)**.** Let X be a scheme of finite type over S. define a contravariant functor $\mathfrak{Quot}_{E/X/S} : \mathrm{Sch}_S \to \mathrm{Sets}$, which

$$T \mapsto \{\textit{All } (\mathcal{F}, q) \textit{ parameterized by } T\}.$$

Let $\psi : T' \to T$ be S-schemes, there is an induced morphism $\mathfrak{Quot}_{E/X/S}(T) \to \mathfrak{Quot}_{E/X/S}(T')$ defined by pull back along ψ. Since the properness and flatness are stable under base extension and the tensor product is right exact, this morphism in Sets is well-defined.

Remark 5. This is perhaps a quite general version, in particular, if we take X to be any projective scheme, the functor $\mathfrak{Quot}_{\mathcal{O}_X/X/S}$ associate any S-scheme to T the set of closed subschemes $Y \subset X_T$ that are proper and flat over T. We denote this functor by $\mathfrak{Hilb}_{X/S}$.

1.2. **Stratification by Hilbert polynomials.** Let $(X,\mathcal{L})$ be a pair in which X is a projective scheme over k with very ample invertible sheaf $\mathcal{L}$. Recall that any coherent sheaf F on X, the Hilbert polynomial related to F is defined by the function

$$\Phi_F(m)=\chi(F(m))=\sum_{i=0}^{n}(-1)^i\dim_k H^i(X,F\otimes L^{\otimes m})$$

where the dimensions of cohomologies are always finite. More generally, One may only assume that X is of finite type over k, $\mathcal{L}$ to be any line bundle, and $\mathrm{Supp}(F)$ with reduced induced scheme structure is proper to keep the finiteness. And a well-known theorem tells us there exists a polynomial P_F associated to F such that for any $m\gg 0$, one has $P(m)=\Phi(m)$.

Now let $X\to S$ be a projective morphism of noetherian schemes, with relatively very ample line bundle $\mathcal{L}$. Let $\mathcal{F}$ be any coherent sheaf on X. Take a point $s\in S$, we get a polynomial Φ_s as the Hilbert polynomial of $\mathcal{F}_s=\mathcal{F}|_{X_s}:=\mathcal{F}\otimes\mathrm{Spec}\ k(s)$ to the fibre X_s over s, where $k(s)$ is the residue field of S. Notice that we select the very ample line bundle to be $\mathcal{L}_s$ on the fiber X_s of each point to calculate Hilbert polynomials. One important reason for flatness is the following proposition.

Proposition 6. *By the assumption above, if $\mathcal{F}$ is flat over S, then the function $s\mapsto\Phi_s$ from S to $\mathbb{Q}[x]$ is locally constant on S.*

Proof. See Thm 9.9 in Chapter III [1]. □

This shows that there is a natural decomposition

$$\mathfrak{Quot}_{E/X/S}=\coprod_{\Phi\in\mathbb{Q}[\lambda]}\mathfrak{Quot}^{\Phi,\mathcal{L}}_{E/X/S}$$

where $\mathfrak{Quot}^{\Phi,\mathcal{L}}_{E/X/S}$ associate to any T the set of all equivalence classes of families $(\mathcal{F},q)$ such that at each $t\in T$ the Hilbert polynomial of $\mathcal{F}_t$ calculating with $\mathcal{L}_t$ is Φ on the fibre X_t. So if $\mathfrak{Quot}_{E/X/S}$ is representable by a scheme denote by $Quot_{E/X/S}$, this implies that

$$Quot_{E/X/S}=\coprod_{\Phi\in\mathbb{Q}[\lambda]}Quot^{\Phi,\mathcal{L}}_{E/X/S}.$$

1.3. **Grassmannian.** Let's see the simplest situation of Quot functor, we take $X=S=\mathrm{Spec}\ k$ for any arbitrary field k, then E is just a finite-dimensional vector space and Φ can only be an integer $d<n=\dim(V)$, the choice of $\mathcal{L}$ doesn't matter at all, then $\mathfrak{Quot}^{d,\mathcal{L}}_{V/k/k}(Y)$ is the set of all locally free quotients of $V\otimes_k\mathcal{O}_Y$ of rank d. And given such a quotient, take relatively $\mathcal{S}pec$ of its symmetry algebra, we get a d-dimensional vector subbundle of $\mathcal{S}pec\ \mathrm{Sym}(V\otimes_k\mathcal{O}_Y)\cong\mathbb{A}^n_Y$.

Let $U=\mathrm{Spec}\ A$ be an affine open subset of Y. If M is a $n\times d$-matrix with entries of any A, and $I\subset\{1,\dots,n\}$ with cardinality $\#(I)=d$, the I-minor of M_I of M means the $d\times d$ minor of M whose columns are indexed by I. Consider the $d\times n$-matrix X^I whose I-minor X^I_I is identity matrix $1_{d\times d}$, while the remaining entries of X^I are independent variables $x^I_{p,q}$ over A. Let $A[X^I]$ denote the polynomial ring in the variables $x^I_{p,q}$. and $U^I=\mathrm{Spec}\ A[X^I]$ which is non-canonically isomorphic to $\mathbb{A}^{(n-d)d}_U$.

For any rank d vector subbundle of $\mathbb{A}^n_Y$, it relates to a $n\times d$-matrix M with coefficients in A, on each fiber of rank d. One can find a $I\subset\{1,\dots,n\}$ with $\#(I)=n-d$, such that $\det(M_I)\neq 0$(may not unique), so let I run though every such set, we have a 'cover'. For any other $J\subset\{1,\dots,n\}$ with $\#(J)=n-d$, let $P^I_J=\det(X^I_J)\in A[X^I]$ where X^I_J is J-minor of X^I. Let $U^I_J=\mathrm{Spec}\ A[X^I,1/P^I_J]$ be the open subscheme where P^I_J invertible.

We define gluing map $\theta_{I,J}:A[X^J,1/P^J_I]\to A[X^I,1/P^I_J]$ as follows. The image of x^J_p,q is given by the entries of matrix formula $\theta_{I,J}(X^J)=(X^I_J)^{-1}X^I$. In particular, we have $\theta_{I,J}(P^J_I)=1/P^I_J$, so the map extends to $A[X^J,1/P^J_I]$. One can check this $\theta_{I,J}$ satisfies the cocycle condition, if $A=k$ we denote this scheme by $\mathrm{Grass_k(n,d)}$, it's easy to see that for any Y we get $\mathrm{Grass}(n,d)\times_k Y$. Also, the way we glue up induces one-to-one correspondence between $\mathfrak{Quot}^{d,\mathcal{L}}_{V/k/k}(Y)$ and $\mathrm{Hom}_k(Y,\mathrm{Grass}(n,d))$.

Moreover, let $Y' \to Y$ be a morphism of schemes, induced $\mathfrak{Quot}^{d,\mathcal{L}}_{V/k/k}(Y) \to \mathfrak{Quot}^{d,\mathcal{L}}_{V/k/k}(Y')$ is given by pull back along the morphism, $\mathrm{Hom}_k(Y, \mathrm{Grass}(n,d)) \to \mathrm{Hom}_k(Y', \mathrm{Grass}(n,d))$ is given by composition. Under the above correspondence, one can check these two maps are the same. To conclude, we prove the following proposition.

Proposition 7 (Grassmannian as a Quot scheme)**.** *Following above assumption, the Quot functor* $\mathfrak{Quot}^{d,\mathcal{L}}_{V/k/k}$ *is represented by* $\mathrm{Grass}_k(n,d)$.

Remark 8. We denote $\mathfrak{Grass}_k(n,d) := \mathfrak{Quot}^{d,\mathcal{L}}_{V/k/k}$. Once $\mathfrak{Quot}^{d,\mathcal{L}}_{V/k/k}$ is represented by $\mathbb{G} = \mathrm{Grass}_k(n,d)$, the identity map $\mathrm{id}_{\mathbb{G}} \in \mathrm{Hom}(\mathbb{G}, \mathbb{G})$ gives us a universal quotient $V \otimes_k \mathcal{O}_{\mathbb{G}} \to \mathcal{U} \to 0$ on $\mathbb{G}$. The reason why called universal is that for any scheme X over k and any quotient of $V \otimes \mathcal{O}_X$ on X gives us a morphism $i : X \to \mathbb{G}$, then this quotient can be recovered by $i^*\mathcal{U}$. Note that in general for any noetherian scheme S the Quot functor $\mathfrak{Quot}^{d,\mathcal{O}_S}_{E/S/S}$ is represented by $\mathrm{Grass}_{\mathbb{Z}}(n,d) \times_{\mathbb{Z}} S$ where E is any locally free sheaf of rank n. The proof is similar. Also note that $\mathrm{Grass}_{\mathbb{Z}}(n,d)$ is proper over $\mathbb{Z}$ and projective given by $\bigwedge^d V \otimes \mathcal{O}_{\mathbb{G}} \to \det(\mathcal{U})$.

1.4. **Idea and statement.** Actually, The development in history is inverse to our introduction above: A.Grothendieck generalize the conception of Grassmannian, by considering quotients of $V \otimes_k \mathcal{O}_X$ instead of quotients of V itself, while we treat Grassmannian as the simplest situation.

Theorem 9 (Grothendieck)**.** *Let S be a noetherian scheme and $f : X \to S$ be a projective morphism with f-ample line bundle $\mathcal{L}$ on X. Let $\mathcal{H}$ be a coherent $\mathcal{O}_X$-module and $P \in \mathbb{Q}[\lambda]$ a polynomial. The functor*

$$\mathcal{Q} := \mathfrak{Quot}^{P,\mathcal{L}}_{\mathcal{H}/X/S}$$

is represented by a projective S-scheme denoted $\pi : \mathrm{Quot}_{X/S}(\mathcal{H}, P) \to S$.

In the rest of this section, we only prove the theorem. We state some results by cohomology(with or without proof). We use m-regularity to 'limit' the dimension of the Grassmannian we embed $\mathcal{Q}$ into and after embedding into Grassmannian we use flatten stratification to show that the image is actually locally closed. Finally, we can prove the valuation criterion for properness, then everything is done.

1.5. **Some preparation.** Now let X be a projective scheme over a field k and $\mathcal{O}(1)$ be a very ample sheaf.

Definition 10 (m-regularity)**.** Let m be an integer. A coherent sheaf $\mathcal{F}$ is said to be m-regular, if

$$H^i(X, F(m-i)) = 0 \; for \; all \; i > 0.$$

Lemma 11. *If $\mathcal{F}$ is m-regular, then the following holds:*

1. *$\mathcal{F}$ is m'-regular for all integer $m' \leq m$.*
2. *$\mathcal{F}(m)$ is globally generated.*
3. *The natural homomorphism $H^0(X, \mathcal{F}(m)) \otimes H^0(X, \mathcal{O}_X(n)) \to H^0(X, \mathcal{F}(m+n))$ are surjective for $n \geq 0$.*

Definition 12. The Mumford-Castelnuovo regularity of a coherent sheaf $\mathcal{F}$ is the number $\mathrm{reg}(\mathcal{F}) = \inf\{m \in \mathbb{Z} | \mathcal{F} \; is \; m\text{-}regular\}$.

This definition always makes sense by Serre's vanishing theorem.

Definition 13 (Boundedness)**.** A family of isomorphism classes of coherent sheaves on X is bounded if there is a k-scheme S of finite type and a coherent $\mathcal{O}_{S\times X}$-sheaf $\mathcal{F}$ such that the given family is contained in the set $\{closed \; fibres \; \mathcal{F}_s \; where \; s \in S\}$.

Lemma 14. *The following properties of a family of sheaves $\{\mathcal{F}_\iota\}_{\iota \in I}$ are equivalent:*

1. *The family is bounded.*

(2) *The set of Hilbert polynomials $\{P(\mathcal{F}_\iota)\}_{\iota\in I}$ is finite and there is a uniform bound* $\operatorname{reg}(\mathcal{F}_\iota) \leq$ *for all $\iota \in I$.*
(3) *The set of Hilbert Polynomials $\{P(\mathcal{F}_\iota)\}_{\iota\in I}$ is finite and there is a coherent sheaf $\mathcal{F}$ such that all $\mathcal{F}_\iota$ admits surjective homomorphisms $\mathcal{F} \to \mathcal{F}_\iota$.*

Theorem 15 (flat stratification)**.** *Let S be a neotherian scheme, and let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^n_S$ over S. Then the set I of Hilbert polynomials of restrictions of $\mathcal{F}$ to fibres of $\mathbb{P}^n_S \to S$ is a finite set. Moreover, for each $f \in I$ there exists a locally closed subscheme $S_f \subset S$, such that the following condition satisfied:*

(1) *The underlying set S_f consists of all points $s \in S$ where the Hilbert polynomials of the restriction to the fiber are f. In particular, the subsets $S_f \subset S$ is disjoint and $\coprod_{f\in I} S_f \twoheadrightarrow S$.*
(2) *Let $S' = \coprod_{f\in I} S_f$ and let $i : S' \to S$ be the morphism induced by inclusions. Then the sheaf $i^*\mathcal{F}$ is flat over S' with universal property: For any $\phi : T \to S$ the pullback $\phi^*(\mathcal{F})$ on $\mathbb{P}^n_T$ is flat if and only if ϕ factor through $i : S' \to S$.*
(3) *Given the set I of Hilbert polynomials a totally order, defining $f < g$ if whenever $f(n) < g(n)$ for all $n \gg 0$. Then the closure of the subset S_f is contained in the union of all S_g where $f \leq g$.*

1.6. **Proof of Theorem 8.** We follow the notation in the statement in Theorem 8.

Step 1: First assume that $S = \text{Spec }(k)$ and that $X = \mathbb{P}^N_k$ and fix $T \in \text{Ob}(\text{Sch}_k)$. By Lemma 13, there's an unifrom bound $m \in \mathbb{Z}_{>0}$, such that if $[\phi : \mathcal{H}_T \to \mathcal{F}] \in \mathcal{Q}(T)$ is any quotient and if $\mathcal{K}$ is the kernal, then for all $t \in T$ the sheaves $\mathcal{K}_t, \mathcal{H}_t, \mathcal{F}_t$ are m-regular. Applying the functor $f_{T*}(- \otimes \mathcal{O}_(m))$ one gets a short exact sequence

$$0 \to f_{T*}(\mathcal{K}(m)) \to \mathcal{O}_T \otimes_k H^0(\mathcal{H}(m)) \to f_{T*}(\mathcal{F}(m)) \to 0$$

of locally free sheaves, and all the higher image sheaves vanish. Moreover, for any $m' \geq m$ there is an exact sequence

$$f_{T*}\mathcal{K}(m) \otimes H^0(m' - m) \to \mathcal{O}_T \otimes H^0(\mathcal{H}(m')) \to f_{T*}(\mathcal{F}(m)) \to 0,$$

where the first map is given by the multiplication of global sections. Thus $f_{T*}\mathcal{K}(m)$ complete determined the graded module $\bigoplus_{m'\geq m} f_{T*}\mathcal{F}(m')$ which in turn determines $\mathcal{F}$. This argument shows that sending $[\mathcal{H}_T \to \mathcal{F}]$ to $\mathcal{O}_T \otimes H^0(\mathcal{H}(m)) \to f_{T*}(\mathcal{F}(m))$ gives an injective morphism of functors

$$\mathfrak{Quot}^{P,\mathcal{L}}_{\mathcal{H}/X/S} \to \mathfrak{Grass}_k(H^0(\mathcal{H}(m)), P(m)).$$

Then we have to show the image $\mathcal{Q}(T) \subset \mathfrak{Grass}_k(H^0(\mathcal{H}(m)), P(m))$ of this injective is actually a (locally closed) subscheme. Since the latter functor is represented, we get a morphism $\psi : T \to \mathbb{G} = \text{Grass}_k(H^0(\mathcal{H}(m)), P(m))$. Consider all these sheaf by graded module of graded ring $S = \bigoplus_{\nu\geq 0} H^0(\mathbb{P}^N_k, (\nu))$ and $\Gamma_*\mathcal{H} = \bigoplus_{\nu\geq 0} H^0(\mathbb{P}^N_k, \mathcal{H}(\nu))$. The subbundle $\mathcal{A}$ generates a submodule $\mathcal{A} \cdot S \subset \Gamma_*\mathcal{H}$, Let $\mathcal{P}$ be the $\mathcal{O}_{\mathbb{P}^N_{\mathbb{G}}}$-module associated to the graded $\mathcal{O}_{\mathbb{G}}$-module $\Gamma_*\mathcal{H}/A \cdot S$. One can directly check $\mathcal{Q}$ is represented by the locally closed subscheme $\mathbb{G}_P \subset \mathbb{G}$ which is the component of the flattening stratification for $\mathcal{P}$ with Hilbert polynomial P, denoted by Q.

Step 2: Let S and X be arbitrary. Choosing a closed immersion $i : X \to \mathbb{P}^N_S$ and replacing $\mathcal{H}$ by $i_*(\mathcal{H})$, we reduce to the case $X = \mathbb{P}^N_S$. By Serre's theorem there exist presentations

$$\mathcal{O}_{\mathbb{P}^N_S}(-m'')^{n''} \to \mathcal{O}_{\mathbb{P}^N_S}(-m')^{n'} \to \mathcal{H} \to 0.$$

Any quotient of $\mathcal{H}$ can be considered as a quotient of $\mathcal{O}_{\mathbb{P}^N_T}(-m')^{n'}$. Conversely, a quotient of $\mathcal{F}$ of $\mathcal{O}_{\mathbb{P}^N_T}(-m')^{n'}$ factors through $\mathcal{H}$ if and only if the composite homomorphism $\mathcal{O}_{\mathbb{P}^N_T}(-m'')^{n''} \to \mathcal{O}_{\mathbb{P}^N_T}(-m')^{n'} \to \mathcal{F}$ vanishes. The latter is equivalent to the vanishing of the homomorphism $\mathcal{O}_T \otimes_k H^0(\mathcal{O}_{\mathbb{P}^N_k}(l - m'')) \to f_{T*}\mathcal{F}(l)$ for large enough integer l, so $\mathfrak{Quot}^{P,\mathcal{L}}_{\mathcal{H}/X/S}$ is represented by the vanishing locus(which is a closed subscheme) of this morphism in

$$\text{Quot}_{X/S}(\mathcal{O}_{\mathbb{P}^N_S}(-m')^{n'}, P) = \text{Quot}_{\mathbb{P}^N_k/k}(\mathcal{O}(-m')^{n'}, P) \times_k S.$$

It remains to show that Q is projective. Since we already show that Q is quasi-projective(Grassmannian is projective), it is sufficient to show that Q is proper. We want to check the valuation criterion of properness i.e. let R be any discrete valuation ring with fraction field K,

$$\mathfrak{Quot}^{\Phi,\mathcal{L}}_{E/X/S}(\text{Spec } R) \to \mathfrak{Quot}^{\Phi,\mathcal{L}}_{E/X/S}(\text{Spec } K)$$

is bijective. To see this, given any coherent quotient $q : E_K \to \mathcal{F}$ on X_K defines an element $(\mathcal{F}, q)$ of $\mathfrak{Quot}^{\Phi,\mathcal{L}}_{E/X/S}(\text{Spec } K)$, let $\bar{\mathcal{F}}$ be the image of $E_R \to j_*(E_K) \to j_*(\mathcal{F})$ where $j : X_K \to X_R$ is the open inclusion. Let $\bar{q} : E_R \to \bar{\mathcal{F}}$, and $\bar{q}$ maps to q induces surjective. And The preimage is in fact unique, since we ask the fiber on the closed point to be flat, this is the same as being torsion-free. This demanding The 'torsion' part of $\bar{\mathcal{F}}$ which only has support on closed fiber is zero, so the non-torsion part is uniquely determined.

2. Basics on deformation

2.1. **Original ideas.**

We now leave algebraic geometry for a while, to show something original.

Let M be compact complex manifold, with finitely many local charts $\{U_i\}_I$ and gluing functions $\{f_{ij}\}_{i,j\in I}$, for simplicity we denote $f_{ij} : U_{ij} \to U_{ji}$ where U_{ij} is a open subset of the trivialization of U_i and each f_{ij} is a biholomorphic function. By 'deform M', we want to fix an domain $0 \in U \subset \mathbb{C}$ find a family of gluing functions i.e. $\{F_{ij} : U_i \times U \to U_j\}$ such that

(1) $F_{ij}(-, 0) = f_{ij}$
(2) for any $t \in U$, $\{F_{ij}(-, t)\}_{i,j\in I}$ is also a gluing function, this means that we glue up to another manifold along this functions.

Note that this is the simplest situation, we can replace $U \subset \mathbb{C}$ by $U \subset \mathbb{C}^n$ containing the origin point. One question is if $\{F_{ij}(-, t)\}_{i,j\in I}$ is just constant about $t \in U$, it's a trivial deformation, or our family is 'isomorphic' to the trivial one, we get nothing from it. A elementary idea is to consider $\frac{\partial F}{\partial t}$. Using the condition of gluing function to calculate the differentials, the given data defines a 1-cocycle of a holomorphic vector field, which lies in some first cohomology group of some holomorphic sheaf. Along such ideas, people algebraicalize them.

2.2. **Some examples.**

Proposition 16 (The Infinitesimal Property)**.** *Let k be an algebraically closed field, and let A be a finitely generated k-algebra such that* Spec A *is nonsingular variety over k. Let $0 \to I \to B' \to B \to 0$ be an exact sequence, where B' is a k-algebra, and I is an ideal with $I^2 = 0$. Finally suppose given a k-algebra homomorphism $f : A \to B$. Then there exists a k-algebra homomorphism $g : A \to B'$ making a commutative diagram*

$$\begin{tikzcd}
 & 0 \arrow[d] \\
 & I \arrow[d] \\
 & B \arrow[d] \\
A \arrow[ur, dashed, "g"] \arrow[r, "f"] & B' \arrow[d] \\
 & 0
\end{tikzcd}$$

Proof. We prove this proposition by following steps(after the guidance in [1]):

Step 1: By the construction of Kahler differentials, we have the canonical isomorphism:

$$\mathrm{Der_k(A,\cdot)} \cong \mathrm{Hom(\Omega_{A/k},\cdot)}.$$

Categorically we have tuple (Ω, d) represent the functor $M \mapsto \mathrm{Der}_k(A, M)$. Given two lifting of f, namely g, g', let $\theta = g - g'$, we have $\theta|_k = 0$, and

$$\begin{aligned}\theta(aa') &= g(aa') - g'(aa')\\ &= g(a)(g(a') - g'(a')) + g'(a')(g(a) - g'(a))\\ &= g(a)\theta(a') + g'(a')\theta(a),\end{aligned}$$

which says $\theta \in \mathrm{Der}_k(A, I)$, or equivalently we say $\theta \in \mathrm{Hom}_A(\Omega_{A/k}, I)$. Conversely, if we fix an element $\theta \in \mathrm{Hom}_A(\Omega_{A/k}, I)$, denote the same θ for $\theta \circ d$,,which is the corresponding element of $\mathrm{Der}_k(A, I)$ and denote $g' = g + \theta$, we have

$$\begin{aligned}g'(a)g'(a') &= (g(a) + \theta(a))(g(a') + \theta(a'))\\ &= g(aa') + g(a)\theta(a') + g(a')\theta(a) + \theta(a)\theta(a')\\ &= g(aa') + g(a)\theta(a') + g(a')\theta(a)\\ &= g(aa') + \theta(aa'),\end{aligned}$$

and $g'|_k = g|_k = \mathrm{id}$, so it's a morphism of k-algebra. Now we conclude that if we fix a lifting of f, then we have a one-to-one correspondence between all the lifting maps of f and $\mathrm{Hom}_A(\Omega_{A/k}, I)$ as sets. Later on, we use this correspondence to do some adjustments in order to find the right lifting.

Step 2: Since A is finitely generated over k, we have the following exact sequence $0 \to J \to P \to A \to 0$, where $P = k[x_1, \ldots, x_n]$ and J is the kernel. We want to construct the following map $h : P \to B'$ making a commutative diagram,

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
J \arrow[d] & I \arrow[d] \\
P \arrow[r, "h"] \arrow[d, "p"'] & B \arrow[d] \\
A \arrow[r, "f"] \arrow[d] & B' \arrow[d] \\
0 & 0
\end{tikzcd}$$

Actually, there should be a lot of choices of such h by the previous result, but we only need to decide on one and we obtain all of them up to a derivation. Let $h(x_i)$ be the an (arbitrary) element of $f(p(x_i))+I$, and make it multiplicative. Note $h(J) \subset I$ and $h(J^2) \subset I^2 = 0$, so h induces $\bar{h} : J/J^2 \to I$.

Step 3: Since Spec A is nonsingular, we have the following exact sequence

$$0 \to J/J^2 \to \Omega_{P/k} \otimes A \to \Omega_{A/k} \to 0.$$

Applying the functor $\mathrm{Hom}(\cdot, I)$ we have

$$0 \to \mathrm{Hom}_A(\Omega_{A/k}, I) \to \mathrm{Hom}_P(\Omega_{P/k}, I) \to \mathrm{Hom}_A(J/J^2, I) \to 0,$$

We explain that in the middle we have $\mathrm{Hom}_A(\Omega_{P/k} \otimes A, I) = \mathrm{Hom}_P(\Omega_{P/k}, I)$, this is clear because $A = P/J$. And the last term of this sequence should be $\mathrm{Ext}^1(\Omega_{A/k}, I)$, since $\Omega_{A/k}$ is a free module of A, we have $\mathrm{Ext}^1(\Omega_{A/k}, I) = 0$. Let $\theta \in \mathrm{Hom}_P(\Omega_{P/k}, I)$ be a preimage of $\bar{h}$, and let $h' = h - \theta$ be a homomorphism from P to B, we have $h'(J) = 0$, so it induces a lifting $h' : A \to B$. $\square$

Remark 17. We always have such lifting for P obviously, but for arbitrary A we use the key correspondent and the information from nonsingularity. And on the geometric side, given Spec $B/I \to$ Spec A with A regular, we have such lifting on the infinitesimal neighborhood of a closed subscheme determined by I, i.e. a morphism Spec $B/I^n \to$ Spec A for any n, this is so-called infinitesimal lifting property. If take $B = k[\varepsilon]/(\varepsilon^2)$ and $B' = k$, it's like doing a Taylor expansion in Calculus.

As an application of the infinitesimal lifting property, we consider the following general problem. Let X be a scheme of finite type over k, and let $\mathscr{F}$ be a coherent sheaf on X. We seek to classify schemes X' over k, which have a sheaf of ideals $\mathscr{I}$ such that $\mathscr{I}^2 = 0$ and $(X', \mathcal{O}_{X'}/\mathscr{I}) \cong (X, \mathcal{O}_X)$, and such that $\mathscr{I}$ with its resulting structure of $\mathcal{O}_X$-module is isomorphic to the given sheaf $\mathscr{F}$. Such a pair $(X', \mathscr{I})$ we call an *infinitesimal extension* of the scheme X by the sheaf $\mathscr{F}$. One such extension the trivial one is obtained as follows. Take $\mathcal{O}_{X'} = \mathcal{O}_X \oplus \mathscr{F}$ as sheaves of abelian groups, and define multiplication by $(a \oplus f) \cdot (a' \oplus f') = aa' \oplus (af' + a'f)$. Then the topological space with the sheaf of rings $\mathcal{O}_{X'}$ is an infinitesimal of X by $\mathscr{F}$. According to Hartshorne, the general situation might be quite complicated. Now we just prove the following special case(also appeared in Hartshorne's book as an exercise):

Proposition 18 (affine situation)**.** *If X is affine and nonsingular, then any infinitesimal extension of X by a coherent sheaf $\mathscr{F}$ is isomorphic to the trivial one.*

Proof. Since everything is affine, we are able to translate it into a purely algebraic description: There is ring isomorphism $A \cong B/I$ where $I^2 = 0$ and note that I is a B/I-module and by this isomorphism I is also a A-module. We need to prove that $B \cong A \oplus I$ as ring and the multiplication of $A \oplus I$ is given by $(a \oplus f) \cdot (a' \oplus f') = aa' \oplus (af' + a'f)$. The given information determined a split exact sequence of abelian groups:

$$0 \to I \to B \to A \to 0,$$

where the split map is given by the infinitesimal lifting, so $B \cong A \oplus I$ as abelian groups, to complete the proof only to figure out the multiplication:

$$(a, i)(a', i') = aa' + ai' + a'i + ii' = (aa', ai' + a'i).$$

This proves the result we want. □

Definition 19 (infinitesimal deformation)**.** Let $Y \subseteq X$ be a closed subscheme, where X is a finite type scheme over a field k and $D =$ Spec $k[x]/(x^2)$ be the dual numbers. An infinitesimal deformation of Y as a closed subscheme of X is a closed subscheme $Y' \subseteq X \times_k D$, and whose closed fiber is Y.

Proposition 20. *Infinitesimal deformation of $Y \subset X$ is classified by $H^0(Y, \mathcal{N}_Y)$. where*

$$\mathcal{N}_Y = \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{I}_Y/\mathcal{I}_Y^2, \mathcal{O}_Y).$$

Sketch of proof. We prove it in the following way: Reduce to affine cases, Let $X =$ Spec A and $Y = V(I)$, fix $\phi \in Hom_{A/I}(I/I^2, A/I)$ we want to define an infinitesimal deformation of Y, which is $Y' \subset X \times_k D$ where $D = k[x]/(x^2)$, with Y' flat over D and the closed fiber, is Y. We define I' to be the ideal of $A[x]/(x^2)$, of forms $a + \phi(a)x$ and $b \in I$ such that $\phi(c) = 0$ where $a, b \in I$. □

Remark 21. Recall that $\mathcal{N}_Y$ is constructed as the normal bundle of Y in X. The geometric image is much clearer, since on one hand elements in $H^0(Y, \mathcal{N}_Y)$ are normal vectors, and on the other hand ring of dual number D carries information on Zariski tangent space. and this is also called first-order deformation.

2.3. **Deformation calculates tangent space.** Now we turn to the study of some infinitesimal properties of the Quot-scheme. Let Y be k-scheme. Recall that given a morphism

$$\varphi \in \mathrm{Hom}_{Sch}(\mathrm{Spec}(k[\varepsilon]/(\varepsilon^2)), Y)$$

is equivalent to give a k-rational point y and an tangent vector in Zariski tangent space T_yY at y. To generalize this, we introduce the category of Artinian local k-algebras with residue field k, denoted by $(Artin/k)$.

Let $Q = \mathrm{Quot}_{X/S}(\mathcal{H}, P)$ and $\sigma : A' \to A$ be a surjective morphism in $(Artin/k)$ and give the following commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(A) & \xrightarrow{\sigma} & Q \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle \pi} \\ \mathrm{Spec}(A') & \xrightarrow{\psi} & S \end{array}$$

Since Q is represented Quot functor, morphism q gives a short exact sequence

$$0 \to K \to \mathcal{H}_A \to F \to 0$$

of coherent sheaves on $X_A = \mathrm{Spec}\ A \times_k X$ with $\mathcal{H}_A = A \otimes_{\mathcal{O}_S} \mathcal{H}$.

We ask whether the morphism q can be extended to a morphism $q' : \mathrm{Spec}\ A' \to Q$. If yes, how many different extensions are there? Without loss of generality, we may assume $\mathfrak{m}_{A'} I = 0$ where I is the kernel of σ.

Suppose that q' exists. It gives an exact sequence

$$0 \to K' \to \mathcal{H}_{A'} \to F' \to 0$$

on $X_{A'}$ with $A \otimes_{A'} F' = F$. Let $F_0 = A/\mathfrak{m}_A \otimes_A F$ and etc. Then we have such a commutative diagram whose columns and rows are exact because of the flatness of $\mathcal{H}_{A'}$ and F':

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & I \otimes_k K_0 & \xrightarrow{1 \otimes i_0} & I \otimes_k \mathcal{H} & \xrightarrow{1 \otimes q_0} & I \otimes_k F_0 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle j} & & \downarrow & & \\ 0 & \longrightarrow & K' & \xrightarrow{i'} & \mathcal{H}_{A'} & \xrightarrow{q'} & F' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle \sigma} & & \downarrow & & \\ 0 & \longrightarrow & K & \xrightarrow{i} & \mathcal{H}_A & \xrightarrow{q} & F & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

In the first row, we have used the isomorphism $I \otimes_{A'} F' \cong I \otimes_k F_0$ etc. In fact, F' is the cokernel of the induced homomorphism $\hat{i} : K \to \mathcal{H}_{A'}/(1 \otimes i_0)(I \otimes_k K_0)$. Conversely, any $\mathcal{O}_{X_{A'}}$-homomorphism $\hat{i}$ gives i when composed with σ defines an A'-flat extension F' of F. Thus the existence of F' is equivalent to the existence of $\hat{i}$ as above which in turn is equivalent to the splitting of the extension

$$0 \to I \otimes_k F_0 \to B \to K \to 0,$$

where B is the middle homology of the complex

$$0 \to I \otimes K_0 \stackrel{j \cdot 1 \otimes i_0}{\to} \mathcal{H}_{A'} \stackrel{q \cdot \sigma}{\to} F \to 0.$$

Check that though B a priori is an $\mathcal{O}_{X_{A'}}$-module it is in fact annihilated by I, so that B can be considered as an $\mathcal{O}_{X_A}$-module. The extension class

$$\mathfrak{o}(\sigma, q, \psi) \in \mathrm{Ext}^1_{X_A}(K, I \otimes_k F_0)$$

defined by B is the *obstruction* to extend q to q'. Since K is a A-flat and $I \otimes F_0$ is annihilated by $\mathfrak{m}_A$, there is a natural isomorphism

$$\mathrm{Ext}^1_{X_A}(K, I \otimes_k F_0) \cong \mathrm{Ext}^1_{X_s}(K_0, F_0) \otimes_k I.$$

Lemma 22. *An extension q' of q exist if and only if $\mathfrak{o}(\sigma, q, \psi)$ vanishes. If this is the case, the possible extension is given by* $\mathrm{Hom}_{X_s}(K_0, F_0) \otimes I$.

Proof. The first statement is proved above, the second one is true because different splitting map differs by a homomorphism $K \to I \otimes_k F_0$. By flatness of K, this is equal to $\mathrm{Hom}_{X_s}(K_0, F_0) \otimes_k I$. □

Theorem 23. *Let $f : X \to S$ be a projective morphism of k-schemes of finite type and $\mathcal{O}_X(1)$ an f-ample line bundle on X. Let $\mathcal{H}$ be an S-flat coherent $\mathcal{O}_X$-module. P a polynomial and $\pi : Q = \mathrm{Quot}_{\mathrm{X/S}}(\mathcal{H}, P) \to S$ the associated relative Quot-scheme. Let $s \in S$ and $q_0 \in \pi^{-1}(s)$ be k-rational points corresponding to a quotient $\mathcal{H}_s \to \mathcal{F}$ with kernel $\mathcal{K}$. Then there is a short exact sequence*

$$0 \to \mathrm{Hom}_{X_s}(\mathcal{K}, \mathcal{F}) \to T_{q_0}Q \xrightarrow{T\pi} T_sS \xrightarrow{\mathfrak{o}} \mathrm{Ext}^1{}_{X_s}(\mathcal{K}, \mathcal{F}).$$

Proof. Just take $A = k$ and $A' = k[\varepsilon]/(\varepsilon^2)$, combining the correspondence in the beginning of 2.3. □

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