$n = 100 \qquad p = 16/100 = 0.16 \qquad q = 1-p = 0.84$

$\sigma = \sqrt{npq} = \sqrt{100 \times 0.16 \times 0.84} = \sqrt{13.44} = 3.66$

check $np = 100 \times 0.16 = 16 \qquad nq = 100 \times 0.84 = 84$

Both are greater than 15, it can be approx by normal distribution

$$Z = \frac{x - np}{\sqrt{npq}} = \frac{x-16}{3.67}$$

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-1/2 (x-m/\sigma)^2} = \frac{1}{\sqrt{2\pi}\; 3.67} \times e^{-1/2 \times \left(\frac{x-16}{3.67}\right)^2}$$

leave it like t

8 2 independent random variates x and Y are normally distributed with mean and σ : $X \sim N(52, 3) \quad Y \sim N(50, 2)$

Find the random chosen pair X and Y differ by 1.7 or more

$P(|X-Y| \geq 1.7)$

$X - Y : \quad m = m_1 - m_2$

$\sigma^2 = \sigma_1^2 + \sigma_2^2 = 3^2 + 2^2 = 13 \qquad \sigma = \sqrt{13}$

$X - Y \sim N(2, \sqrt{3})$

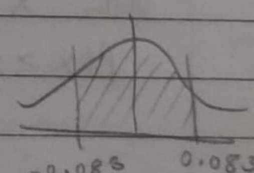
$P(|X-Y| \geq 1.7) :- \quad Z = (x-2)/\sqrt{13} \qquad Z_1 = (1.7-2)/\sqrt{13} = 0$

$P(-1.7 < \mu < 1.7) = 1 - P(-1.7 < \mu < 1.7)$

$= 1 - P(-0.083 < Z < 0.083)$

$= 1 - 2P(Z < 0.083)$

$= 1 - 2 \times 0.0319 = 0.9362$



- Exponential distribution

Continuous rV X having a pdf :-

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } x \geq 0 \\ 0, & \text{if } x < 0 \end{cases}$$

is said to have an exponential distribution

Examples: time b/w 2 successive job arrivals, duration of telephone calls, life time of a product, service time at a server in a queue, time required for repair of component.

Mean $E(X^r) = \frac{\Gamma(r+1)}{\lambda^r}$

gamma fn

Variance $\sigma^2 = E(X^2) - [E(X)]^2 = 1/\lambda^2$

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Cumulative Distribution Function :-

$$F(x) = \int_0^x f(t)\,dt = 1 - e^{-\lambda x} \quad \text{for } x \geq 0$$

$$P(\alpha < x < \beta) = \int_\alpha^\beta f(x)\,dx \quad \text{where } F'(x) = f(x)$$

$$F(x) = \int_{-\infty}^{x} f(x)\,dx$$

$F(x) = 0$ when $x < 0$

$\Gamma{n+1} = n\Gamma{n} = n!$

$$\int_0^\infty e^{-x} x^n\,dx = \Gamma{n+1} \qquad F(a) = \int_{-\infty}^{a} f(x)\,dx$$

For any $a \geq 0$

$P(X \geq a) = P(X > a) = 1 - F(a) = e^{-\lambda a}$

$P(a \leq X \leq b) = P(a \leq X < b) = P(a < X < b)$

$= P(a < X \leq b)$

$= F(b) - F(a) = e^{-\lambda a} - e^{-\lambda b}$

Survival Function: Gives probability that the system survives more than x units of time and is given by

$$P(X > x) = 1 - F(x) = \begin{cases} 1 & \text{if } x < 0 \\ e^{-\lambda x} & \text{if } x \geq 0 \end{cases}$$

- Questions :-

1. Let the mileage (in 1000 of miles) of a particular tire be a random variable X having the probability density f(x)

$$f(x) = \begin{cases} \frac{1}{20} e^{-x/20}, & \text{if } x > 0 \\ 0, & \text{if } x \geq 0 \end{cases}$$

Find the probability that one of these tires will last

a) at most 10000 miles

b) anywhere from 16000 to 24000 miles

c) at least 30000 miles

d) Find the mean

e) Find the variance of the given probability density

$e^{-\infty} = \frac{1}{e^{\infty}} = \frac{1}{\infty} = 0$

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Mean $= 1/\lambda$

a) $P(x \le 10) = \int_0^{10} f(x)\,dx = \int_0^{10} \frac{1}{20} e^{-x/20}\,dx$

$= \left[1/20 \; e^{-x/20} (-20/1) \right]_0^{10} = 1 - e^{-1/2} = 0.3934$

b) $P(16 \le x \le 24) = \int_{16}^{24} f(x)\,dx = \int_{16}^{24} 1/20\, e^{-x/120}\,dx$

$= -e^{-x/120}\,dx = 0.148$

c) $P(x \ge 30) = \int_{30}^{\infty} f(x)\,dx = \int_{30}^{\infty} 1/20\, e^{-x/20}\,dx$

$= \left[-e^{-x/20} \right]_{30}^{\infty} = e^{-3/2}$

$= 0.2231$

d) Mean $= \mu = \int_{-\infty}^{\infty} x \cdot f(x)\,dx = \int_0^{\infty} x \cdot 1/20\, e^{-x/20}\,dx$

$= -\int_0^{\infty} x \cdot d(e^{-x/20}) = \left[-xe^{-x/20} - 20\, e^{-x/20} \right]_0^{\infty}$

$= 0 - (-20) = 20 = 1/\lambda$

e) Variance $= \sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx = \int_{-\infty}^{\infty} x^2 f(x)\,dx - \mu^2$

Consider $\int_{-\infty}^{\infty} x^2 f(x)\,dx = \int_0^{\infty} x^2\, 1/20\, e^{-x/20}\,dx$

$= \left[-x^2 e^{-x/20} \right]_0^{\infty} + 2.20 \int_0^{\infty} 1/20\, e^{-x/20}\,dx$

$= 0 + 2.20.\mu = 2.20.20 = 2.20^2$

$\sigma^2 = \int_0^{\infty} x^2 f(x)\,dx - \mu^2 = 2 \times 20^2 - 20^2 = 20^2 = 1/\lambda^2$

2 If a random variable x has the exponential distribution with mean $\mu = 1/\lambda = 1/2$. Calculate the probabilities that.
a) x will lie b/w 1 and 3 b) x is greater than 0.5
c) x is at most 4.

Given $\lambda = 2$

Pdf of x is $f(x) = \begin{cases} 2e^{-2x} & x \ge 0 \\ 0 & x < 0 \end{cases}$

a) $P(1 < x < 3) = \int_1^3 2e^{-2x}\,dx = -\left[e^{-2x}\right]_1^3 = e^{-2} - e^{-6} = 0.1328$

b) $P(x > 0.5) = \int_{0.5}^{\infty} 2e^{-2x}\,dx = -\left[e^{-2x}\right]_{0.5}^{\infty} = -(0 - e^{-1}) = 0.3678$

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c) $P(x<4) = \int_{-\infty}^{4} f(x)\,dx = \int_{-\infty}^{0} 0\,dx + \int_{0}^{4} 2e^{-2x}\,dx = -(e^{-8} - e^{0})$
$= 1 - e^{-8} = 0.999$

3 The length of time for one person to be served at a cafeteria is a random variable X having an exponential distribution with a mean of 4 minutes. Find the probability that a person is served in less than 3 mins on at least 4 of the next 6 days.

→ P(person served at a cafeteria is less than 3 mins) $= P(T<3) = 1 - P(T \geq 3)$

$\mu = 1/\lambda = 4 \qquad \lambda = 1/4 \qquad ED = \frac{1}{4} e^{-1/4 x}$

$P(T<3) = 1 - \int_{3}^{\infty} \frac{1}{4} e^{-1/4 x}\,dt = \left[1 - \frac{1}{4} e^{-t/4} \left(\frac{-4}{1}\right)\right]_{3}^{\infty}$

$= 1 - e^{-3/4} = 0.5276$

Using binomial distribution, P(person is served in less than 3 minutes on at least 4 of the next 6 days) $= P(X \geq 4)$

$= \sum_{x=4}^{6} {}^{6}C_{x} (0.5276)^{x} (0.4724)^{6-x} = 0.3968$

4 Let T be the time (in years) to failure of certain components of a system. The random variable T has exponential distribution with mean time to failure $\beta = 5$. If 5 of these components are in different system find the probability that at least 2 are still functioning at the end of 8 years.

→ Pdf of T is $f(x) = \begin{cases} \frac{1}{5} e^{-1/5 x} & , x \geq 0 \\ 0 & , x < 0 \end{cases}$ $\qquad 1/\lambda = 5$, $\lambda = 1/5$

P(component is functioning at the end of 8 years) $= P(T>8)$

$= \int_{8}^{\infty} f(x)\,dx = \frac{1}{5} \int_{8}^{\infty} e^{-x/5}\,dx = -(0 - e^{-8/5}) = 0.2018$

Let X represent the no. of components functioning at the end of 8 years

Using binomial distribution, $p = 0.2018 \quad q = 1 - p = 0.7982 \quad n = 5$

Required probability $= 1 - P(X<2) = 1 - [P(X=0) + P(X=1)]$

$= 1 - [{}^{5}C_{0} p^{0} q^{5} + {}^{5}C_{1} p^{1} q^{4}] = 0.2666$

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5 The life (in years) of a certain electrical switch has an exponential distribution with an average life of $1/\lambda = 2$. If 100 of these switches are installed in different system, find the probability that at most 30 fail during the first year. <1

→ Pdf of T is $f(x) = \begin{cases} \frac{1}{2} e^{-1/2 x} & x \geq 0 \\ 0 & x < 0 \end{cases}$ $\quad 1/\lambda = 2$, $\lambda = 1/2$

P (no. of switches failing in 1^{st} year) $= P(x < 1)$
$= \int_0^1 \frac{1}{2} e^{-1/2 x}\, dx = 1 - e^{-1/2} = 0.3935$

Let x represent the no. of switches failing during 1^{st} year
Using binomial distribution, $p = 0.3935$ $\quad q = 0.6065$ $\quad n = 100$
$P(X \leq 30) = \sum_{x=0}^{30} {}^{100}C_x \, p^x q^{n-x} = 0.03342$

6 The time x (seconds) that it takes a certain online computer terminal (the time elapsed b/w the end of user's inquiry and the beginning of the system's response to that inquiry) has an exponential distribution with expected time 20 seconds. Compute the probabilities

a) $P(X \leq 30)$ b) $P(x \geq 20)$ c) $P(20 \leq x \leq 30)$
d) For what value of t is $P(x \leq t) = 0.5$

→ $\lambda = 1/20$ pdf for x is $f(x) = \begin{cases} \frac{1}{20} e^{-t/20}, & t \geq 0 \\ 0, & t < 0 \end{cases}$

a) $P(x \leq 30) = \int_0^{30} \frac{1}{20} e^{-t/20}\, dt = 1 - e^{-3/2} = 0.7768$

b) $P(x \geq 20) = \int_{20}^{\infty} \frac{1}{20} e^{-1/20\, t}\, dt = -(0 - e^{-1}) = 1/e = 0.3678$

c) $P(20 \leq x \leq 30) = \int_{20}^{30} \frac{1}{20} e^{-t/20}\, dt = e^{-1} - e^{-3/2} = 0.1447$

d) $P(x \leq t) = 0.5$
$\int_0^t \frac{1}{20} e^{-t/20}\, dt = 0.5 \longrightarrow -[e^{-t/20}]_0^t = 0.5$
$e^{-t/20} = 1 - 0.5 = 0.5$
$e^{t/20} = 2 \quad t = 20 \log_e 2$
$= 13.8629$

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- Discrete Uniform Distribution

Every one of n values has equal probability $1/n$.

A r.v is said to follow a uniform distribution if x takes all integral values 1, 2, 3, ..., n with $P(X=x_i) = 1/n$

$\forall x_i = 1, 2, \dots n$. It is denoted by $U(1,n)$

$E(x) = \sum p_i x_i = \sum_{i=0}^{n} x_i \cdot 1/n$

$= 1\times 1/n + 2\times 1/n + 3\times 1/n + \dots + n\times 1/n$

$= 1/n\,(1+2+3 \dots + n) = n(n+1)/2n = (n+1)/2$

$E(x^2) = 1/n\,(1^2 + 2^2 + \dots + n^2)$

$= 1/n \quad n(n+1)(2n+1)/6 = (n+1)(2n+1)/6$

$Var(x) = E(x^2) - (E(x))^2 = (n^2 - 1)/12$

- Questions :-

1 Roll a six faced fair die. Suppose x denote the number appear on the top of a die.

a) Find the probability that an even number appear on the top

b) Find the probability that the number appear on the top is less than 3

c) Compute mean and variance of x.

→ Let x denote the number appear on the top of a die. Then the r.v. X take the values $x = 1,2,3,4,5,6$ and x follows $U(1,6)$ distribution.

The probability mass function rv X is $P(X=x) = 1/6$, $x = 1,2,3,4,5,6$

a) $P(x = \text{even number}) = P(x=2) + P(x=4) + P(x=6) = 0.5$

b) $P(x<3) = P(x=1) + P(x=2) = 1/3$

c) Mean $= (1+n)/2 = 3.5$ n = no. of faces

Variance $= (n^2-1)/12 = 35/12 = 2.9167$

2 A telephone no. is selected at random from a directory. Suppose X denote the last digit of selected telephone number. Find the probability that the last digit of the selected no. is

a) 6 b) less than 3 iii) greater than or equal to 8

d) Find mean and variance

→ Let x denote the last digit of randomly selected phone no.
Then rv X take the values $x = 0, 1, 2 \ldots 9$
All the numbers $0, 1, 2, \ldots, 9$ are equally likely. Thus the random variable X follows a discrete uniform distribution $U(0,9)$

$P(X = x) = 1/10 \qquad x = 0,1,2,3,4,5,6,7,8,9$

a) $P(X = 6) = 1/10 = 0.1$

b) $P(X < 3) = P(X = 1) + P(X = 0) + P(X = 2)$
$= 3/10 = 0.3$

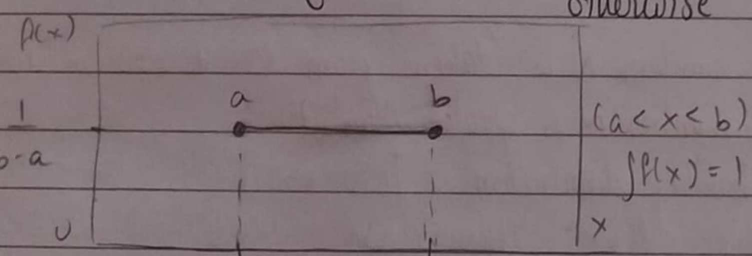
c) $P(X \geq 8) = P(X = 8) + P(X = 9) = 2/10 = 0.2$

d) Mean $= (n+1)/2 = (10+1)/2 = 5.5$
Variance $= (n^2 - 1)/12 = 99/12 = 8.25$

- Continuous Uniform Distribution

The continuous r.v X is said to be uniformly distributed, or having rectangular distribution on the interval $[a,b]$

$$f(x) = \begin{cases} 1/(b-a) & a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$$



Height of rectangle $= 1/(b-a)$
Length of base $= b - a$
Value of pdf = area of rectangle

$$E(x) = \int_a^b x f(x)\, dx = \int_a^b x \frac{1}{b-a} = \frac{b+a}{2}$$

$$E(x^2) = \int_a^b x^2 f(x)\, dx = (b^2 + ab + a^2)/3$$

~~Var (x) = E(x²)~~

$$Var(x) = E(x^2) - [E(x)]^2 = (b-a)^2/12$$

Questions

1. Suppose in a quiz there are 30 participants and a question is given to all of them and the time allowed to answer is 25 seconds. Find the number of probable participants respond within 6 seconds.

→ Given interval of probability distribution = [0 seconds, 25 seconds]

Density of the probability $f(x) = \frac{1}{25-0} = 1/25$

The probability $P(x<6) = \int_0^6 f(x)dx = 6/25$

There are 30 participants, so the no. of participants likely to answer it in 6 seconds $= 6/25 \times 30 \approx 7$

2. Suppose a flight is about to land and the announcement says that the expected time to land is 30 minutes. Find the probability of getting flight b/w 25 to 30 minutes.

→ Given interval of probability distribution = [0 minutes, 30 minutes]

Density of the probability $f(x) = 1/(30-0) = 1/30$

The probability $P(25 < x < 30) = \int_{25}^{30} f(x)\,dx = 5/30 = 1/6$

Hence probability of getting flight land between 25 mins to 30 mins = 0.16

3. Suppose a random number N is taken from 690 to 850 in uniform distribut^n. Find the probability number N is greater than 790.

→ Given interval of probability distribution = [690, 850]

Density of probability $f(x) = 1/(850-690) = 1/160$

$P(790 < x < 850) = \int_{790}^{850} f(x)dx = 60/160 = 3/8$

$P(N > 790) = 0.375$

4. Suppose a train is delayed by approx 60 minutes. What is the probability that train will reach by 57 minutes to 60 minutes?

→ [0 mins, 60 mins]

$f(x) = \frac{1}{60-0} = 1/60$

$P(57 < x < 60) = \int_{57}^{60} f(x)\,dx = 3/60 = 0.05$

5 If x is uniformly distributed in $-2 \le x \le 2$
Find a) $P(x<1)$ b) $P(|x-1| \ge 1/2)$
c) Mean and Variance

→ $U[-2, 2]$ density $= 1/(b-a) = 1/(2-(-2)) = 1/4$

i) $P(x<1) = \int_{-2}^{1} f(x)\,dx = \frac{1}{4}[x]_{-2}^{1} = 3/4$

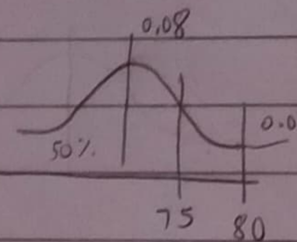
ii) $P[|x-1| \ge 1/2] = 1 - P(|x-1| < 1/2)$
$= 1 - P(1/2 < x < 3/2) = 1 - \int_{0.5}^{1.5} f(x)\,dx = 1 - 1/4 = 3/4$

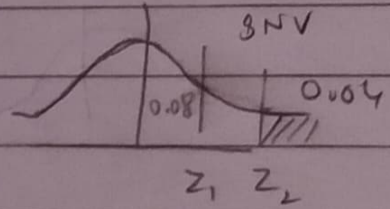
iii) Mean $= [2+(-2)]/2 = 0$
Variance $= [2-(-2)]^2/12 = 4/3$

• Find the mean and the standard deviation of a ND of marks in an examinat^{n} where 58% of the candidates obtained marks below 75, 4% got above 80 and the rest b/w 75 and 80

→ $Z_1 = \frac{x_1 - m}{\sigma} = \frac{75-m}{\sigma}$

$Z_2 = \frac{x_2 - m}{\sigma} = \frac{80-m}{\sigma}$





for area 0.08 corresponding value $Z_1 = 0.2$
For area 0.04 corresponding value $Z_2 = 1.75$

$\frac{75-m}{\sigma} = 0.2 = Z_1 \rightarrow m = 75 + 0.2\sigma$

$Z_2 = 1.75 = (80-m)/\sigma$
$-m = 80 + 1.75\sigma$ $\rightarrow m = 74.3548$ $\sigma = 3.2258$