ITC Mod 1

- ## Communication Channels

Noise

① → Channel Encoder ② → Mod ③ → Transmission Medium ④ → Σ ⑤ → De Mod ⑥ → Channel Decoder ⑦ →

← Transmitter → | ← Physical Channel → | ← Receiver →

1) Binary Input
Mod : Modulator
7) Binary output

3-5 : Modulation Channel
2-6 : Coding Channel (Discrete)
1-7 : Discrete Data Communication Channel

- ## Representation of a channel

input alphabet $A = \{a_1, a_2, a_3, \ldots, a_r\}$
output alphabet $B = \{b_1, b_2, b_3, \ldots, b_s\}$

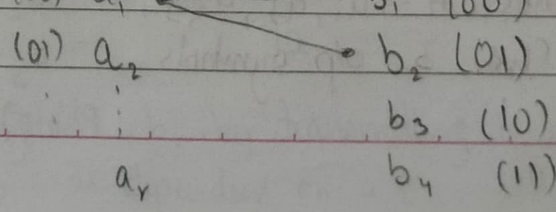
Consists of 's' symbols and a set of conditional probabilities $P(b_j/a_i)$
$i = 1, 2, 3, \ldots r$ & $j = 1, 2, \ldots s$

Channel Matrix or Noise Matrix : cond^n probabilities which are represented in a matrix form [input symbols = row-wise, output = column]

$$P(b_j/a_i) \text{ or } P(B/A) = \underbrace{\begin{array}{c|ccc} & b_1 & b_2 & \cdots\ b_s \\ \hline a_1 & P(b_1/a_1) & P(b_2/a_1) & P(b_s/a_1) \\ a_2 & P(b_1/a_2) & P(b_2/a_2) & P(b_s/a_2) \\ \vdots & & & \\ a_r & P(b_1/a_r) & P(b_2/a_r) & P(b_s/a_r) \end{array}}_{\text{o/p}}$$

I/p

Channel or Noise diagram : channel represent with conditional probabilities relating i/p and o/p symbols and the diagram



$P_{11} = P(b_1/a_1)$

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$\sum_{j=1}^{s} P(b_j/a_i) = 1$: Theorem of total probability

$\sum_{i=1}^{r} P(a_i) = 1$

Baye's Rule :-

$$P(a_i/b_j) = \frac{P(b_j/a_i)\,P(a_i)}{P(b_j)} \qquad P(A/B) = \frac{P(A\cap B)}{P(B)}$$

$$P(b_j/a_i) = \frac{P(a_i/b_j)\,P(b_j)}{P(a_i)} \qquad = \frac{P(A,B)}{P(B)}$$

- Joint Probability → JPM

Joint Probability Matrix : elements in matrix are joint probabilities between input and output symbols.

$$P(A,B) = \underbrace{P(B/A)}_{\text{channel matrix}} \times \underbrace{P(A)}_{\text{Preferred}} = P(A/B) \times P(B)$$

Properties :-

1) By adding elements of JPM column wise, we can obtain probability of output symbols

$$\sum_{i=1}^{r} P(a_i, b_j) = P(b_j)$$

2) Theorem of total probability : By adding elements of JPM row wise, we can obtain probability of input symbols

$$\sum_{j=1}^{s} P(a_i, b_j) = P(a_i)$$

3) Sum of all elements of JPM is equal to unity

$$\sum_{i=1}^{r}\sum_{j=1}^{s} P(a_i, b_j) = 1$$

- Questions

1 In a communication system, a transmitter has 3 input symbols $A = \{a_1, a_2, a_3\}$ and receiver also has 3 o/p symbols

$a_i \backslash b_j$	b_1	b_2	b_3
a_1	1/12	*	5/36
a_2	5/36	1/9	5/36
a_3	*	1/6	*
$P(b_j)$:	1/3	14/36	*

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$B = \{b_1, b_2, b_3\}$. The matrix shows JPM with some marginal probabilities. i) Find the missing probabilities *

ii) Find $P(b_3/a_1)$ and $P(a_1/b_3)$

iii) Are the events a_1 and b_1 statistically independent? Why?

Theorem of ~~tat~~ total probability : $\sum_{i=1}^{r} P(a_i) = 1$

$\sum_{j=1}^{3} P(b_j) = 1$ for o/p

i) $S = 3$ $\quad \sum_{j=1}^{3} P(b_j) = P(b_1) + P(b_2) + P(b_3) = 1$

$= 1/3 + 14/36 + P(b_3) = 1$

$P(b_3) = 5/18$

From 1st property of JPM, $\sum_{i=1}^{r} P(a_i, b_j) = P(b_j)$

$P(b_2) = P(a_1, b_2) + P(a_2, b_2) + P(a_3, b_2)$

$14/36 = 1/9 + 1/36 + P(a_1, b_2)$

$P(a_1, b_2) = 1/9$

$P(b_1) = P(a_1, b_1) + P(a_2, b_1) + P(a_3, b_1)$

$1/3 = 1/12 + 5/36 + P(a_3, b_1)$

$P(a_3, b_1) = 1/9$

$P(b_3) = P(a_1, b_3) + P(a_2, b_3) + P(a_3, b_3)$

$5/18 = 5/36 + 5/36 + P(a_3, b_3)$

$P(a_3, b_3) = 0$ $\qquad$ draw the matrix

ii) Conditional probability :-

$P(b_j / a_i) = \frac{P(a_i, b_j)}{P(a_i)}$ $\qquad \sum_{j=1}^{S} P(a_i, b_j) = P(a_i)$

$P(b_3 / a_1) = \frac{P(a_1, b_3)}{P(a_1)} = \frac{5/36}{1/3} = \frac{5}{12}$

$P(a_1 / b_3) = \frac{P(a_1, b_3)}{P(b_3)} = \frac{5/36}{5/18} = \frac{1}{2}$

iii) Independent if $P(a_1 \cap b_1) = P(a_1, b_1) = P(a_1) P(b_1)$

$P(a_1) \cdot P(b_1) = \frac{1}{3} \times \frac{1}{3} = \frac{1}{9}$

$P(a_1, b_1) = \frac{1}{12}$ $\qquad P(a_1, b_1) \neq P(a_1) P(b_1)$

We will receive b_1 by transmitting a_1 with some probability, hence b_1 is dependent on a_1.

- Entropy Functions of Communication Channel

$$H(A) = \sum_{i=1}^{r} P(a_i) \log\left[\frac{1}{P(a_i)}\right] \text{ bits/symbol}$$

$$H(B) = \sum_{j=1}^{s} P(b_j) \log\left[\frac{1}{P(b_j)}\right] \text{ bits/symbol}$$

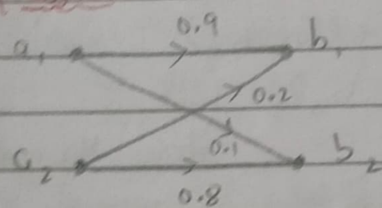
$$H(B/A) = \sum_{i=1}^{r}\sum_{j=1}^{s} P(a_i, b_j) \log\left[1/P(b_j/a_i)\right] \text{ bits/symbol}$$

$$H(A/B) = \sum_{i=1}^{r}\sum_{j=1}^{s} P(b_j, a_i) \log\left[1/P(a_i/b_j)\right] \text{ bits/symbol}$$

$$H(A,B) = \underbrace{\sum_{i=1}^{r}\sum_{j=1}^{s} P(a_i, b_j)}_{\text{same}} \log\left[1/P(a_i, b_j)\right] \text{ bits/symbol}$$

$$H(A,B) = H(B/A) + H(A) = H(A/B) + H(B)$$

- Questions:

1. 

A binary channel is described by the state diagram and given $P(a_1) = 0.7$, $P(a_2) = 0.3$

Find $H(A)$, $H(B)$, $H(A,B)$, $H(A/B)$, $H(B/A)$

→ Channel matrix : $P(B/A) =$

$$\begin{array}{c c} & \begin{array}{cc} b_1 & b_2 \end{array} \\ \begin{array}{c} a_1 \\ a_2 \end{array} & \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{bmatrix} \end{array}$$

JPM $P(A,B) = \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{bmatrix} \begin{matrix} \times 0.7 \\ \times 0.3 \end{matrix} = \begin{bmatrix} 0.63 & 0.07 \\ 0.06 & 0.24 \end{bmatrix}$

i) $H(A) = \sum_{i=1}^{r} P(a_i) \log_2 [1/P(a_i)] = 0.7 \log_2 1/0.7 + 0.3 \log_2 1/0.3$

$= 0.8813$ bits/symbols

$0.63 + 0.07$

$0.06 + 0.24$

ii) $H(B) = \sum_{j=1}^{s} P(b_j) \log_2 [1/P(b_j)]$

$P(b_j) = 0.63 + 0.06$
$= 0.69$

$P(b_1) = 0.31 \log_2 \frac{1}{0.31}$

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$H(B) = 0.69 \log_2 1/0.69 + 0.31 \log_2 1/0.31 = 0.8931$ bits/symbols

iii) $H(A,B) = \sum_{i=1}^{r} \sum_{j=1}^{s} P(a_i, b_j) \log_2 1/P(a_i, b_j)$

$= 0.63 \log_2 \frac{1}{0.63} + 0.07 \log_2 \frac{1}{0.07} + 0.06 \log_2 \frac{1}{0.06} + 0.24 \log_2 \frac{1}{0.24}$

$= 1.426$ bits/symbol

iv) $H(B/A) = H(A,B) - H(A) = 1.426 - 0.8813 = 0.5447$ bits/symbols

v) $H(A/B) = H(A,B) - H(B) = 1.426 - 0.8931 = 0.533$ bits/symbols

ITC Mod 1

- **Self Information :-**

Information : message or intelligence

Information Source : A source which produces these messages.

→ Analog Information Source (continuous valued)

→ Discrete Information Source

Units of Information :-

$I = \log_2 (1/P)$ ... bits

$I = \log_3 (1/P)$ ... trinits / ternary units

$I = \log_4 (1/P)$ ... quaternary units / quadits

$I = \log_e (1/P)$ ... Neper units / Nats

$I = \log_{10} (1/P)$ ... decits / Hartley

Self Info : amount of info or knowledge that a message contains about the event.

Entropy : Average Self Information.

$$H(S) = \sum_{i=1}^{q} P_i \log_2 (1/P_i) \quad \text{bits/symbol}$$

Symbol rate : no. of symbols emitted from source per second

↳ 'r_s' ⟶ unit : symbols/sec

Information rate : average source information rate

$$R_s = r_s H(S) \quad \text{bits/secs or BPS}$$

$$\log_a b = \frac{1}{\log_b a} \qquad \frac{\log_x b}{\log_x a} = \log_a b \qquad \log_e 10 = \ln$$

- **Questions :-**

1 Consider a source $S = \{s_1, s_2, s_3\}$ with $P = \{1/2, 1/4, 1/4\}$

→ To find self info : $I_k = \log_2 1/P_k$

$s_1 : I_1 = \log_2 \frac{1}{1/2} = 1$ bit

$s_2 : I_2 = \log_2 \frac{1}{1/4} = 2$ bits

$s_3 : I_3 = \log_2 4 = 2$ bits

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2 Random card selected (red). How much info you have received? How much more info you need to completely specify the card

→ Total no. of cards = 52 Total red = 26

P(red) = 26/52 = 1/2 → $\log_2 1/P_i$

Amount of info contained = $\log_2 2 = 1$ bit

" " for 1 unique card = $\log_2 1/(1/52) = \log_2 52 = 5.7$ bits

More info amount = 5.7 − 1 = 4.7 bits

3 TV picture has array of 500 rows and 600 columns dots. These dots may take on any one of 10 distinguishable levels. What is the amount of info provided by 1 picture.

→ Total no. of dots = 500 × 600 = 3×10^5

Total no. of pictures possible = $(10\times10\times\ldots10) = 10^{3\times10^5}$

Probability of each picture = $1/10^{3\times10^5}$

Amount of information in each picture $I = \log_2 10^{3\times10^5} = 996.578$ Kbits

4 A discrete source emits one of six symbols once every m-sec. The symbol probabilities : 1/2, 1/4, 1/8, 1/16, 1/32 and 1/32. Find source entropy and info rate

→ $H(s) = \sum_{i=1}^{6} P_i \log(1/P_i) = \frac{1}{2}\log_2 2 + \frac{1}{4}\log_2 4 + \frac{1}{8}\log_2 8 + \frac{1}{16}\log_2 16 + \frac{1}{32}\log_2 32 + \frac{1}{32}\log_2 32$

$= 1.9375$ bits/message-symbol

Information rate $R_s = r_s H(s)$

$r_s = 1$ message symbol / m-sec $= 10^3$ msg symbol/sec

$R_s = 10^3 \times 1.9375 = 1937.5$ bits/sec

5 150 symbols emitted, 32 with $P = 1/64$ and other 118 with $P = 1/236$. Source emits 2000 symbols/sec. Find Avg information

→ $H(S) = \sum_{i=1}^{150} P_i \log_2 1/P_i = \sum_{i=1}^{32} P_i \log 1/P_i + \sum_{i=33}^{118} P_i \log 1/P_i$

$= [1/64 \times \log 64] \times 32 + 118 \times [1/236 \times \log 236]$

$= 6.9413$ bits/message symbol

$r_s = 2000$ symbols/sec

$R_s = r_s H(s) = 13882.6$ bits/sec

ace : 1 × 4 → colors

spade, heart, diamond, clubs

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6 A code is composed of dots and dashes. Assuming that a dash is 3 times as long as a dot and has one-third the probability of occurrence. Calculate :-

3 dots = 1 dash
1 dot = 1/3 dash

i) The info in a dot and a dash

ii) Entropy iii) Avg info rate if dot lasts for 10m-sec, b/w symbols

→ $P_{dot} + P_{dash} = 1 = P_{dot} + 1/3 P_{dot}$

$P_{dot} = 3/4 \quad P_{dash} = 1/4$

i) $I_{dot} = \log 1/P_{dot} = 0.415$ bits

$I_{dash} = \log 1/P_{dash} = 2$ bits

ii) $H(s) = 3/4 \log 4/3 + 1/4 \log 4 = 0.8113$ bits/symbol

iii) $r_s = 4$ symbols / 100 m-sec $= 40$ symbols/sec

$R_s = 40 \times 0.8113 = 32.452$ bits/sec

7 Pair of dice tossed simultaneously.

$A = \{(x_1, x_2)$ such that $x_1 + x_2 \leq 7\}$

$B = \{(x_1, x_2)$ such that $x_1 > x_2\}$

Which event conveys more information?

→ $A = \{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (2,1), (2,2), (2,3), (2,4), (2,5), (3,1), (3,2), (3,3), (3,4), (4,1), (4,2), (4,3), (5,1), (5,2), (6,1)\}$

$B = \{(2,1), (3,1), (3,2), (4,1), (4,2), (4,3), (5,1), (5,2), (5,3), (5,4), (6,1), (6,2), (6,3), (6,4), (6,5)\}$

$P(A) = 21/36 = 7/12 \quad P(B) = 15/36 = 5/12$

$I_A = \log 12/7 = 0.776$ bits

$I_B = \log 12/5 = 1.263$ bits

$I_B > I_A$

↳ conveys more info

amount of info.
It is the measure of mutual dependence b/w 2 variables

- Mutual Information :

Info lost b/w transmitter and receiver, due to noise, in channel

$I(A,B) = H(A) - H(A/B)$ bits/symbol

Properties :

1. Symmetric $I(A,B) = I(B,A)$
2. Non negative : $I(A,B) \geq 0$
3. $I(A,B) = H(B) - H(B/A)$: output's entropy
4. Joint entropy : $I(A,B) = H(A) + H(B) - H(A,B)$

   consider it as intersection

Questions :

1. A transmitter has an alphabet consisting of 5 letters $\{a_1, a_2, a_3, a_4, a_5\}$ and $B\{b_1, b_2, b_3, b_4\}$

$$P(A,B) = \begin{array}{c|cccc} & b_1 & b_2 & b_3 & b_4 \\ \hline a_1 & 0.25 & 0 & 0 & 0 \\ a_2 & 0.10 & 0.30 & 0 & 0 \\ a_3 & 0 & 0.05 & 0.10 & 0 \\ a_4 & 0 & 0 & 0.05 & 0.10 \\ a_5 & 0 & 0 & 0.05 & 0 \end{array}$$

Joint Probabilities

Compute different entropies

$\rightarrow$ $H(A)$, $H(B)$, $H(A,B)$, $H(B/A)$ and $H(A/B)$,

$I(A,B) = H(A) - H(A/B)$

From property 2 (theorem of total prob.) of JPM,

$P(a_1) = 0.25$ $P(a_2) = 0.40$ $P(a_3) = 0.15$

$P(a_4) = 0.15$ $P(a_5) = 0.05$

$P(b_1) = 0.35$ $P(b_2) = 0.35$ $P(b_3) = 0.20$ $P(b_4) = 0.10$

from property 1

Entropy of input : $H(A) = \sum_{i=1}^{5} P(a_i) \log_2 1/P(a_i) = 2.066$ bits/symbols

Entropy of output : $H(B) = \sum_{j=1}^{4} P(b_j) \log_2 1/P(b_j) = 1.857$ bits/symbols

$H(A,B) = \sum_{j=1}^{5} \sum_{i=1}^{4} P(a_i, b_j) \log 1/P(a_i, b_j)$

$= 0.25 \log 1/0.25 + 0.10 \log 1/0.10 + 0.30 \log_2 1/0.30$
$+ 0.05 \log_2 1/0.05 + 0.10 \log_2 1/0.10 + 0.05 \log_2 1/0.05$
$+ 0.05 \log_2 1/0.05 + 0.10 \log_2 1/0.10$

$= 2.666$ bits/symbol

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$H(B/A) = H(A,B) - H(A) = 0.6$ bits / symbols

$H(A,B) = H(A/B) + H(B) \longrightarrow H(A/B) = 0.809$ bits/symbols

$I(A,B) = H(A) + H(B) - H(A,B) = H(A) - H(A/B)$

$= 2.066 - 0.809 = 1.257$ bits/symbols

2) 5 symbols transmitted with probabilities 0.2, 0.3, 0.2, 0.1 and 0.2

Given the channel matrix

Calculate i) $H(B)$

ii) $H(A,B)$

$$P(B/A) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/4 & 3/4 & 0 & 0 \\ 0 & 1/3 & 2/3 & 0 \\ 0 & 0 & 1/3 & 2/3 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$\rightarrow$ JPM equation: $P(a_i, b_j) = P(a_i)\, P(b_j / a_i)$

Multiply rows with probabilities

- Shannon's Theorem: C: channel capacity

$R \le C$: there exists a coding technique which enables transmission over channel with small probability of error even in presence of noise

↓ error

$R > C$: reliable transmission of info not possible & no coding technique can control the errors.

Channel efficiency : $\eta_{ch} = \frac{R_t}{C} \times 100\%$

$$\eta_{ch} = \frac{I(A,B)}{Max[I(A,B)]} \times 100\%$$

Channel Redundancy : $R_{\eta_{ch}} = 1 - \eta_{ch}$

- Special Channels →
  - Symmetric / Uniform: all rows & columns have same elements arranged in diff order
  - Binary Symmetric (BSC): 2 i/p, 2 o/p
  - Binary Erasure (BEC): errors totally erased
  - Noiseless : only 1 non zero element in each column
  - Deterministic : " " each row
  - Cascaded: 2 channels connected in cascade