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- Questions

1 If x_1 has mean 4 and variance 9 and x_2 has mean -2 and variance 4 and they both are independent, find $E(2x_1 + x_2 - 3)$ and $V(2x_1 + x_2 - 3)$

→ Given: $E(x_1) = 4$, $V(x_1) = 9$
$E(x_2) = -2$ and $V(x_2) = 4$

$E(2x_1 + x_2 - 3) = E(2x_1 + x_2) - 3 = 2E(x_1) + E(x_2) - 3$
$= 2(4) + (-2) - 3 = 3$

$V(2x_1 + x_2 - 3) = V(2x_1 + x_2) = 2^2 V(x_1) + V(x_2) = 4(9) + 4 = 40$

2 Given the following probability function of a discrete r.v. X.

X :	0	1	2	3	4	5	6	7
$P(X=x)$:	0	c	2c	2c	3c	c^2	$2c^2$	$7c^2+c$

Find: i) c ii) $P(x \geq 6)$ iii) $P(x < 6)$
iv) k if $P(x \leq k) > 1/2$ where $k \in N$
v) $P(1.5 < x < 4.5 / x > 2)$ vi) $E(x)$

→ i) $\sum P_i = 1$ $0 + c + 2c \times 2 + 3c + c^2 + 2c^2 + 7c^2 + c = 1$
$9c + 10c^2 - 1 = 0$ $c = 1/10 = 0.1$

X :	0	1	2	3	4	5	6	7
$P(c=0.1)$:	0	0.1	0.2	0.2	0.3	0.01	0.02	0.17

ii) $P(x \geq 6) = P(x=6) + P(x=7)$
$= 0.02 + 0.17 = 0.19$

iii) $P(x<6) = 1 - P(x \geq 6) = 1 - 0.19 = 0.81$

iv) The smallest k $P(X \leq k) > 1/2$
$P(x \leq 2) = 0.3$ $P(X \leq 3) = 1/2 = 0.5$ $P(x \leq 4) = 0.8$
smallest k is 4

v) $P(1.5 < x < 4.5 \mid x > 2)$
$P(A/B)$ A: $1.5 < x < 4.5$ B: $x > 2$

$$P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{P(2 < x < 4.5)}{P(x > 2)}$$

$$= \frac{P(3) + P(4)}{P(3)+P(4)+P(5)+P(6)+P(7)} = \frac{0.5}{0.7} = \frac{5}{7}$$

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vi) $E(x) = \sum p_i x_i = 0\times0 + 1\times0.1 + 2\times0.2 + 3\times0.2$
$+ 4\times0.3 + 5\times0.01 + 6\times0.02 + 7\times0.17 = 3.66$

vii) $V(x) = E(x^2) - [E(x)]^2 = \sum p_i(x_i^2) + [3.66]^2$

$E(x^2) = 1\times0.1 + 0.2\times4 + 9\times0.2 + 16\times0.3 + 25\times0.01$
$+ 36\times0.02 + 49\times0.17 = 16.8$

$V(x) = 16.8 - [3.66]^2 = 3.4044$

- Median and Mode

For continuous distribution, the median M divides the area under the curve from $x=a$ to $x=b$ into 2 equal parts

If M is median, $\int_a^M f(x)\,dx = \int_M^b f(x)\,dx = \frac{1}{2}$

By solving any one of the equations, we can get M

Using theory of maxima, mode is obtained by solving the equation

$\frac{dy}{dx} = 0$ i.e. $f'(x) = 0$ with the condition

that $\frac{d^2y}{dx^2} < 0$ i.e. $f''(x) < 0$

and that x lies in the interval $[a,b]$ of x.

- Questions :-

1 Prob. distribution of a discrete random variable is given by

x :	0	1	2	3
$P(X=x)$:	k	0.3	0.5	k

If $Y = X^2 + 2X$, find the prob distribution of Y, mean and variance of Y. What is the prob that $1 < y < 10$

$\Sigma P_i = 1$ $\quad k + 0.3 + 0.5 + k = 1$

$k = 0.1$

X :	0	1	2	3
$P(X=x)$:	0.1	0.3	0.5	0.1

$Y = x^2 + 2x$

Y :	0	3	8	15
$P(Y=y)$:	0.1	0.3	0.5	0.1

Prob. Distribution

$P(1 < Y < 10) = P(y=3) + P(y=8) = 0.3 + 0.5 = 0.8$

mean $\bar{Y} = \Sigma P_i Y_i = 0(0.1) + 3(0.3) + 8(0.5) + 15(0.1) = 6.4$

$E(Y^2) = \Sigma P_i Y_i^2 = 0^2(0.1) + 9(0.3) + 64(0.5) + 225(0.1) = 57.2$

Variance $V(Y) = E(Y^2) - (E(Y))^2$

$= 57.2 - 40.96 = 16.24$

2 A rv x has the following probability distribution

x :	-2	-1	0	1	2
$P(X=x)$:	1/5	1/5	2/5	2/15	1/15

Find the prob distribution of i) $V = x^2 + 1$ ii) $W = x^2 + 2x + 3$

i)

V :	5	2	1	2	5
$P(V)$:	1/5	1/5	2/5	2/15	1/15

We can write it as :-

V :	1	2	5
$P(v)$:	2/5	1/3	4/15

$\frac{1}{5} + \frac{1}{15} = \frac{4}{15}$

$\frac{1}{5} + \frac{2}{15} = \frac{1}{3}$

ii)

W :	3	2	3	6	11
$P(W)$:	1/5	1/5	2/5	2/15	1/5

We can write it as :-

W :	2	3	6	11
$P(W)$:	1/5	3/5	2/15	1/15

3 If the mean of the following distribution is 16, find m, n and variance

X :	8	12	16	20	24
$P(X=x)$:	1/8	m	n	1/4	1/12

$E(X) = \Sigma P_i X_i = 8 \times \frac{1}{8} + 12 \times m + 16 \times n + 20 \times \frac{1}{4} + 24 \times \frac{1}{12}$

$16 = 8 + 12m + 16n \longrightarrow 12m + 16n = 8$

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$\sum p_i = 1 \longrightarrow 1 = 1/8 + m + n + 1/4 + 1/12$

$m + n = 13/24$

$m = 1/6 \quad n = 3/8$

$Var(x) = E(x^2) + [E(x)]^2$

$E(x^2) = \sum p_i(x_i^2) = 64 \times 1/8 + 144 \times 1/6 + 256 \times 13/24 + 400 \times 1/4 + 24 \times 24 \times 1/12$

$= 318.666$

$Var(x) = 318.666 + 256 = 574.666$

4 A function is defined as $f(x) = \begin{cases} 0, & \text{for } x < 2 \\ (2x+3)/18, & \text{for } 2 \le x \le 4 \\ 0, & \text{for } x > 4 \end{cases}$

Show that $f(x)$ is a probability density function and find the probability that $2 < x < 3$

→ Conditional pdf :- $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$\int_{\infty}^{2} 0\,dx + \int_{2}^{4} \frac{(2x+3)}{18}\,dx + \int_{4}^{\infty} 0\,dx = \frac{1}{9}\left[\frac{x^2}{2}\right]_2^4 + \frac{1}{6}[x]_2^4$

$= 16/18 - 4/18 + 4/6 - 2/6 = 18/18 = 1$

$P(2 < x < 3) = \int_{2}^{3} \frac{2x+3}{18} = \left[\frac{1}{9}\frac{x^2}{2} + \frac{1}{6}x\right]_2^3$

$= \frac{1}{9} \times \frac{9}{2} + \frac{1 \times 5}{6} - \frac{1}{9} \times \frac{4}{2} + \frac{1}{6} \times 2$

$= \left[\frac{1}{2} + \frac{1}{2}\right] - \left[\frac{2}{9} + \frac{1}{3}\right] = 1 - \frac{5}{9} = \frac{4}{9} = 0.444$

5 A continuous random variable has probability density fn.

$f(x) = 6(x - x^2), \quad 0 \le x \le 1$

Find i) mean ii) Variance iii) median iv) mode v) harmonic mean vi) $P(|x - m| < \sigma)$

vii) determine a number b such that

$P(x \le b) = P(x \ge b)$

viii) $P(\mu - 2\sigma < X < \mu + 2\sigma)$

where m = median, μ = mean, σ = S.D.

$f(x) = k(x - x^2) \quad 0 \le x \le 1$

$\int_0^1 f(x)\,dx = 1 \qquad k\int_0^1 (x - x^2)\,dx = 1$

$k\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = 1 \longrightarrow k\left[\frac{1}{2} - \frac{1}{3}\right] = 1 \longrightarrow k \times \frac{1}{6} = 1$

$k = 6 \longrightarrow 6 \times 1/6 = 1$

i) mean = $E(x) = \sum P_i x_i = \int f(x)\, x\, dx$

$\cancel{= \int 6(x - x^2)\, f(x)\, dx}$

$= \int_0^1 6(x - x^2)\, x\, dx = 6\left[\frac{x^3}{3} - \frac{x^4}{4}\right]$

$= 6\left[1/3 - 1/4\right] = 6 \times 1/12 = 1/2$

ii) $V(x) = E(x^2) - [E(x)]^2$

$E(x^2) = \int_0^1 x^2 f(x)\,dx = 6\int_0^1 x^2(x - x^2)\,dx$

$= 6\left[\frac{x^4}{4} - \frac{x^5}{5}\right]_0^1 = 6\left[\frac{1}{4} - \frac{1}{5}\right] = \frac{6}{20}$

$V(x) = 6/20 - (1/2)^2 = 1/20$

iii) $\int_0^M 6(x - x^2)\,dx = \frac{1}{2}$

$6\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^M = \frac{1}{2} \rightarrow 3M^2 - 2M^3 = 1/2$

$M = 1/2$ or $(1 \pm \sqrt{3})/2$

$M = 1/2$ lies in $(0, 1)$, so $M = 1/2$

iv) $f'(x) = 0 \quad f''(x) < 0$

$f'(x) = 6(1 - 2x) = 0 \qquad x = 1/2$

$f''(x) = -12 \qquad$ Mode $= 1/2$

v) Harmonic mean : $\frac{1}{H} = \int_0^1 \frac{1}{x}\, 6(x - x^2)\,dx$

$= 6\int_0^1 (1 - x)\,dx = 6\left[x - x^2/2\right]_0^1$

$= 6\left[1 - 1/2\right] = 3$

$H = 1/3$

vi) $P(|x-M| < \sigma) = P(-\sigma < x-M < \sigma)$

$= P(M-\sigma < x < M+\sigma)$

$= P(0.2764 < x < 0.7236)$

$= 6\int_{0.2764}^{0.7236} (x-x^2)\,dx = 6\left[\frac{x^2}{2}-\frac{x^3}{3}\right]_{0.2764}^{0.7236}$

$= 6[0.1355 - 0.0311] = 0.6264$

6) If X is a continuous r.v with probability density fn. given by $f(x) = k(x-x^3)$; $0 \le x \le 1$

Find i) k ii) mean iii) variance iv) median v) mode

→ i) $\int_0^1 k(x-x^3)\,dx = 1 \rightarrow k\left[\frac{x^2}{2}-\frac{x^4}{4}\right]_0^1 = 1$

$k\left[\frac{1}{2}-\frac{1}{4}\right] = 1 \longrightarrow k = 4$

ii) mean $= E(x) = \sum p_i x_i = \int x f(x)\,dx = \int_0^1 (x^2-x^4)4\,dx$

$= 4\left[\frac{x^3}{3}-\frac{x^5}{5}\right]_0^1 = 4\left[\frac{1}{3}-\frac{1}{5}\right] = \frac{8}{15}$

iii) variance $= V(x) = \int x^2 f(x)\,dx = \int_0^1 (x^3-x^5)4\,dx$

$= 4\left[\frac{x^4}{4}-\frac{x^5}{5}\right]_0^1 = 4\left[\frac{1}{4}-\frac{1}{5}\right] = \frac{1\times\cancel{4}}{5\times\cancel{4}} = \frac{1}{5}$

iv) $\int_0^M f(x)\,dx = \frac{1}{2} \longrightarrow 4\int_0^M (x-x^3)\,dx = \frac{1}{2}$

$4\left[\frac{x^2}{2}-\frac{x^4}{4}\right]_0^M = \frac{1}{2} \longrightarrow 2M^2 - M^4 = 1/2$

$M = 1/2$

v) mode : $f'(x) = 0 \longrightarrow 4(1-3x^2) = 0$

$4 - 12x^2 = 0 \longrightarrow x^2 = 1/3$

$f''(x) = 4(-6x) < 0 \longrightarrow -24x < 0$

mode $= 1/\sqrt{3} = \sqrt{3}/3$

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7 A continuous r.v. X has p.d.f $f(x) = kx^3 e^{-x}$, $x \geq 0$
Find k, mean and variance.

$\rightarrow$ $\int_0^\infty kx^3 e^{-x}\, dx = 1 \longrightarrow k\int_0^\infty x^3 e^{-x}\, dx = 1 = k\sqrt{4}$
$= 6k \rightarrow k = 1/6$

mean $= \int f(x)\, x\, dx = k\int_0^\infty x^4 e^{-x}\, dx = \frac{1}{\cancel{6}} \times 5\times 4 \times \cancel{3} \times 2$
$= 5\times 4 = 20$

variance $= \int f(x)\, x^2\, dx = 1/6 \int_0^\infty x^5 e^{-x}\, dx = \frac{1}{\cancel{6}} \times 5\times4\times3\times2\times\cancel{6}$
$= 120$

8 Let x be a continuous random variable with pdf $f(x) = kx(1-x)^3$
$0 \leq x \leq 1$. Find k, mean and variance.

$\rightarrow$ $\int_0^1 kx(1-x^3-2x+2x^2)\, dx = \int_0^1 k(x-x^4-2x^2+2x^3)\, dx = 1$

$= k\left[\frac{x^2}{2} - \frac{x^5}{5} - \frac{2x^3}{3} + \frac{x^4}{2}\right]_0^1 = k\left[\frac{1}{2} - \frac{1}{5} - \frac{2}{3} + \frac{1}{2}\right] = 1$

$k = 15/2$

mean $= \int f(x)\, x\, dx = \int_0^1 k(x^2 - x^5 - 2x^3 + 2x^4)\, dx$

$= \left[\frac{x^3}{3} - \frac{x^6}{6} - \frac{2x^4}{4} + \frac{2x^5}{5}\right]_0^1 \times \frac{15}{2}$

$= \left[\frac{1}{3} - \frac{1}{6} - \frac{1}{2} + \frac{2}{5}\right] \times \frac{15}{2} = \frac{1}{2}$

Variance $= \int f(x)\, x^2\, dx = k\int_0^1 (x^3 - x^6 - 2x^4 + 2x^5)\, dx$

$= \frac{15}{2}\left[\frac{x^4}{4} - \frac{x^7}{7} - \frac{2x^5}{5} + \frac{2x^6}{6}\right]_0^1 = \frac{15}{2}\left[\frac{1}{4} - \frac{1}{7} - \frac{2}{5} + \frac{1}{3}\right] = \frac{17}{56}$

$\longrightarrow$ Joint Probability