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PSOT Mod 5

- **Queuing theory**

Helps to determine the blance b/w
a> cost of offering the service ↑ services ↑ expensive ↓ time waiting
b> cost incurred due to delay in offering service can't let customers go.

- **Performance Measure of a Queing System**

1 Average (or expected) time spent by a customer in queue & system.
If this is more, then the queueing system is bad.

- W_q : avg time an arriving customer has to wait in queue before being served.
- W_s : avg time an arriving customer spends in the system, including waiting and service.

2 Average (expected) no. of customers in queue or/and system.
L_q : avg no. of customers waiting for service in the queue (queue length)
L_s : avg no. of customers in the system (either waiting for services in the queue or being served)

3 Value of time both for customers and servers
P_w : probability that an arriving customer has to wait before being served (also called locking probability)
rho $\rho = \lambda / \mu$: % of time a server is busy serving customers (system utilization)
P_n : Probability of n customers waiting for service in queueing system.
P_d : Probability that an arriving customer is not allowed to enter in the queueing (system capacity is full)

4 n = no. of customers in the system
λ = avg customer arrival rate or avg no. of arrivals per unit in the queueing system.
μ = average service rate or average no. of customers served per unit time at the place of service.

should be given in question

$\frac{\lambda}{\mu} = \rho = \frac{\text{avg service completion time } (1/\mu)}{\text{avg interarrival time } (1/\lambda)}$ = traffic intensity or server utilization factor.

P_0 = probability of no customer in the system

s = no. of service channels (service facilities or servers)

N = max no. of customers allowed in the system

$P_w = 1 - P_0 = \lambda/\mu$

For achieving a ~~state~~ steady state condition, $\lambda/\mu < 1$

- **Relation Among Performance Measures**

$L_s = L_q$ + customer being served $= L_q + \lambda/\mu$

$W_q = L_q/\lambda$

$W_s = W_q + 1/\mu$

Probability of being in the system (waiting and being served) longer than time t is given by :-

$$P(W_s > t) = e^{-(\mu-\lambda)t}$$

$$P(W_s \le t) = 1 - P(W_s > t)$$

Probability of only waiting for service longer than time t is given by :-

$$P(W_q > t) = \lambda/\mu \, e^{-(\mu-\lambda)t}$$

Probability of exactly n customers in the system is given by

$$P_n = P_0 \left(\frac{\lambda}{\mu}\right)^n = \left(1 - \frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^n$$

Probability that no. of customers in system, n exceeds a given no., r is given by :- $P(n > r) = \left(\frac{\lambda}{\mu}\right)^{r+1}$

- **Classification of Queueing Models**

$\{(a/b/c) : (d/e)\}$

a: arrivals distribution b: service time distribution

c: no. of servers (service channels).

d = capacity of the system (queue plus service)
e = queue (or service) discipline

- Single Server Queueing Theory

Model I : {(M/M/1) : (∞/FCFS)}

↳ 1 person serving ↳ ∞ queue is possible ↳ First Come First Serve

Exponential Service (Poisson)

$P_0 = 1 - \lambda/\mu = 1 - \rho$

$P_n = \left(\frac{\lambda}{\mu}\right)^n \left(1 - \frac{\lambda}{\mu}\right) = \rho^n (1 - \rho)$;

$L_s = \frac{\rho}{1-\rho} = \frac{\lambda}{\mu - \lambda}$; $\rho = \frac{\lambda}{\mu}$ customer in line + customer being served

$L_q = \frac{\lambda}{\mu - \lambda} - \frac{\lambda}{\mu} = \frac{\lambda^2}{\mu(\mu - \lambda)}$ customers waiting in queue (queue length)

$W_q = \lambda\left(1 - \frac{\lambda}{\mu}\right)\frac{1}{(\mu-\lambda)^2} = \frac{\lambda}{\mu(\mu-\lambda)}$ or $\frac{L_q}{\lambda}$ expected waiting time

$W_s = W_q + \frac{1}{\mu} = \frac{\lambda}{\mu(\mu-\lambda)} + \frac{1}{\mu} = \frac{1}{\mu - \lambda}$ or $\frac{L_s}{\lambda}$ waiting and service

variance (fluctuation of queue length)

$Var(n) = \frac{\rho}{(1-\rho)^2} = \frac{\lambda\mu}{(\mu-\lambda)^2}$

$P(n > 1) = 1 - P_0 - P_1$ queue non empty, 2 people (1 served, 1 in queue)

$= 1 - \left(1 - \frac{\lambda}{\mu}\right) - \left(1 - \frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right) = \left(\frac{\lambda}{\mu}\right)^2$

$P(n \geq k) = \left(\frac{\lambda}{\mu}\right)^k$ and $P(n > k) = \left(\frac{\lambda}{\mu}\right)^{k+1}$

$L = \frac{\text{expected length of waiting line}}{P(n>1)} = \frac{\mu}{\mu - \lambda}$ expected length of non-empty queue

Questions

1. With the usual notations, find the average number of customers in the system and in the queue if the system is $M/M/1/\infty$ and $\lambda = 10$, $\mu = 15$

→ $L_q = \dfrac{\lambda^2}{\mu(\mu-\lambda)} = \dfrac{100}{15(15-10)} = \dfrac{100}{75} = 1.33$ avg no of customers in queue

$L_s = \dfrac{\lambda}{\mu-\lambda} = \dfrac{10}{5} = 2$ " " in system

2. With the usual notation find the avg waiting time per customer in the queue and in the system for $M/M/1/\infty$ model if $\lambda = 9$, $\mu = 15$ per hour.

→ $W_q = \dfrac{\lambda}{\mu(\mu-\lambda)} = \dfrac{9}{15(15-9)} = 0.1$ hour $= 6$ min avg waiting time in queue

$W_s = \dfrac{1}{\mu-\lambda} = \dfrac{1}{6}$ hours $= 10$ min " in system

3. Find probability that a customer has to wait in an $M/M/1/\infty$ model if $\lambda = 8$, $\mu = 10$ per hour.

→ P(customer has to wait) $= P(n > 0) = 1 - P_0 = \lambda/\mu = 8/10$

4. What is probability that in an $M/M/1/\infty$ model with 6 persons arriving per hour and 8 persons being served per hour, there will be more than 8 persons in the system

→ $\lambda = 6$ $\mu = 8$

$P(n > 8) = \left(\dfrac{\lambda}{\mu}\right)^{n+1} = \left(\dfrac{6}{8}\right)^{9} = 0.075$

5. Find the probability that a customer has to wait more than 20 minutes to be out of the service stn with $\lambda = 8$ per hour and $\mu = 11$ per hour if its $M/M/1/\infty$ model. Also find probability that a customer has to wait in queue more than 15 mins.

→ $\lambda = 8$ $\mu = 11$

i) $P(W_s > 1/3) = e^{-(\lambda-\mu)t} = e^{-(11-8)1/3} = e^{-1} = 0.3678$

ii) $P(W_q > 1/4) = \dfrac{\lambda}{\mu} e^{-(\mu-\lambda)t} = \dfrac{8}{11} e^{-3 \times 1/4} = 0.3435$

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6 TV repairman has exponential distribution of time spent on jobs with a mean of 30min. Arrival of sets follows Poisson distribution with appx avg rate of 10 per 8 hours day. Whats repairman's expected idle time each day? How many jobs are ahead of average set just brought in?

→ $\lambda = \frac{10}{8} = \frac{5}{4}$ sets / hour $\quad \mu = \left(\frac{1}{30}\right) 60 = 2$ sets / hour

No. of hours for which repairman remains busy in an 8-hours day (traffic intensity) is given by.

busy time $= \rho = \lambda/\mu = \frac{5}{4 \times 2} = \frac{5}{8}$ (1 hour)

free time $= 1 - \rho = 1 - \frac{5}{8} = \frac{3}{8}$ (1 hour)

8 hours $\longrightarrow 8 \times 3/8 = 3$ hours

$L_s = \frac{\lambda}{\mu - \lambda} = \frac{5/4}{2-(5/4)} = \frac{5}{3} = 2$ (approx) TV sets expected in system

7 In a railway yard, goods train arrive at a rate of 30 trains per day. Exponential distribut^n of inter arrival time, service time distribut^n too with avg of 36 min.
Calculate: a) expected queue size (line length)
b) probability that queue size exceeds 10.
If input of trains increases to an average of 33 per day, what will be change in (a) and (b)

→ $\lambda = 30$ trains per day $\quad \mu = 40$ trains per day

$\mu = 36 \times 24/60$

service rate per day

$\rho = \lambda/\mu = \frac{30}{40} = 0.75$ traffic intensity

Expected queue size $L_q = \frac{\lambda^2}{\mu(\mu-\lambda)} = \frac{30 \times 30}{40(10)} = \frac{9}{4}$

$P(n>10) = \left(\frac{\lambda}{\mu}\right)^{k+1} = \left(\frac{30}{40}\right)^{11} = \left(\frac{3}{4}\right)^{11}$

$\lambda = 33 \quad \mu = 40$

$\rho = 0.825$

$L_q = 33^2 / 40(40-33) = 33^2/280$

$P(n>10) = (33/40)^{11} = 0.120$

8 Arrivals at telephone booth: Poisson, avg time = 10 min b/w arrivals. Length of phone calls exponentially distributed with mean = 3 min.

a) What is the probability that a person arriving at booth will have to wait?

b) What is the probability that it'll take a customer more than 10 min altogether to wait for the phone and complete his call?

c) The telephone department will install a second booth when convinced that an arrival has to wait for at least 3 min for a phone call. By how much should the flow of arrivals increase in order to justify a second booth.

$\lambda = 6$ per hour as its $1/10$ per minute

$\mu = 1/3$ per min $= 1/3 \times 60 = 20$ person per hour

a) $P(n>0) = 1 - P_0 = \lambda/\mu = \frac{0.1}{0.33} = 0.3$

b) $P(W_s \geq 10) = e^{-(\mu-\lambda)t} = e^{-2.3}$

c) $W_q = \frac{\lambda'}{(\mu-\lambda')\mu} = 3$ (given)

λ' : increased arrival rate

$(0.33-\lambda')\, 3 \times 0.33 = \lambda'$

$\lambda' = 0.16$ increase in $\lambda = 0.16 - 0.10 = 0.06$ arrivals / min

9 Warehouse has only one loading dock, manned by 3 person crew. Trucks arrive at avg rate of 4 trucks per hour and is Poisson distributed. Loading a truck takes 10 min on an avg and can be assumed to be exponentially

distributed. The operating cost of a truck is ₹20 / hour and crew = ₹6 / hour. Would you advice the truck owner to add another crew of three persons?

→ $\lambda = 4$ trucks / hour $\mu = 6$ per hour : crew

Existing crew :- Total hourly cost = loading crew cost + operating cost

= {(no. of loaders) × (Hourly wage rate)}

+ {total trucks in system × hourly operating cost}

$= 6 \times 3 + \frac{1}{\mu - \lambda} \times 20 \times 4 = ₹58$ / hour

New Crew :- μ : 3 → 6 6 → 12

Total hourly cost $= 6 \times 6 + \frac{1}{12-4} \times 4 \times 20 = ₹46$ per hour

Truck owner must add a crew of another 3 loaders

10 A road transport company has 1 reservation clerk on duty at a time. He handles info of bus schedule & makes reservations. Customers arrive at 8 per hour & clerk service 12 customers / hour

a) what is avg no. of customers waiting for service of the clerk? L_q

b) What's avg time a customer has to wait before being served? W_q

c) Management wants to install a comp. system for handling info & reservations. This is expected to reduce the service from 5 to 3 min. Additional cost of having new system works out to ₹50 / day & cost of goodwill of having to wait is estimated to be 12 paise /min spent waiting, before being served, should the company install company system? Assume 8 hours working day.

→ $\lambda = 8$ / hour customers $\mu = 12$ customers/hour

b) $W_q = \frac{\lambda}{\mu(\mu-\lambda)} = \frac{8}{12(12-8)} = \frac{1}{6}$ hours = 10 mins

a) $L_q = \frac{\lambda^2}{\mu(\mu-\lambda)} = \frac{8 \times 8}{12 \times 4} = \frac{4}{3}$

c) Existing :- avg cost for a customer's waiting time in the system is $0.12 \times W_q$

→ 12 paise / hour

before $\mu = 12$ 5 min $5 \times 12 = 60$

$\mu = 20$ for 3 min $3 \times 20 = 60$

1 hour $\rightarrow$ 8

8 hours $\rightarrow$ 64

There are 8 arrivals / hour or in 8 hours, $8 \times 8 = 64$ customers request service at a goodwill cost of $0.12 \times W_q^{\to 10} \times 64$

$= 76.8$

By installing a computer system, computer will ↑ a clerk's service rate upto $\mu = 20$ customer per hour (3 customers per min) Thus, avg time spent by a customer waiting will be

$$W_q = \frac{\lambda}{\mu(\mu-\lambda)} = \frac{8}{20(20-8)} = \frac{1}{30} \text{ hours} = 2 \text{ mins}$$

goodwill

and avg daily queueing cost will be reduced to $64 \times (0.12 \times W_q)$ and an additional cost of having computer would be ₹50 / day. Thus avg total daily cost TC = Comp cost + goodwill cost

$$= 50 + (64 \times 0.12 \times W_q)^2$$

$$= 50 + \underset{15.3}{\cancel{}} = \cancel{}\ 65.3$$

- Model 2: $\{(M/M/1) : (N/FCFS)\}$

Exponential Service - Finite (or Limited) Queue

$$P_0 = \begin{cases} \left(\dfrac{1-\rho}{1-\rho^{N+1}}\right) & ; \dfrac{\lambda}{\mu} \neq 1 \\ \dfrac{1}{N+1} & ; \rho = 1 \end{cases}$$

$$P_n = \begin{cases} \left(\dfrac{1-\rho}{1-\rho^{N+1}}\right)\rho^n & ; n \leq N;\ \lambda/\mu \neq 1 \\ \dfrac{1}{N+1} & ; \dfrac{\lambda}{\mu} = 1 \end{cases}$$

$$L_s = \sum_{n=0}^{N} n P_n = \begin{cases} \dfrac{\rho}{1-\rho} - \dfrac{(N+1)\rho^{N+1}}{1-\rho^{N+1}} & ; \rho \neq 1 (\lambda \neq \mu) \\ \dfrac{N}{2} & ; \rho = 1 (\lambda = \mu) \end{cases}$$

$$L_q = L_s - (1 - P_0)$$

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$W_s = (L_s + 1)/\mu$

$W_q = W_s - 1/\mu = L_s/\mu$

Potential customer lost = time for which system is busy.

$P_N = P_0 \rho^N$

Effective arrival rate, $\lambda_{eff} = \lambda(1 - P_N)$

Effective traffic intensity, $\rho_{eff} = \lambda_e/\mu$

- Questions

1 In an M/M/1/4 Queueing system with Poisson Model $\lambda = 5$ & $\mu = 15$

a) Find Probability that there's nobody in the system. Also find the probability if $\mu = \lambda$

b) Find Probability that there'll be 4 customers in system. Also find this probability if $\mu = \lambda$.

→ $\lambda = 5 \quad \mu = 15 \quad \rho = \lambda/\mu = 1/3 \quad N = 4$

a) $P_0 = \dfrac{1-\rho}{1-\rho^{N+1}} = \dfrac{1-1/3}{1-(1/3)^5} = 0.669$

if $\lambda = \mu \rightarrow P_0 = 1/N+1 = 1/5 = 0.2$

b) $P_4 = P_0 \rho^N = 0.669\,(1/3)^4 = 0.0082$

if $\lambda = \mu \quad P_4 = 1/N+1 = 0.2$

2 A small Aerodrome has capacity of handling one plane at a time. The area available can accommodate only 2 planes in waiting for take off. Aeroplanes arrive at the rate of 4 planes per day (in 8 hour shift), aerodrome can handle on an avg 4 planes/day

P_1, P_2, P_3 ← Find probabilities that there will be 1, 2 or 3 planes at aerodrome

L_s & L_q ← Find expected no. of planes at aerodrome & in queue.

→ $N = 3 \quad \lambda = 4 \quad \mu = 4$

i) $P_1 = \dfrac{1}{N+1} = \dfrac{1}{3+1} = \dfrac{1}{4} \quad \lambda = \mu$

$P_2 = 1/4 \quad P_3 = 1/4$

ii) $L_s = N/2 = 3/2 \quad P_0 = 1/N+1 = 1/4$

$L_q = L_s - (1 - P_0) = 3/2 - (3/4) = 3/4$

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3 1 operation theatre, arrival rate = 3 patients / day (8 hours)
expected time of an operation = 0.25 day. The hospital has capacity to take care of 3 patients (including the one on the operation table) at the most.

L_q ← a) Find the average no. of patients in the hospital in the queue
$\lambda = 4$, $\mu = 4$ ← b) How'll the result change if arrival rate is 4 patients per day

→ $\lambda = 3$ $\mu = 4$ $N = 3$ $\rho = 3/4$

$L_q = L_s - (1 - P_0)$ $L_s = \frac{\rho}{1-\rho} - \frac{(N+1)\rho^{N+1}}{1-\rho^{N+1}}$

$\rho = \lambda/\mu = 0.75$ $L_s = 1.1485$

$L_q = 1.1485 - (1 - P_0)$

$P_0 = \frac{1-\rho}{1-\rho^{N+1}} = \frac{0.25}{1-(0.75)^4} = 0.366$

$L_q = 0.5145$

$\lambda = 4$ $\mu = 4$ $N = 3$ $\rho = 1$

$L_q = L_s(1 - P_0)$

$L_s = \frac{N}{2} = 1.5$ $P_0 = \frac{1}{N+1} = 0.75$

$L_q = \frac{3}{2} - \left(1 - \frac{1}{4}\right) = \frac{3}{4}$

4 Scooter mech can accommodate 5 scooters at a time, 4 in queue and 1 being repaired. The no. of scooters arriving follows a Poisson distribution with mean 6 per hour. Repair time = exponential distribution with mean of 4/hour.

i) What % of time the mechanic is idle?
ii) What % of customers are turned away?
iii) What is effective arrival rate
iv) What is expected no. of scooters waiting for repair?
v) What is avg time a customer will be at the auto centre?

→ $N = 5$ $\lambda = 6$ $\mu = 4$

i) $P_0 = \left(\frac{1-\rho}{1-\rho^{N+1}}\right)$

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ii) $P_N = \left(\frac{1-\rho}{1-\rho^{N+1}}\right)\rho^N = 0.3652 = 36.52\%$.

iii) $\lambda_{eff} = \lambda(1-P_N)$

iv) $L_q = L_s - (1-P_0)$

$L_s = \frac{\rho}{1-\rho} - \frac{(N+1)\rho^{N+1}}{1-\rho^{N+1}}$

v) $W_s = (L_s+1)/\mu$

5 Consider a single server queueing system with Poisson input & exp. service times. Mean arrival rate = 3 calling units/hour, expected service time = 0.25 hour and max^m permissible calling units in the system is 2. Derive steady state probability distribution of no. of calling units in system, & calculate expected no. in the system.

→ $\lambda = 3$ units/hour $\quad \mu = \frac{1}{0.25} = 4$ units/hour $\quad N = 2 \quad \rho = \frac{\lambda}{\mu} = \frac{3}{4} = 0.75$

$P_n = \frac{(1-\rho)\rho^n}{1-\rho^{N+1}} = \frac{0.25(0.75)^n}{1-(0.75)^{2+1}} = (0.43)(0.75)^n$

$P_0 = \frac{(1-\rho)}{1-\rho^{N+1}} = 0.431$

Steady state pd of n customers (calling units) in system

$L_s = \frac{\rho}{1-\rho} - \frac{(N+1)\rho^{N+1}}{1-\rho^{N+1}} = \frac{0.75}{0.25} - \frac{3(0.75)^3}{1-(0.75)^3}$

$= 0.8108$

or $L_s = \sum_{n=1}^{N} n P_n = \sum_{n=1}^{2} n (0.43)(0.75)^n$

$= 0.81$

6 If in a period of 2 hours in a day (08:00 to 10:00 am) trains arrive at the yard every 20 min but the service time continues to remain 36 min, then calculate for this period
a) The probability that the yard is empty and
b) avg no. of trains in the system on assumption that the line capacity of the yard is only limited to 4 trains

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→ $\lambda = \frac{1}{20}$ trains/min $\quad \mu = \frac{1}{36}$ trains/min $\quad N = 4$

$\rho = \lambda/\mu = 36/20 = 1.8$

a) $P_0 = \dfrac{1-\rho}{1-\rho^{N+1}} = 0.042187$

b) $L_s = \dfrac{\rho}{1-\rho} - \dfrac{(N+1)\rho^{N+1}}{1-\rho^{N+1}}$