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Practical work 1

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Work developed in the scope of the course

Computational Modelling

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1 Problem

A 1D problem governed by the Helmholtz equation $\frac{\partial^2 p}{\partial x^2} + (\frac{\omega}{c})^2 p = 0$, in which p is the acoustic pressure, c the sound propagation velocity and ω the angular frequency. In this problem, there is the following scenario, in which the propagation medium is composed of two different fluids, with the following arrangement and boundary conditions:

$P = 1Pa$

$v_n = 0\ m/s$

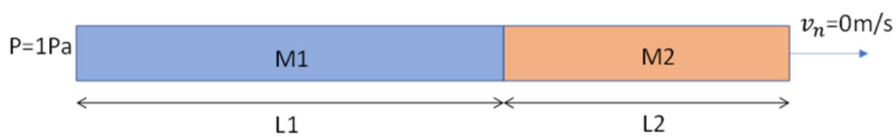

Figure 1: Graphical representation of two fluids with boundary conditions

Both fluids have the same density, and the sound propagation velocity in each fluid is given by:

A1 = 6, A2 = 6;

FLUID M1: $c = 1500.0 + 10.0 \times A1 - A2 = 1554\ m/s$

FLUID M2: $c = (1800.0 + 20.0 \times A2 + A1) \times (1 + 0.01 * i * (2 + \sqrt{A1})) = 1926 + 85.6971724460040i\ m/s$

The dimensions L1 and L2 are given by:

$$L1 = 20 + A1/0.5 = 32\ m$$

$$L2 = 25 + A2/0.5 = 37\ m$$

The excitation frequency is: $500.0 + A2 \times 20.0 + A1 = 626\ Hz$

2 FDM & FEM theoretical questions

2.1 For a generic discretization (with spacing Δx), write the Finite Difference Method (FDM) equations at an internal node of M1, at an internal node of M2, and at the extreme nodes where the boundary conditions are prescribed

Node i: $\dfrac{M1_{i-1} - 2 \cdot M1_i + M1_{i+1}}{\Delta x^2} + K_1^2 \cdot M1_i = 0 \quad \dfrac{M2_{i-1} - 2 \cdot M2_i + M2_{i+1}}{\Delta x^2} + K_2^2 \cdot M2_i$

2.2 Consider now a finite element discretization, with 50 linear elements in fluid M1 and 50 in fluid M2. For the node connecting M1 and M2, write the acoustic stiffness and mass matrices of the two elements connected to the node, and establish the corresponding equation of the Finite Element Method (FEM) at that node.

k_{e_i} – the acoustic stiffness matrice;
m_{e_i} – the acoustic mass matrice;

$$k_{e1} = \begin{pmatrix} \frac{1}{0{,}64} & -\frac{1}{0{,}64} \\ -\frac{1}{0{,}64} & \frac{1}{0{,}64} \end{pmatrix} = \begin{pmatrix} 1.5625 & -1.5625 \\ -1.5625 & 1.5625 \end{pmatrix}$$

$$m_{e1} = \begin{pmatrix} \frac{0{,}64}{3} & \frac{0{,}64}{6} \\ \frac{0{,}64}{6} & \frac{0{,}64}{3} \end{pmatrix} = \begin{pmatrix} 0.2133 & 0.1067 \\ 0.1067 & 0.2133 \end{pmatrix}$$

$$k_{e2} = \begin{pmatrix} \frac{1}{0{,}74} & -\frac{1}{0{,}74} \\ -\frac{1}{0{,}74} & \frac{1}{0{,}74} \end{pmatrix} = \begin{pmatrix} 1.3514 & -1.3514 \\ -1.3514 & 1.3514 \end{pmatrix}$$

$$m_{e2} = \begin{pmatrix} \frac{0{,}74}{3} & \frac{0{,}74}{6} \\ \frac{0{,}74}{6} & \frac{0{,}74}{3} \end{pmatrix} = \begin{pmatrix} 0.2467 & 0.1233 \\ 0.1233 & 0.2467 \end{pmatrix}$$

3 FDM questions

3.1 Starting from the Matlab codes used in the classes for the FDM, adapting it, and also making use of the analytical solution given in the appendix:

3.1.1 Calculate the response using 200 points distributed throughout the domain using the finite difference method, and compare the response with the given analytical solution.

Setting initial values with discretisation of 200 points for the finite difference method:

```
clear;
A1 = 6;
A2 = 6;
% Dimensions:
L1=20+A1/0.5;
L2=25+A2/0.5;
% The length of the domain
```

```
L=L1+L2;
% the number of points for this discretization
npts=200;
% a spacing between points of
dx=L/(npts-1);
%  point coordinates:
x=0:dx:dx*(npts-1);
```

The boundary conditions of the spatial domain (Dirichlet on the left and Neumann on the right) for the problem can also be stored in variables:

```
pressure_left=1; %Dirichlet condition
velocity_right=0; %neumann condition
```

Finding the solution with FDM and plotting results:

```
P=fdm_1D(freq,c1,c2,L1,L2,pressure_left, velocity_right, npts, rho);
plot(x,real(P),x,imag(P),'LineWidth',2);grid
    on;xlabel('x(m)');ylabel('P(Pa)');legend('real','imag.')
```

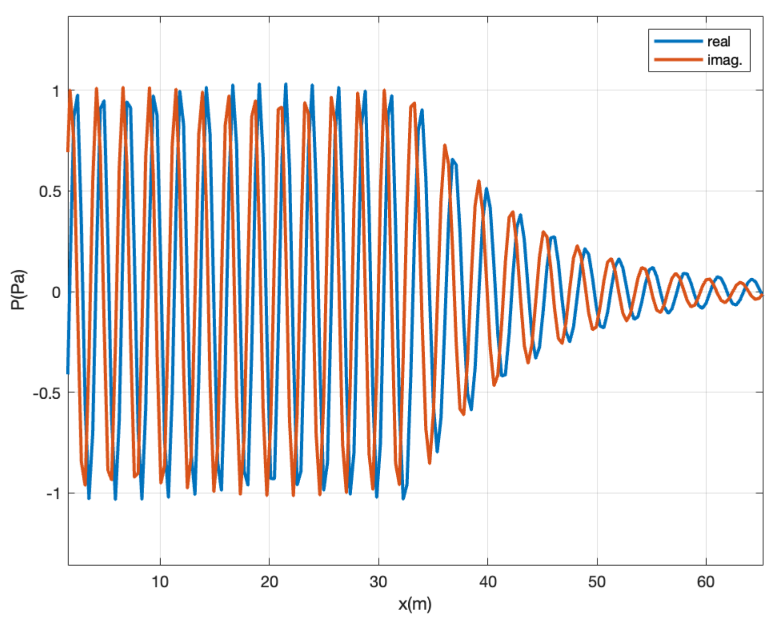

Figure 2: Numerical solution with FDM for 200 points

From x(m) $\approx$ 32 can be seen damping oscillations of pressure as there is a point of intersection between fluids.

Comparing the response with the given analytical solution

Real part has the same beginning, but difference between two solutions gradually

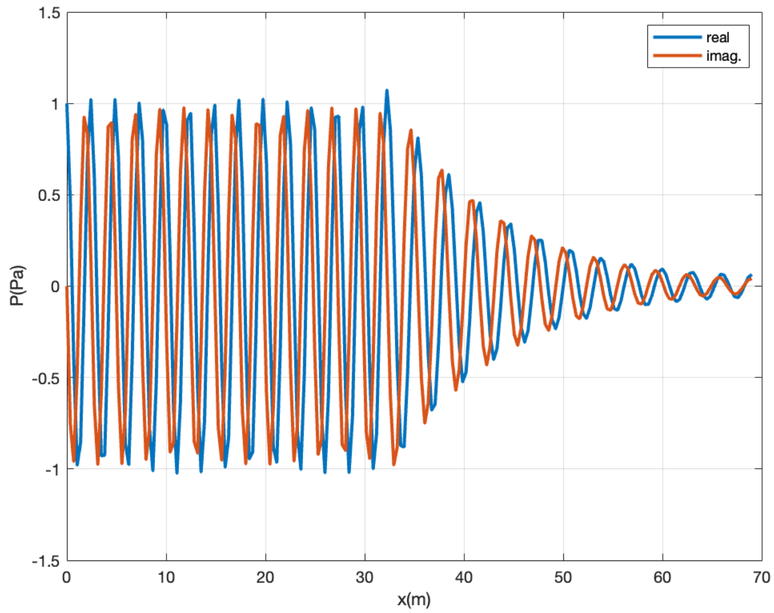

Figure 3: Analytical solution with FDM for 200 points

starts to appear. Imaginary part of the numerical solution has higher range and goes off in the end points

3.1.2 For discretizations starting at 100 points and then progressively increasing this number (use at least 10 different discretizations), calculate the error in the numerical estimation of the pressure (P) at the interface between the two media, and represent the result as a convergence curve (in a Log-Log scale), representing the number of points in the horizontal axis and the absolute error in the vertical axis.

For convergence analysis the absolute error is given by:

$$\epsilon = |p_{analytical} - p_{numeric}|$$

```
ndiscr = 20;
x_plot = 100:100:100*ndiscr;
y_plot = zeros(1,ndiscr);
...
%  the numerical estimation of the pressure (P) at the interface between the two
    media
   x1=0;
   for jj=floor(L1/dx):npts-1
       if(x(jj)>=L1)
```

```
        x1=jj-1;
        break
    end
end

a=L1-x(x1); b=x(x1+1)-L1;
p_num =(P(x1)*b+P(x1+1)*a)/dx;
p_analytical =(P_s(x1)*b+P_s(x1+1)*a)/dx;
y_plot(ii) = abs(p_analytical-p_num);
```

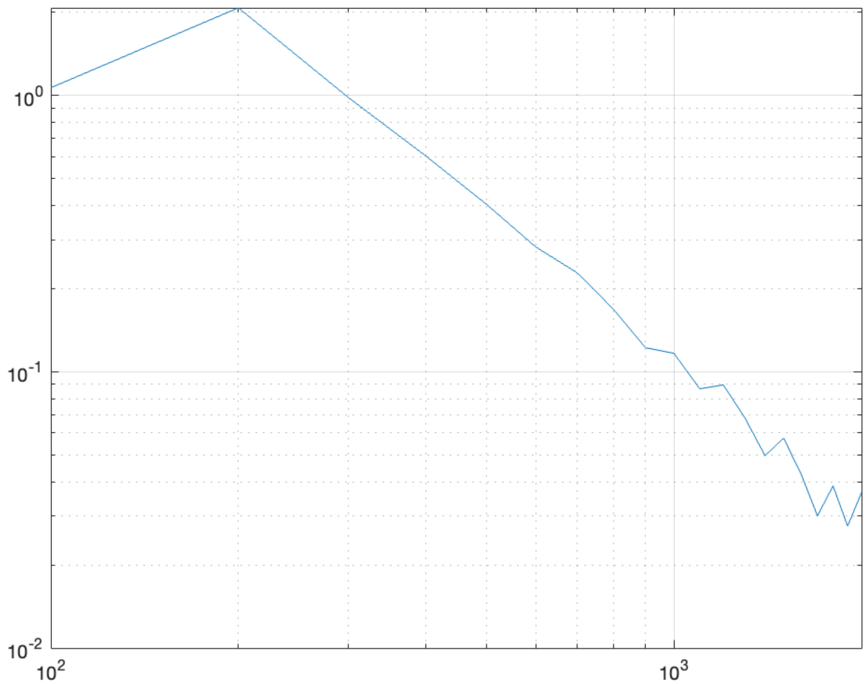

Figure 4: Log-Log scale of the convergence curve with 20 different discretizations from 100 up to 2000 points

From the starting point of 100 points and to 200 points absolute error was rising. In can be explained with not enough discretisation for the current case. With further increment of points graph continued to decrease the error between solutions. The higher amount of points the graph has (after 1000 points), the more fluctuations appear still having a decline of error values.

4 FEM questions

4.1 Starting from the Matlab codes used in the classes for the FEM, adapting them, and also making use of the analytical solution given in the appendix:

1. Discretize the domain using a maximum element size of 0.5m, and compute the solution using the FEM. Compare the response with the given analytical solution.

```
P = fem_1D(freq,c1,c2,L1,L2,L1/dx,L2/dx,rho)
```

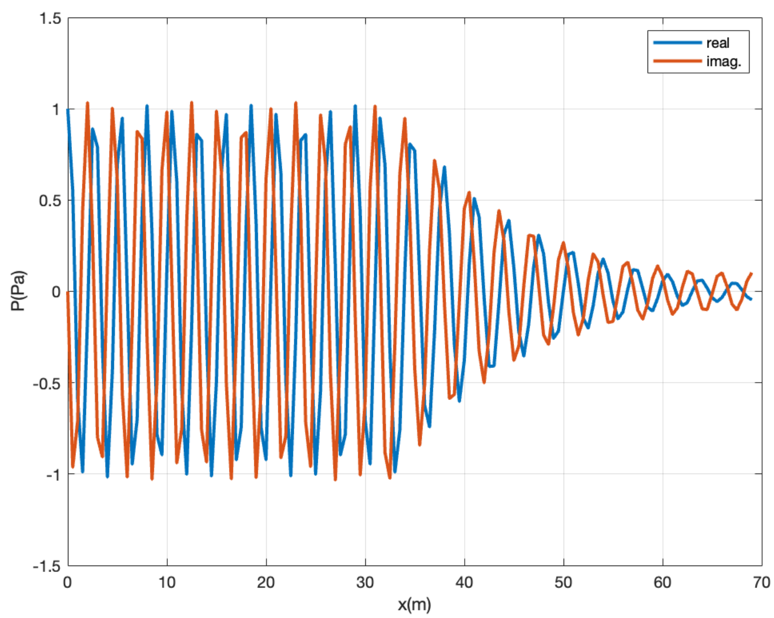

Figure 5: FEM numerical solution of pressure over two fluids with a element size of 0.5 m

Comparing the response with the given analytical solution

Current discretisation is not enough for the given case, that is why graph of the numerical solution is not as smooth as analytical one.

2. For successively finer discretizations, starting with an element size of approximately 1.0 m and then reducing the element size (use at least 10 different discretizations), calculate the error in the numerical estimation of the pressure (P) at interface between the two media, and represent the result as a convergence curve (in a Log-Log scale), representing the total number of nodes in the horizontal axis and the absolute error in the vertical axis.

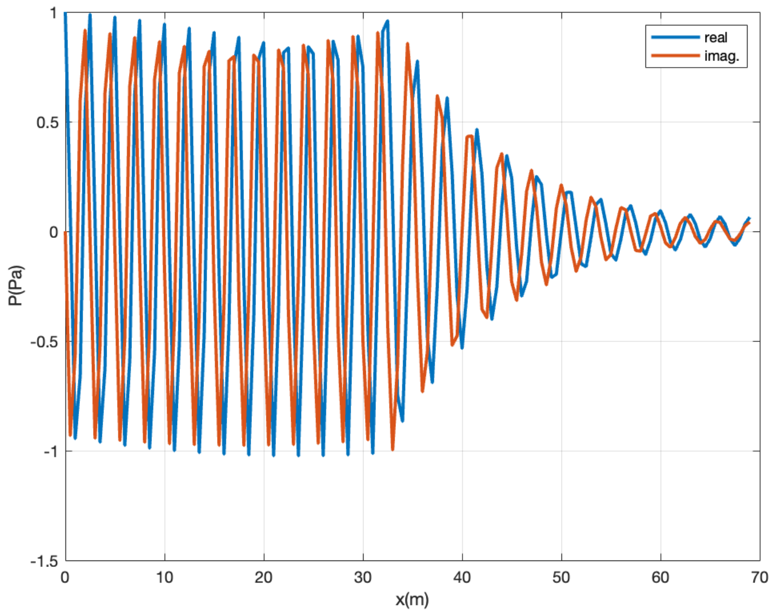

Figure 6: FEM analytical solution of pressure over two fluids with a element size of 0.5 m

For convergence analysis the absolute error is given by:

$$\epsilon = |p_{analytical} - p_{numeric}|$$

```
ndiscr = 20;
x_plot = 100:100:100*ndiscr;
y_fem_plot = zeros(1,ndiscr);
% different discretization for each length
for ii=1:ndiscr
    npts=x_plot(ii);
    n1=floor(npts*L1/L); n2=ceil(npts*L2/L);
    dx1=L1/n1;    dx2=L2/n2;
    x_range1=0:dx1:L1;
    x_range2=L1:dx2:L;
    x = [x_range1 x_range2];
    x(npts+1)=[];

   P=fem_1D(freq,c1,c2,L1,L2,n1,n2,rho);
    P_s=solution_1D(freq,c1,c2,L1,L2,x);
    %  the numerical estimation of the pressure (P) at the interface between the
        two media
    y_fem_plot(ii) = abs(P_s(n1)-P(n1));
```

Each fluid interval has an amount of points proportional to the percentage of its part on the whole interval.

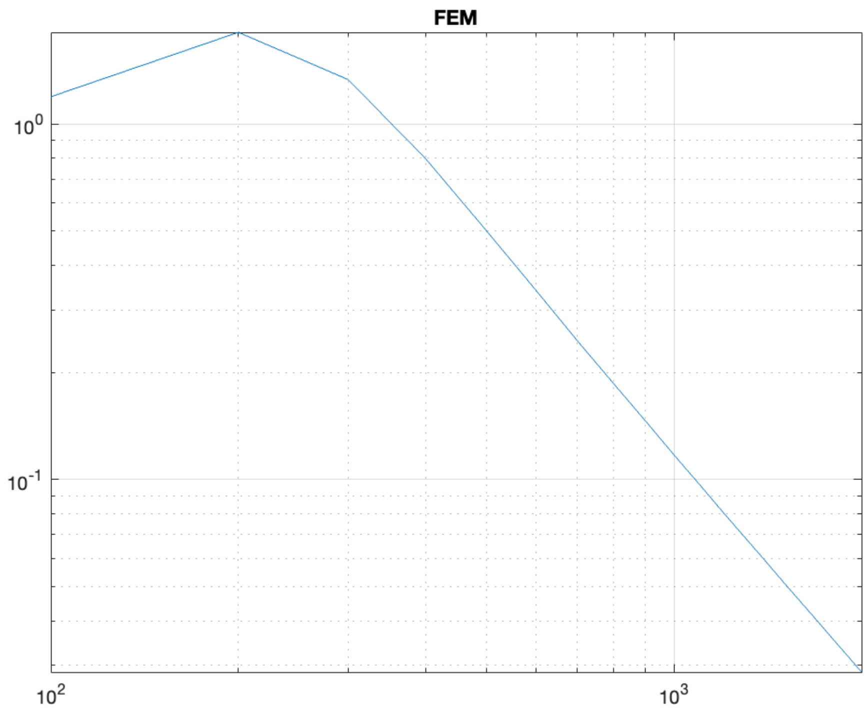

Figure 7: Log-Log scale of the convergence curve with 20 different discretizations from 100 to 2000 points

From the starting point of 100 points and to 200 points absolute error raised. In can be explained with the small discretisation for the current case. With further increment of points graph continued to decrease the error between solutions. The graph on higher amount of points keeps going down with a tendency to some fluctuations.

4.2 Compare the convergence curves calculated for the FDM and FEM, superimposing both curves in the same plot. Comment the obtained results.

In the initial discretisation of 100 points to 200 points it has a significant increase of error. It can be explained with the small discretisation for the current case. Further discretizations have shown a gradual decrease of error.

FEM curve should have had faster decline than FDM, in that case I assume that my calculations might be incorrect with identifying point of intersection between two fluids.

Nevertheless, both curves show a declining tendency of absolute error between solutions. That indicates that with an increase of number of elements on the interval of two

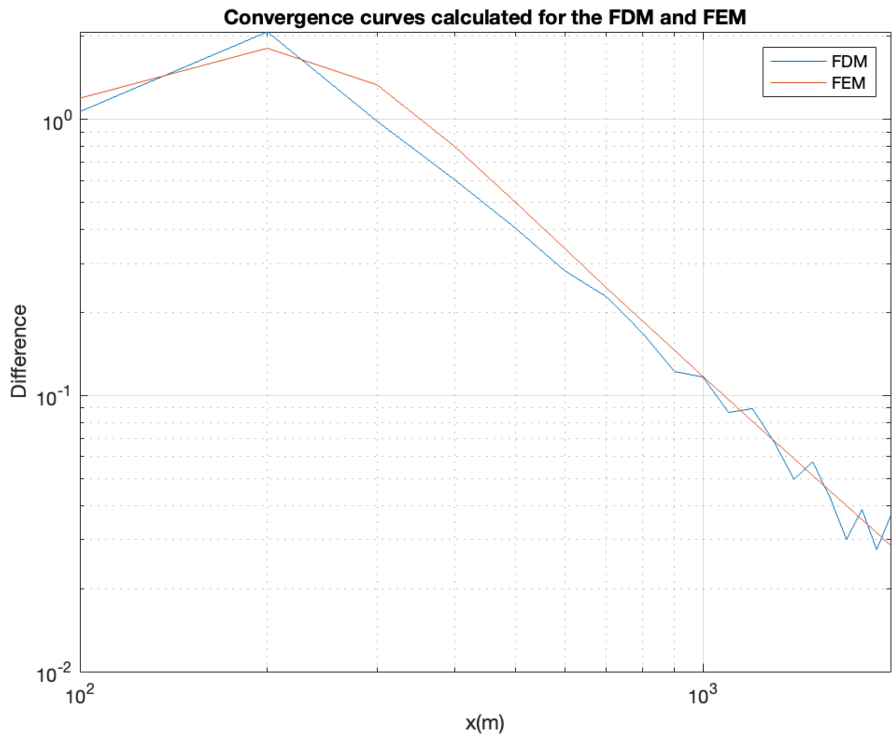

Figure 8: Log-Log scale of the convergence curve with 20 different discretizations from 100 to 2000 points

fluids we have better performance of the absolute error between numerical and analytical solutions. Also, there is a tendency to some fluctuations on higher discretizations for FDM case (>1000 points).

4.3 Using the FEM with adequate discretizations, compute the response of the system for frequencies between 1 Hz and the excitation frequency given above, considering two different cases for the properties of medium M2:

$$h = \frac{c}{8f} - \text{for defining the adequate discretization}$$

For this case the chosen discretization was 600 points.

1. The propagation velocity given above;
2. Only the REAL part of the propagation velocity given above.

These plots show different behaviour of pressure depending on the input velocity. Following the golden colour plot at low frequencies both graphs have sharp rises of pressure

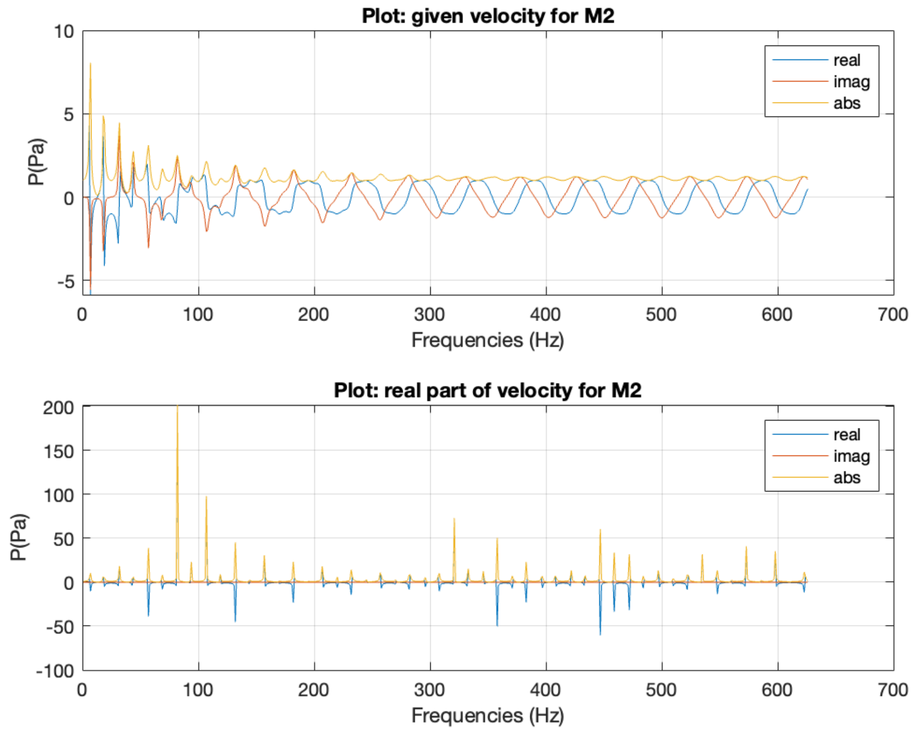

Figure 9: Graphical results at the interface point between M1 and M2 in terms of the real, imaginary and absolute values of the acoustic pressure for each frequency

value, but with given velocity for M2 that peak gradually fades to a periodic wave. On the other hand, in the second case there present some minor peaks and it does not show the sign of periodicity along the whole frequency range.

For each case, present the results graphically:

1. at the interface point between M1 and M2 in terms of the real, imaginary and absolute values of the acoustic pressure for each frequency.

2. in a colormap plot, for position x frequency, in terms of a sound pressure level with a reference 1 Pa (20*log10(P/1.0)).

```
pcolor(x,freqs, 20*log10(abs(C)));
colormap(jet);
colorbar;
shading interp;
```

Comparing colormaps for cases with given and only real part of velocities for M2 fluid we can see that there is a different gradation of sound pressure level. For first first case sound pressure changes up to -40 at the end of input interval. While for the second case values tend to stay the same changing mainly from -20 to 20.

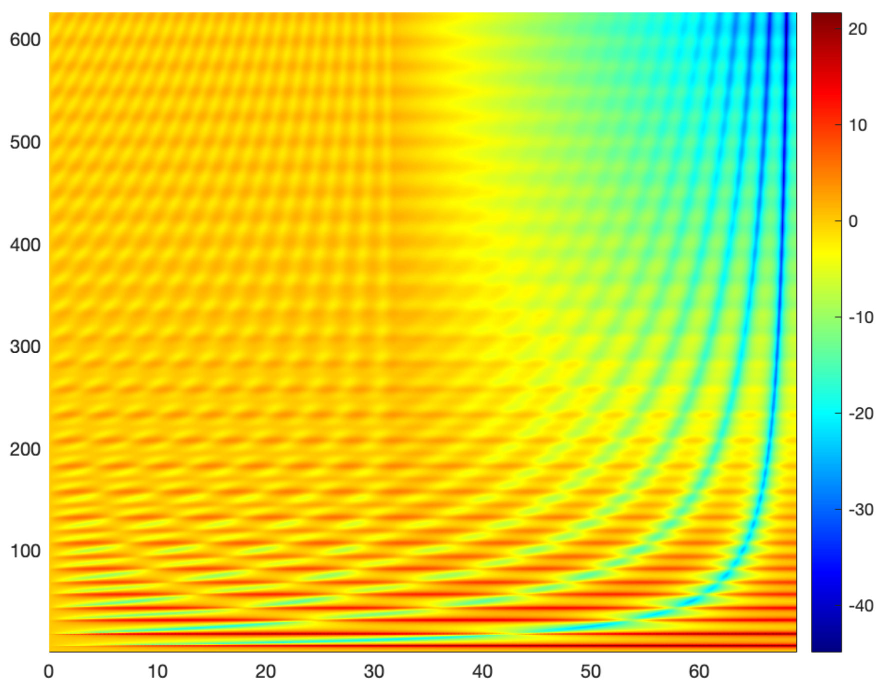

Figure 10: Colormap plot, for position x frequency, in terms of a sound pressure level for M2 with only the given propagation velocity

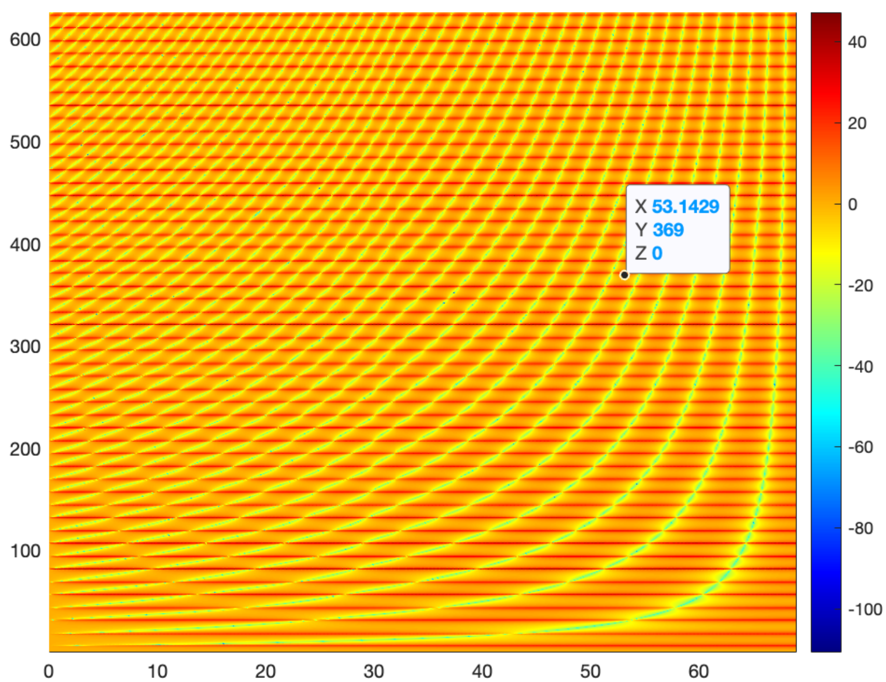

Figure 11: Colormap plot, for position x frequency, in terms of a sound pressure level for M2 with only the REAL part of the propagation velocity given above