数值分析 · 课程作业 · 第五章

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1. (1) 求一次拟合多项式. 有 $\varphi_0(x)=1,\ \varphi_1(x)=x$, 所以

$$\boldsymbol{G}=\begin{bmatrix}1 & -1.00\\ 1 & -0.75\\ 1 & -0.50\\ 1 & -0.25\\ 1 & 0\\ 1 & 0.25\\ 1 & 0.50\\ 1 & 0.75\\ 1 & 1.00\end{bmatrix},\ \boldsymbol{y}=\begin{bmatrix}-0.2209\\ 0.3295\\ 0.8826\\ 1.4392\\ 2.0003\\ 2.5645\\ 3.1334\\ 3.7061\\ 4.2836\end{bmatrix},\ \boldsymbol{G}^{\mathrm{T}}\boldsymbol{G}=\begin{bmatrix}9 & 0\\ 0 & 3.75\end{bmatrix},\ \boldsymbol{G}^{\mathrm{T}}\boldsymbol{y}=\begin{bmatrix}18.1183\\ 8.44367\end{bmatrix}.$$

解法方程组 $\boldsymbol{G}^{\mathrm{T}}\boldsymbol{G}\begin{bmatrix}a\\ b\end{bmatrix}=\boldsymbol{G}^{\mathrm{T}}\boldsymbol{y}$ 得

$$a\approx 2.01314,\ b\approx 2.25165,$$

故所求一次拟合多项式为

$$y=2.01314+2.25165x,$$

(2) 求二次拟合多项式. 有 $\varphi_0(x)=1,\ \varphi_1(x)=x,\ \varphi_2(x)=x^2$, 所以

$$\boldsymbol{G}=\begin{bmatrix}1 & -1.00 & 1\\ 1 & -0.75 & 0.5625\\ 1 & -0.50 & 0.2500\\ 1 & -0.25 & 0.0625\\ 1 & 0 & 0\\ 1 & 0.25 & 0.0625\\ 1 & 0.50 & 0.2500\\ 1 & 0.75 & 0.5625\\ 1 & 1.00 & 1\end{bmatrix},\ \boldsymbol{G}^{\mathrm{T}}\boldsymbol{G}=\begin{bmatrix}9 & 0 & 3.75\\ 0 & 3.75 & 0\\ 3.75 & 0 & 2.7656\end{bmatrix},\ \boldsymbol{G}^{\mathrm{T}}\boldsymbol{y}=\begin{bmatrix}18.1183\\ 8.4437\\ 7.5870\end{bmatrix}.$$

解法方程组 $\boldsymbol{G}^{\mathrm{T}}\boldsymbol{G}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \boldsymbol{G}^{\mathrm{T}}\boldsymbol{y}$ 得

$$a \approx 2.00009,\ b \approx 2.25165,\ c \approx 0.0313426,$$

故所求二次拟合多项式为

$$y = 2.00009 + 2.25165x + 0.0313426x^2.$$

2. (1) 方程组的偏差平方和为

$$Q = (3x_1 + 2x_2 - 2)^2 + (4x_1 - 5x_2 - 3)^2 + (2x_1 + x_2 - 11)^2 + (-x_1 + 3x_2 - 10)^2,$$

方程组的最小二乘解满足

$$\frac{\partial Q}{\partial x_1} = 0,\ \frac{\partial Q}{\partial x_2} = 0,$$

$$\begin{cases} 3(3x_1 + 2x_2 - 2) + 4(4x_1 - 5x_2 - 3) + 2(2x_1 + x_2 - 11) - (-x_1 + 3x_2 - 10) = 0, \\ 2(3x_1 + 2x_2 - 2) - 5(4x_1 - 5x_2 - 3) + (2x_1 + x_2 - 11) + 3(-x_1 + 3x_2 - 10) = 0. \end{cases}$$

所以最小二乘解为

$$x_1 = \frac{12}{7},\ x_2 = \frac{10}{7}.$$

3. 有 $\varphi_0(x) = 1,\ \varphi_1(x) = x^2$, 所以

$$\boldsymbol{G} = \begin{bmatrix} 1 & 361 \\ 1 & 625 \\ 1 & 961 \\ 1 & 1444 \\ 1 & 1936 \end{bmatrix},\ \boldsymbol{y} = \begin{bmatrix} 19.0 \\ 32.3 \\ 49.0 \\ 73.3 \\ 97.8 \end{bmatrix},\ \boldsymbol{G}^{\mathrm{T}}\boldsymbol{G} = \begin{bmatrix} 5 & 5327 \\ 5327 & 7277699 \end{bmatrix},\ \boldsymbol{G}^{\mathrm{T}}\boldsymbol{y} = \begin{bmatrix} 271.4 \\ 369322 \end{bmatrix}.$$

解法方程组 $\boldsymbol{G}^{\mathrm{T}}\boldsymbol{G}\begin{bmatrix} a \\ b \end{bmatrix} = \boldsymbol{G}^{\mathrm{T}}\boldsymbol{y}$ 得

$$a \approx 0.972579,\ b \approx 0.0500351,$$

故所求经验公式为

$$y = 0.972579 + 0.0500351x^2,$$

其均方误差为

$$\sqrt{\frac{1}{5}\sum_{i=1}^{5}(y_i - f(x_i))^2} \approx 0.054815.$$

4. 将经验公式变换为

$$\ln y = \ln a + bx.$$

设 $\hat{y} = \ln y,\ \hat{a} = \ln a$, 于是

$$\hat{y} = \hat{a} + bx.$$

有

$$\varphi_0(x) = 1,\ \varphi_1(x) = x,$$

计算得

$$\boldsymbol{G} = \begin{bmatrix} 1 & 2.2 \\ 1 & 2.7 \\ 1 & 3.5 \\ 1 & 4.1 \\ 1 & 4.8 \end{bmatrix},\ \hat{\boldsymbol{y}} = \begin{bmatrix} 4.174439 \\ 4.09434 \\ 3.97029 \\ 3.91202 \\ 3.82864 \end{bmatrix},\ \boldsymbol{G}^{\mathrm{T}}\boldsymbol{G} = \begin{bmatrix} 5 & 17.3 \\ 17.3 & 64.23 \end{bmatrix},\ \boldsymbol{G}^{\mathrm{T}}\hat{\boldsymbol{y}} = \begin{bmatrix} 19.9797 \\ 68.5511 \end{bmatrix}.$$

解法方程组 $\boldsymbol{G}^{\mathrm{T}}\boldsymbol{G}\begin{bmatrix} \hat{a} \\ b \end{bmatrix} = \boldsymbol{G}^{\mathrm{T}}\hat{\boldsymbol{y}}$ 得

$$\hat{a} \approx 4.4538,\ a \approx 85.953,\ b \approx -0.13233,$$

故所求经验公式为

$$y = 85.953\mathrm{e}^{-0.13233x}.$$

9. 作变换 $x = \dfrac{t+1}{2}$, 则 $f(\dfrac{t+1}{2}) = \sin\dfrac{(t+1)\pi}{2} \triangleq g(t) \in C[-1,1]$. 采用勒让德多项式作最佳平方逼近, 有

$$P_0(t) = 1,\ P_1(t) = t,\ P_2(t) = \frac{3t^2-1}{2},$$

最佳平方逼近的系数为

$$a_k^* = \frac{2k+1}{2}\int_{-1}^{1} P_k(t)g(t)\mathrm{d}t, \quad k = 0,1,2.$$

计算得

$$
\begin{aligned}
a_0^* &= \frac{1}{2}\int_{-1}^{1} \sin\frac{(t+1)\pi}{2}\mathrm{d}t = \frac{2}{\pi},\\
a_1^* &= \frac{3}{2}\int_{-1}^{1} t\sin\frac{(t+1)\pi}{2}\mathrm{d}t = 0,\\
a_2^* &= \frac{5}{2}\int_{-1}^{1} \frac{3t^2-1}{2}\sin\frac{(t+1)\pi}{2}\mathrm{d}t = \frac{10}{\pi}\left(1-\frac{12}{\pi^2}\right).
\end{aligned}
$$

故

$$
P_2^*(t) = \frac{2}{\pi} + \frac{10}{\pi}\left(1-\frac{12}{\pi^2}\right)\frac{3t^2-1}{2},
$$

于是所求的二次最佳平方逼近多项式为

$$
\begin{aligned}
P_2^*(2x-1) &= \frac{2}{\pi} + \frac{10}{\pi}\left(1-\frac{12}{\pi^2}\right)\frac{3(2x-1)^2-1}{2}\\
&= \frac{2}{\pi} + \frac{10}{\pi}\left(1-\frac{12}{\pi^2}\right)(6x^2-6x+1)\\
&\approx -4.12251x^2 + 4.12251x - 0.050465.
\end{aligned}
$$

10. 采用勒让德多项式作最佳平方逼近, 有

$$
P_0(x) = 1,\ P_1(x) = x,\ P_2(x) = \frac{3x^2-1}{2},\ P_3(x) = \frac{5x^3-3x}{2},
$$

最佳平方逼近的系数为

$$
a_k^* = \frac{2k+1}{2}\int_{-1}^{1} P_k(x)f(x)\mathrm{d}x, \quad k = 0,1,2,3.
$$

计算得

$$
\begin{aligned}
a_0^* &= \frac{1}{2}\int_{-1}^{1}(3x^4+2x^3+1)\mathrm{d}x = \frac{1}{2}\left(\frac{3}{5}x^5+\frac{x^4}{2}+x)\right|_{-1}^{1} = \frac{8}{5},\\
a_1^* &= \frac{3}{2}\int_{-1}^{1}x(3x^4+2x^3+1)\mathrm{d}x = \frac{3}{2}\left(\frac{x^6}{2}+\frac{2}{5}x^5+\frac{x^2}{2})\right|_{-1}^{1} = \frac{6}{5},\\
a_2^* &= \frac{5}{2}\int_{-1}^{1}\frac{3x^2-1}{2}(3x^4+2x^3+1)\mathrm{d}x = \frac{5}{2}\left[\frac{3}{2}\left(\frac{3}{7}x^7+\frac{x^6}{3}+\frac{x^3}{3})\right|_{-1}^{1} - \frac{8}{5}\right] = \frac{12}{7},\\
a_3^* &= \frac{7}{2}\int_{-1}^{1}\frac{5x^3-3x}{2}(3x^4+2x^3+1)\mathrm{d}x = \frac{7}{2}\left[\frac{5}{2}\left(\frac{3}{8}x^8+\frac{2}{7}x^7+\frac{x^4}{4})\right|_{-1}^{1} - \frac{6}{5}\right] = \frac{4}{5},
\end{aligned}
$$

于是所求的一次和三次最佳平方逼近多项式分别为

$$
\begin{aligned}
P_1^*(x) &= \frac{8}{5} + \frac{6}{5}x, \\
P_3^*(x) &= \frac{8}{5} + \frac{6}{5}x + \frac{12}{7}\frac{3x^2-1}{2} + \frac{4}{5}\frac{5x^3-3x}{2} \\
&= 2x^3 + \frac{18}{7}x^2 + \frac{26}{35}.
\end{aligned}
$$