数值分析 · 课程作业 · 第三章

姓名：潘林越　班级：数学与应用数学 2020-2 班　学号：15194694

1. (2) 设 $\boldsymbol{A}=(a_{ij})_{n\times n}$, $\begin{bmatrix} a_{11} & \boldsymbol{a}_1 \\ 0 & \boldsymbol{A}_2 \end{bmatrix}=(a'_{ij})_{n\times n}$. 则已知 $|a_{ii}|>\sum\limits_{j\neq i}|a_{ij}|, i=1,2,\cdots,n$，要证 $|a'_{ii}|>\sum\limits_{2\leq j\leq n, j\neq i}|a'_{ij}|, i=2,\cdots,n$. 由高斯顺序消去法，我们有 $l_i=a_{i1}/a_{11}$, $a'_{ij}=a_{ij}-l_ia_{1j}$, $i,j=2,\cdots,n$，于是

$$\text{待证命题}\Leftrightarrow |a_{ii}-l_ia_{1i}|>\sum_{2\leq j\leq n, j\neq i}|a_{ij}-l_ia_{1j}|,\quad i=2,\cdots,n$$

有 $LHS\geq |a_{ii}|-|l_ia_{1i}|$ 以及 $\sum\limits_{2\leq j\leq n, j\neq i}|a_{ij}|+|l_ia_{1j}|\geq RHS$，则原命题加强为

$$\Leftarrow |a_{ii}|-|l_ia_{1i}|>\sum_{2\leq j\leq n, j\neq i}(|a_{ij}|+|l_ia_{1j}|),\quad i=2,\cdots,n$$

$$\Leftrightarrow |a_{ii}|>|l_i|\sum_{2\leq j\leq n}|a_{1j}|+\sum_{2\leq j\leq n, j\neq i}|a_{ij}|,\quad i=2,\cdots,n$$

有 $\sum\limits_{2\leq j\leq n}|a_{1j}|<|a_{11}|$，则原命题加强为

$$\Leftarrow |a_{ii}|\geq|l_ia_{11}|+\sum_{2\leq j\leq n, j\neq i}|a_{ij}|=|a_{i1}|+\sum_{2\leq j\leq n, j\neq i}|a_{ij}|=\sum_{1\leq j\leq n, j\neq i}|a_{ij}|,\quad i=2,\cdots,n$$

由已知，此命题成立，证毕.

2. (1) $\overline{A}=\left[\begin{array}{ccc|c} 4 & 2 & 1 & 7 \\ 2 & -5 & 2 & -1 \\ 1 & 2 & 6 & 9 \end{array}\right]\xrightarrow[r_3-\frac{1}{4}r_1]{r_2-\frac{1}{2}r_1}\left[\begin{array}{ccc|c} 4 & 2 & 1 & 7 \\ 0 & -6 & \frac{3}{2} & -\frac{9}{2} \\ 0 & \frac{3}{2} & \frac{23}{4} & \frac{29}{4} \end{array}\right]\xrightarrow{r_3+\frac{1}{4}r_2}\left[\begin{array}{ccc|c} 4 & 2 & 1 & 7 \\ 0 & -6 & \frac{3}{2} & -\frac{9}{2} \\ 0 & 0 & \frac{49}{8} & \frac{49}{8} \end{array}\right]$

$\Rightarrow x_1=x_2=x_3=1.$

(2) $\overline{A}=\left[\begin{array}{ccc|c} -5 & 1 & -1 & -6 \\ 1 & 5 & -2 & -1 \\ -1 & -2 & 4 & 3 \end{array}\right]\xrightarrow[r_3-\frac{1}{5}r_1]{r_2+\frac{1}{5}r_1}\left[\begin{array}{ccc|c} -5 & 1 & -1 & -6 \\ 0 & \frac{26}{5} & -\frac{11}{5} & -\frac{11}{5} \\ 0 & -\frac{11}{5} & \frac{21}{5} & \frac{21}{5} \end{array}\right]$

$\xrightarrow{r_3+\frac{11}{26}r_2}\left[\begin{array}{ccc|c} -5 & 1 & -1 & -6 \\ 0 & \frac{26}{5} & -\frac{11}{5} & -\frac{11}{5} \\ 0 & 0 & \frac{85}{26} & \frac{85}{26} \end{array}\right]\Rightarrow x_1=1,x_2=0,x_3=1.$

观察可知，每步消元结果的系数矩阵都对称且严格对角占优，右下方矩阵对称，列主元均在对角线处，消元过程符合上题结论.

3. (1) $\boldsymbol{A}\boldsymbol{x}=\boldsymbol{b}\Leftrightarrow\begin{cases}x_1-\frac{1}{2}x_2=\frac{1}{2}\\ x_2-\frac{2}{3}x_3=\frac{1}{3}\\ x_3-\frac{3}{4}x_4=1\\ x_4=\frac{4}{5}\end{cases}\Leftrightarrow\begin{cases}x_4=\frac{4}{5}\\ x_3=\frac{8}{5}\\ x_2=\frac{7}{5}\\ x_1=\frac{6}{5}\end{cases}$

4. (1) $\left|\begin{array}{ccc}4 & 2 & 1\\ \frac{1}{2} & \multicolumn{1}{|c}{-6} & \frac{3}{2}\\ \frac{1}{4} & \multicolumn{1}{|c}{-\frac{1}{4}} & \multicolumn{1}{|c}{\frac{49}{8}}\end{array}\right.$ 于是 $\boldsymbol{A}=\boldsymbol{L}\boldsymbol{U}=\begin{bmatrix}1&0&0\\ \frac{1}{2}&1&0\\ \frac{1}{4}&-\frac{1}{4}&1\end{bmatrix}\begin{bmatrix}4&2&1\\ 0&-6&\frac{3}{2}\\ 0&0&\frac{49}{8}\end{bmatrix}$.

解 $\boldsymbol{L}\boldsymbol{y}=\boldsymbol{b}$ 得 $\boldsymbol{y}=(7,-\frac{9}{2},\frac{49}{8})^{\mathrm{T}}$，解 $\boldsymbol{U}\boldsymbol{x}=\boldsymbol{y}$ 得 $\boldsymbol{x}=(1,1,1)^{\mathrm{T}}$.

(2) $\left|\begin{array}{ccc}-5 & 1 & -1\\ -\frac{1}{5} & \multicolumn{1}{|c}{\frac{26}{5}} & -\frac{11}{5}\\ \frac{1}{5} & \multicolumn{1}{|c}{-\frac{11}{26}} & \multicolumn{1}{|c}{\frac{85}{26}}\end{array}\right.$ 于是 $\boldsymbol{A}=\boldsymbol{L}\boldsymbol{U}=\begin{bmatrix}1&0&0\\ -\frac{1}{5}&1&0\\ \frac{1}{5}&-\frac{11}{26}&1\end{bmatrix}\begin{bmatrix}-5&1&-1\\ 0&\frac{26}{5}&-\frac{11}{5}\\ 0&0&\frac{85}{26}\end{bmatrix}$.

解 $\boldsymbol{L}\boldsymbol{y}=\boldsymbol{b}$ 得 $\boldsymbol{y}=(-6,-\frac{11}{5},\frac{85}{26})^{\mathrm{T}}$，解 $\boldsymbol{U}\boldsymbol{x}=\boldsymbol{y}$ 得 $\boldsymbol{x}=(1,0,1)^{\mathrm{T}}$.

7. $||\boldsymbol{x}||_1=1+5+2=8, ||\boldsymbol{x}||_2=\sqrt{1^2+5^2+2^2}=\sqrt{30}, ||\boldsymbol{x}||_\infty=5, ||\boldsymbol{A}||_1=\max\{4,4,4+2\}=6, ||\boldsymbol{A}||_2=\sqrt{\lambda_{\max}(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{A})}=\sqrt{18+2\sqrt{65}}, ||\boldsymbol{A}||_\infty=\max\{4,4+4,2\}=8.$

10. 由 $||\boldsymbol{x}||$ 是 $\mathbb{R}^n$ 上的一种向量范数可知，

(1) $||\boldsymbol{x}||_P=||\boldsymbol{P}\boldsymbol{x}||\geq 0$，当且仅当 $\boldsymbol{P}\boldsymbol{x}=\boldsymbol{0}$ 即 $\boldsymbol{x}=\boldsymbol{0}$ 时等号成立;

(2) $\forall\lambda\in\mathbb{R}, ||\lambda\boldsymbol{x}||_P=||\lambda\boldsymbol{P}\boldsymbol{x}||=|\lambda|\cdot||\boldsymbol{P}\boldsymbol{x}||=|\lambda|\cdot||\boldsymbol{x}||_P$;

(3) $||\boldsymbol{x}+\boldsymbol{y}||_P=||\boldsymbol{P}\boldsymbol{x}+\boldsymbol{P}\boldsymbol{y}||\leq||\boldsymbol{P}\boldsymbol{x}||+||\boldsymbol{P}\boldsymbol{y}||=||\boldsymbol{x}||_P+||\boldsymbol{y}||_P$.

所以 $||\boldsymbol{x}||_P$ 是 $\mathbb{R}^n$ 上的一种向量范数.

14. (1) $\boldsymbol{r}(\boldsymbol{x}_1)=\boldsymbol{A}\boldsymbol{x}_1-\boldsymbol{b}=(-0.001343,-0.001572)^{\mathrm{T}}, ||\boldsymbol{r}(\boldsymbol{x}_1)||_\infty=0.001572,$

$\boldsymbol{r}(\boldsymbol{x}_2)=\boldsymbol{A}\boldsymbol{x}_2-\boldsymbol{b}=(-1\times10^{-6},0)^{\mathrm{T}}, ||\boldsymbol{r}(\boldsymbol{x}_2)||_\infty=10^{-6},$

$\delta\boldsymbol{x}_1=\boldsymbol{x}-\boldsymbol{x}_1=(0.001,0.001)^{\mathrm{T}}, \delta\boldsymbol{x}_2=\boldsymbol{x}-\boldsymbol{x}_2=(0.659,-0.913)^{\mathrm{T}}$

(2) $||\boldsymbol{A}||_\infty=1.572, \boldsymbol{A}^{-1}=\begin{bmatrix}659,000 & -563,000\\ -913,000 & 780,000\end{bmatrix}, ||\boldsymbol{A}^{-1}||_\infty=1,693,000,$ $\mathrm{cond}(\boldsymbol{A})_\infty=2,661,396$

(3) 方程组的条件数过大，很大范围的解都能近似满足方程组，此时残向量的大小无法说明解的精确程度.

15. (1) $\boldsymbol{M}_J=\frac{1}{4}\begin{bmatrix}0&1&1&0\\1&0&1&1\\1&1&0&1\\0&1&1&0\end{bmatrix}, \boldsymbol{M}_G=\frac{1}{256}\begin{bmatrix}0&64&64&0\\0&16&80&64\\0&20&36&80\\0&9&29&36\end{bmatrix}, \rho(\boldsymbol{M}_J)=\frac{1+\sqrt{17}}{8}\approx$ $0.640388<1, \rho(\boldsymbol{M}_G)\approx 0.418433<1$，则雅可比迭代与高斯-塞德尔迭代均收敛.

雅可比迭代格式为 $\begin{cases}x_1^{(k+1)}=\frac{1}{4}(x_2^{(k)}+x_3^{(k)}+1.8)\\x_2^{(k+1)}=\frac{1}{4}(x_1^{(k)}+x_3^{(k)}+x_4^{(k)}+2)\\x_3^{(k+1)}=\frac{1}{4}(x_1^{(k)}+x_2^{(k)}+x_4^{(k)}+3)\\x_4^{(k+1)}=\frac{1}{4}(x_2^{(k)}+x_3^{(k)}+0.2)\end{cases}$，

高斯-塞德尔迭代格式为 $\begin{cases}x_1^{(k+1)}=\frac{1}{4}(x_2^{(k)}+x_3^{(k)})+0.45\\x_2^{(k+1)}=\frac{1}{16}(x_2^{(k)}+5x_3^{(k)}+4x_4^{(k)})+0.6125\\x_3^{(k+1)}=\frac{1}{64}(5x_2^{(k)}+9x_3^{(k)}+20x_4^{(k)})+1.01563\\x_4^{(k+1)}=\frac{1}{256}(9x_2^{(k)}+29x_3^{(k)}+36x_3^{(k)})+0.457031\end{cases}$.

(3) $\boldsymbol{M}_J=\begin{bmatrix}0&-2&2\\-1&0&-1\\-2&-2&0\end{bmatrix}, \boldsymbol{M}_G=\begin{bmatrix}0&-2&2\\0&2&-3\\0&0&2\end{bmatrix}, \rho(\boldsymbol{M}_J)=0<1, \rho(\boldsymbol{M}_G)=$ $2>1$，则雅可比迭代收敛, 高斯-塞德尔迭代不收敛.

雅可比迭代格式为 $\begin{cases}x_1^{(k+1)}=-2x_2^{(k)}+2x_3^{(k)}+1\\x_2^{(k+1)}=-x_1^{(k)}-x_3^{(k)}+1\\x_3^{(k+1)}=-2x_1^{(k)}-2x_2^{(k)}+1\end{cases}$.

18. (1) $\boldsymbol{M}_J=\begin{bmatrix}0&-\frac{1}{a}&-\frac{1}{a}\\-\frac{1}{a^2}&0&0\\-\frac{1}{a^2}&0&0\end{bmatrix}, \rho(\boldsymbol{M}_J)=\frac{\sqrt{2}}{|a|^{\frac{3}{2}}},$

当 $\frac{\sqrt{2}}{|a|^{\frac{3}{2}}}<1$ 即 $|a|>\sqrt[3]{2}$ 时，雅可比迭代收敛.

(2) $\boldsymbol{M}_J=\begin{bmatrix}0&-\frac{1}{a}&-\frac{3}{a}\\-\frac{1}{a}&0&-\frac{2}{a}\\\frac{3}{a}&-\frac{2}{a}&0\end{bmatrix}, \rho(\boldsymbol{M}_J)=\frac{2}{|a|},$

当 $\frac{2}{|a|}<1$ 即 $|a|>2$ 时，雅可比迭代收敛.