Control Systems

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Contents

***Abstract*—This manual is an introduction to control systems based on GATE problems.Links to sample Python codes are available in the text.**

Download python codes using

```
svn co https://github.com/gadepall/school/trunk/
    control/codes
```

1 Stability

1.1 Second order System

1.1. Consider the following second order system with the transfer function

$$G(s) = \frac{1}{1 + 2s + s^2} \tag{1.1.1}$$

Is the system stable?
Solution: The poles of

$$G(s) = \frac{1}{1 + 2s + s^2} \tag{1.1.2}$$

are at

$$s = -1 \tag{1.1.3}$$

i.e., the left half of s-plane. Hence the system is stable.

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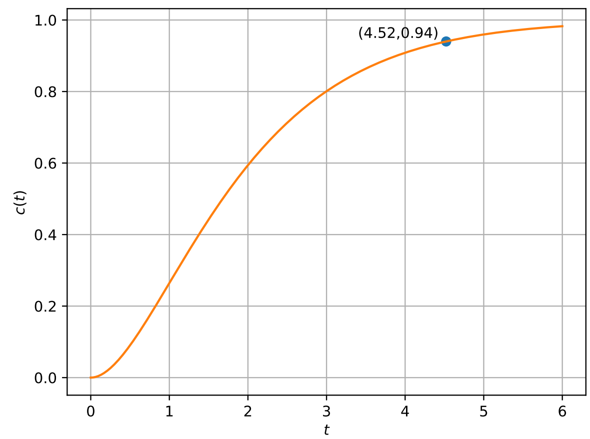

Fig. 1.2

1.2. Find and sketch the step response $c(t)$ of the system.
Solution: For step-response, we take input as unit-step function u(t)

$$C(s) = U(s).G(s) = \left[\frac{1}{s}\right]\left[\frac{1}{1 + 2s + s^2}\right] \tag{1.2.1}$$

$$= \frac{1}{s(1+s)^2} \tag{1.2.2}$$

$$= \frac{1}{s} - \frac{1}{(1+s)} - \frac{1}{(1+s)^2} \tag{1.2.3}$$

Taking the inverse Laplace transform,

$$c(t) = L^{-1}\left[\frac{1}{s}\right] - L^{-1}\left[\frac{1}{1+s}\right] - L^{-1}\left[\frac{1}{(1+s)^2}\right] \tag{1.2.4}$$

$$= \left(1 - e^{-t} - te^{-t}\right) u(t) \tag{1.2.5}$$

The following code plots $c(t)$ in Fig. 1.2

```
codes/ee18btech11002/plot.py
```

1.3. Find the steady state response of the system using the final value theorem. Verify using 1.2.5
Solution: To know the steady response value

of c(t), using final value theorem,

$$\lim_{t\to\infty} c(t) = \lim_{s\to 0} sC(s) \tag{1.3.1}$$

We get

$$\lim_{s\to 0} s\left(\frac{1}{s}\right)\left(\frac{1}{1+s+s^2}\right) = \frac{1}{1+0+0} = 1 \tag{1.3.2}$$

Using 1.2.5,

$$\lim_{t\to\infty} c(t) = \lim_{t\to\infty} \left(1 - e^{-t} - te^{-t}\right) u(t) \tag{1.3.3}$$
$$= (1 - 0 - 0) = 1 \tag{1.3.4}$$

1.4. Find the time taken for the system output c(t) to reach 94% of its steady state value.
Solution: Now, 94% of 1 is 0.94, so we should now solve for a positive t such that

$$1 - e^{-t} - te^{-t} = 0.94 \tag{1.4.1}$$

The following code

```
codes/ee18btech11002/solution.py
```

provides the necessary solution as

$$t = 4.5228 \tag{1.4.2}$$

2 Routh Hurwitz Criterion

2.1 Marginal Stability

2.1.1. Consider a unity feedback system as shown in Fig. 2.1.1, with an integral compensator $\frac{k}{s}$ and open-loop transfer function

$$G(s) = \frac{1}{s^2 + 3s + 2} \tag{2.1.1.1}$$

where k greater than 0. Find its closed loop transfer function.

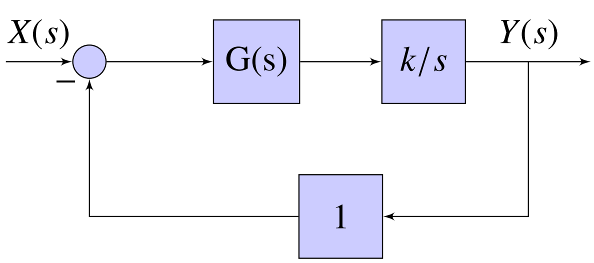

Fig. 2.1.1

Solution: $\because H(s) = 1$ in Fig. 2.1.1, due to unity feedback, the transfer function is given by

$$\frac{Y(s)}{X(s)} = \frac{G(s)}{1 + G(s)H(s)} \tag{2.1.1.2}$$
$$\implies T(s) = \frac{k}{s^3 + 3s^2 + 2s} \tag{2.1.1.3}$$

2.1.2. Find the *characteristic* equation for $G(s)$.
Solution: The characteristic equation is

$$1 + G(s)H(s) = 0 \tag{2.1.2.1}$$
$$\implies 1 + \left[\frac{k}{s^3 + 3s^2 + 2s}\right] = 0 \tag{2.1.2.2}$$
$$\text{or, } s^3 + 3s^2 + 2s + k = 0 \tag{2.1.2.3}$$

2.1.3. Using the tabular method for the Routh hurwitz criterion, find $k > 0$ for which there are two poles of unity feedback system on jω axis.
Solution: This criterion is based on arranging the coefficients of characteristic equation into an array called Routh array. For any characteristic equation

$$q(s) = a_0 s^n + a_1 s^{n-1} + \ldots.. + a_{n-1}s + a_n = 0 \tag{2.1.3.1}$$

the Routh array can be constructed as

$$\left|\begin{matrix} s^n \\ s^{n-1} \\ s^{n-2} \\ \vdots \end{matrix}\right\| \begin{matrix} a_0 & a_2 & a_4 & \cdots & \\ a_1 & a_3 & a_5 & \cdots & \\ b_1 & b_2 & b_3 & \cdots & \\ \vdots & \vdots & \vdots & \ddots & \vdots\cdots \end{matrix}\right| \tag{2.1.3.2}$$

where

$$b_1 = \frac{a_1 a_2 - a_0 a_3}{a_1} \tag{2.1.3.3}$$
$$b_2 = \frac{a_1 a_4 - a_0 a_5}{a_1} \tag{2.1.3.4}$$
$$c_1 = \frac{b_1 a_3 - a_1 b_2}{b_1} \tag{2.1.3.5}$$
$$c_2 = \frac{b_1 a_5 - a_1 b_3}{b_1} \tag{2.1.3.6}$$

For poles to lie on imaginary axis any one entire row of hurwitz matrix should be zero. Constructing the routh array for the characteristic equation obtained in 2.1.2.1,

$$s^3 + 3s^2 + 2s + k = 0 \tag{2.1.3.7}$$

$$\begin{vmatrix} s^3 \\ s^2 \\ s^1 \\ s^0 \end{vmatrix} \begin{vmatrix} 1 & 2 \\ 3 & k \\ \frac{6-k}{3} & 0 \\ k & 0 \end{vmatrix} \tag{2.1.3.8}$$

For poles on $\jmath\omega$ axis any one of the row should be zero.

$$\therefore \frac{6-k}{3} = 0 \text{ or } k = 0 \tag{2.1.3.9}$$

$$\implies k = 6 \quad \because k > 0 \tag{2.1.3.10}$$

2.1.4. Repeat the above using the determinant method.
Solution: The *Routh matrix* can be expressed as

$$\mathbf{R} = \begin{pmatrix} a_0 & a_2 & a_4 & \cdots & \\ a_1 & a_3 & a_5 & \cdots & \\ 0 & a_0 & a_2 \cdots & & \\ 0 & a_1 & a_3 \cdots & & \\ \vdots & \vdots & \vdots & \ddots & \vdots \cdots \end{pmatrix} \tag{2.1.4.1}$$

and the corresponding Routh determinants are

$$D_1 = |a_0| \tag{2.1.4.2}$$

$$D_2 = \begin{vmatrix} a_0 & a_2 \\ a_1 & a_3 \end{vmatrix} \tag{2.1.4.3}$$

$$D_3 = \begin{vmatrix} a_0 & a_2 & a_4 \\ a_1 & a_3 & a_5 \\ 0 & a_0 & a_2 \end{vmatrix} \tag{2.1.4.4}$$

$$\cdots \tag{2.1.4.5}$$

If at least any one of the Determinents are zero then the poles lie on imaginary axes. From (2.1.2.1),

$$D_1 = 1 \neq 0 \tag{2.1.4.6}$$

$$D2 = \begin{vmatrix} 1 & 2 \\ 3 & k \end{vmatrix} = k - 6 = 0 \implies k = 6 \tag{2.1.4.7}$$

2.1.5. Verify your answer using a python code for both the determinant method as well as the tabular method.
Solution: The following code

```
codes/ee18btech11005/ee18btech11005.py
```

provides the necessary soution.

- For the system to be stable all coefficients should lie on left half of s-plane. Because if any pole is in right half of s-plane then there will be a component in output that increases without bound,causing system to be unstable. All the coefficients in the characteristic equation should be positive.This is necessary condition but not sufficient.Because it may have poles on right half of s plane. Poles are the roots of the characteristic equation.
- A system is stable if all of its characteristic modes go to finite value as t goes to infinity.It is possible only if all the poles are on the left half of s plane. The characteristic equation should have negative roots only. So the first column should always be greater than zero.That means no sign changes.
- A system is unstable if its characteristic modes are not bounded. Then the characteristic equation will also have roots in the right side of s-plane.That means it has sign changes.

2.2 *Stability*

2.2.1. The characteristic equation of linear time invariant system is given by

$$\nabla(s) = s^4 + 3s^3 + 3s^2 + s + k = 0 \tag{2.2.1.1}$$

Find the condition for the system to be BIBO stable using the Routh Array.

2.2.2. **solution**

$$\nabla(s) = s^4 + 3s^3 + 3s^2 + s + k = 0 \tag{2.2.2.1}$$

2.2.3. Modify the Python code in Problem 2.1.5 to verify your solution by choosing two different values of k.
For a system to be stable all coefficients of characteristic equation should lie on left half of s-plane because if any of the pole is in right half of the s-plane then there will be a component which is exponentially increasing in output,causing system to be unstable.This can be verified by Routh Array Criterion.
The Routh hurwitz criterion:-
This criterion is based on arranging the coefficients of characteristic equation into an array called Routh array.If all the coefficients in the first row of routh array are of same algebric sign then the system is stable.
For any characteristic equation q(s),

$$q(s) = a_0 s^n + a_1 s^{n-1} + \ldots.. + a_{n-1} s + a_n = 0 \tag{2.2.3.1}$$

Routh array can be constructed as follows..,

$$\begin{pmatrix} s^n \\ s^{n-1} \\ s^{n-2} \\ \vdots \end{pmatrix} \begin{pmatrix} a_0 & a_2 & a_4 & \cdots & \\ a_1 & a_3 & a_5 & \cdots & \\ b_1 & b_2 & b_3 & \cdots & \\ \vdots & \vdots & \vdots & \ddots & \vdots \cdots \end{pmatrix}$$

where

$$b_1 = \frac{a_1 a_2 - a_0 a_3}{a_1} \tag{2.2.3.2}$$

$$b_2 = \frac{a_1 a_4 - a_0 a_5}{a_1} \tag{2.2.3.3}$$

$$c_1 = \frac{b_1 a_3 - a_1 b_2}{b_1} \tag{2.2.3.4}$$

$$c_2 = \frac{b_1 a_5 - a_1 b_3}{b_1} \tag{2.2.3.5}$$

$$\begin{pmatrix} s^4 \\ s^3 \\ s^2 \\ s^1 \\ s^0 \end{pmatrix} \begin{pmatrix} 1 & 3 & k \\ 3 & 1 & 0 \\ \frac{8}{3} & k & 0 \\ \frac{\frac{8}{3}-3k}{\frac{8}{3}} & 0 & 0 \\ k & 0 & 0 \end{pmatrix} \tag{2.2.3.6}$$

Given system is stable if

$$\frac{\frac{8}{3} - 3k}{\frac{8}{3}} > 0, k > 0 \tag{2.2.3.7}$$

$$\frac{8}{3} - 3k > 0 \tag{2.2.3.8}$$

$$3k < \frac{8}{3} \tag{2.2.3.9}$$

$$(0 < k < \frac{8}{9}) \tag{2.2.3.10}$$

for example the zeros of polynomial $s^4 + 3s^3 + 3s^2 + s + 0.5 = 0$ are

$$s1 = -0.08373 + 0.45773i \tag{2.2.3.11}$$

$$s2 = -0.08373 - 0.45773i \tag{2.2.3.12}$$

$$s3 = -1.41627 + 0.55075i \tag{2.2.3.13}$$

$$s4 = -1.41627 - 0.55075i \tag{2.2.3.14}$$

3 State-Space Model

3.1. State the general model of a state space system specifying the dimensions of the matrices and vectors.

Solution: The model is given by

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \tag{3.1.1}$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \tag{3.1.2}$$

with parameters listed in Table 3.1.

Variable	**Size**	**Description**
u	$p \times 1$	input(control) vector
y	$q \times 1$	output vector
x	$n \times 1$	state vector
A	$n \times n$	state or system matrix
B	$n \times p$	input matrix
C	$q \times n$	output matrix
D	$q \times p$	feedthrough matrix

TABLE 3.1

3.2. Find the transfer function $\mathbf{H}(s)$ for the general system.

Solution: Taking Laplace transform on both sides we have the following equations

$$s\mathbf{I}X(s) - x(0) = \mathbf{A}X(s) + \mathbf{B}U(s) \tag{3.2.1}$$

$$(s\mathbf{I} - \mathbf{A})X(s) = \mathbf{B}U(s) + x(0) \tag{3.2.2}$$

$$X(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}U(s) + (s\mathbf{I} - \mathbf{A})^{-1}x(0) \tag{3.2.3}$$

and

$$Y(s) = \mathbf{C}X(s) + D\mathbf{I}U(s) \tag{3.2.4}$$

Substituting from (3.2.3) in the above,

$$Y(s) = (\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + D\mathbf{I})U(s) + \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}x(0) \tag{3.2.5}$$

3.3. Find $H(s)$ for a SISO (single input single output) system.

Solution:

$$H(s) = \frac{Y(s)}{U(s)} = C(sI - A)^{-1}B + DI \tag{3.3.1}$$

3.4. Given

$$H(s) = \frac{1}{s^3 + 3s^2 + 2s + 1} \tag{3.4.1}$$
$$D = 0 \tag{3.4.2}$$
$$\mathbf{B} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{3.4.3}$$

find **A** and **C** such that the state-space realization is in *controllable canonical form.*
Solution:

$$\because \frac{Y(s)}{U(s)} = \frac{Y(s)}{V(s)} \times \frac{V(s)}{U(s)}, \tag{3.4.4}$$

letting

$$\frac{Y(s)}{V(s)} = 1, \tag{3.4.5}$$

results in

$$\frac{U(s)}{V(s)} = s^3 + 3s^2 + 2s + 1 \tag{3.4.6}$$

giving

$$U(s) = s^3V(s) + 3s^2V(s) + 2sV(s) + V(s) \tag{3.4.7}$$

so equation 0.1.13 can be written as

$$\begin{pmatrix} sV(s) \\ s^2V(s) \\ s^3V(s) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{pmatrix} \begin{pmatrix} V(s) \\ s(s) \\ s^2V(s) \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} U \tag{3.4.8}$$

So

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{pmatrix} \tag{3.4.9}$$

$$Y = X_1(s) = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} V(s) \\ sV(s) \\ s^2V(s) \end{pmatrix} \tag{3.4.10}$$

$$\mathbf{C} = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \tag{3.4.11}$$

3.5. Obtain **A** and **C** so that the state-space realization in in *observable canonical form.*
Solution: Given that

$$H(s) = \frac{1}{s^3 + 3s^2 + 2s + 1} \tag{3.5.1}$$

$$\frac{Y(s)}{U(s)} = \frac{1}{s^3 + 3s^2 + 2s + 1} \tag{3.5.2}$$
$$Y(s) \times (s^3 + 3s^2 + 2s + 1) = U(s) \tag{3.5.3}$$

$$s^3Y(s) + 3s^2Y(s) + 2sY(s) + Y(s) = U(s) \tag{3.5.4}$$
$$s^3Y(s) = U(s) - 3s^2Y(s) - 2sY(s) - Y(s) \tag{3.5.5}$$
$$Y(s) = -3s^{-1}Y(s) - 2s^{-2}Y(s) + s^{-3}(U(s) - Y(s)) \tag{3.5.6}$$

let $Y = aU + X_1$
by comparing with equation 1.5.6 we get a=0 and

$$Y = X_1 \tag{3.5.7}$$

inverse laplace transform of above equation is

$$y = x_1 \tag{3.5.8}$$

so from above equation 1.5.6 and 1.5.7

$$X_1 = -3s^{-1}Y(s) - 2s^{-2}Y(s) + s^{-3}(U(s) - Y(s)) \tag{3.5.9}$$
$$sX_1 = -3Y(s) - 2s^{-1}Y(s) + s^{-2}(U(s) - Y(s)) \tag{3.5.10}$$

inverse laplace transform of above equation

$$\dot{x_1} = -3y + x_2 \tag{3.5.11}$$

where

$$X_2 = -2s^{-1}Y(s) + s^{-2}(U(s) - Y(s)) \tag{3.5.12}$$
$$sX_2 = -2Y(s) + s^{-1}(U(s) - Y(s)) \tag{3.5.13}$$

inverse laplace transform of above equation

$$\dot{x_2} = -2y + x_3 \tag{3.5.14}$$

where

$$X_3 = s^{-1}(U(s) - Y(s)) \tag{3.5.15}$$
$$sX_3 = U(s) - Y(s) \tag{3.5.16}$$

inverse laplace transform of above equation

$$\dot{x_3} = u - y \tag{3.5.17}$$

so we get four equations which are

$$y = x_1 \tag{3.5.18}$$
$$\dot{x_1} = -3y + x_2 \tag{3.5.19}$$
$$\dot{x_2} = -2y + x_3 \tag{3.5.20}$$
$$\dot{x_3} = u - y \tag{3.5.21}$$

sub $y = x_1$ in 1.5.19,1.5.20,1.5.21 we get

$$y = x_1 \tag{3.5.22}$$
$$\dot{x_1} = -3x_1 + x_2 \tag{3.5.23}$$
$$\dot{x_2} = -2x_1 + x_3 \tag{3.5.24}$$
$$\dot{x_3} = u - x_1 \tag{3.5.25}$$

so above equations can be written as

$$\begin{pmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \end{pmatrix} = \begin{pmatrix} -3 & 1 & 0 \\ -2 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} U \tag{3.5.26}$$

So

$$\mathbf{A} = \begin{pmatrix} -3 & 1 & 0 \\ -2 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix} \tag{3.5.27}$$

$$y = x_1 = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \tag{3.5.28}$$

$$\mathbf{C} = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \tag{3.5.29}$$

3.6. Find the eigenvaues of $\mathbf{A}$ and the poles of $H(s)$ using a python code.
Solution: The following code

```
codes/ee18btech11004.py
```

gives the necessary values. The roots are the same as the eigenvalues.

3.7. Theoretically, show that eigenvaues of $\mathbf{A}$ are the poles of $H(s)$. **Solution:**
as we know tthat the characteristic equation is det(sI-A)

$$\mathbf{sI} - \mathbf{A} = \begin{pmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{pmatrix} - \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{pmatrix} = \begin{pmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 1 & 2 & s+3 \end{pmatrix} \tag{3.7.1}$$

therfore

$$det(sI - A) = s(s^2 + 3s + 2) + 1(1) = s^3 + 3s^2 + 2s + 1 \tag{3.7.2}$$

so from equation 1.6.2 we can see that charcteristic equation is equal to the denominator of the transefer function

4 Nyquist Plot

4.1. The open loop transfer function of a unity feedback system is given by

$$G(s) = \frac{\pi e^{-0.25s}}{s} \tag{4.1.1}$$

4.2. Find $\text{Re}\{G(\jmath\omega)\}$ and $\text{Im}\{G(\jmath\omega)\}$.
Solution: From (4.1.1),

$$G(j\omega) = \frac{\pi}{\omega}(-\sin 0.25\omega - j\cos 0.25\omega) \tag{4.2.1}$$

$$\implies \text{Re}\{G(\jmath\omega)\} = \frac{\pi}{\omega}(-\sin 0.25\omega) \tag{4.2.2}$$
$$\text{Im}\{G(\jmath\omega)\} = \frac{\pi}{\omega}(-j\cos 0.25\omega) \tag{4.2.3}$$

4.3. Sketch the Nyquist plot.
Solution: The Nyquist plot is a graph of $\text{Re}\{G(\jmath\omega)\}$ vs $\text{Im}\{G(\jmath\omega)\}$. The following python code generates the Nyquist plot in Fig. 4.3

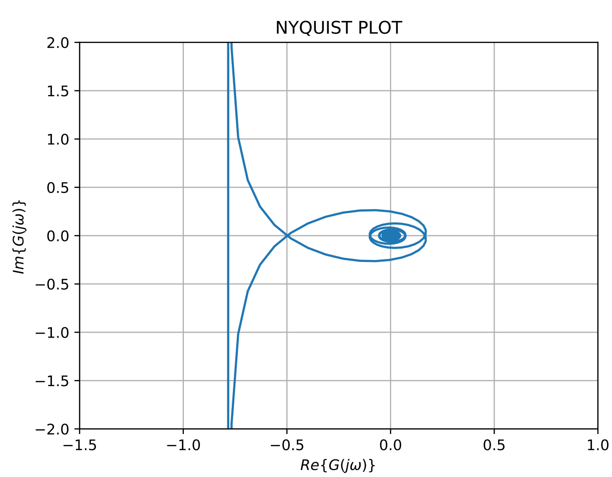

Fig. 4.3

4.4. Find the point at which the Nyquist plot of G(s) passes through the negative real axis

Solution: Nyquist plot cuts the negative real axis at ω for which

$$\angle G(\jmath\omega) = -\pi \tag{4.4.1}$$

From (4.1.1),

$$G(\jmath\omega) = \frac{\pi e^{-\frac{\jmath\omega}{4}}}{\jmath\omega} = \frac{\pi e^{-\jmath\left(\frac{\omega}{4}+\frac{\pi}{2}\right)}}{\omega} \tag{4.4.2}$$

$$\implies \angle G(\jmath\omega) = -\left(\frac{\omega}{4} + \frac{\pi}{2}\right) \tag{4.4.3}$$

From (4.4.3) and (4.4.1),

$$\frac{\omega}{4} + \frac{\pi}{2} = \pi \tag{4.4.4}$$

$$\implies \omega = 2\pi \tag{4.4.5}$$

Also, from (4.1.1),

$$\left|G(\jmath\omega)\right| = \frac{\pi}{|\omega|} \tag{4.4.6}$$

$$\implies \left|G(\jmath 2\pi)\right| = \frac{1}{2} \tag{4.4.7}$$

4.5. Find the value of P defined in Table 4.5 from Fig. 4.3.

Variable	Value	Description
Z	0	Poles of $\frac{G(s)}{1+G(s)H(s)}$ in right half of s plane
N	0	Poles of $G(s)H(s)$ in right half of s plane
P	0	No of clockwise encirclements of $G(s)H(s)$ about -1+j0 in the Nyquist plot

TABLE 4.5

Solution: $P = 0$.

4.6. Find the value of N defined in Table 4.5 from (4.1.1)
Solution: $\because H(s) = 1$, $G(s)H(s) = G(s)$. Also, $G(s)$ has a pole at $s = 0$, hence $N = 0$.

4.7. Use the Nyquist Stability criterion to determine if the system in (4.4.3) is stable.
Solution: According to the Nyquist criterion, the system is stable if

$$Z = P - N = 0, \tag{4.7.1}$$

where Z is defined in Table 4.5. $\because Z = 0$ from (4.7.1) , the system is stable.

5 Compensators