Control Systems

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CONTENTS

***Abstract*—This manual is an introduction to control systems based on GATE problems.Links to sample Python codes are available in the text.**

Download python codes using

```
svn co https://github.com/gadepall/school/trunk/
    control/codes
```

1 Stability

1.1 Second order System

1.1. Consider the following second order system with the transfer function

$$G(s) = \frac{1}{1+2s+s^2} \tag{1.1.1}$$

Is the system stable?
Solution: The poles of

$$G(s) = \frac{1}{1+2s+s^2} \tag{1.1.2}$$

are at

$$s = -1 \tag{1.1.3}$$

i.e., the left half of s-plane. Hence the system is stable.

1.2. Find and sketch the step response $c(t)$ of the system.

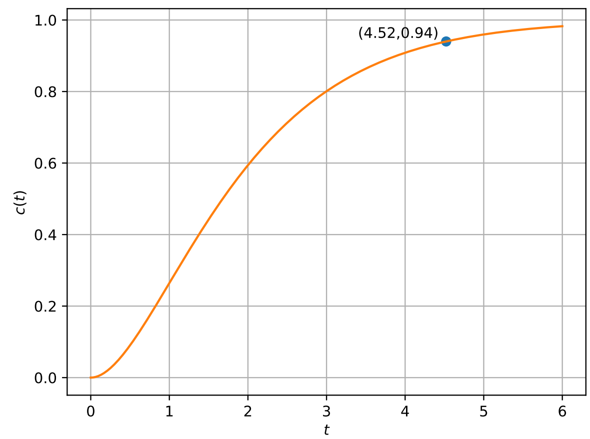

Fig. 1.2

Solution: For step-response, we take input as unit-step function u(t)

$$C(s) = U(s).G(s) = \left[\frac{1}{s}\right]\left[\frac{1}{1+2s+s^2}\right] \tag{1.2.1}$$

$$= \frac{1}{s(1+s)^2} \tag{1.2.2}$$

$$= \frac{1}{s} - \frac{1}{(1+s)} - \frac{1}{(1+s)^2} \tag{1.2.3}$$

Taking the inverse Laplace transform,

$$c(t) = L^{-1}\left[\frac{1}{s}\right] - L^{-1}\left[\frac{1}{1+s}\right] - L^{-1}\left[\frac{1}{(1+s)^2}\right] \tag{1.2.4}$$

$$= \left(1 - e^{-t} - te^{-t}\right)u(t) \tag{1.2.5}$$

The following code plots $c(t)$ in Fig. 1.2

```
codes/ee18btech11002/plot.py
```

1.3. Find the steady state response of the system using the final value theorem. Verify using 1.2.5
Solution: To know the steady response value

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of c(t), using final value theorem,

$$\lim_{t\to\infty} c(t) = \lim_{s\to 0} sC(s) \tag{1.3.1}$$

We get

$$\lim_{s\to 0} s\left(\frac{1}{s}\right)\left(\frac{1}{1+s+s^2}\right) = \frac{1}{1+0+0} = 1 \tag{1.3.2}$$

Using 1.2.5,

$$\lim_{t\to\infty} c(t) = \lim_{t\to\infty}\left(1 - e^{-t} - te^{-t}\right)u(t) \tag{1.3.3}$$
$$= (1-0-0) = 1 \tag{1.3.4}$$

1.4. Find the time taken for the system output c(t) to reach 94% of its steady state value.
Solution: Now, 94% of 1 is 0.94, so we should now solve for a positive t such that

$$1 - e^{-t} - te^{-t} = 0.94 \tag{1.4.1}$$

The following code

```
codes/ee18btech11002/solution.py
```

provides the necessary solution as

$$t = 4.5228 \tag{1.4.2}$$

2 Routh Hurwitz Criterion

2.1. Consider a unity feedback system as shown in Fig. 2.1, with an integral compensator $\frac{k}{s}$ and open-loop transfer function

$$G(s) = \frac{1}{s^2+3s+2} \tag{2.1.1}$$

where k greater than 0. Find its closed loop transfer function.

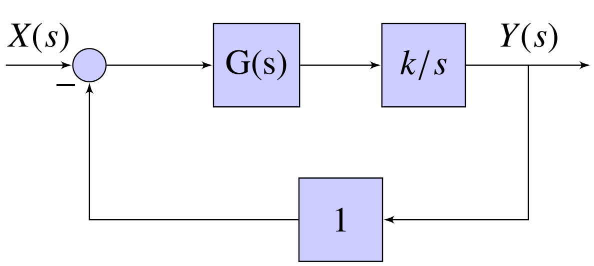

Fig. 2.1

Solution: $\because H(s) = 1$ in Fig. 2.1, due to unity feedback, the transfer function is given by

$$\frac{Y(s)}{X(s)} = \frac{G(s)}{1+G(s)H(s)} \tag{2.1.2}$$
$$\implies T(s) = \frac{k}{s^3+3s^2+2s} \tag{2.1.3}$$

2.2. Find the *characteristic* equation for $G(s)$.
Solution: The characteristic equation is

$$1 + G(s)H(s) = 0 \tag{2.2.1}$$
$$\implies 1 + \left[\frac{k}{s^3+3s^2+2s}\right] = 0 \tag{2.2.2}$$
$$\text{or, } s^3 + 3s^2 + 2s + k = 0 \tag{2.2.3}$$

2.3. Using the tabular method for the Routh hurwitz criterion, find $k > 0$ for which there are two poles of unity feedback system on jω axis.
Solution: This criterion is based on arranging the coefficients of characteristic equation into an array called Routh array. For any characteristic equation

$$q(s) = a_0 s^n + a_1 s^{n-1} + \ldots.. + a_{n-1}s + a_n = 0 \tag{2.3.1}$$

the Routh array can be constructed as

$$\begin{vmatrix} s^n \\ s^{n-1} \\ s^{n-2} \\ \vdots \end{vmatrix}\begin{vmatrix} a_0 & a_2 & a_4 & \cdots \\ a_1 & a_3 & a_5 & \cdots \\ b_1 & b_2 & b_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots\ldots \end{vmatrix} \tag{2.3.2}$$

where

$$b_1 = \frac{a_1a_2 - a_0a_3}{a_1} \tag{2.3.3}$$
$$b_2 = \frac{a_1a_4 - a_0a_5}{a_1} \tag{2.3.4}$$
$$c_1 = \frac{b_1a_3 - a_1b_2}{b_1} \tag{2.3.5}$$
$$c_2 = \frac{b_1a_5 - a_1b_3}{b_1} \tag{2.3.6}$$

For poles to lie on imaginary axis any one entire row of hurwitz matrix should be zero. Constructing the routh array for the characteristic equation obtained in 2.2.1,

$$s^3 + 3s^2 + 2s + k = 0 \tag{2.3.7}$$

$$\begin{vmatrix} s^3 \\ s^2 \\ s^1 \\ s^0 \end{vmatrix} \begin{vmatrix} 1 & 2 \\ 3 & k \\ \frac{6-k}{3} & 0 \\ k & 0 \end{vmatrix} \tag{2.3.8}$$

For poles on $\jmath\omega$ axis any one of the row should be zero.

$$\therefore \frac{6-k}{3} = 0 \text{ or } k = 0 \tag{2.3.9}$$

$$\implies k = 6 \quad \because k > 0 \tag{2.3.10}$$

2.4. Repeat the above using the determinant method.
Solution: The *Routh matrix* can be expressed as

$$\mathbf{R} = \begin{pmatrix} a_0 & a_2 & a_4 & \cdots & \\ a_1 & a_3 & a_5 & \cdots & \\ 0 & a_0 & a_2 \cdots & & \\ 0 & a_1 & a_3 \cdots & & \\ \vdots & \vdots & \vdots & \ddots & \vdots \cdots \end{pmatrix} \tag{2.4.1}$$

and the corresponding Routh determinants are

$$D_1 = |a_0| \tag{2.4.2}$$

$$D_2 = \begin{vmatrix} a_0 & a_2 \\ a_1 & a_3 \end{vmatrix} \tag{2.4.3}$$

$$D_3 = \begin{vmatrix} a_0 & a_2 & a_4 \\ a_1 & a_3 & a_5 \\ 0 & a_0 & a_2 \end{vmatrix} \tag{2.4.4}$$

$$\cdots \tag{2.4.5}$$

If at least any one of the Determinents are zero then the poles lie on imaginary axes. From (2.2.1),

$$D_1 = 1 \neq 0 \tag{2.4.6}$$

$$D2 = \begin{vmatrix} 1 & 2 \\ 3 & k \end{vmatrix} = k - 6 = 0 \implies k = 6 \tag{2.4.7}$$

2.5. Verify your answer using a python code for both the determinant method as well as the tabular method.
Solution: The following code

```
codes/ee18btech11005/ee18btech11005.py
```

provides the necessary soution.

- For the system to be stable all coefficients should lie on left half of s-plane. Because if any pole is in right half of s-plane then there will be a component in output that increases without bound,causing system to be unstable. All the coefficients in the characteristic equation should be positive.This is necessary condition but not sufficient.Because it may have poles on right half of s plane. Poles are the roots of the characteristic equation.
- A system is stable if all of its characteristic modes go to finite value as t goes to infinity.It is possible only if all the poles are on the left half of s plane. The characteristic equation should have negative roots only. So the first column should always be greater than zero.That means no sign changes.
- A system is unstable if its characteristic modes are not bounded. Then the characteristic equation will also have roots in the right side of s-plane.That means it has sign changes.

3 State-Space Model

3.1. State the general model of a state space system specifying the dimensions of the matrices and vectors.
Solution: The model is given by

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \tag{3.1.1}$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \tag{3.1.2}$$

with parameters listed in Table 3.1.

Variable	Size	Description
$\mathbf{u}$	$p \times 1$	input(control) vector
$\mathbf{y}$	$q \times 1$	output vector
$\mathbf{x}$	$n \times 1$	state vector
$\mathbf{A}$	$n \times n$	state or system matrix
$\mathbf{B}$	$n \times p$	input matrix
$\mathbf{C}$	$q \times n$	output matrix
$\mathbf{D}$	$q \times p$	feedthrough matrix

TABLE 3.1

3.2. Find the transfer function $\mathbf{H}(s)$ for the general system.
Solution: Taking Laplace transform on both

sides we have the following equations

$$s\mathbf{I}X(s) - x(0) = \mathbf{A}X(s) + \mathbf{B}U(s) \tag{3.2.1}$$

$$(s\mathbf{I} - \mathbf{A})X(s) = \mathbf{B}U(s) + x(0) \tag{3.2.2}$$

$$X(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}U(s) + (s\mathbf{I} - \mathbf{A})^{-1}x(0) \tag{3.2.3}$$

and

$$Y(s) = \mathbf{C}X(s) + D\mathbf{I}U(s) \tag{3.2.4}$$

Substituting from (3.2.3) in the above,

$$Y(s) = (\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + D\mathbf{I})U(s) + \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}x(0) \tag{3.2.5}$$

3.3. Find $H(s)$ for a SISO (single input single output) system.
Solution:

$$H(s) = \frac{Y(s)}{U(s)} = C(sI - A)^{-1}B + DI \tag{3.3.1}$$

3.4. Given

$$H(s) = \frac{1}{s^3 + 3s^2 + 2s + 1} \tag{3.4.1}$$

$$D = 0 \tag{3.4.2}$$

$$\mathbf{B} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{3.4.3}$$

find **A** and **C**.
Solution:

$$\because \frac{Y(s)}{U(s)} = \frac{Y(s)}{V(s)} \times \frac{V(s)}{U(s)}, \tag{3.4.4}$$

letting

$$\frac{Y(s)}{V(s)} = 1, \tag{3.4.5}$$

results in

$$\frac{U(s)}{V(s)} = s^3 + 3s^2 + 2s + 1 \tag{3.4.6}$$

giving

$$U(s) = s^3V(s) + 3s^2V(s) + 2sV(s) + V(s) \tag{3.4.7}$$

so equation 0.1.13 can be written as

$$\begin{pmatrix} sV(s) \\ s^2V(s) \\ s^3V(s) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{pmatrix} \begin{pmatrix} V(s) \\ s(s) \\ s^2V(s) \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} U \tag{3.4.8}$$

So

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{pmatrix} \tag{3.4.9}$$

$$Y = X_1(s) = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} V(s) \\ sV(s) \\ s^2V(s) \end{pmatrix} \tag{3.4.10}$$

$$\mathbf{C} = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \tag{3.4.11}$$

4 Compensators

5 Nyquist Plot