Balancing a Chemical Equation

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Contents

***Abstract*—This manual shows how to balance chemical equations using matrices.**

Download python codes using

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svn co https://github.com/gadepall/school/trunk/
    training
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1 GATE Problems

1. Consider a unity feedback system as shown in the figure,shown with an integral compensator $\frac{k}{s}$ and open-loop transfer function

$$G(s) = \frac{1}{s^2 + 3s + 2} \tag{1.1.1}$$

where $k > 0$. Find the positive value of k for which there are two poles of unity feedback system on the $\jmath\omega$ axis.

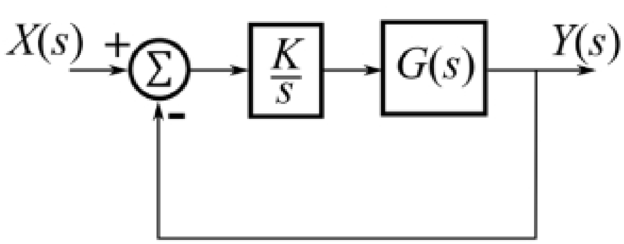

Fig. 1.1

Solution: The open loop transfer function

$$G(s) = \frac{1}{s^2 + 3s + 2} \tag{1.1.2}$$

Hence,

$$H(s) = \frac{Y(s)}{X(s)} = \frac{G(s)k/s}{1 + G(s)k/s} \tag{1.1.3}$$

$$= \frac{k}{1 + ks(s^2 + 3s + 2)} \tag{1.1.4}$$

The poles of $H(s)$ are obtained from

$$s^3 + 3s^2 + 2s + k = 0 \tag{1.1.5}$$

The corresponding Routh array is given by

$$\begin{vmatrix} s^3 \\ s^2 \\ s^1 \\ s^0 \end{vmatrix}\begin{vmatrix} 1 & 2 \\ 3 & k \\ \frac{6-k}{3} & 0 \\ k & 0 \end{vmatrix} \tag{1.1.6}$$

2. Let the state-space representation of an LTI system be.

$$\mathbf{x}\dot{(t)} = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t) \tag{1.2.1}$$

$$y(t) = \mathbf{C}\mathbf{x}(t) + Du(t) \tag{1.2.2}$$

where $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are matrices, D is a scalar, $u(t)$ is input to the system and $y(t)$ is output to the system. Let

$$\mathbf{B} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{1.2.3}$$

$$D = 0 \tag{1.2.4}$$

Find $\mathbf{A}$ and $\mathbf{C}$, given that

$$H(s) = \frac{1}{s^3 + 3s^2 + 2s + 1} \tag{1.2.5}$$

3. The block diagram of a system is illustrated in the figure shown, where $X(s)$ is the input and $Y(s)$ is the output. Find the transfer function $H(s) = \frac{Y(s)}{X(s)}$
4. For an LTI system, the Bode plot for its gain is as illustrated in Fig. 1.4. Find the number of system poles N_p and number of system zeros

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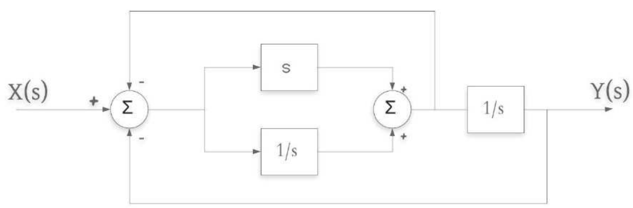


Fig. 1.3

N_z in the frequency range 1 Hz $\leq f \leq 10^7$ Hz.

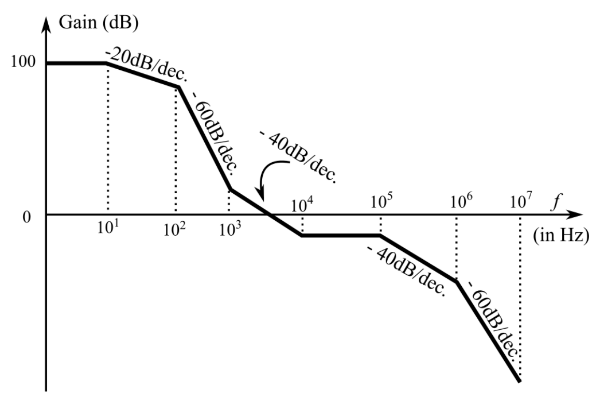


Fig. 1.4

Solution: Let

$$H(s) = k\frac{(s - z_1)(s - z_2)...(s - z_{m-1})(s - z_m)}{(s - p_1)(s - p_2)....(s - p_{n-1})(s - p_n)} \tag{1.4.1}$$

$$\begin{aligned} Gain &= 20\log|H(s)| \\ &= 20\log|k| + 20\log|s - z_1| \\ &+ 20\log|s - z_2| + \cdots + 20\log|s - z_m| \\ &- 20\log|s - p_1| - 20\log|s - p_2| \\ &- \cdots - 20\log|s - p_n| \end{aligned} \tag{1.4.2}$$

- When a pole is encountered the slope always decreases by -20 dB/decade
- When a zero is encountered the slope always increases by +20 dB/decade
- At f = 10 Hz , change in slope = -20dB/sec, Hence we have 1 pole here
- At f = 10^2 Hz, Change in slope = -40dB/sec, Hence we have 2 poles here
- At f = 10^3 Hz, Change in slope = +20dB/sec, Hence we have 1 zero here
- At f = 10^4 Hz, Change in slope = +40dB/sec, Hence we have 2 zeros here
- At f = 10^5 Hz, Change in slope = -40dB/sec, Hence we have 2 poles here
- At f = 10^6 Hz, Change in slope = -20dB/sec, Hence we have 1 pole here

$$N_p = 6 \tag{1.4.3}$$
$$N_z = 3 \tag{1.4.4}$$

5. The transfer function $C(s)$ of a compensator is given below.

$$C(s) = \frac{(1 + \frac{s}{0.1})(1 + \frac{s}{100})}{(1 + \frac{s}{1})(1 + \frac{s}{10})} \tag{1.5.1}$$

Find the frequency range in which the phase (lead) introduced by the compensator reaches the maximum

Solution: The three types of compensators — lag, lead and lag-lead are most commonly used electrical compensators.

- Compensate a unstable system to make it stable.
- Compensating networks can also increase the steady state accuracy of the system.
- A lead compensator can increase the stability or speed of response of a system.
- lag compensator can reduce (but not eliminate) the steady-state error.

Solution: The transfer function of a lag-lead compensator is given by

$$\frac{(1 + \alpha sT_1)(1 + \beta sT_2)}{(1 + sT_1)(1 + sT_2)} \tag{1.5.2}$$

The compensator in (1.5.1) can be expressed as

$$C(s) = \frac{(1 + 0.01s)(1 + 10s)}{(1 + 0.1s)(1 + s)} \tag{1.5.3}$$

Comparing with

$$\alpha = 10, T_1 = 1, \beta = 0.1 \text{ and } T_2 = 0.1. \tag{1.5.4}$$

6. Consider the following second order system with the transfer function:

$$G(s) = \frac{1}{1 + 2s + s^2}$$

with input unit step

$$R(s) = \frac{1}{s}$$

Let C(s) be the corresponding output. The time taken by the system output c(t) to reach 94% of its steady state value, rounded off to two decimal places is

a) 5.25
b) 4.50
c) 2.81
d) 3.89

Solution:- The approach for finding the solution is as follows:

- finding C(s)
- finding c(t)
- finding the time at which c(t) attains 94% of its steady state value

We are given G(s) and R(s), to find C(s), we can simply multiply these two

$$C(s) = R(s).G(s) = (\frac{1}{s})(\frac{1}{1+2s+s^2})C(s) = \frac{1}{s(1+s)^2} \tag{1.6.1}$$

To find c(t), we have to do inverse Laplace transform on C(s)

$$c(t) \longleftrightarrow C(s)$$

Inverse Laplace transform can be calculated by the formula:

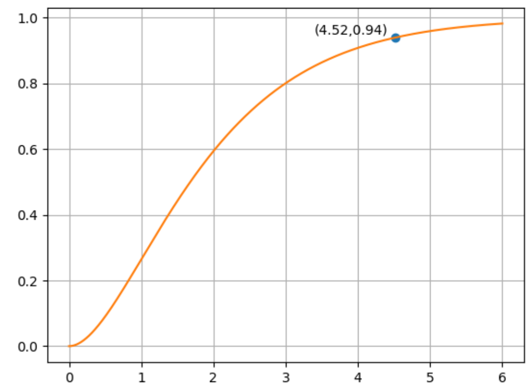
$$f(t) = \frac{1}{2\pi j}\int_{a-j\infty}^{a+j\infty} F(s)e^{st}ds$$

From the above formula, the inverse Laplace for some common expressions are:

$$u(t) \longleftrightarrow \frac{1}{s}$$

$$e^{-at}u(t) \longleftrightarrow \frac{1}{s+a}$$

$$te^{-at}u(t) \longleftrightarrow \frac{1}{(s+a)^2}$$



We found C(s) as:

$$C(s) = \frac{1}{s(1+s)^2}$$

Now, we will use partial fractions to make applying Inverse Laplace easy.

$$C(s) = \frac{1}{s(1+s)^2} = \frac{A}{s} + \frac{B}{(1+s)} + \frac{C}{(1+s)^2}$$

We get,

$$A = 1 \quad A+B=0 \quad 2A+B+C=0$$
$$A = 1 \quad B=-1 \quad C=-1$$

Therefore,

$$C(s) = \frac{1}{s} - \frac{1}{(1+s)} - \frac{1}{(1+s)^2}$$

$$c(t) = L^{-1}(\frac{1}{s} - \frac{1}{(1+s)} - \frac{1}{(1+s)^2})$$

From the properties of inverse Laplace transform,

$$L^{-1}(F_1(s)+F_2(s)+F_3(s)) = L^{-1}(F_1(s)) + L^{-1}(F_2(s)) + L^{-1}(F_3(s))$$

Therefore;

$$c(t) = L^{-1}(\frac{1}{s}) - L^{-1}(\frac{1}{(1+s)}) - L^{-1}(\frac{1}{(1+s)^2})$$

Using the Known inverse transforms:

$$c(t) = (1 - e^{-t} - te^{-t}).u(t)$$

To know the steady state value of c(t), we calculate

$$\lim_{t\to\infty} c(t) = (1+0+0).(1) = 1$$

Now, 94% of 1 is 0.94, so we should now solve for a positive t such that

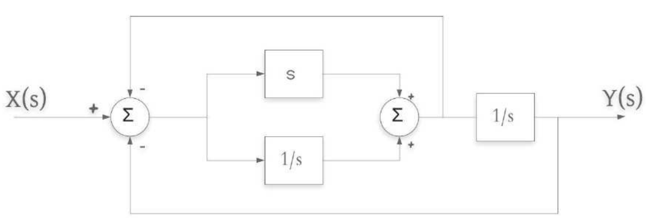
$$(1 - e^{-t} - te^{-t}) = 0.94$$

After calculation, t turns out to be

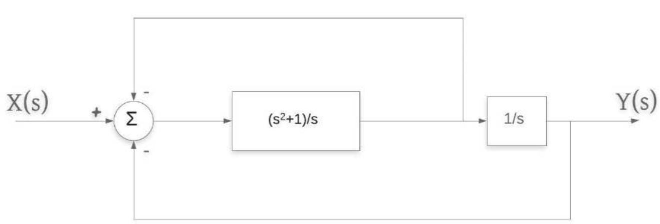
$$t = 4.5221$$

Therefore, answer is option (b) We can also find the solution by plotting c(t):

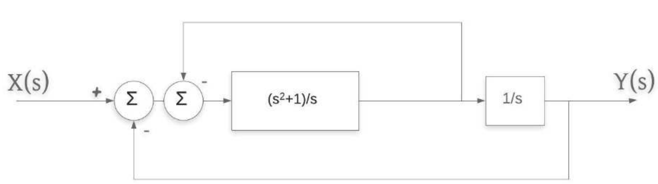
7. The block diagram of a system is illustrated in the figure shown, where X(s) is the input and Y(s) is the output. The transfer function H(s)=$\frac{Y(s)}{X(s)}$ is
(A) H(s)=$\frac{s^2+1}{s^3+s^2+s+1}$
(B) H(s)=$\frac{s^2+1}{s^3+2s^2+s+1}$
(C) H(s)=$\frac{s^2+1}{s^2+s+1}$
(D) H(s)=$\frac{s^2+1}{2s^2+1}$
Solution:- Here we have two transfer function



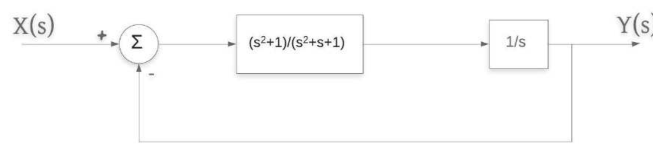
s and $\frac{1}{s}$ in parallel with a adder as shown in figure. After solving these two parallel transfer function by just adding both of them we will get



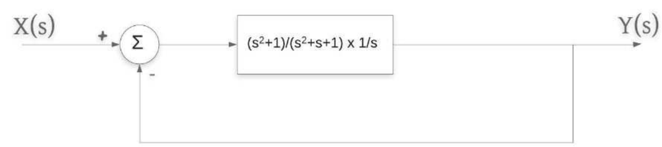
Now we will convert three input adder into two input adder as shown in figure given below.



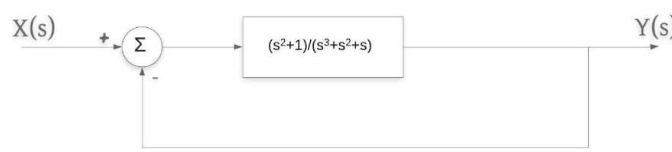
Now we have Negative Unity Feedback System(NUFS) in closed loop transfer function.
Let's say we have transfer function G(s) with Negative Unity Feedback System in closed loop then we will solve this by $\frac{G(s)}{1+G(s)}$.
Here we have



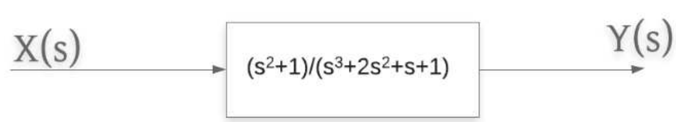
Here we have two transfer function in series



Now we have one more transfer function with negative unity feedback.



Again we will solve this then we will get



Now

X(s)($\frac{s^2+1}{s^3+2s^2+s+1}$)=Y(s)

$\frac{Y(s)}{X(s)}=\frac{s^2+1}{s^3+2s^2+s+1}$

The correct option is (B)

8. Let the state-space representation of an LTI system be.

$\dot{x}(t) = Ax(t) + Bu(t)$
y(t)=Cx(t)+Du(t)

A,B,C are matrices, D is scalar, u(t) is input to the system and y(t) is output to the system. let

$$b1 = \begin{vmatrix} 0 & 0 & 1 \end{vmatrix}$$

$b1^T = B$
and D=0. Find A and C.

$$H(s) = \frac{1}{s^3 + 3s^2 + 2s + 1}$$

Solution:- STATE MODEL
Let U1(t) and U2(t) are the inputs of the MIMO system and y1(t),y2(t) are the output of the system and x1(t) and x2(t) are the state variables.
so output equation is,

$$y1(t) = C_{11}\times x1(t)+C_{12}\times x2(t)+d_{11}\times U1(t)+d_{12}\times U2(t) \quad (1.8.1)$$

$$y2(t) = C_{21}\times x1(t)+C_{22}\times x2(t)+d_{21}\times U1(t)+d_{22}\times U2(t) \quad (1.8.2)$$

$$\begin{bmatrix} y1(t) \\ y2(t) \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} \\ C_{11} & C_{12} \end{bmatrix} \times \begin{bmatrix} x1(t) \\ x2(t) \end{bmatrix} + \begin{bmatrix} d_{11} & d_{12} \\ d_{11} & d_{12} \end{bmatrix} \times \begin{bmatrix} U1(t) \\ U2(t) \end{bmatrix}$$

therefore Y(t)=C.X(t)+D.U(t)

$$x\dot{1}(t) = a_{11}\times x1(t)+a_{12}\times x2(t)+b_{11}\times U1(t)+b_{12}\times U2(tx \quad (1.8.3)$$

$$x\dot{2}(t) = a_{21}\times x1(t)+a_{22}\times x2(t)+b_{21}\times U1(t)+b_{22}\times U2(t) \quad (1.8.4)$$

$$\begin{bmatrix} x\dot{1}(t) \\ x\dot{2}(t) \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{11} & a_{12} \end{bmatrix} \times \begin{bmatrix} x1(t) \\ x2(t) \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{11} & b_{12} \end{bmatrix} \times \begin{bmatrix} U1(t) \\ U2(t) \end{bmatrix}$$

$therefore\ X\dot{(}t)$=A.X(t)+B.U(t)

9. FINDING TRANSFER FUNCTION

$So,\ X\dot{(}t)$=A.X(t)+B.U(t) $be \quad equation$ 1

and Y(t)=C.X(t)+D.U(t) $be \quad equation$ 2

by applying laplace transforms on both sides of equation 1

we get

S.X(S)-X(0)=A.X(S)+B.U(S)

S.X(S)-A.X(S)=B.U(S)+X(0)

(SI-A)X(S)=X(0)+B.U(S)

$$X(S) = X(0)([SI-A])^-1+B.([SI-A])^-1.U(S)$$

$Laplace\ transform\ of\ equation$ 2 $and\ sub\ X(s)$

Y(S)=C.X(S)+D.U(S)

$$Y(S) = C.[X(0)([SI - A])^-1 + B.([SI - A])^-1.U(S)] + D.U(S)$$

If X(0)=0

$$thenY(S) = C.[B.([SI-A])^-1.U(S)] + D.U(S)$$

$$\frac{Y(S)}{U(S)} = C.[B.([SI - A])^-1] + D = H(S)$$

As we know that

$$Y(s) = H(s)\times U(s) = (\frac{1}{s^3 + 3s^2 + 2s + 1})\times U(s)$$

let $X(S) = \dfrac{U(S)}{denominator}$

$$Y(S) = X(S) \times numerotor$$

$$s^3X(s) + 3s^2X(s) + 2sX(s) + X(s) = U(S)$$

$$\begin{bmatrix} sx(s) \\ s^2x(s) \\ s^3x(s) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix} \times \begin{bmatrix} x(s) \\ sx(s) \\ s^2x(s) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times U$$

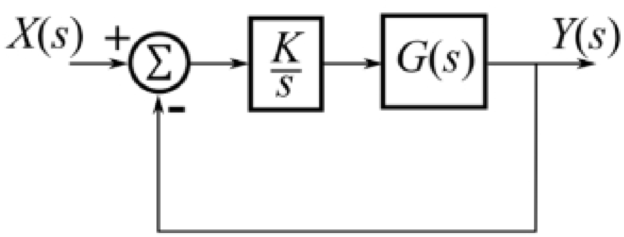
therfore $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix}$

Since $Y(S) = X(S) \times numerator$
therefore Y(S)=X(S);

$$Y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} x(s) \\ sx(s) \\ s^2x(s) \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

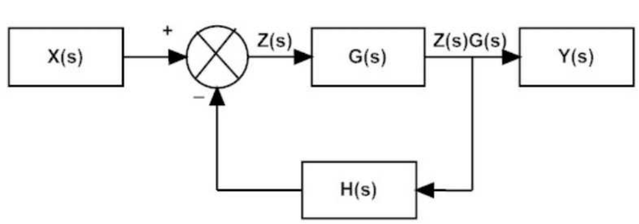
10. Consider a unity feedback system as shown in the figure,shown with an integral compensator k/s and open-loop transfer function
G(s) = $\frac{1}{s^2+3s+2}$
where k>0. The positive value of k for which there are two poles of unity feedback system on jω axis is equal to——(rounded off to two decimal places)



Solution:- A transfer function is the relative function between input and output.
In a negative feedback system an intermediate signal is defined as Z.
Y(s) = Z(s).G(s)
Z(s) = X(s) - Y(s).H(s) $\Rightarrow$ X(s) = Z(s)+Y(s).H(s)

X(s) = Z(s)+Z(s).G(s).H(s)

$\frac{Y(s)}{X(s)} = \frac{Z(s).G(s)}{Z(s)+Z(s).G(s).H(s)}$
So,the transfer function of negative feedback is $\frac{G(s)}{1+G(s).H(s)}$
Since unit feedback H(s) = 1
Now the transfer function of unity negative feedback is $\frac{G(s)}{1+G(s)}$
The net transfer function in the given



11. is.....

$\frac{Y(s)}{X(s)} = \frac{G(s)*k/s}{1+G(s)*k/s}$

The characteristic equation is 1 + (G(s)x$\frac{k}{s}$) = 0
that is..,

C.E = 1 + $\frac{k}{s(s^2+3s+2)}$ = 0 $\Rightarrow s(s^2+3s+2)+k = 0$

$\Rightarrow s^3 + 3s^2 + 2s + k = 0$
Routh-Hurwitz Criterion:-

This criterion is based on arranging the coefficients of characteristic equation intoan array called Routh array.
$q(s) = a_0s^n+a_1s^{n-1}+a_2s^{n-2}+.....+a_{n-1}s+a_n = 0$

$$\begin{vmatrix} s^n \\ s^{n-1} \\ s^{n-2} \\ \vdots \end{vmatrix}\begin{vmatrix} a_0 & a_2 & a_4 & \cdots \\ a_1 & a_3 & a_5 & \cdots \\ b_1 & b_2 & b_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots\cdots \end{vmatrix} \tag{1.11.1}$$

where

$$b_1 = \frac{a_1a_2 - a_0a_3}{a_1} \quad b_2 = \frac{a_1a_4 - a_0a_5}{a_1} \tag{1.11.2}$$

$$c_1 = \frac{b_1a_3 - a_1b_2}{b_1} \quad c_2 = \frac{b_1a_5 - a_1b_3}{b_1} \tag{1.11.3}$$

For poles to lie on imaginary axis any one entire row of hurwitz matrix should be zero.

For the given characteristic equation

$$= s^3 + 3s^2 + 2s + k = 0 \tag{1.11.4}$$

$$\begin{vmatrix} s^3 \\ s^2 \\ s^1 \\ s^0 \end{vmatrix}\begin{vmatrix} 1 & 2 \\ 3 & k \\ \frac{6-k}{3} & 0 \\ k & 0 \end{vmatrix} \tag{1.11.5}$$

$Forpoleson$jω axis any one of the row should be zero
$\Rightarrow \frac{6-k}{3}$ = 0 or k = 0
But given k>0 ...
therefore, 6-k=0 $\Rightarrow$ k = 6
To find the location of poles on jω axis

Auxillary equation of the given CE is $3s^2 + k = 0$

$$\Rightarrow 3s^2 + 6 = 0 \tag{1.11.6}$$
$$\Rightarrow s = \pm j2 \tag{1.11.7}$$

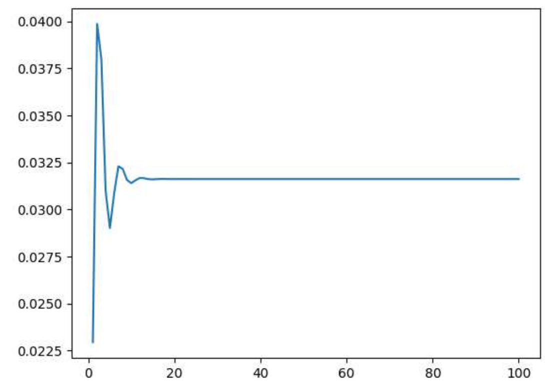
12. The output response of a system is denoted as y(t), and its Laplace transform is given by
$$Y(s) = \frac{10}{s(s^2 + s + 100(2)^{0.5})} \tag{1.12.1}$$
The steady state value of y(t) is
a) $100(2)^{0.5}$
b) $\frac{1}{10(2)^{0.5}}$
c) $10(2)^{0.5}$
d) $\frac{1}{100(2)^{0.5}}$
Solution:-

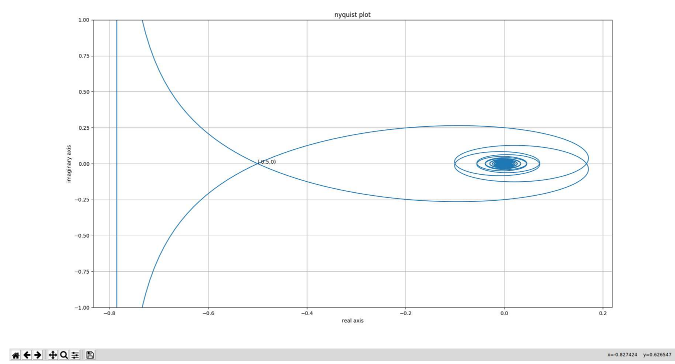
The final value theorem states that
$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s)$$

This is valid only when sY(s) has poles that lie in the negative half of the real side.

If the quadratic equation $ax^2 + bx + c$ has complex roots then the real part of those roots will be -b/2a
Hence, verified that the roots of $s^2+s+100(2)^{0.5}$ have a negative real part which is -0.5. So, Final value theorem is applicable.
Steady state value of y(t) =
$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) = \lim_{s\to 0} \frac{10s}{s(s^2 + s + 100(2)^{0.5})}$$
$$= \frac{10}{100(2)^{0.5}} = \frac{1}{10(2)^{0.5}}$$
We can see that y(t) is approaching a constant value 0.031 which is verifies our answer!

13. • The open loop transfer function of a unity feedback system is given by
$$G(s) = \frac{\pi e^{-0.25s}}{s}$$
in G(s) plane,the Nyquist plot of G(s) passes through the negative real axis at the point (A)(-0.5,j0) (B)(-0.75,j0) (C)(-1.25,j0) (D)(-1.5,j0)
Solution:-





$$G(s) = \frac{\pi e^{-0.25s}}{s}$$
Nyquist plot cuts the negative real Axis at ω = phase cross over frequency
$$G(j\omega) = \frac{\pi}{\omega}(-\sin 0.25\omega - j\cos 0.25\omega) \tag{1.13.1}$$
$$\angle G(j\omega) = -90^\circ - 0.25\omega \times 180^\circ\pi \tag{1.13.2}$$
$$\angle G(j\omega)|_{\omega=\omega_{pc}} = -180^\circ \tag{1.13.3}$$

by solving for $\omega we get \omega_{pc} = 2\pi$
magnitude at any point is$X = |G(j\omega)| = \frac{\pi}{\omega}$
substituting $\omega = 2\pi$ in magnitude we get X=0.5
hence it intersects at (-0.5,0j) so answer is A
we can verify with the following plot that it intersects at (-0.5,0j)

14. The characteristic equation of linear time

invariant system is given by

$$\nabla(s) = s^4 + 3s^3 + 3s^2 + s + k = 0$$

The system is BIBO stable if

A.$0 < k < \frac{12}{9}$

B.k¿3

C.$0 < k < \frac{8}{9}$

D.k¿6 Solution:-

Given data:

$$\nabla(s) = s^4 + 3s^3 + 3s^2 + s + k = 0$$

s^4	1	3	K
s^3	3	1	0
s^2	8/3	k	0
s	(8/3-3K)/(8/3)	0	0
s^0	k	0	0

$$\Rightarrow \frac{\frac{8}{3} - 3k}{\frac{8}{3}} > 0 \tag{1.14.1}$$

$$\tag{1.14.2}$$

$$\tag{1.14.3}$$

$$\Rightarrow \frac{8}{3} - 3k > 0 \tag{1.14.4}$$

$$\tag{1.14.5}$$

$$\Rightarrow 3k < \frac{8}{9} \tag{1.14.6}$$

$$\tag{1.14.7}$$

$$\Rightarrow (0 < k < \frac{8}{9}) \tag{1.14.8}$$

for example the zeros of polynomial $s^4 + 3s^3 + 3s^2 + s + 0.5 = 0$ are

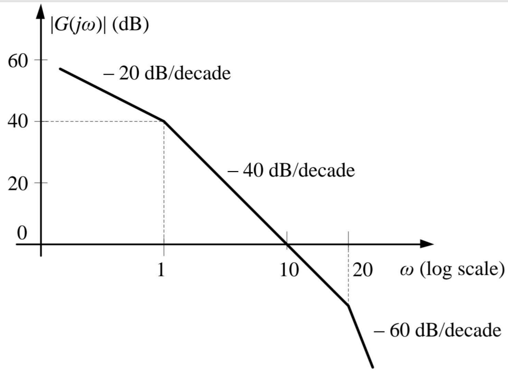
$$s1 = -0.08373 + 0.45773i \tag{1.14.9}$$

$$s2 = -0.08373 - 0.45773i \tag{1.14.10}$$

$$s3 = -1.41627 + 0.55075i \tag{1.14.11}$$

$$s4 = -1.41627 - 0.55075i \tag{1.14.12}$$

15. The asymptotic Bode magnitude plot of minimum phase transfer function G(s) is show below.



Consider the following two statements.

Statement 1: Transfer function G(s) has 3 poles and one zero

Statement 2: At very high frequency ($\omega \to \infty$), the phase angle $\angle G(j\omega) = -3\pi/2$

Which of the following is correct ?

(A) Statement 1 is true and Statement 2 is false.

(B) Statement 1 is false and Statement 2 is true.

(C) Both the statements are true.

(D) Both the statements are false.

Solution:- Since, each pole corresponds to -20 dB/decade and each zero corresponds to +20 dB/decade.

Therefore, from the given Bode plot we can get the Transfer equation,

$$G(s) = \frac{k}{s(1+s)(20+s)}$$

Now, from the Transfer equation we can conclude that, there are three poles (0, -1 and -20) and no zeros.

$\therefore$ Statement 1 is false ....(1)

Calculating phase

Since we know that,

phase ϕ is the sum of all the phases corresponding to each pole and zero.

phase corresponding to pole is =

$$-tan^{-1}(\frac{imaginary}{real})$$

phase corresponding to zero is =

$$tan^{-1}(\frac{imaginary}{real})$$

now take,

$$s = j\omega$$

$$\Rightarrow G(j\omega) = \frac{k}{j\omega(1 + j\omega)(20 + j\omega)}$$

Therefore,

$$\phi = -tan^{-1}(\frac{\omega}{0}) - tan^{-1}(\omega) - tan^{-1}(\frac{\omega}{20})$$

$$\phi = -90^\circ - tan^{-1}(\omega) - tan^{-1}(\frac{\omega}{20})$$

$$\because \omega \to \infty$$

$$\phi = -90^\circ - 90^\circ - 90^\circ$$

$$\phi = -270^\circ$$

$$\phi = -3\pi/2$$

$\therefore$ Statement 2 is true

........(2)

thus, from (1) and (2) option (B) is correct.

16. The transfer function of phase lead compensator is given by

$$D(s) = \frac{3(s + \frac{1}{3T})}{(s + \frac{1}{T})}$$

The frequency (in rad/sec), at which $\angle D(j\omega)$ is maximum, is

(a) $\sqrt{\frac{1}{T^2}}$ **(b)** $\sqrt{\frac{1}{3T^2}}$ **(c)** $\sqrt{3T}$

(d) $\sqrt{3T^2}$

Solution:- The basic requirement of the phase lead network is that all poles and zeros of the transfer function of the network must lie on (-)ve real axis interlacing each other with a zero located as the nearest point to origin. The given transfer fuction is

$$D(s) = \frac{3(s + \frac{1}{3T})}{(s + \frac{1}{T})}$$

Now substituting $s = j\omega$ in D(s), we get

$$D(j\omega) = \frac{3(j\omega + \frac{1}{3T})}{(j\omega + \frac{1}{T})}$$

The phase of this transfer function $\phi(\omega)$ is given by

$$\phi(\omega) = \tan^{-1}(3\omega T) - \tan^{-1}(\omega T)$$

$\phi(\omega)$ has its maximum at ω_c such that $\phi'(\omega_c) = 0$,

$$\phi'(\omega_c) = \frac{3T}{1 + (3\omega_c T)^2} - \frac{T}{1 + (\omega_c T)^2}$$

After simplification,

$$\omega_c^2 T^2 = \frac{1}{3}$$

$$\omega_c = \sqrt{\frac{1}{3T^2}}$$

Hence **(b)** is the correct option.

font=• Consider a state-variable model of a system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\alpha & -2\beta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} r \quad (1.16.1)$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad (1.16.2)$$

where y is the output, and r is the input. The damping ratio ζ and the undamped natural frequency ω_n (rad/sec) of the system are given by:

(A) $\zeta = \frac{\beta}{\sqrt{\alpha}}, \omega_n = \sqrt{\alpha}$

(B) $\zeta = \sqrt{\alpha}, \omega_n = \frac{\beta}{\sqrt{\alpha}}$

(C) $\zeta = \frac{\alpha}{\sqrt{\beta}}, \omega_n = \sqrt{\beta}$

(D) $\zeta = \sqrt{\beta}, \omega_n = \sqrt{\alpha}$

Solution:- Transformation of State Equations

to a Single Differential Equation [font=•]
THE STATE EQUATIONS $\dot{x}$ =AX+BR FOR A LINEAR SECOND-ORDER SYSTEM WITH A SINGLE INPUT ARE A PAIR OF COUPLED FIRST-ORDER DIFFERENTIAL EQUATIONS IN THE TWO STATE VARIABLES:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} r. \quad (1.16.3)$$

OR

$$\frac{dx_1}{dt} = a_{11}x_1 + a_{12}x_2 + b_1 r.$$

$$\frac{dx_2}{dt} = a_{21}x_1 + a_{22}x_2 + b_2 r.$$

THE STATE-SPACE SYSTEM REPRESENTATION MAY BE TRANSFORMED INTO A SINGLE DIFFERENTIAL EQUATION INEITHER OF THE TWO STATE-VARIABLES. TRANSFORMATION OF STATE EQUATIONS TO A SINGLE DIFFERENTIAL EQUATION [FONT=•] TAKING THE LAPLACE TRANSFORM OF THE STATE EQUATIONS

$$(sI - A)X(s) = BR(s) \quad (1.16.4)$$

$$X(s) = (sI - A)^{-1}BR(s) \quad (1.16.5)$$

$$X(s) = \frac{1}{det[sI - A]} \begin{bmatrix} s - a_{22} & a_{12} \\ a_{21} & s - a_{11} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} R(s) \quad (1.16.6)$$

$$det[sI - A]X(s) = \begin{bmatrix} s - a_{22} & a_{12} \\ a_{21} & s - a_{11} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} R(s) \quad (1.16.7)$$

TRANSFORMATION OF STATE EQUATIONS TO A SINGLE DIFFERENTIAL EQUATION [FONT=•] FROM THIS WE CAN WRITE

$$\Rightarrow \frac{d^2x_1}{dt^2} - (a_{11} + a_{22})\frac{dx_1}{dt} + (a_{11}a_{22} - a_{12}a_{21})x_1 \quad (1.16.8)$$

$$= b_1\frac{du}{dt} + (a_{12}b_2 - a_{22}b_1)r. \quad (1.16.9)$$

and (1.16.10)

$$\Rightarrow \frac{d^2x_2}{dt^2} - (a_{11} + a_{22})\frac{dx_2}{dt} + (a_{11}a_{22} - a_{12}a_{21})x_2 \quad (1.16.11)$$

$$= b_2\frac{du}{dt} + (a_{21}b_1 - a_{11}b_2)r. \quad (1.16.12)$$

WHICH CAN BE WRITTEN IN TERMS OF THE TWO PARAMETERS ω_n AND ζ AS FOLLOWS: TRANSFORMATION OF STATE EQUATIONS TO A SINGLE DIFFERENTIAL EQUATION [FONT=•]

$$\frac{d^2x_1}{dt^2} + 2\zeta\omega_n\frac{dx_1}{dt} + \omega_n^2 x_1 = b_1\frac{du}{dt} + (a_{12}b_2 - a_{22}b_1)r.$$

AND

$$\frac{d^2x_2}{dt^2} + 2\zeta\omega_n\frac{dx_2}{dt} + \omega_n^2 x_1 = b_2\frac{du}{dt} + (a_{21}b_1 - a_{11}b_2)r.$$

WHERE ζ IS THE SYSTEM(DIMENSIONLESS) DAMPING RATIO AND THE UNDAMPED NATURAL FREQUENCY WITH UNITS OF RADIAN/SECOND IS ω_n.
BY COMPARING ABOVE EQUATIONS WE GET THAT:

$$\omega_n = \sqrt{a_{11}a_{22} - a_{12}a_{21}}$$

AND

$$\zeta = -\frac{(a_{11} + a_{22})}{\omega_n} = \frac{-(a_{11} + a_{22})}{2\sqrt{a_{11}a_{22} - a_{12}a_{21}}}$$

17. MATCH THE TRANSFER FUNCTIONS OF THE SECOND-ORDER SYSTEMS WITH THE NATURE OF THE SYSTEMS GIVEN BELOW

- (A)P-1,Q-2,R-3

- (B)P-2,Q-1,R-3
- (C)P-3,Q-2,R-1
- (D)P-3,Q-1,R-2

18. The number of roots of the polynomial, $s^7 + s^6 + 7s^5 + 14s^4 + 31s^3 + 73s^2 + 25s + 200$, in the open left half of the complex plane is
(A) 3
(B) 4
(C) 5
(D) 6

19. The unit step response of y(t) of a unity feedback system with an open loop transfer function

$$G(s)H(S) = \frac{K}{(s+1)^2(s+2)}$$

is shown in figure. The value of K is(up to two decimal places).

20. An input p(t) = sin(t) is applied to the system $G(s) = \frac{s-1}{s+1}$. The corresponding steady state output is y(t) = sin(t + φ), where the phase φ (in degrees), when restricted to $0^\circ \leq \varphi \leq 360^\circ$, is ?

21. Consider the transfer function $G(s) = \frac{2}{(s+1)(s+2)}$. The Phase Margin of $G(s)$ in degrees is