Control Systems

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Contents

***Abstract*—This manual is an introduction to control systems based on GATE problems.Links to sample Python codes are available in the text.**

Download python codes using

```
svn co https://github.com/gadepall/school/trunk/
    control/codes
```

1 Mason's Gain Formula

1.1. The Block diagram of a system is illustrated in the figure shown, where $X(s)$ is the input and $Y(s)$ is the output. Draw the equivalent signal flow graph.
Solution: Signal flow graph of given above block diagram is

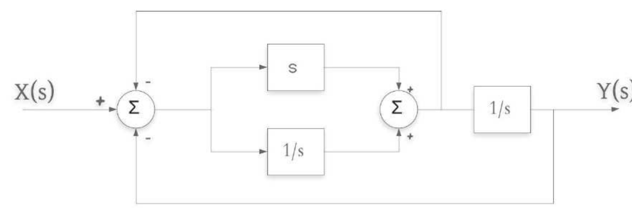

Fig. 1.1.1

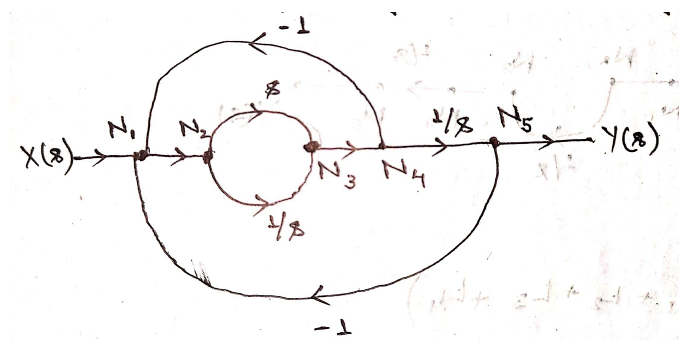

Fig. 1.1.2: signal flow graph

1.2. Draw all the forward paths and compute the respective gains. **Solution:** Here,

$$P_1 = \frac{s}{s} = 1 \tag{1.2.1}$$

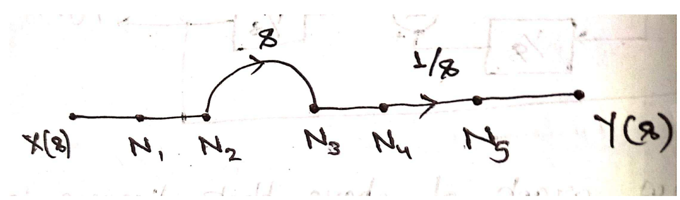

Fig. 1.2.3: P1

$$P_2 = (1/s)(1/s) = 1/s^2 \tag{1.2.2}$$

1.3. Draw the loops and calculate the respective gains.
Solution:

$$L_1 = (-1)(s) = -s \tag{1.3.1}$$

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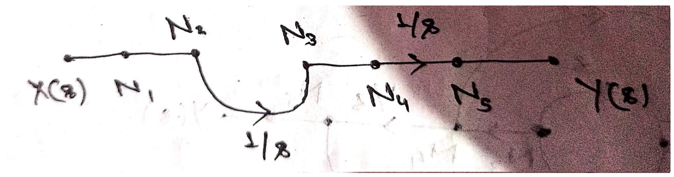

Fig. 1.2.4: P2

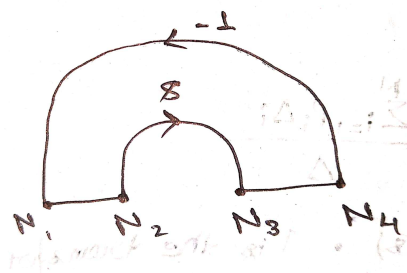

Fig. 1.3.5: L1

$$L_2 = \frac{s}{-s} = -1 \tag{1.3.2}$$

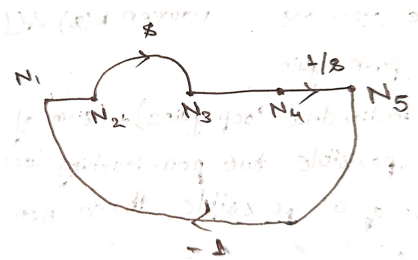

Fig. 1.3.6: L2

$$L_3 = (\frac{1}{s}) * (-1) = \frac{-1}{s} \tag{1.3.3}$$

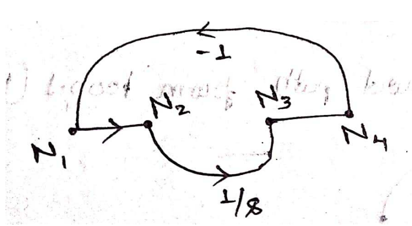

Fig. 1.3.7: L3

$$L_4 = (\frac{1}{s}) * (\frac{1}{s})8(-1) = \frac{-1}{s^2} \tag{1.3.4}$$

1.4. State Mason's Gain formula and explain the parameters through a table.

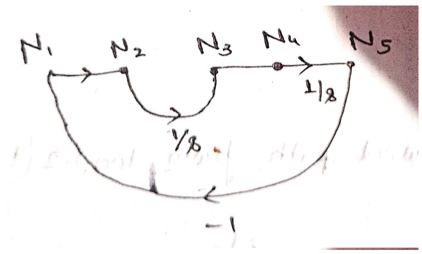

Fig. 1.3.8: L4

Solution: According to Mason's Gain Formula,

$$T = \frac{Y(s)}{X(s)} \tag{1.4.1}$$

$$T = \frac{\sum_{i=1}^{N} P_i \Delta_i}{\Delta} \tag{1.4.2}$$

1.5. Find the transfer function using Mason's Gain Formula.
Solution:
Now,
Pi is the ith forward path.
$\Delta = 1 - (L_1 + L_2 + L_3 + L_4)$

$$L_1 = (-1)(s) = -s \tag{1.5.1}$$

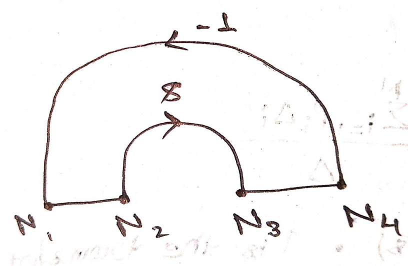

Fig. 1.5.9: L1

$$L_2 = \frac{s}{-s} = -1 \tag{1.5.2}$$

$$L_3 = (\frac{1}{s}) * (-1) = \frac{-1}{s} \tag{1.5.3}$$

$$L_4 = (\frac{1}{s}) * (\frac{1}{s})8(-1) = \frac{-1}{s^2} \tag{1.5.4}$$

$\Delta = 1 - (-s - 1 - \frac{1}{s} - \frac{1}{s^2})$ $\Delta = \frac{s^3+2s^2+s+1}{s^2}$
$\Delta_1 = 1$
$\Delta_2 = 1$

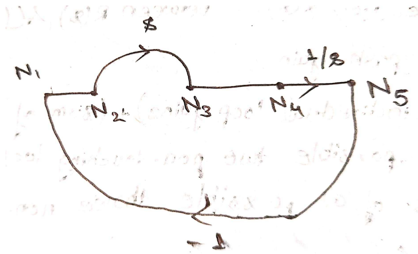

Fig. 1.5.10: L2

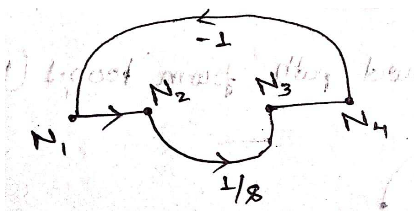

Fig. 1.5.11: L3

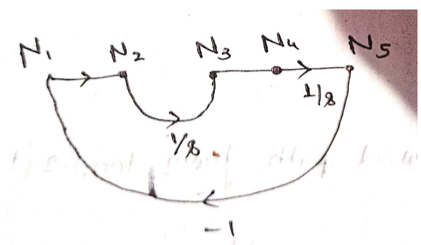

Fig. 1.5.12: L4

Fig. 1.5.13: Delta1

Fig. 1.5.14: Delta2

$\Delta_3 = 1$
$\Delta_4 = 1$

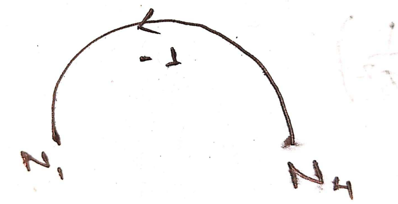

Fig. 1.5.15: Delta3

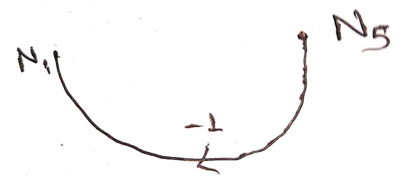

Fig. 1.5.16: Delta4

Here,

$$T = \frac{\sum_{i=1}^{N} (P_i)(\Delta_i)}{\Delta} \tag{1.5.5}$$

$$T = \frac{P_1\Delta_1 + P_2\Delta_2 + P_3\Delta_3 + P_4\Delta_4}{\Delta} \tag{1.5.6}$$

$$T = \frac{1 * 1 + (\frac{1}{s^2}) * 1 + 0 * 1 + 0 * 1}{\frac{s^3+2s^2+s+1}{s^2}} \tag{1.5.7}$$

$$H(s) = \frac{s^2 + 1}{s^3 + 2s^2 + s + 1} \tag{1.5.8}$$

2 Bode Plot

2.1 Introduction

2.1. For an LTI system, the Bode plot for its gain defined as

$$G(s) = 20 \log |H(s)| \tag{2.1.1}$$

is as illustrated in the Fig. 2.1. Express $G(f)$ in terms of f.

Solution: Let us consider a generalized transfer gain

$$H(s) = k\frac{(s - z_1)(s - z_2)...(s - z_{m-1})(s - z_m)}{(s - p_1)(s - p_2)....(s - p_{n-1})(s - p_n)} \tag{2.1.2}$$

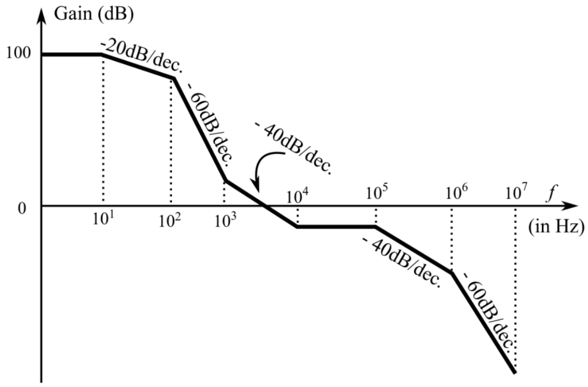

Fig. 2.1

$$Gain = 20\log|H(s)| = 20\log|k| + 20\log|s - z_1| + 20\log|s - z_2| + \cdots + 20\log|s - z_m| - 20\log|s - p_1| - 20\log|s - p_2| - \cdots - 20\log|s - z_n| \quad (2.1.3)$$

Let us consider the term $20\log|s - z_1|$ and let $s = j\omega$

$$20\log|s - z_1| = 20\log\left|\sqrt{\omega^2 + z_1^2}\right| \quad (2.1.4)$$

Based on log scale plot approximations,to the left of z_1 $\omega << z_1$ and towards right $\omega >> z_1$
For $\omega < z_1$

$$20\log|s - z_1| = 20\log\left|\sqrt{\omega^2 + z_1^2}\right| = 20\log|z_1| \quad (2.1.5)$$

$$= constant \quad (2.1.6)$$

i.e. Slope = 0

For $\omega > z_1$

$$20\log|s - z_1| = 20\log\left|\sqrt{\omega^2 + z_1^2}\right| = 20\log|\omega| \quad (2.1.7)$$

i.e Slope = 20

When a zero is encountered the slope always increases by 20 dB/decade
By performing similar analysis for $-20\log|s - p_1|$, we conclude that
When a pole is encountered the slope always decreases by 20 dB/decade
So, Poles are encountered at $f = 10, 10^2, 10^5, 10^6$
Similarly Zeros are encountered at $f = 0, 10^3, 10^4$
Final Transfer function is

$$H(f) = \frac{K(f + 10^3)(f + 10^4)^2}{(f + 10^1)(f + 10^2)^2(f + 10^5)^2(f + 10^6)} \quad (2.1.8)$$

$$G(f) = 20\log\frac{K(f + 10^3)(f + 10^4)^2}{(f + 10^1)(f + 10^2)^2(f + 10^5)^2(f + 10^6)} \quad (2.1.9)$$

2.2. Express the slope of $G(f)$ in terms of f.
Solution:

$$Slope = \nabla G(f) = \frac{d(G(f))}{df} \quad (2.2.1)$$

$$\nabla G(f) = \begin{cases} 0 & 0 < f < 10^1 \\ -20 & 10 < f < 10^2 \\ -60 & 10^2 < f < 10^3 \\ -40 & 10^3 < f < 10^4 \\ 0 & 10^4 < f < 10^5 \\ -40 & 10^5 < f < 10^6 \\ -60 & 10^6 < f < 10^7 \end{cases} \quad (2.2.2)$$

2.3. Express the change of slope of $G(f)$ in terms of f.
Solution:
$\Delta(\nabla G(f))$ = Change of slope G(f) at f

$$\Delta(\nabla G(f)) = \begin{cases} -20 & f = 10^1 \\ -40 & f = 10^2 \\ +20 & f = 10^3 \\ +40 & f = 10^4 \\ -40 & f = 10^5 \\ -20 & f = 10^6 \end{cases} \quad (2.3.1)$$

2.4. Find the number of poles and zeros of $H(s)$.
Solution:
When a zero is encountered the slope always increases by 20 dB/decade
When a pole is encountered the slope always decreases by 20 dB/decade

$$N_p = 6 \quad (2.4.1)$$

$$N_z = 3 \quad (2.4.2)$$

2.5. Find the location of the poles and zeros of $H(s)$.The number of system poles N_p and number of system zeros N_z in the frequency range 1 Hz ≤ f ≤ 10^7 Hz is.
Solution:
When a zero is encountered the slope always increases by 20 dB/decade
When a pole is encountered the slope always decreases by 20 dB/decade
So, Poles are encountered at
$f = 10, 10^2, 10^5, 10^6$
Similarly Zeros are encountered at
$f = 0, 10^3, 10^4$

2.6. Obtain the transfer function of $H(s)$.
Solution: $s = j\omega = j2\pi f$

$$H(s) = \frac{K(s + j2\pi 10^3)(s + j2\pi 10^4)^2}{(s + j2\pi 10^1)(s + j2\pi 10^2)^2(s + j2\pi 10^5)^2(s + j2\pi 10^6)} \quad (2.6.1)$$

2.7. Obtain the Bode plot and the slope plot for $H(s)$ and verify with Fig. 2.1
Solution: Bode Plot of obtained Transfer Function is

2.2 Example

2.2.1. The asymptotic Bode magnitude plot of minimum phase transfer function G(s) is show below. Express $20\log|G(\jmath\omega)|$ as a function of ω using Fig. 2.2.1.

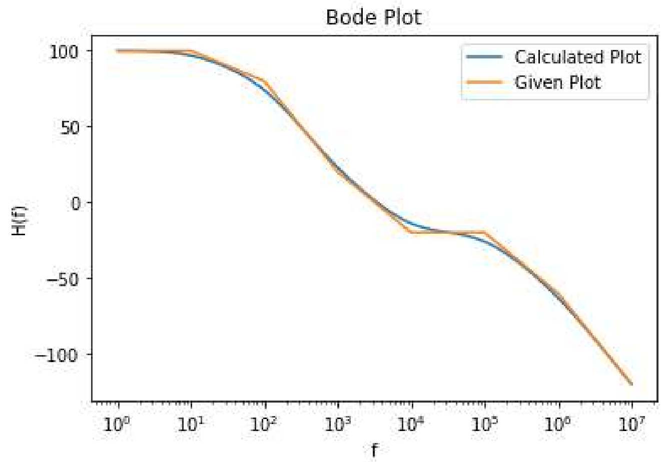

Fig. 2.7

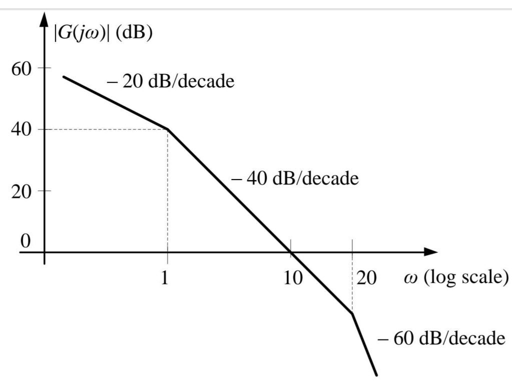

Fig. 2.2.1

2.2.2. Express the slope of $20\log|G(\jmath\omega)|$ as a function of ω.

2.2.3. Express the change of slope of $20\log|G(\jmath\omega)|$ as a function of ω.

2.2.4. Find the poles and zeros of $G(s)$.

2.2.5. Find $G(s)$

2.2.6. Obtain the Bode plot of $G(s)$ through a python code and compare with the line plot of the expression that you obtained in Problem 2.2.1

2.2.7. Verify if at very high frequency ($\omega \to \infty$), the phase angle $\angle G(j\omega) = -3\pi/2$ **Solution:**
Since, each pole corresponds to -20 dB/decade and each zero corresponds to +20 dB/decade. Therefore, from the given Bode plot we can get the Transfer equation,

$$G(s) = \frac{k}{s(1 + s)(20 + s)} \quad (2.2.7.1)$$

Now, from the Transfer equation we can conclude that, there are three poles (0, -1 and -20) and no zeros.

$\therefore$ Statement 1 is false ....(1)

Calculating phase:

Since we know that,
phase ϕ is the sum of all the phases corresponding to each pole and zero.
phase corresponding to pole is =

$$-tan^{-1}(\frac{imaginary}{real}) \tag{2.2.7.2}$$

phase corresponding to zero is =

$$tan^{-1}(\frac{imaginary}{real}) \tag{2.2.7.3}$$

Now take,

$$s = j\omega \tag{2.2.7.4}$$

$$\Rightarrow G(j\omega) = \frac{k}{j\omega(1 + j\omega)(20 + j\omega)} \tag{2.2.7.5}$$

Therefore,

$$\phi = -tan^{-1}(\frac{\omega}{0}) - tan^{-1}(\omega) - tan^{-1}(\frac{\omega}{20}) \tag{2.2.7.6}$$

$$\phi = -90^\circ - tan^{-1}(\omega) - tan^{-1}(\frac{\omega}{20}) \tag{2.2.7.7}$$

$$\because \omega \to \infty \tag{2.2.7.8}$$

$$\phi = -90^\circ - 90^\circ - 90^\circ \tag{2.2.7.9}$$

$$\phi = -270^\circ \tag{2.2.7.10}$$

$$\phi = -3\pi/2 \tag{2.2.7.11}$$

$\therefore$ Statement 2 is true ....(2)
thus, from (1) and (2) option (B) is correct.

2.2.8.

3 Second order System

3.1 Damping

3.1.1. List the different kinds of damping for a second order system defined by

$$H(s) = \frac{\omega^2}{s^2 + 2\zeta\omega + \omega^2} \tag{3.1.1.1}$$

where ω is the natural frequency and ζ is the damping factor.
Solution: The details are available in Table 3.1.1

Damping Ratio	**Damping Type**
$\zeta > 1$	Overdamped
$\zeta = 1$	Critically Damped
$0 < \zeta < 1$	Underdamped
$\zeta = 0$	Undamped

TABLE 3.1.1

3.1.2. Classify the following second-order systems according to damping.
a) $H(s) = \frac{15}{s^2+5s+15}$
b) $H(s) = \frac{25}{s^2+10s+25}$
c) $H(s) = \frac{35}{s^2+18s+35}$

Solution: For

$$H(s) = \frac{25}{s^2 + 10s + 25}, \tag{3.1.2.1}$$

$$\omega^2 = 25, 2\zeta\omega = 10 \tag{3.1.2.2}$$

$$\implies \omega = 1, \zeta = 1 \tag{3.1.2.3}$$

and the system is critically damped. Similarly, the damping factors for other systems in Problem 3.1.2 are calculated and listed in Table 3.1.2

$H(s)$	ω	ζ	**Damping Type**
$\frac{35}{s^2+18s+35}$	$\sqrt{35}$	$\sqrt{\frac{81}{35}} > 1$	Overdamped
$\frac{25}{s^2+10s+25}$	5	1	Critically Damped
$\frac{15}{s^2+5s+15}$	$\sqrt{15}$	$\sqrt{\frac{5}{12}} < 1$	Underdamped

TABLE 3.1.2

3.1.3. By choosing an appropriate input, illustrate the effect of damping using a Python code to sketch the response.

3.2 Example

3.1. Consider the following second order system with the transfer function

$$G(s) = \frac{1}{1 + 2s + s^2} \tag{3.1.1}$$

Is the system stable?
Solution: The poles of

$$G(s) = \frac{1}{1 + 2s + s^2} \tag{3.1.2}$$

are at

$$s = -1 \tag{3.1.3}$$

i.e., the left half of s-plane. Hence the system is stable.

3.2. Find and sketch the step response $c(t)$ of the system.
Solution: For step-response, we take input as unit-step function u(t)

$$C(s) = U(s).G(s) = \left[\frac{1}{s}\right]\left[\frac{1}{1 + 2s + s^2}\right] \tag{3.2.1}$$

$$= \frac{1}{s(1 + s)^2} \tag{3.2.2}$$

$$= \frac{1}{s} - \frac{1}{(1 + s)} - \frac{1}{(1 + s)^2} \tag{3.2.3}$$

Taking the inverse Laplace transform,

$$c(t) = L^{-1}\left[\frac{1}{s}\right] - L^{-1}\left[\frac{1}{1 + s}\right] - L^{-1}\left[\frac{1}{(1 + s)^2}\right] \tag{3.2.4}$$

$$= \left(1 - e^{-t} - te^{-t}\right) u(t) \tag{3.2.5}$$

The following code plots $c(t)$ in Fig. 3.2

```
codes/ee18btech11002/plot.py
```

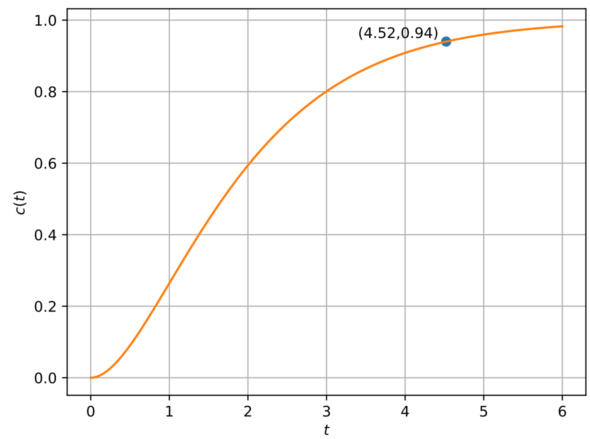

Fig. 3.2

3.3. Find the steady state response of the system using the final value theorem. Verify using 3.2.5
Solution: To know the steady response value of c(t), using final value theorem,

$$\lim_{t\to\infty} c(t) = \lim_{s\to 0} sC(s) \tag{3.3.1}$$

We get

$$\lim_{s\to 0} s\left(\frac{1}{s}\right)\left(\frac{1}{1 + s + s^2}\right) = \frac{1}{1 + 0 + 0} = 1 \tag{3.3.2}$$

Using 3.2.5,

$$\lim_{t\to\infty} c(t) = \lim_{t\to\infty} \left(1 - e^{-t} - te^{-t}\right) u(t) \tag{3.3.3}$$

$$= (1 - 0 - 0) = 1 \tag{3.3.4}$$

3.4. Find the time taken for the system output c(t) to reach 94% of its steady state value.
Solution: Now, 94% of 1 is 0.94, so we should now solve for a positive t such that

$$1 - e^{-t} - te^{-t} = 0.94 \tag{3.4.1}$$

The following code

```
codes/ee18btech11002/solution.py
```

provides the necessary solution as

$$t = 4.5228 \tag{3.4.2}$$

4 Routh Hurwitz Criterion

4.1 Routh Array

4.1.1. Generate the Routh array for the polynomial,

$$f(s) = s^7 + s^6 + 7s^5 + 14s^4 + 31s^3 + 73s^2 + 25s + 200 \tag{4.1.1.1}$$

Solution:

$$\begin{vmatrix} s^7 \\ s^6 \\ s^5 \end{vmatrix}\begin{vmatrix} 1 & 7 & 31 & 25 \\ 1 & 14 & 73 & 200 \\ -7 & -42 & -175 & 0 \end{vmatrix} \tag{4.1.1.2}$$

$$\begin{vmatrix} s^7 \\ s^6 \\ s^5 \\ s^4 \end{vmatrix}\begin{vmatrix} 1 & 7 & 31 & 25 \\ 1 & 14 & 73 & 200 \\ -7 & -42 & -175 & 0 \\ 8 & 48 & 200 & 0 \end{vmatrix} \tag{4.1.1.3}$$

$$\begin{vmatrix} s^7 \\ s^6 \\ s^5 \\ s^4 \\ s^3 \end{vmatrix}\begin{vmatrix} 1 & 7 & 31 & 25 \\ 1 & 14 & 73 & 200 \\ -7 & -42 & -175 & 0 \\ 8 & 48 & 200 & 0 \\ 0 & 0 & 0 & \end{vmatrix} \tag{4.1.1.4}$$

When such a case is encountered, we take the derivative of the expression formed the the coefficients above it i.e derivative of $8s^4+48s^2+200$.

$$\frac{d}{dx}(8s^4 + 48s^2 + 200) = 32s^3 + 96s$$

The coefficients of obtained expression are placed in the table.

$$\begin{vmatrix} s^7 \\ s^6 \\ s^5 \\ s^4 \\ s^3 \end{vmatrix}\begin{vmatrix} 1 & 7 & 31 & 25 \\ 1 & 14 & 73 & 200 \\ -7 & -42 & -175 & 0 \\ 8 & 48 & 200 & 0 \\ 32 & 96 & 0 & \end{vmatrix} \tag{4.1.1.5}$$

$$\begin{vmatrix} s^7 \\ s^6 \\ s^5 \\ s^4 \\ s^3 \\ s^2 \end{vmatrix}\begin{vmatrix} 1 & 7 & 31 & 25 \\ 1 & 14 & 73 & 200 \\ -7 & -42 & -175 & 0 \\ 8 & 48 & 200 & 0 \\ 32 & 96 & 0 & \\ 24 & 200 & 0 & \end{vmatrix} \tag{4.1.1.6}$$

$$\begin{vmatrix} s^7 \\ s^6 \\ s^5 \\ s^4 \\ s^3 \\ s^2 \\ s^1 \end{vmatrix}\begin{vmatrix} 1 & 7 & 31 & 25 \\ 1 & 14 & 73 & 200 \\ -7 & -42 & -175 & 0 \\ 8 & 48 & 200 & 0 \\ 32 & 96 & 0 & \\ 24 & 200 & 0 & \\ -170.67 & 0 & & \end{vmatrix} \tag{4.1.1.7}$$

$$\begin{vmatrix} s^7 \\ s^6 \\ s^5 \\ s^4 \\ s^3 \\ s^2 \\ s^1 \\ s^0 \end{vmatrix}\begin{vmatrix} 1 & 7 & 31 & 25 \\ 1 & 14 & 73 & 200 \\ -7 & -42 & -175 & 0 \\ 8 & 48 & 200 & 0 \\ 32 & 96 & 0 & \\ 24 & 200 & 0 & \\ -170.67 & 0 & & \\ 200 & & & \end{vmatrix} \tag{4.1.1.8}$$

So, the above one is the Routh-Hurwitz Table.

4.1.2. Find the number of roots of the polynomial in the right half of the s-plane.
Solution: The number of roots of the polynomial that are in the right half-plane is equal to the number of sign changes in the first column. From 4.1.1.8, the polynomial in (4.1.1.1) has 4 roots lie on right-side of Imaginary Axis.

4.1.3. Write a Python code for generating each stage of the Routh Table.
Solution: The following code

```
codes/ee18btech11014/
    Routh_Hurwitz_Stages.py
```

generates the various stages.

4.1.4. Find the roots of the polynomial in in (4.1.1.1) and verify that 4 roots are in the right half s-plane.
Solution: The following code generates the necessary roots.

```
codes/ee18btech11014/Roots.py
```

4.2 Marginal Stability

4.2.1. Consider a unity feedback system as shown in Fig. 4.2.1, with an integral compensator $\frac{k}{s}$ and open-loop transfer function

$$G(s) = \frac{1}{s^2 + 3s + 2} \tag{4.2.1.1}$$

where k greater than 0. Find its closed loop transfer function.
Solution: $\because H(s) = 1$ in Fig. 4.2.1, due to unity feedback, the transfer function is given by

$$\frac{Y(s)}{X(s)} = \frac{G(s)}{1 + G(s)H(s)} \tag{4.2.1.2}$$

$$\implies T(s) = \frac{k}{s^3 + 3s^2 + 2s} \tag{4.2.1.3}$$

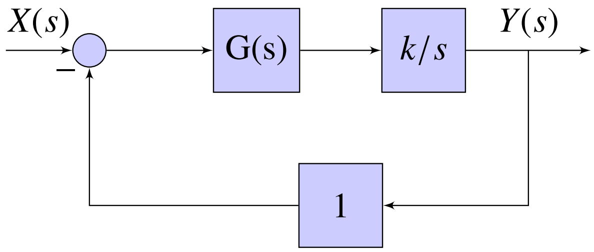

Fig. 4.2.1

4.2.2. Find the *characteristic* equation for $G(s)$.
Solution: The characteristic equation is

$$1 + G(s)H(s) = 0 \tag{4.2.2.1}$$

$$\implies 1 + \left[\frac{k}{s^3 + 3s^2 + 2s}\right] = 0 \tag{4.2.2.2}$$

$$\text{or, } s^3 + 3s^2 + 2s + k = 0 \tag{4.2.2.3}$$

4.2.3. Using the tabular method for the Routh hurwitz criterion, find $k > 0$ for which there are two poles of unity feedback system on $\jmath\omega$ axis.
Solution: This criterion is based on arranging the coefficients of characteristic equation into an array called Routh array. For any characteristic equation

$$q(s) = a_0 s^n + a_1 s^{n-1} + ... + a_{n-1} s + a_n = 0 \tag{4.2.3.1}$$

the Routh array can be constructed as

$$\begin{vmatrix} s^n \\ s^{n-1} \\ s^{n-2} \\ \vdots \end{vmatrix} \begin{vmatrix} a_0 & a_2 & a_4 & \cdots \\ a_1 & a_3 & a_5 & \cdots \\ b_1 & b_2 & b_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \cdots \end{vmatrix} \tag{4.2.3.2}$$

where

$$b_1 = \frac{a_1 a_2 - a_0 a_3}{a_1} \tag{4.2.3.3}$$

$$b_2 = \frac{a_1 a_4 - a_0 a_5}{a_1} \tag{4.2.3.4}$$

$$c_1 = \frac{b_1 a_3 - a_1 b_2}{b_1} \tag{4.2.3.5}$$

$$c_2 = \frac{b_1 a_5 - a_1 b_3}{b_1} \tag{4.2.3.6}$$

For poles to lie on imaginary axis any one entire row of hurwitz matrix should be zero. Constructing the routh array for the characteristic equation obtained in 4.2.2.1,

$$s^3 + 3s^2 + 2s + k = 0 \tag{4.2.3.7}$$

$$\begin{vmatrix} s^3 \\ s^2 \\ s^1 \\ s^0 \end{vmatrix} \begin{vmatrix} 1 & 2 \\ 3 & k \\ \frac{6-k}{3} & 0 \\ k & 0 \end{vmatrix} \tag{4.2.3.8}$$

For poles on $\jmath\omega$ axis any one of the row should be zero.

$$\therefore \frac{6-k}{3} = 0 \text{ or } k = 0 \tag{4.2.3.9}$$

$$\implies k = 6 \quad \because k > 0 \tag{4.2.3.10}$$

4.2.4. Repeat the above using the determinant method.
Solution: The *Routh matrix* can be expressed as

$$\mathbf{R} = \begin{pmatrix} a_0 & a_2 & a_4 & \cdots \\ a_1 & a_3 & a_5 & \cdots \\ 0 & a_0 & a_2 \cdots \\ 0 & a_1 & a_3 \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \cdots \end{pmatrix} \tag{4.2.4.1}$$

and the corresponding Routh determinants are

$$D_1 = |a_0| \tag{4.2.4.2}$$

$$D_2 = \begin{vmatrix} a_0 & a_2 \\ a_1 & a_3 \end{vmatrix} \tag{4.2.4.3}$$

$$D_3 = \begin{vmatrix} a_0 & a_2 & a_4 \\ a_1 & a_3 & a_5 \\ 0 & a_0 & a_2 \end{vmatrix} \tag{4.2.4.4}$$

$$\cdots \tag{4.2.4.5}$$

If at least any one of the Determinents are zero then the poles lie on imaginary axes. From (4.2.2.1),

$$D_1 = 1 \neq 0 \tag{4.2.4.6}$$

$$D2 = \begin{vmatrix} 1 & 2 \\ 3 & k \end{vmatrix} = k - 6 = 0 \implies k = 6 \tag{4.2.4.7}$$

4.2.5. Verify your answer using a python code for both the determinant method as well as the tabular method.
Solution: The following code

```
codes/ee18btech11005.py
```

provides the necessary soution.

- For the system to be stable all coefficients should lie on left half of s-plane. Because if any pole is in right half of s-plane then there will be a component in output that increases without bound,causing system to be unstable. All the coefficients in the characteristic equation should be positive.This is necessary condition but not sufficient.Because it may have poles on right half of s plane. Poles are the roots of the characteristic equation.
- A system is stable if all of its characteristic modes go to finite value as t goes to infinity.It is possible only if all the poles are on the left half of s plane. The characteristic equation should have negative roots only. So the first column should always be greater than zero.That means no sign changes.
- A system is unstable if its characteristic modes are not bounded. Then the characteristic equation will also have roots in the right side of s-plane.That means it has sign changes.

4.3 Stability

4.3.1. The characteristic equation of linear time invariant system is given by

$$\nabla(s) = s^4 + 3s^3 + 3s^2 + s + k = 0 \quad (4.3.1.1)$$

Find the condition for the system to be BIBO stable using the Routh Array.
solution

$$\nabla(s) = s^4 + 3s^3 + 3s^2 + s + k = 0 \quad (4.3.1.2)$$

The Routh hurwitz criterion:-

$$\begin{vmatrix} s^4 \\ s^3 \\ s^2 \\ s^1 \\ s^0 \end{vmatrix} \begin{vmatrix} 1 & 3 & k \\ 3 & 1 & 0 \\ \frac{8}{3} & k & 0 \\ \frac{\frac{8}{3}-3k}{\frac{8}{3}} & 0 & 0 \\ k & 0 & 0 \end{vmatrix} \quad (4.3.1.3)$$

From the above array, the given system is stable if

$$k > 0$$
$$\frac{\frac{8}{3} - 3k}{\frac{8}{3}} > 0 \quad (4.3.1.4)$$
$$\implies 0 < k < \frac{8}{9} \quad (4.3.1.5)$$

4.3.2. Modify the Python code in Problem 4.2.5 to verify your solution by choosing two different values of k.
Solution: The following code

```
codes/ee18btech11008.py
```

provides the necessary soution for $k = 0.5, 3$.

- $k = 0.5 < \frac{8}{9}$ has no sign changes in first column of its routh array.So the system is stable.
- $k = 3 > \frac{8}{9}$ has 2 sign changes in first column of its routh array.So the system is unstable.

5 State-Space Model

5.1 Controllability and Observabiity

5.1. State the general model of a state space system specifying the dimensions of the matrices and vectors.
Solution: The model is given by

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \quad (5.1.1)$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \quad (5.1.2)$$

5.2. Find the transfer function $\mathbf{H}(s)$ for the general system.
Solution: Taking Laplace transform on both sides we have the following equations

$$s\mathbf{I}\mathbf{X}(s) - \mathbf{x}(0) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}\mathbf{U}(s) \quad (5.2.1)$$
$$(s\mathbf{I} - \mathbf{A})\mathbf{X}(s) = \mathbf{B}\mathbf{U}(s) + \mathbf{x}(0) \quad (5.2.2)$$
$$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s) \quad (5.2.3)$$
$$+ (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{x}(0) \quad (5.2.4)$$

and

$$\mathbf{Y}(s) = \mathbf{C}\mathbf{X}(s) + D\mathbf{I}\mathbf{U}(s) \quad (5.2.5)$$

Substituting from (5.2.4) in the above,

$$\mathbf{Y}(s) = (\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + D\mathbf{I})\mathbf{U}(s)$$
$$+ \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{x}(0) \quad (5.2.6)$$

5.3. Find $H(s)$ for a SISO (single input single output) system.
Solution:

$$H(s) = \frac{Y(s)}{U(s)} = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + D\mathbf{I} \quad (5.3.1)$$

5.4. Given

$$H(s) = \frac{1}{s^3+3s^2+2s+1} \tag{5.4.1}$$
$$D = 0 \tag{5.4.2}$$
$$\mathbf{B} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{5.4.3}$$

find **A** and **C** such that the state-space realization is in *controllable canonical form*.
Solution:

$$\because \frac{Y(s)}{U(s)} = \frac{Y(s)}{V(s)} \times \frac{V(s)}{U(s)}, \tag{5.4.4}$$

letting

$$\frac{Y(s)}{V(s)} = 1, \tag{5.4.5}$$

results in

$$\frac{U(s)}{V(s)} = s^3+3s^2+2s+1 \tag{5.4.6}$$

giving

$$U(s) = s^3V(s) + 3s^2V(s) + 2sV(s) + V(s) \tag{5.4.7}$$

so the above equation can be written as

$$\begin{pmatrix} sV(s) \\ s^2V(s) \\ s^3V(s) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{pmatrix} \begin{pmatrix} V(s) \\ sV(s) \\ s^2V(s) \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} U \tag{5.4.8}$$

Letting

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{pmatrix} \tag{5.4.9}$$
$$\mathbf{X}_1 = \begin{pmatrix} sV(s) \\ s^2V(s) \\ s^3V(s) \end{pmatrix} \tag{5.4.10}$$
$$\mathbf{X} = \begin{pmatrix} V(s) \\ sV(s) \\ s^2V(s) \end{pmatrix}, \tag{5.4.11}$$

$$\mathbf{X}_1(s) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}U(s) \tag{5.4.12}$$
$$Y = \mathbf{C}\mathbf{X}_1(s) \tag{5.4.13}$$

where

$$\mathbf{C} = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \tag{5.4.14}$$

5.5. Obtain **A** and **C** so that the state-space realization in in *observable canonical form*.
Solution: Given that

$$H(s) = \frac{1}{s^3+3s^2+2s+1}, \tag{5.5.1}$$
$$\frac{Y(s)}{U(s)} = \frac{1}{s^3+3s^2+2s+1} \tag{5.5.2}$$
$$\implies Y(s) = -3s^{-1}Y(s) - 2s^{-2}Y(s) + s^{-3}\left(U(s) - Y(s)\right) \tag{5.5.3}$$

after some algebra.
let $Y = aU + X_1$
by comparing with equation 1.5.6 we get a=0 and

$$Y = X_1 \tag{5.5.4}$$

inverse laplace transform of above equation is

$$y = x_1 \tag{5.5.5}$$

so from above equation 1.5.6 and 1.5.7

$$X_1 = -3s^{-1}Y(s) - 2s^{-2}Y(s) + s^{-3}(U(s) - Y(s)) \tag{5.5.6}$$
$$sX_1 = -3Y(s) - 2s^{-1}Y(s) + s^{-2}(U(s) - Y(s)) \tag{5.5.7}$$

inverse laplace transform of above equation

$$\dot{x_1} = -3y + x_2 \tag{5.5.8}$$

where

$$X_2 = -2s^{-1}Y(s) + s^{-2}(U(s) - Y(s)) \tag{5.5.9}$$
$$sX_2 = -2Y(s) + s^{-1}(U(s) - Y(s)) \tag{5.5.10}$$

inverse laplace transform of above equation

$$\dot{x_2} = -2y + x_3 \tag{5.5.11}$$

where

$$X_3 = s^{-1}(U(s) - Y(s)) \tag{5.5.12}$$
$$sX_3 = U(s) - Y(s) \tag{5.5.13}$$

inverse laplace transform of above equation

$$\dot{x_3} = u - y \tag{5.5.14}$$

so we get four equations which are

$$x_1 = y \quad (5.5.15)$$
$$\dot{x_1} = -3y + x_2 \quad (5.5.16)$$
$$\dot{x_2} = -2y + x_3 \quad (5.5.17)$$
$$\dot{x_3} = u - y \quad (5.5.18)$$

sub $y = x_1$ in 1.5.19,1.5.20,1.5.21 we get

$$x_1 = y \quad (5.5.19)$$
$$\dot{x_1} = -3x_1 + x_2 \quad (5.5.20)$$
$$\dot{x_2} = -2x_1 + x_3 \quad (5.5.21)$$
$$\dot{x_3} = u - x_1 \quad (5.5.22)$$

so above equations can be written as

$$\begin{pmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \end{pmatrix} = \begin{pmatrix} -3 & 1 & 0 \\ -2 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} U \quad (5.5.23)$$

So

$$\mathbf{A} = \begin{pmatrix} -3 & 1 & 0 \\ -2 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix} \quad (5.5.24)$$

$$y = x_1 = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad (5.5.25)$$

$$\mathbf{C} = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \quad (5.5.26)$$

5.6. Find the eigenvaues of **A** and the poles of $H(s)$ using a python code.
Solution: The following code

```
codes/ee18btech11004.py
```

gives the necessary values. The roots are the same as the eigenvalues.

5.7. Theoretically, show that eigenvaues of **A** are the poles of $H(s)$.
Solution: As we know that the characteristic equation is det(sI-A)

$$\mathbf{sI} - \mathbf{A} = \begin{pmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{pmatrix} - \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{pmatrix} \quad (5.7.1)$$
$$= \begin{pmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 1 & 2 & s+3 \end{pmatrix} \quad (5.7.2)$$

therefore

$$det(sI - A) = s(s^2 + 3s + 2) + 1(1) \quad (5.7.3)$$
$$= s^3 + 3s^2 + 2s + 1 \quad (5.7.4)$$

so from equation 1.6.2 we can see that characteristic equation is equal to the denominator of the transfer function

5.2 *Second Order System*

5.2.1. Consider a state-variable model of a system

$$\begin{pmatrix} \dot{x_1} \\ \dot{x_2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\alpha & -2\beta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} r \quad (5.2.1.1)$$

$$y = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad (5.2.1.2)$$

where y is the output, and r is the input.

5.2.2. List the various state matrices in (5.2.1.1)

5.2.3. Find the the system transfer function $H(s)$.
Solution: From (**??**) and , (**??**), the transfer function for the state space model is

$$H(s) = C(sI - A)^{-1}B + D \quad (5.2.3.1)$$
$$= \frac{\begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} s+2\beta & 1 \\ -\alpha & s \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}}{s(s+2\beta)+\alpha} \quad (5.2.3.2)$$
$$= \frac{b_1(s+2\beta)+b_2}{s^2+2s\beta+\alpha} \quad (5.2.3.3)$$
$$\implies H(s) = \frac{b_1 s}{s^2+2s\beta+\alpha} + \frac{2b_1\beta+b_2}{s^2+2s\beta+\alpha} \quad (5.2.3.4)$$

5.2.4. Find the Damping ratio ζ and the Undamped natural frequency ω_n of the system.
Solution: Generally for a second order system the transfer function is given by 3.1.1.1

$$H(s) = \frac{\omega_n^2}{s^2 + 2s\zeta\omega_n + \omega_n^2} \quad (5.2.4.1)$$

Comparing the denominator of the above with (5.2.3.4),

$$2\zeta\omega_n = 2\beta, \quad (5.2.4.2)$$
$$\omega_n^2 = \alpha \quad (5.2.4.3)$$
$$\implies \zeta = \frac{\beta}{\sqrt{\alpha}}, \omega_n = \sqrt{\alpha} \quad (5.2.4.4)$$

5.2.5. Using Table 3.1.1, explain how the damping conditions depend upon α and β.

6 Nyquist Plot

6.1. The open loop transfer function of a unity feedback system is given by

$$G(s) = \frac{\pi e^{-0.25s}}{s} \tag{6.1.1}$$

6.2. Find $\text{Re}\{G(\jmath\omega)\}$ and $\text{Im}\{G(\jmath\omega)\}$.
Solution: From (6.1.1),

$$G(j\omega) = \frac{\pi}{\omega}(-\sin 0.25\omega - j\cos 0.25\omega) \tag{6.2.1}$$

$$\implies \text{Re}\{G(\jmath\omega)\} = \frac{\pi}{\omega}(-\sin 0.25\omega) \tag{6.2.2}$$

$$\text{Im}\{G(\jmath\omega)\} = \frac{\pi}{\omega}(-j\cos 0.25\omega) \tag{6.2.3}$$

6.3. Sketch the Nyquist plot.
Solution: The Nyquist plot is a graph of $\text{Re}\{G(\jmath\omega)\}$ vs $\text{Im}\{G(\jmath\omega)\}$. The following python code generates the Nyquist plot in Fig. 6.3

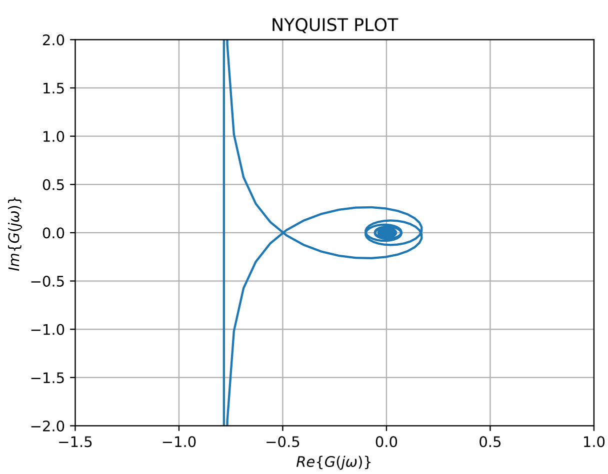

Fig. 6.3

6.4. Find the point at which the Nyquist plot of G(s) passes through the negative real axis
Solution: Nyquist plot cuts the negative real axis at ω for which

$$\angle G(\jmath\omega) = -\pi \tag{6.4.1}$$

From (6.1.1),

$$G(\jmath\omega) = \frac{\pi e^{-\frac{\jmath\omega}{4}}}{\jmath\omega} = \frac{\pi e^{-\jmath\left(\frac{\omega}{4}+\frac{\pi}{2}\right)}}{\omega} \tag{6.4.2}$$

$$\implies \angle G(\jmath\omega) = -\left(\frac{\omega}{4} + \frac{\pi}{2}\right) \tag{6.4.3}$$

From (6.4.3) and (6.4.1),

$$\frac{\omega}{4} + \frac{\pi}{2} = \pi \tag{6.4.4}$$

$$\implies \omega = 2\pi \tag{6.4.5}$$

Also, from (6.1.1),

$$\left|G(\jmath\omega)\right| = \frac{\pi}{|\omega|} \tag{6.4.6}$$

$$\implies \left|G(\jmath 2\pi)\right| = \frac{1}{2} \tag{6.4.7}$$

6.5. Use the Nyquist Stability criterion to determine if the system in (6.4.3) is stable.

Variable	Value	Description
Z	0	Poles of $\frac{G(s)}{1+G(s)H(s)}$ in right half of s plane
P	0	Poles of $G(s)H(s)$ in right half of s plane
N	0	No of clockwise encirclements of $G(s)H(s)$ about -1+j0 in the Nyquist plot

TABLE 6.5

Solution: Consider Table 6.5. According to the Nyquist stability criterion,

a) If the open-loop transfer function $G(s)$ has a zero pole of multiplicity l, then the Nyquist plot has a discontinuity at ω =0. During further analysis it should be assumed that the phasor travels l times clock-wise along a semicircle of infinite radius. After applying this rule, the zero poles should be neglected, i.e. if there are no other unstable poles, then the open-loop transfer function $G(s)$ should be considered stable.
b) If the open-loop transfer function $G(s)$ is stable, then the closed-loop system is unstable for any encirclement of the point -1. If the open-loop transfer function $G(s)$ is unstable, then there must be one counter clock-wise encirclement of -1 for each pole of $G(s)$ in

the right-half of the complex plane.

c) The number of surplus encirclements (N + P greater than 0) is exactly the number of unstable poles of the closed-loop system.

d) However, if the graph happens to pass through the point $-1+\jmath 0$, then deciding upon even the marginal stability of the system becomes difficult and the only conclusion that can be drawn from the graph is that there exist zeros on the $\jmath\omega$ axis.

From (6.1.1), $G(s)$ is stable since it has a single pole at $s = 0$. Further, from Fig. 6.3, the Nyquist plot doesnot encircle $s = -1$. From Theorem 6.5b, we may conclude that the system is stable.

7 Compensators

7.1. The Transfer function of Phase Lead Compensator is given by

$$D(s) = \frac{3(s + \frac{1}{3T})}{(s + \frac{1}{T})} \tag{7.1.1}$$

Find out the frequency (in rad/sec), at which $\angle D(j\omega)$ is maximum?

Solution: The basic requirement of the phase lead network is that all poles and zeros of the transfer function of the network must lie on negative real axis interlacing each other with a zero located as the nearest point to origin.
Substituting $s = j\omega$ in D(s), we get

$$D(j\omega) = \frac{3(j\omega + \frac{1}{3T})}{(j\omega + \frac{1}{T})} \tag{7.1.2}$$

The phase of this transfer function $\phi(\omega)$ is given by,

$$\phi(\omega) = \tan^{-1}(3\omega T) - \tan^{-1}(\omega T) \tag{7.1.3}$$

$\phi(\omega)$ has its maximum at ω_c Where $\phi'(\omega_c) = 0$,

$$\phi'(\omega_c) = 0 = \frac{3T}{1 + (3\omega_c T)^2} - \frac{T}{1 + (\omega_c T)^2} \tag{7.1.4}$$

After solving and Simplification , we have

$$\omega_c^2 T^2 = \frac{1}{3} \tag{7.1.5}$$

$$\omega_c = \sqrt{\frac{1}{3T^2}} \tag{7.1.6}$$

7.2. Verify your result through a plot.
Solution: The following plots the Phase value of the transfer function,

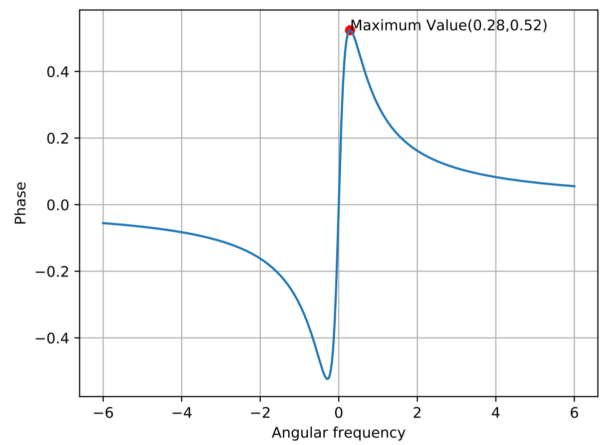

Fig. 7.2

Applications:

a) Phase lead Compensators can be used as High pass filters,Differentiators.
b) They are used to reduce steady state errors.
c) Increases Phase Margin , relative stability.

7.3. What is purpose of of a Phase Lead Compensator?

7.4. Through an example, show how the compensator in Problem 7.1 can be used in a control system.