Control Systems

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Contents

***Abstract*—This manual is an introduction to control systems based on GATE problems.Links to sample Python codes are available in the text.**

Download python codes using

```
svn co https://github.com/gadepall/school/trunk/
    control/codes
```

1 Stability

1.1 Second order System

1.1. Consider the following second order system with the transfer function

$$G(s) = \frac{1}{1+2s+s^2} \tag{1.1.1}$$

Is the system stable?

Solution: The poles of

$$G(s) = \frac{1}{1+2s+s^2} \tag{1.1.2}$$

are at

$$s = -1 \tag{1.1.3}$$

i.e., the left half of s-plane. Hence the system is stable.

1.2. Find and sketch the step response $c(t)$ of the system.

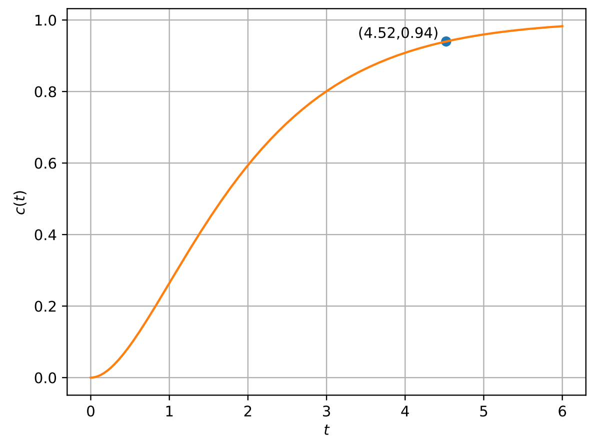

Fig. 1.2

Solution: For step-response, we take input as unit-step function u(t)

$$C(s) = U(s).G(s) = \left[\frac{1}{s}\right]\left[\frac{1}{1+2s+s^2}\right] \tag{1.2.1}$$

$$= \frac{1}{s(1+s)^2} \tag{1.2.2}$$

$$= \frac{1}{s} - \frac{1}{(1+s)} - \frac{1}{(1+s)^2} \tag{1.2.3}$$

Taking the inverse Laplace transform,

$$c(t) = L^{-1}\left[\frac{1}{s}\right] - L^{-1}\left[\frac{1}{1+s}\right] - L^{-1}\left[\frac{1}{(1+s)^2}\right] \tag{1.2.4}$$

$$= \left(1 - e^{-t} - te^{-t}\right)u(t) \tag{1.2.5}$$

The following code plots $c(t)$ in Fig. 1.2

```
codes/ee18btech11002/plot.py
```

1.3. Find the steady state response of the system using the final value theorem. Verify using 1.2.5

Solution: To know the steady response value

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of c(t), using final value theorem,

$$\lim_{t\to\infty} c(t) = \lim_{s\to 0} sC(s) \tag{1.3.1}$$

We get

$$\lim_{s\to 0} s\left(\frac{1}{s}\right)\left(\frac{1}{1+s+s^2}\right) = \frac{1}{1+0+0} = 1 \tag{1.3.2}$$

Using 1.2.5,

$$\lim_{t\to\infty} c(t) = \lim_{t\to\infty} \left(1 - e^{-t} - te^{-t}\right) u(t) \tag{1.3.3}$$

$$= (1 - 0 - 0) = 1 \tag{1.3.4}$$

1.4. Find the time taken for the system output c(t) to reach 94% of its steady state value.
Solution: Now, 94% of 1 is 0.94, so we should now solve for a positive t such that

$$1 - e^{-t} - te^{-t} = 0.94 \tag{1.4.1}$$

The following code

```
codes/ee18btech11002/solution.py
```

provides the necessary solution as

$$t = 4.5228 \tag{1.4.2}$$

2 Routh Hurwitz Criterion

2.1. Consider a unity feedback system as shown in Fig. 2.1, with an integral compensator $\frac{k}{s}$ and open-loop transfer function

$$G(s) = \frac{1}{s^2+3s+2} \tag{2.1.1}$$

where k greater than 0. Find its closed loop transfer function.

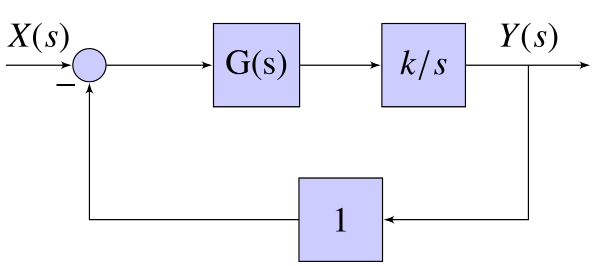

Fig. 2.1

Solution: $\because H(s) = 1$ in Fig. 2.1, due to unity feedback, the transfer function is given by

$$\frac{Y(s)}{X(s)} = \frac{G(s)}{1+G(s)H(s)} \tag{2.1.2}$$

$$\implies T(s) = \frac{k}{s^3+3s^2+2s} \tag{2.1.3}$$

2.2. Find the *characteristic* equation for $G(s)$.
Solution: The characteristic equation is

$$1 + G(s)H(s) = 0 \tag{2.2.1}$$

$$\implies 1 + \left[\frac{k}{s^3+3s^2+2s}\right] = 0 \tag{2.2.2}$$

$$\text{or, } s^3 + 3s^2 + 2s + k = 0 \tag{2.2.3}$$

2.3. Using the tabular method for the Routh hurwitz criterion, find $k > 0$ for which there are two poles of unity feedback system on jω axis.
Solution: This criterion is based on arranging the coefficients of characteristic equation into an array called Routh array. For any characteristic equation

$$q(s) = a_0 s^n + a_1 s^{n-1} + \ldots.. + a_{n-1}s + a_n = 0 \tag{2.3.1}$$

the Routh array can be constructed as

$$\left|\begin{matrix} s^n \\ s^{n-1} \\ s^{n-2} \\ \vdots \end{matrix}\right|\left|\begin{matrix} a_0 & a_2 & a_4 & \cdots & \\ a_1 & a_3 & a_5 & \cdots & \\ b_1 & b_2 & b_3 & \cdots & \\ \vdots & \vdots & \vdots & \ddots & \vdots\ldots \end{matrix}\right| \tag{2.3.2}$$

where

$$b_1 = \frac{a_1a_2 - a_0a_3}{a_1} \tag{2.3.3}$$

$$b_2 = \frac{a_1a_4 - a_0a_5}{a_1} \tag{2.3.4}$$

$$c_1 = \frac{b_1a_3 - a_1b_2}{b_1} \tag{2.3.5}$$

$$c_2 = \frac{b_1a_5 - a_1b_3}{b_1} \tag{2.3.6}$$

For poles to lie on imaginary axis any one entire row of hurwitz matrix should be zero. Constructing the routh array for the characteristic equation obtained in 2.2.1,

$$s^3 + 3s^2 + 2s + k = 0 \tag{2.3.7}$$

$$\begin{vmatrix} s^3 \\ s^2 \\ s^1 \\ s^0 \end{vmatrix}\begin{vmatrix} 1 & 2 \\ 3 & k \\ \frac{6-k}{3} & 0 \\ k & 0 \end{vmatrix} \tag{2.3.8}$$

For poles on $\jmath\omega$ axis any one of the row should be zero.

$$\therefore \frac{6-k}{3} = 0 \text{ or } k = 0 \tag{2.3.9}$$

$$\implies k = 6 \quad \because k > 0 \tag{2.3.10}$$

2.4. Repeat the above using the determinant method.
Solution: The *Routh matrix* can be expressed as

$$\mathbf{R} = \begin{pmatrix} a_0 & a_2 & a_4 & \cdots & \\ a_1 & a_3 & a_5 & \cdots & \\ 0 & a_0 & a_2 \cdots & & \\ 0 & a_1 & a_3 \cdots & & \\ \vdots & \vdots & \vdots & \ddots & \vdots \cdots \end{pmatrix} \tag{2.4.1}$$

and the corresponding Routh determinants are

$$D_1 = |a_0| \tag{2.4.2}$$

$$D_2 = \begin{vmatrix} a_0 & a_2 \\ a_1 & a_3 \end{vmatrix} \tag{2.4.3}$$

$$D_3 = \begin{vmatrix} a_0 & a_2 & a_4 \\ a_1 & a_3 & a_5 \\ 0 & a_0 & a_2 \end{vmatrix} \tag{2.4.4}$$

$$\cdots \tag{2.4.5}$$

If at least any one of the Determinents are zero then the poles lie on imaginary axes. From (2.2.1),

$$D_1 = 1 \neq 0 \tag{2.4.6}$$

$$D2 = \begin{vmatrix} 1 & 2 \\ 3 & k \end{vmatrix} = k - 6 = 0 \implies k = 6 \tag{2.4.7}$$

2.5. Verify your answer using a python code for both the determinant method as well as the tabular method. **Solution:** The following code

```
codes/ee18btech11005/ee18btech11005.py
```

provides the necessary soution.

3 Compensators

4 Nyquist Plot