Control Systems

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CONTENTS

Abstract—**This manual is an introduction to control systems based on GATE problems.Links to sample Python codes are available in the text.**

Download python codes using

svn co https://github.com/gadepall/school/trunk/
    control/codes

1 STABILITY

1.1 Second order System

1.1. Consider the following second order system with the transfer function

$$G(s) = \frac{1}{1 + 2s + s^2} \qquad (1.1.1)$$

Is the system stable?
Solution: The poles of

$$G(s) = \frac{1}{1 + 2s + s^2} \qquad (1.1.2)$$

are at

$$s = -1 \qquad (1.1.3)$$

i.e., the left half of s-plane. Hence the system is stable.

1.2. Find and sketch the step response $c(t)$ of the system.

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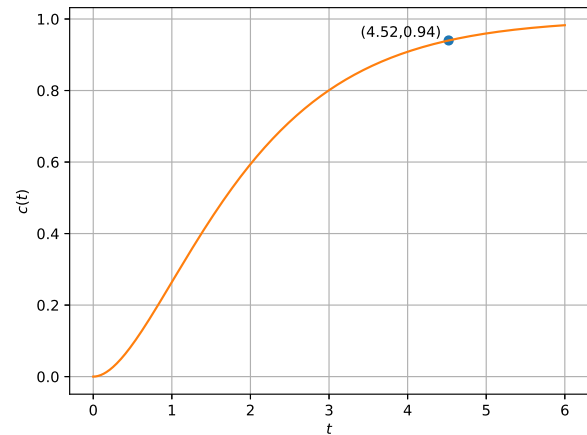
Fig. 1.2

Solution: For step-response, we take input as unit-step function u(t)

$$C(s) = U(s).G(s) = \left[\frac{1}{s}\right]\left[\frac{1}{1 + 2s + s^2}\right] \qquad (1.2.1)$$

$$= \frac{1}{s(1 + s)^2} \qquad (1.2.2)$$

$$= \frac{1}{s} - \frac{1}{(1 + s)} - \frac{1}{(1 + s)^2} \qquad (1.2.3)$$

Taking the inverse Laplace transform,

$$c(t) = L^{-1}\left[\frac{1}{s}\right] - L^{-1}\left[\frac{1}{1 + s}\right] - L^{-1}\left[\frac{1}{(1 + s)^2}\right] \qquad (1.2.4)$$

$$= (1 - e^{-t} - te^{-t})\,u(t) \qquad (1.2.5)$$

The following code plots $c(t)$ in Fig. 1.2

codes/ee18btech11002/plot.py

1.3. Find the steady state response of the system using the final value theorem. Verify using 1.2.5
Solution: To know the steady response value

of c(t), using final value theorem,

$$\lim_{t \to \infty} c(t) = \lim_{s \to 0} sC(s) \qquad (1.3.1)$$

We get

$$\lim_{s \to 0} s \left(\frac{1}{s}\right)\left(\frac{1}{1 + s + s^2}\right) = \frac{1}{1 + 0 + 0} = 1 \qquad (1.3.2)$$

Using 1.2.5,

$$\lim_{t \to \infty} c(t) = \lim_{t \to \infty} \left(1 - e^{-t} - te^{-t}\right) u(t) \qquad (1.3.3)$$

$$= (1 - 0 - 0) = 1 \qquad (1.3.4)$$

1.4. Find the time taken for the system output c(t) to reach 94% of its steady state value.
Solution: Now, 94% of 1 is 0.94, so we should now solve for a positive t such that

$$1 - e^{-t} - te^{-t} = 0.94 \qquad (1.4.1)$$

The following code

codes/ee18btech11002/solution.py

provides the necessary solution as

$$t = 4.5228 \qquad (1.4.2)$$

2 ROUTH HURWITZ CRITERION

2.1. Consider a unity feedback system as shown in Fig. 2.1, with an integral compensator $\frac{k}{s}$ and open-loop transfer function

$$G(s) = \frac{1}{s^2 + 3s + 2} \qquad (2.1.1)$$

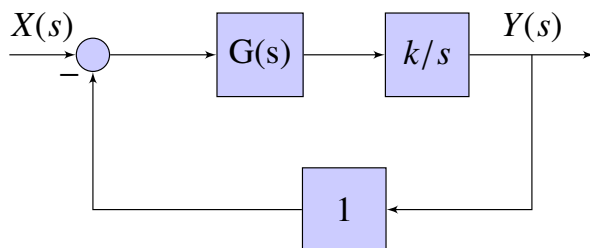where k greater than 0. Find its closed loop transfer function.

Fig. 2.1

Solution: $\because H(s) = 1$ in Fig. 2.1, due to unity

feedback, the transfer function is given by

$$\frac{Y(s)}{X(s)} = \frac{G(s)}{1 + G(s)H(s)} \qquad (2.1.2)$$

$$\implies T(s) = \frac{k}{s^3 + 3s^2 + 2s} \qquad (2.1.3)$$

2.2. Find the *characteristic* equation for $G(s)$.
Solution: The characteristic equation is

$$1 + G(s)H(s) = 0 \qquad (2.2.1)$$

$$\implies 1 + \left[\frac{k}{s^3 + 3s^2 + 2s}\right] = 0 \qquad (2.2.2)$$

$$\text{or, } s^3 + 3s^2 + 2s + k = 0 \qquad (2.2.3)$$

2.3. Using the tabular method for the Routh hurwitz criterion, find $k > 0$ for which there are two poles of unity feedback system on $j\omega$ axis.
Solution: This criterion is based on arranging the coefficients of characteristic equation into an array called Routh array. For any characteristic equation

$$q(s) = a_0 s^n + a_1 s^{n-1} + \ldots + a_{n-1}s + a_n = 0 \qquad (2.3.1)$$

the Routh array can be constructed as

$$\begin{vmatrix} s^n \\ s^{n-1} \\ s^{n-2} \\ \vdots \end{vmatrix} \begin{vmatrix} a_0 & a_2 & a_4 & \cdots \\ a_1 & a_3 & a_5 & \cdots \\ b_1 & b_2 & b_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \cdots \end{vmatrix} \qquad (2.3.2)$$

where

$$b_1 = \frac{a_1 a_2 - a_0 a_3}{a_1} \qquad (2.3.3)$$

$$b_2 = \frac{a_1 a_4 - a_0 a_5}{a_1} \qquad (2.3.4)$$

$$c_1 = \frac{b_1 a_3 - a_1 b_2}{b_1} \qquad (2.3.5)$$

$$c_2 = \frac{b_1 a_5 - a_1 b_3}{b_1} \qquad (2.3.6)$$

For poles to lie on imaginary axis any one entire row of hurwitz matrix should be zero. Constructing the routh array for the characteristic equation obtained in 2.2.1,

$$s^3 + 3s^2 + 2s + k = 0 \qquad (2.3.7)$$

$$\begin{array}{c|cc} s^3 & 1 & 2 \\ s^2 & 3 & k \\ s^1 & \frac{6-k}{3} & 0 \\ s^0 & k & 0 \end{array} \qquad (2.3.8)$$

For poles on $\jmath\omega$ axis any one of the row should be zero.

$$\therefore \frac{6-k}{3} = 0 \text{ or } k = 0 \qquad (2.3.9)$$

$$\implies k = 6 \quad \because k > 0 \qquad (2.3.10)$$

2.4. Repeat the above using the determinant method.

Solution: The *Routh matrix* can be expressed as

$$\mathbf{R} = \begin{pmatrix} a_0 & a_2 & a_4 & \cdots \\ a_1 & a_3 & a_5 & \cdots \\ 0 & a_0 & a_2 \cdots \\ 0 & a_1 & a_3 \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \cdots \end{pmatrix} \qquad (2.4.1)$$

and the corresponding Routh determinants are

$$D_1 = |a_0| \qquad (2.4.2)$$

$$D_2 = \begin{vmatrix} a_0 & a_2 \\ a_1 & a_3 \end{vmatrix} \qquad (2.4.3)$$

$$D_3 = \begin{vmatrix} a_0 & a_2 & a_4 \\ a_1 & a_3 & a_5 \\ 0 & a_0 & a_2 \end{vmatrix} \qquad (2.4.4)$$

$$\cdots \qquad (2.4.5)$$

If at least any one of the Determinents are zero then the poles lie on imaginary axes. From (2.2.1),

$$D_1 = 1 \neq 0 \qquad (2.4.6)$$

$$D2 = \begin{vmatrix} 1 & 2 \\ 3 & k \end{vmatrix} = k - 6 = 0 \implies k = 6 \quad (2.4.7)$$

2.5. Verify your answer using a python code for both the determinant method as well as the tabular method.

Solution: The following code

codes/ee18btech11005/ee18btech11005.py

provides the necessary soution.

- For the system to be stable all coefficients should lie on left half of s-plane. Because if any pole is in right half of s-plane then there will be a component in output that increases without bound,causing system to be unsta-ble. All the coefficients in the characteristic equation should be positive.This is necessary condition but not sufficient.Because it may have poles on right half of s plane. Poles are the roots of the characteristic equation.

- A system is stable if all of its characteristic modes go to finite value as t goes to infin-ity.It is possible only if all the poles are on the left half of s plane. The characteristic equation should have negative roots only. So the first column should always be greater than zero.That means no sign changes.

- A system is unstable if its characteristic modes are not bounded. Then the charac-teristic equation will also have roots in the right side of s-plane.That means it has sign changes.

3 STATE-SPACE MODEL

3.1. State the general model of a state space system specifying the dimensions of the matrices and vectors.

Solution: The model is given by

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \qquad (3.1.1)$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \qquad (3.1.2)$$

with parameters listed in Table 3.1.

Variable	Size	Description
$\mathbf{u}$	$p \times 1$	input(control) vector
$\mathbf{y}$	$q \times 1$	output vector
$\mathbf{x}$	$n \times 1$	state vector
$\mathbf{A}$	$n \times n$	state or system matrix
$\mathbf{B}$	$n \times p$	input matrix
$\mathbf{C}$	$q \times n$	output matrix
$\mathbf{D}$	$q \times p$	feedthrough matrix

TABLE 3.1

3.2. Find the transfer function $\mathbf{H}(s)$ for the general system.

Solution: Taking Laplace transform on both

sides we have the following equations

$$s\mathbf{I}X(s) - x(0) = \mathbf{A}X(s) + \mathbf{B}U(s)$$
(3.2.1)

$$(s\mathbf{I} - \mathbf{A})X(s) = \mathbf{B}U(s) + x(0)$$
(3.2.2)

$$X(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}U(s) + (s\mathbf{I} - \mathbf{A})^{-1}x(0)$$
(3.2.3)

and

$$Y(s) = \mathbf{C}X(s) + \mathbf{D}IU(s)$$
(3.2.4)

Substituting from (3.2.3) in the above,

$$Y(s) = (\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}I)U(s)$$
$$+ \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}x(0) \quad (3.2.5)$$

3.3. Find $H(s)$ for a SISO (single input single output) system.
Solution:

$$H(s) = \frac{Y(s)}{U(s)} = C(sI - A)^{-1}B + DI \quad (3.3.1)$$

3.4. Given

$$H(s) = \frac{1}{s^3 + 3s^2 + 2s + 1}$$
(3.4.1)

$$D = 0$$
(3.4.2)

$$\mathbf{B} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$
(3.4.3)

find $\mathbf{A}$ and $\mathbf{C}$.
Solution:

$$\because \frac{Y(s)}{U(s)} = \frac{Y(s)}{V(s)} \times \frac{V(s)}{U(s)},$$
(3.4.4)

letting

$$\frac{Y(s)}{V(s)} = 1,$$
(3.4.5)

results in

$$\frac{U(s)}{V(s)} = s^3 + 3s^2 + 2s + 1$$
(3.4.6)

giving

$$U(s) = s^3V(s) + 3s^2V(s) + 2sV(s) + V(s)$$
(3.4.7)

so equation 0.1.13 can be written as

$$\begin{pmatrix} sV(s) \\ s^2V(s) \\ s^3V(s) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{pmatrix} \begin{pmatrix} V(s) \\ s(s) \\ s^2V(s) \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} U$$
(3.4.8)

So

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{pmatrix}$$
(3.4.9)

$$Y = X_1(s) = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} V(s) \\ sV(s) \\ s^2V(s) \end{pmatrix}$$
(3.4.10)

$$\mathbf{C} = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$$
(3.4.11)

3.5. Find the eigenvaues of $\mathbf{A}$ and the roots of the characteristic equation of $H(s)$ using a python code.
Solution:

4 COMPENSATORS

5 NYQUIST PLOT