Natural Deduction in Computer Science

1. Introduction to Natural Deduction

$Natural deduction involves a collection of proof rules$ $Given a set of premises and a conclusion (w$

- **Premises:** Assumed to be true
- **Sequent:** Represents the relationship between premises and the conclusion

2. Basic Proof Rules

And Introduction ($\wedge_i$)**:** $\frac{p \quad q}{p \wedge q}$

0. **And Elimination** ($\wedge_e$)**:** $p \wedge q \vdash p$ and $p \wedge q \vdash q$
0. **Double Negation:** $\neg\neg p$ introduction and elimination
0. **Implies Elimination (Modus Ponens):** $p, p \to q \vdash q$
0. **Implies Elimination (Modus Tollens):** $\neg q, p \to q \vdash \neg p$
0. **OR Introduction** ($\vee_i$)**:** $p \vdash p \vee q$ and $q \vdash p \vee q$
0. **Law of Excluded Middle:** $(p \vee \neg p)$

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3. Semantics and Proof Theory

[cite$_start$]

- **Semantic Entailment** ($\models$)**:** $\Gamma \models \phi$ means if all Γ evaluate to true, so does ϕ
- **Soundness:** If something is provable, it is semantically true $(\Gamma \vdash \phi \Rightarrow \Gamma \models \phi)$
[cite$_start$]
- **Completeness:** If something is semantically true, it is provable $(\Gamma \models \phi \Rightarrow \Gamma \vdash \phi)$
- **Provably Equivalent:** Two formulas are provably equivalent if they can be proven from each other using inference rules

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4. Normal Forms

- **Conjunctive Normal Form (CNF):** A conjunction of disjunctions of literals $F = \bigwedge_{i=1}^{n} \bigvee_{j=1}^{m} L_{i,j}$

Disjunctive Normal Form (DNF): A disjunction of conjunctions of literals $F = \bigvee_{i=1}^{m} \bigwedge_{j=1}^{n} L_{i,j}$
|**Conversion to CNF:**

Replace $p \to q$ with $\neg p \lor q$
Use De Morgan's laws to move negations inward: $\neg(p \land q) \equiv \neg p \lor \neg q$|
Distribute $\lor$ over $\land$

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5. Horn Formulas

$]AformulaF isa$**Horn Formula**$ifitisinCNFandeverydisjunctioncontainsatmostonepositiveliteral$

Satisfiability (SAT): Determining if a Horn formula is SAT takes at most $O(n^2)$ steps

Algorithm: Mark literals as true. $Ifaclause(P_1 \land P_2 \cdots \land P_m) \to Q$ has all P_i marked, mark Q $Ifacontradiction(P_1 \land \cdots \land P_m) \to \bot$ is reached, it is **Unsat**|

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6. Resolution

Let C_1, C_2 be clauses $]If$p $\in C_1$ and $\neg p \in C_2$, the **resolvent** is:

$$R = (C_1 \setminus \{p\}) \cup (C_2 \setminus \{\neg p\})$$

[cite$_start$]**Soundness of Resolution:**$Ifanemptyclause(\Box)$ is found in the set of resolvents, the formula F is **Unsat**