Black-Scholes-Merton Model

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CHAPTER 1

Black-Scholes-Merton model

In the field of financial mathematics, the Black-Scholes-Merton model stands as a cornerstone for option pricing theory. Developed by Fischer Black, Myron Scholes, and Robert Merton, this model provides a systematic framework for determining the theoretical price of European-style options. At the heart of the model lies the concept of hedging, which aims to construct a risk-free portfolio through the use of the underlying asset and a risk-free bond. A crucial aspect of the Black-Scholes-Merton model is the application of "Greeks"—the partial derivatives of the option price with respect to various parameters. Key Greeks include Delta, which measures sensitivity to changes in the underlying asset's price; Gamma, which assesses the rate of change of Delta; and Vega, which indicates sensitivity to volatility. These measures enable traders and risk managers to understand and mitigate the risks associated with holding options, making the Black-Scholes-Merton model an essential tool in modern finance. The Black-Scholes-Merton model is a mathematical framework for pricing European-style options. It was developed by Fischer Black, Myron Scholes, and Robert Merton in the early 1970s.

1.0.1. Assumptions. The model is based on several key assumptions:

(1) The stock price follows a geometric Brownian motion with constant drift and volatility.

(2) There are no arbitrage opportunities.

(3) The market is frictionless (no transaction costs or taxes).

(4) The risk-free interest rate is constant and known.

(5) The stock pays no dividends during the option's life.

1.0.2. Stock Price Dynamics. The price of the underlying stock S_t is assumed to follow the stochastic differential equation (SDE):

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \tag{1.1}$$

where μ is the drift rate, σ is the volatility, and W_t is a standard Wiener process.

1.1. Derivation of the Black-Scholes Partial Differential Equation

To derive the Black-Scholes PDE, we start with a European call option whose price $C(S,t)$ is a function of the stock price S and time t.

1.1.1. Ito's Lemma. Applying Ito's Lemma to $C(S,t)$, we get:

$$dC = \frac{\partial C}{\partial t}dt + \frac{\partial C}{\partial S}dS + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}dS^2. \tag{1.2}$$

Substitute $dS = \mu S dt + \sigma S dW_t$ and $dS^2 = \sigma^2 S^2 dt$:

$$dC = \frac{\partial C}{\partial t}dt + \frac{\partial C}{\partial S}(\mu S dt + \sigma S dW_t) + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}\sigma^2 S^2 dt. \tag{1.3}$$

Simplifying, we obtain:

$$dC = \left(\frac{\partial C}{\partial t} + \mu S\frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 C}{\partial S^2}\right)dt + \sigma S\frac{\partial C}{\partial S}dW_t. \tag{1.4}$$

1.1.2. Risk-Neutral Valuation. To eliminate the stochastic term, construct a risk-free portfolio by taking a long position in the option and a short position in $\Delta = \frac{\partial C}{\partial S}$ shares of the stock. The portfolio value Π is:

$$\Pi = C - \Delta S. \tag{1.5}$$

The change in the portfolio value is:

$$d\Pi = dC - \Delta dS. \tag{1.6}$$

Substitute the expressions for dC and dS, and set the stochastic term to zero:

$$d\Pi = \left(\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 C}{\partial S^2}\right)dt. \tag{1.7}$$

Since the portfolio is risk-free, it must earn the risk-free rate r:

$$d\Pi = r\Pi dt. \tag{1.8}$$

Equating the two expressions for $d\Pi$:

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 C}{\partial S^2} = r(C - S\frac{\partial C}{\partial S}). \tag{1.9}$$

Rearranging terms, we obtain the Black-Scholes partial differential equation (PDE):

$$\frac{\partial C}{\partial t} + rS\frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 C}{\partial S^2} - rC = 0. \tag{1.10}$$

1.2. Black-Scholes Formula for European Call and Put Options

The solution to the Black-Scholes PDE for a European call option is given by:

$$C(S_0,t) = S_0 N(d_1) - Ke^{-r(T-t)}N(d_2), \tag{1.11}$$

where:

$$d_1 = \frac{\ln(S_0/K) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}, \tag{1.12}$$

$$d_2 = d_1 - \sigma\sqrt{T-t}, \tag{1.13}$$

and $N(x)$ is the cumulative distribution function of the standard normal distribution.

The price of a European put option is given by:

$$P(S_0, t) = Ke^{-r(T-t)}N(-d_2) - S_0 N(-d_1). \tag{1.14}$$

1.3. The Greek Letters

The Greek letters (Greeks) are sensitivities of the option price to various parameters. They are important for risk management and hedging strategies.

1.3.1. Delta (Δ). Delta measures the sensitivity of the option price to changes in the price of the underlying asset:

$$\Delta = \frac{\partial C}{\partial S}. \tag{1.15}$$

For a European call option:

$$\Delta_{\text{call}} = N(d_1). \tag{1.16}$$

For a European put option:

$$\Delta_{\text{put}} = N(d_1) - 1. \tag{1.17}$$

1.3.2. Gamma (Γ). Gamma measures the sensitivity of delta to changes in the price of the underlying asset:

$$\Gamma = \frac{\partial \Delta}{\partial S} = \frac{\partial^2 C}{\partial S^2}. \tag{1.18}$$

For both call and put options:

$$\Gamma = \frac{N'(d_1)}{S\sigma\sqrt{T-t}}, \tag{1.19}$$

where $N'(d_1)$ is the probability density function of the standard normal distribution evaluated at d_1.

1.3.3. Vega (ν). Vega measures the sensitivity of the option price to changes in the volatility of the underlying asset:

$$\nu = \frac{\partial C}{\partial \sigma}. \tag{1.20}$$

For both call and put options:

$$\nu = S_0\sqrt{T-t}N'(d_1). \tag{1.21}$$

1.3.4. Theta (Θ). Theta measures the sensitivity of the option price to the passage of time (time decay):

$$\Theta = \frac{\partial C}{\partial t}. \tag{1.22}$$

For a European call option:

$$\Theta_{\text{call}} = -\frac{S_0 \sigma N'(d_1)}{2\sqrt{T-t}} - rKe^{-r(T-t)}N(d_2). \tag{1.23}$$

For a European put option:

$$\Theta_{\text{put}} = -\frac{S_0 \sigma N'(d_1)}{2\sqrt{T-t}} + rKe^{-r(T-t)}N(-d_2). \tag{1.24}$$

1.3.5. Rho (ρ). Rho measures the sensitivity of the option price to changes in the risk-free interest rate:

$$\rho = \frac{\partial C}{\partial r}. \tag{1.25}$$

For a European call option:

$$\rho_{\text{call}} = K(T-t)e^{-r(T-t)}N(d_2). \tag{1.26}$$

For a European put option:

$$\rho_{\text{put}} = -K(T-t)e^{-r(T-t)}N(-d_2). \tag{1.27}$$

Remark 1.1. The Black-Scholes-Merton model is a cornerstone of modern financial theory, providing a theoretical framework for pricing European options. Understanding the Greeks is essential for managing the risks associated with options trading.

Exercises

(1) **Black-Scholes-Merton Model**

 (i). **Model Assumptions:** List and explain the key assumptions of the Black-Scholes-Merton model. How do these assumptions impact the applicability of the model in real-world scenarios?

 (ii). **Stock Price Dynamics:** The stock price S_t follows a geometric Brownian motion described by the SDE $dS_t = \mu S_t dt + \sigma S_t dW_t$. Define each term in this equation and explain its significance.

 (iii). **Black-Scholes PDE:** Derive the Black-Scholes partial differential equation for a European call option from the stock price SDE using Ito's Lemma and the concept of a risk-free portfolio.

(2) **Option Pricing with the Black-Scholes Formula**

 (i). **European Call Option:** Given the following parameters: $S_0 = 100$, $K = 95$, $r = 0.05$, $\sigma = 0.2$, and $T = 1$ year, calculate the price of a European call option using the Black-Scholes formula.

(ii). **European Put Option:** Using the same parameters as in the previous question, calculate the price of a European put option using the Black-Scholes formula.

(iii). **Put-Call Parity:** Verify the put-call parity relationship for the prices obtained in the previous two questions. Explain the importance of put-call parity in options pricing.

(3) **The Greek Letters**

1.3.6. Delta (Δ).

(i). **Definition and Calculation:** Define Delta (Δ) for an option. Using the parameters from the previous section, calculate the Delta for the European call option.

(ii). **Interpretation:** Explain the economic interpretation of Delta. How does Delta change as the stock price increases or decreases?

1.3.7. Gamma (Γ).

(i). **Definition and Calculation:** Define Gamma (Γ). Using the parameters from the previous section, calculate the Gamma for the European call option.

(ii). **Interpretation:** Explain the economic significance of Gamma. How does Gamma help in managing the risks associated with Delta hedging?

1.3.8. Vega (ν).

(a) **Definition and Calculation:** Define Vega (ν). Using the parameters from the previous section, calculate the Vega for the European call option.

(b) **Interpretation:** Discuss how Vega reflects the option's sensitivity to volatility. Why is Vega important for traders and risk managers?

1.3.9. Theta (Θ).

(i). **Definition and Calculation:** Define Theta (Θ). Using the parameters from the previous section, calculate the Theta for the European call option.

(ii). **Time Decay:** Explain the concept of time decay in options pricing. How does Theta change as the option approaches its expiration date?

1.3.10. Rho (ρ).

(i). **Definition and Calculation:** Define Rho (ρ). Using the parameters from the previous section, calculate the Rho for the European call option.

(ii). **Interest Rate Sensitivity:** Discuss how changes in the risk-free interest rate affect option prices. Why is Rho an important Greek for options pricing?

1.4. Advanced Questions

(i). **Risk-Neutral Valuation:** Explain the concept of risk-neutral valuation and its role in the derivation of the Black-Scholes formula. How does this concept simplify the pricing of derivatives?

(ii). **Implied Volatility:** Define implied volatility. How can the Black-Scholes model be used to determine the implied volatility of an option? Discuss its significance in the context of options trading.

(iii). **Volatility Smiles:** Describe the phenomenon of the volatility smile. What does it indicate about market perceptions and the assumptions of the Black-Scholes model?

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