Stochastic Calculus

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Contents

Chapter 1. Stochastic Calculus 1
1.1. Ito's Lemma 1
1.2. Stochastic Differential Equations (SDEs) 2
1.3. Applications in Finance 2
1.4. Exercises 3

References 5

CHAPTER 1

Stochastic Calculus

Itô's Lemma and stochastic differential equations (SDEs) form the bedrock of stochastic calculus, playing a pivotal role in modeling and analyzing systems subject to random fluctuations. Itô's Lemma provides a powerful tool for differentiating stochastic functions, allowing us to extend traditional calculus to handle stochastic processes such as Brownian motion. By accounting for the randomness inherent in these processes, Itô's Lemma enables the derivation of differential equations that accurately capture the dynamics of complex systems. Stochastic differential equations, in turn, offer a framework for describing how quantities evolve over time in the presence of random noise. These equations find wide-ranging applications in fields such as finance, physics, and engineering, where they serve as essential tools for modeling phenomena influenced by uncertain factors. This introduction sets the stage for exploring the rich theory of stochastic calculus and its practical implications in various domains. A stochastic process is a collection of random variables indexed by time. In finance, these processes are often used to model the evolution of asset prices. One of the most important stochastic processes in finance is the Wiener process (or standard Brownian motion), denoted by W_t.

1.0.1. Wiener Process (Brownian Motion). A Wiener process W_t has the following properties:

(1) $W_0 = 0$

(2) W_t has independent increments

(3) $W_t - W_s \sim \mathcal{N}(0, t-s)$ for $0 \leq s < t$

(4) W_t has continuous paths

1.1. Ito's Lemma

Ito's Lemma is a fundamental result in stochastic calculus, which provides the differential of a function of a stochastic process.

1.1.1. Statement of Ito's Lemma. Let X_t be an Itô process given by

$$dX_t = \mu(X_t, t)dt + \sigma(X_t, t)dW_t, \tag{1.1}$$

where μ and σ are functions of X_t and t. If $f(X_t, t)$ is a twice continuously differentiable function, then $f(X_t, t)$ follows the SDE:

$$df(X_t, t) = \left(\frac{\partial f}{\partial t} + \mu\frac{\partial f}{\partial X} + \frac{1}{2}\sigma^2\frac{\partial^2 f}{\partial X^2}\right)dt + \sigma\frac{\partial f}{\partial X}dW_t. \tag{1.2}$$

1.1.2. Example: Log Transformation. Consider X_t following the geometric Brownian motion:

$$dX_t = \mu X_t dt + \sigma X_t dW_t. \tag{1.3}$$

Let $f(X_t) = \ln(X_t)$. Applying Ito's Lemma:

$$\frac{\partial f}{\partial X} = \frac{1}{X},$$
$$\frac{\partial^2 f}{\partial X^2} = -\frac{1}{X^2}.$$

Therefore,

$$df(X_t) = \left(\frac{1}{X_t}\mu X_t + \frac{1}{2}\sigma^2 X_t^2 \left(-\frac{1}{X_t^2}\right)\right) dt + \frac{1}{X_t}\sigma X_t dW_t = \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma dW_t. \tag{1.4}$$

1.2. Stochastic Differential Equations (SDEs)

SDEs are differential equations in which one or more terms are stochastic processes. They are used to model systems affected by random noise.

1.2.1. Basic Form of an SDE. A simple SDE has the form:

$$dX_t = \mu(X_t, t)dt + \sigma(X_t, t)dW_t, \tag{1.5}$$

where $\mu(X_t, t)$ is the drift term and $\sigma(X_t, t)$ is the diffusion term.

1.2.2. Solution of SDEs. The solution of an SDE can often be found using methods such as:

- **Analytical Methods:** Exact solutions using integration, as in the case of the geometric Brownian motion.
- **Numerical Methods:** Approximate solutions using methods like the Euler-Maruyama method or the Milstein method.

1.2.3. Example: Geometric Brownian Motion. The SDE for geometric Brownian motion is given by:

$$dS_t = \mu S_t dt + \sigma S_t dW_t. \tag{1.6}$$

This SDE models the price of a stock, where μ is the drift rate and σ is the volatility. The solution is:

$$S_t = S_0 \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W_t\right). \tag{1.7}$$

1.3. Applications in Finance

SDEs and Ito's Lemma are widely used in financial modeling. Some common applications include:

1.3.1. Black-Scholes Model. The Black-Scholes model for option pricing uses the geometric Brownian motion to model stock prices. The SDE for the stock price S_t is:

$$dS_t = \mu S_t dt + \sigma S_t dW_t. \tag{1.8}$$

Ito's Lemma is used to derive the Black-Scholes partial differential equation (PDE) for the price of a European call option.

1.3.2. Interest Rate Models. Models such as the Vasicek and Cox-Ingersoll-Ross (CIR) models use SDEs to describe the evolution of interest rates over time. For example, the Vasicek model is given by:

$$dr_t = \alpha(\beta - r_t)dt + \sigma dW_t. \tag{1.9}$$

1.3.3. Stochastic Volatility Models. Stochastic volatility models, such as the Heston model, use SDEs to describe the dynamics of both asset prices and their volatility. These models provide a more realistic description of market behavior compared to constant volatility models.

Remark 1.1. Ito's Lemma and stochastic differential equations are crucial tools in stochastic calculus, providing a framework for modeling and analyzing systems influenced by randomness. Their applications in finance, particularly in modeling asset prices and interest rates, demonstrate their importance in both theoretical and practical contexts.

1.4. Exercises

(1) **Basic Stochastic Integration**. Compute the Itó integral

$$\int_0^T W_t \, dW_t$$

where W_t is a standard Brownian motion over the interval $[0, T]$.

(2) **Stochastic Differential Equation**. Solve the stochastic differential equation (SDE)

$$dX_t = \mu X_t \, dt + \sigma X_t \, dW_t$$

where μ and σ are constants, and W_t is a standard Brownian motion.

(3) **Itó's Lemma Application**. Using Itó's lemma, find the differential $df(W_t)$ where $f(x) = e^x$ and W_t is a standard Brownian motion.

(4) **Quadratic Variation**. Show that the quadratic variation of a standard Brownian motion W_t over $[0, T]$ is T, i.e.,

$$[W]_T = T.$$

(5) **Integration by Parts**. Use the integration by parts formula for Itô integrals to compute

$$\int_0^T W_t \, dW_t.$$

(6) **Martingale Property**. Prove that $X_t = e^{-\frac{1}{2}t + W_t}$ is a martingale, where W_t is a standard Brownian motion.

(7) **Geometric Brownian Motion**. Solve the SDE for geometric Brownian motion

$$dS_t = \mu S_t \, dt + \sigma S_t \, dW_t,$$

where μ and σ are constants, and S_0 is the initial value.

(8) **Ornstein-Uhlenbeck Process**. Solve the SDE for the Ornstein-Uhlenbeck process

$$dX_t = \theta(\mu - X_t) \, dt + \sigma \, dW_t,$$

where θ, μ, σ are constants, and W_t is a standard Brownian motion.

(9) **Stochastic Integral Representation**. Given a standard Brownian motion W_t, express $W_t^2 - t$ as a stochastic integral.

(10) **Expected Value of Itô Integral**. Show that the expected value of the Itô integral

$$\mathbb{E}\left[\int_0^T W_t \, dW_t\right]$$

is zero, where W_t is a standard Brownian motion.

(11) **Properties of Brownian Motion**. Prove that for a standard Brownian motion W_t, the increments $W_{t_2} - W_{t_1}$ and $W_{t_4} - W_{t_3}$ are independent if $0 \leq t_1 < t_2 \leq t_3 < t_4$.

(12) **Distribution of Brownian Motion**. Show that $W_t \sim \mathcal{N}(0, t)$, where W_t is a standard Brownian motion and $\mathcal{N}(0, t)$ denotes a normal distribution with mean 0 and variance t.

Martingales. Prove that $W_t^2 - t$ is a martingale, where W_t is a standard Brownian motion.

(13) **Geometric Brownian Motion** Consider the geometric Brownian motion defined by the SDE

$$dS_t = \mu S_t \, dt + \sigma S_t \, dW_t,$$

where μ and σ are constants, and W_t is a standard Brownian motion. Solve this SDE and find the explicit form of S_t.

(14) **Black-Scholes Equation** Derive the Black-Scholes partial differential equation for the price of a European call option on a non-dividend-paying stock, given that the stock price S_t follows a geometric Brownian motion.

(15) {bf Itó's Lemma for Functions of Two Variables}. Use Itô's lemma to find the differential $df(S_t, t)$ for $f(S_t, t) = \ln(S_t) + t$, where S_t follows the SDE

$$dS_t = \mu S_t \, dt + \sigma S_t \, dW_t.$$

(16) **Expected Value of a Stochastic Integral**. Compute the expected value $\mathbb{E}\left[\int_0^T S_t \, dW_t\right]$, where S_t follows the SDE

$$dS_t = \mu S_t \, dt + \sigma S_t \, dW_t.$$

(17) **Hedging in the Black-Scholes Model** Explain how to construct a delta-hedged portfolio for a European call option and derive the delta of the option using the Black-Scholes model.

(18) **Brownian Bridge** Define the Brownian bridge B_t on $[0, T]$ by $B_t = W_t - \frac{t}{T} W_T$, where W_t is a standard Brownian motion. Show that B_t has mean 0 and covariance $\mathrm{Cov}(B_s, B_t) = s(T - t)/T$ for $0 \leq s \leq t \leq T$.

(19) **Mean-Reverting Process**. Solve the SDE for the Ornstein-Uhlenbeck process

$$dX_t = \theta(\mu - X_t) \, dt + \sigma \, dW_t,$$

where θ, μ, σ are constants, and W_t is a standard Brownian motion. Show that X_t is a mean-reverting process.

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