Numerical Methods

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CHAPTER 1

Numerical Methods

In mathematical finance, elegant analytical solutions aren't always achievable for complex financial models. This is where the power of numerical methods comes into play. By leveraging techniques to approximate solutions to differential equations, we gain valuable insights into financial instruments and market behavior. This section will explore the application of numerical solutions for differential equations in mathematical finance. We'll delve into the limitations of analytical solutions and how numerical methods like the finite difference method or Monte Carlo simulations bridge the gap. We'll see how these techniques are used to price options, manage risk through hedging strategies, and analyze the behavior of various financial products under different market scenarios.

Definition 1.1 (Ordinary Differential Equations (ODEs)). An ordinary differential equation (ODE) is an equation that contains one or more functions of one independent variable and its derivatives.

Definition 1.2 (Initial Value Problems (IVPs)). An initial value problem involves solving an ODE given an initial condition at a specific point. For example:

$$y'(t) = f(t, y(t)), \quad y(t_0) = y_0. \tag{1.1}$$

1.0.1. Euler's Method. Euler's method is a simple numerical method for solving IVP and is given below. It approximates the solution by advancing in small steps of size h:

$$y_{n+1} = y_n + hf(t_n, y_n). \tag{1.2}$$

Algorithm 1 Euler Method

```
Input: Initial value y_0, step size h, number of steps N
Output: Approximation of solution at each step y_1, y_2, ..., y_N
t ← 0
y ← y_0
for i ← 1 to N do
    y ← y + h · f(t, y)        ▷ Update y using Euler method
    t ← t + h                  ▷ Update t
    Store y in a list
end for
```

1.0.2. Improved Euler's Method (Heun's Method). Heun's method, also known as the improved Euler's method, improves the accuracy by averaging the slopes at the beginning and the end of the interval:

$$k_1 = f(t_n, y_n), \tag{1.3}$$

$$k_2 = f(t_n + h, y_n + hk_1), \tag{1.4}$$

$$y_{n+1} = y_n + \frac{h}{2}(k_1 + k_2). \tag{1.5}$$

Algorithm 2 Heun's Method

Input: Initial value y_0, step size h, number of steps N
Output: Approximation of solution at each step $y_1, y_2, \ldots, y_N$
$t \leftarrow 0$
$y \leftarrow y_0$
for $i \leftarrow 1$ to N **do**
    $\tilde{y} \leftarrow y + h \cdot f(t, y)$ ▷ Predictor step
    $y \leftarrow y + \frac{h}{2}\left(f(t, y) + f(t + h, \tilde{y})\right)$ ▷ Corrector step using midpoint
    $t \leftarrow t + h$ ▷ Update t
    Store y in a list
end for

1.0.3. Runge-Kutta Methods. The Runge-Kutta methods are a family of iterative methods that provide higher-order accuracy. The most commonly used is the fourth-order Runge-Kutta method (RK4):

$$k_1 = f(t_n, y_n), \tag{1.6}$$

$$k_2 = f\left(t_n + \frac{h}{2}, y_n + \frac{hk_1}{2}\right), \tag{1.7}$$

$$k_3 = f\left(t_n + \frac{h}{2}, y_n + \frac{hk_2}{2}\right), \tag{1.8}$$

$$k_4 = f(t_n + h, y_n + hk_3), \tag{1.9}$$

$$y_{n+1} = y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4). \tag{1.10}$$

1.1. Systems of ODEs

Systems of ODEs can be solved using methods similar to those for single ODEs. For example, a system of two first-order ODEs:

$$y_1'(t) = f_1(t, y_1, y_2), \tag{1.11}$$

$$y_2'(t) = f_2(t, y_1, y_2), \tag{1.12}$$

can be solved using Euler's method or Runge-Kutta methods by treating y_1 and y_2 as vectors.

Algorithm 3 Fourth-Order Runge-Kutta Method

```
Input: Initial value y_0, step size h, number of steps N
Output: Approximation of solution at each step y_1, y_2, ..., y_N
t ← 0
y ← y_0
for i ← 1 to N do
    k_1 ← h · f(t, y)                              ▷ Step 1
    k_2 ← h · f(t + h/2, y + k_1/2)                ▷ Step 2
    k_3 ← h · f(t + h/2, y + k_2/2)                ▷ Step 3
    k_4 ← h · f(t + h, y + k_3)                    ▷ Step 4
    y ← y + (1/6)(k_1 + 2k_2 + 2k_3 + k_4)         ▷ Update y using weighted average
    t ← t + h                                      ▷ Update t
    Store y in a list
end for
```

1.2. Boundary Value Problems (BVPs)

DEFINITION 1.3. Boundary value problems involve solving an ODE subject to boundary conditions at different points. For example:

$$y''(x) = f(x, y(x), y'(x)), \quad y(a) = \alpha, \quad y(b) = \beta. \tag{1.13}$$

1.2.1. Shooting Method.

The shooting method transforms a BVP into an IVP by guessing the initial slope and iteratively adjusting it to satisfy the boundary conditions.

Algorithm 4 Shooting Method

```
Input: Initial guess for the boundary value problem y(a), y(b)
Output: Approximation of the solution to the boundary value problem
Choose a step size h and a shooting parameter α
Initialize y(a) = y_0 and y'(a) = α
for i ← 1 to maximum number of iterations do
    Solve the initial value problem using any numerical method (e.g., Runge-Kutta)
    Compute y(b) using the solution
    Compute the error ϵ = |y(b) − y_target|
    if ϵ is within tolerance then
        Break
    end if
    Update the shooting parameter α using a root-finding algorithm (e.g., bisection method)
end for
Return the approximation of the solution
```

1.2.2. Finite Difference Method.

The finite difference method approximates derivatives using difference equations. For a second-order ODE:

$$y''(x) \approx \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}, \tag{1.14}$$

where h is the step size. The BVP is discretized and solved as a system of linear equations.

1.3. Partial Differential Equations (PDEs)

Partial differential equations involve multiple independent variables. Numerical methods for PDEs include finite difference methods, finite element methods, and spectral methods.

1.3.1. Finite Difference Method for PDEs. The finite difference method discretizes the PDE on a grid. For example, the heat equation:

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2} \tag{1.15}$$

is discretized as:

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \alpha \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2}. \tag{1.16}$$

1.3.2. Finite Element Method. The finite element method divides the domain into smaller subdomains (elements) and approximates the solution using piecewise polynomials. It is particularly useful for complex geometries.

1.4. Stability and Convergence

1.4.1. Stability. Stability refers to the behavior of numerical errors during the solution process. A method is stable if errors do not grow uncontrollably.

1.4.2. Convergence. Convergence ensures that the numerical solution approaches the exact solution as the step size decreases. A method is convergent if it is both consistent and stable.

1.4.3. Consistency. Consistency means that the discretization error vanishes as the step size approaches zero.

1.5. Examples and Applications

1.5.1. Example: Solving an IVP with Euler's Method. Consider the IVP:

$$y'(t) = -2y(t), \quad y(0) = 1. \tag{1.17}$$

Using Euler's method with $h = 0.1$:

- Initialize: $t_0 = 0$, $y_0 = 1$, $h = 0.1$, $N = 10$.
- Compute:

$$\begin{aligned} y_1 &= y_0 + h(-2y_0) = 1 + 0.1(-2 \cdot 1) = 0.8, \\ y_2 &= y_1 + h(-2y_1) = 0.8 + 0.1(-2 \cdot 0.8) = 0.64, \\ &\vdots \\ y_{10} &= \text{Continue until } t = 1. \end{aligned}$$

1.5.2. Application: Heat Equation. Solve the heat equation:

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L, \quad t > 0 \tag{1.18}$$

with initial condition $u(x, 0) = \sin(\pi x/L)$ and boundary conditions $u(0, t) = 0$, $u(L, t) = 0$, using the finite difference method.

Exercises

(1) **Initial Value Problems (IVPs)**

(i). **Euler's Method:** Use Euler's method to approximate the solution of the following IVP over the interval $0 \leq t \leq 1$ with a step size of $h = 0.2$:

$$y'(t) = 3y(t) + 2t, \quad y(0) = 1. \tag{1.19}$$

Compute the values of $y(t)$ at $t = 0.2, 0.4, 0.6, 0.8, 1.0$.

(ii). **Improved Euler's Method (Heun's Method):** Apply Heun's method to solve the same IVP given above with the same step size $h = 0.2$. Compute the values of $y(t)$ at $t = 0.2, 0.4, 0.6, 0.8, 1.0$.

(iii). **Runge-Kutta Method:** Use the fourth-order Runge-Kutta method (RK4) to solve the IVP:

$$y'(t) = -2ty(t), \quad y(0) = 1, \tag{1.20}$$

over the interval $0 \leq t \leq 1$ with a step size of $h = 0.1$. Compute the values of $y(t)$ at $t = 0.1, 0.2, \ldots, 1.0$.

(2) **Systems of ODEs**

(i). **Euler's Method for Systems:** Use Euler's method to solve the following system of ODEs over the interval $0 \leq t \leq 1$ with a step size of $h = 0.2$:

$$y_1'(t) = y_2(t), \quad y_1(0) = 1, \tag{1.21}$$

$$y_2'(t) = -y_1(t), \quad y_2(0) = 0. \tag{1.22}$$

Compute the values of $y_1(t)$ and $y_2(t)$ at $t = 0.2, 0.4, 0.6, 0.8, 1.0$.

(ii). **Runge-Kutta Method for Systems:** Apply the fourth-order Runge-Kutta method (RK4) to solve the following system of ODEs:

$$y_1'(t) = 2y_1(t) - y_2(t), \quad y_1(0) = 1, \tag{1.23}$$

$$y_2'(t) = y_1(t) + y_2(t), \quad y_2(0) = 0, \tag{1.24}$$

over the interval $0 \leq t \leq 1$ with a step size of $h = 0.1$. Compute the values of $y_1(t)$ and $y_2(t)$ at $t = 0.1, 0.2, \ldots, 1.0$.

(3) **Boundary Value Problems (BVPs)**

(i). **Shooting Method:** Use the shooting method to solve the BVP:

$$y''(x) = -\pi^2 y(x), \quad y(0) = 0, \quad y(1) = 0. \tag{1.25}$$

Use initial guesses for $y'(0)$ and iteratively adjust to meet the boundary condition at $x = 1$.

(ii). **Finite Difference Method:** Apply the finite difference method to solve the following BVP:

$$y''(x) = e^x, \quad y(0) = 0, \quad y(1) = 1, \tag{1.26}$$

using a step size of $h = 0.25$. Formulate the system of linear equations and solve for $y(x)$ at $x = 0.25, 0.5, 0.75$.

(4) **Partial Differential Equations (PDEs)**

(i). **Finite Difference Method for Heat Equation:** Solve the heat equation

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 1, \quad t > 0, \tag{1.27}$$

with initial condition $u(x, 0) = \sin(\pi x)$ and boundary conditions $u(0, t) = 0$ and $u(1, t) = 0$, using the explicit finite difference method with $\alpha = 0.01$, $\Delta x = 0.1$, and $\Delta t = 0.005$. Compute the solution at $t = 0.01, 0.02, \ldots, 0.05$.

(ii). **Finite Element Method:** Describe the steps involved in solving the PDE:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y), \quad (x, y) \in \Omega, \tag{1.28}$$

using the finite element method. Consider the domain Ω as the unit square $[0, 1] \times [0, 1]$ and provide a general outline for setting up the weak formulation, discretization, and assembling the system of equations.

(iii). **Stability Analysis:** Analyze the stability of the explicit finite difference scheme for the heat equation. Determine the condition on the time step Δt in terms of the spatial step Δx and the diffusion coefficient α to ensure stability.

(5) **Stability and Convergence Exercises**

(i). **Consistency:** Show that the forward Euler method is consistent by deriving the local truncation error for the method applied to the IVP $y'(t) = f(t, y(t)), \quad y(0) = y_0$.

(ii). **Convergence:** Prove that if a numerical method is consistent and stable, it is also convergent. Use the Lax Equivalence Theorem to support your argument.

(iii). **Stability of Backward Euler Method:** Analyze the stability of the backward Euler method for solving $y'(t) = \lambda y(t)$ with $\lambda < 0$. Show that the method is unconditionally stable.

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