STA 2060: BROWNIAN MOTION

Brownian motion, named after the botanist Robert Brown, who first observed it in 1827, is a stochastic process that models the random motion of particles suspended in a fluid (such as gas or liquid). The key defining characteristic of Brownian motion is its randomness, which arises from the random collisions between the particles and the molecules of the fluid.

Formally, Brownian motion is defined as a continuous-time stochastic process (W(t) , where $t \geq 0$, with the following properties:

1. **Initial Condition**: $W(0) = 0$, meaning that the particle starts at the origin.

2. **Independent Increments**: For any $0 \leq t_1 < t_2 < \dots < t_n$, the random variables $W(t_1), W(t_2) - W(t_1), \dots, W(t_n) - W(t_{n-1})$ are independent. This property implies that the motion is memoryless, and each increment is independent of previous increments.

3. **Stationary Increments**: The distribution of $W(t_2) - W(t_1)$ depends only on the difference $t_2 - t_1$ and not on the specific values of t_1 and t_2. This property means that the statistical properties of the motion are constant over time.

4. **Gaussian Increments**: For any $0 \leq s < t$, the increment $W(t) - W(s)$ is normally distributed with mean 0 and variance $t - s$. In other words, the displacement of the particle at any time interval follows a Gaussian (normal) distribution.

5. **Continuous Paths**: The paths of Brownian motion are continuous functions of time, with probability 1.

These properties collectively describe the behavior of Brownian motion and make it a fundamental model in various scientific disciplines, including physics, finance, and biology.

Standard Brownian Motion

A continuous-time stochastic process $(B_t)_{t\geq 0}$ is a *standard Brownian motion* if it satisfies the following properties:

1. $B_0 = 0$.

2. *(Normal distribution)* For $t > 0$, B_t has a normal distribution with mean 0 and variance t.
3. *(Stationary increments)* For $s, t > 0$, $B_{t+s} - B_s$ has the same distribution as B_t. That is,

$$P(B_{t+s} - B_s \leq z) = P(B_t \leq z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t}\, dx,$$

   for $-\infty < z < \infty$.
4. *(Independent increments)* If $0 \leq q < r \leq s < t$, then $B_t - B_s$ and $B_r - B_q$ are independent random variables.
5. *(Continuous paths)* The function $t \mapsto B_t$ is continuous, with probability 1.

Here, we introduce a construction of Brownian motion from a symmetric random walk. Divide the half-line $[0, \infty)$ to tiny subintervals of length δ as shown in Figure 11.30.

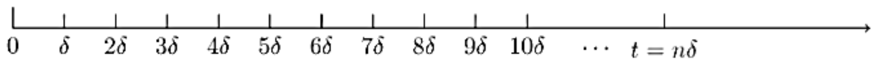

Figure 11.30 - Dividing the half-line $[0, \infty)$ to tiny subintervals of length δ.

Each subinterval corresponds to a time slot of length δ. Thus, the intervals are $(0, \delta]$, $(\delta, 2\delta]$, $(2\delta, 3\delta]$, $\cdots$. More generally, the kth interval is $\big((k-1)\delta, k\delta\big]$. We assume that in each time slot, we toss a fair coin. We define the random variables X_i as follows. $X_i = \sqrt{\delta}$ if the kth coin toss results in heads, and $X_i = -\sqrt{\delta}$ if the kth coin toss results in tails. Thus,

$$X_i = \begin{cases} \sqrt{\delta} & \text{with probability } \frac{1}{2} \\ & \\ -\sqrt{\delta} & \text{with probability } \frac{1}{2} \end{cases}$$

Now, we would like to define the process $W(t)$ as follows. We let $W(0) = 0$. At time $t = n\delta$, the value of $W(t)$ is given by

$$W(t) = W(n\delta) = \sum_{i=1}^{n} X_i.$$

Since $W(t)$ is the sum of n i.i.d. random variables, we know how to find $E[W(t)]$ and $\text{Var}(W(t))$. In particular,

$$\begin{aligned} E[W(t)] &= \sum_{i=1}^{n} E[X_i] \\ &= 0, \\ \text{Var}(W(t)) &= \sum_{i=1}^{n} \text{Var}(X_i) \\ &= n\text{Var}(X_1) \\ &= n\delta \\ &= t. \end{aligned}$$

For any $t \in (0, \infty)$, as n goes to ∞, δ goes to 0. By the central limit theorem, $W(t)$ will become a normal random variable,

$$W(t) \sim N(0, t).$$

Since the coin tosses are independent, we conclude that $W(t)$ has *independent increments*. That is, for all $0 \leq t_1 < t_2 < t_3 \cdots < t_n$, the random variables

$$W(t_2) - W(t_1),\ W(t_3) - W(t_2),\ \cdots,\ W(t_n) - W(t_{n-1})$$

are independent. Remember that we say that a random process $X(t)$ has *stationary increments* if, for all $t_2 > t_1 \geq 0$, and all $r > 0$, the two random variables $X(t_2) - X(t_1)$ and $X(t_2 + r) - X(t_1 + r)$ have the same distributions. In other words, the distribution of the difference depends only on the length of the interval $(t_1, t_2]$, and not on the exact location of the interval on the real line. We now claim that the random process $W(t)$, defined above, has stationary increments. To see this, we argue as follows. For $0 \leq t_1 < t_2$, if we have $t_1 = n_1\delta$ and $t_2 = n_2\delta$, we obtain

$$\begin{aligned}
W(t_1) &= W(n_1\delta) = \sum_{i=1}^{n_1} X_i, \\
W(t_2) &= W(n_2\delta) = \sum_{i=1}^{n_2} X_i.
\end{aligned}$$

Then, we can write

$$W(t_2) - W(t_1) = \sum_{i=n_1+1}^{n_2} X_i.$$

Therefore, we conclude

$$\begin{aligned}
E[W(t_2) - W(t_1)] &= \sum_{i=n_1+1}^{n_2} E[X_i] \\
&= 0, \\
\mathrm{Var}(W(t_2) - W(t_1)) &= \sum_{i=n_1+1}^{n_2} \mathrm{Var}(X_i) \\
&= (n_2 - n_1)\mathrm{Var}(X_1) \\
&= (n_2 - n_1)\delta \\
&= t_2 - t_1.
\end{aligned}$$

Therefore, for any $0 \leq t_1 < t_2$, the distribution of $W(t_2) - W(t_1)$ only depends on the lengths of the interval $[t_1, t_2]$, i.e., how many coin tosses are in that interval. In particular, for any $0 \leq t_1 < t_2$, the distribution of $W(t_2) - W(t_1)$ converges to $N(0, t_2 - t_1)$. Therefore, we conclude that $W(t)$ has *stationary increments*.

Example 8.1 For $0 < s < t$, find the distribution of $B_s + B_t$.

Solution Write $B_s + B_t = 2B_s + (B_t - B_s)$. By independent increments, B_s and $B_t - B_s$ are independent random variables, and thus $2B_s$ and $B_t - B_s$ are independent. The sum of independent normal variables is normal. Thus, $B_s + B_t$ is normally distributed with mean $E(B_s + B_t) = E(B_s) + E(B_t) = 0$, and variance

$$\begin{aligned} Var(B_s + B_t) &= Var(2B_s + (B_t - B_s)) = Var(2B_s) + Var(B_t - B_s) \\ &= 4Var(B_s) + Var(B_{t-s}) = 4s + (t - s) \\ &= 3s + t. \end{aligned}$$

The second equality is because the variance of a sum of independent random variables is the sum of their variances. The third equality uses stationary increments. We have that $B_s + B_t \sim \text{Normal}(0, 3s + t)$. ■

Example 8.2 A particle's position is modeled with a standard Brownian motion. If the particle is at position 1 at time $t = 2$, find the probability that its position is at most 3 at time $t = 5$.

Solution The desired probability is

$$\begin{aligned} P(B_5 \le 3 | B_2 = 1) &= P(B_5 - B_2 \le 3 - B_2 | B_2 = 1) \\ &= P(B_5 - B_2 \le 2 | B_2 = 1) \\ &= P(B_5 - B_2 \le 2) \\ &= P(B_3 \le 2) = 0.876. \end{aligned}$$

The third equality is because $B_5 - B_2$ and B_2 are independent. The penultimate equality is by stationary increments. The desired probability in R is

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> pnorm(2,0,sqrt(3))
[1] 0.8758935
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Example 8.3 Find the covariance of B_s and B_t.

Solution For the covariance,

$$\text{Cov}(B_s, B_t) = E(B_s B_t) - E(B_s)E(B_t) = E(B_s B_t).$$

For $s < t$, write $B_t = (B_t - B_s) + B_s$, which gives

$$\begin{aligned} E(B_s B_t) &= E(B_s(B_t - B_s + B_s)) \\ &= E(B_s(B_t - B_s)) + E\left(B_s^2\right) \\ &= E(B_s)E(B_t - B_s) + E\left(B_s^2\right) \\ &= 0 + Var(B_s) = s. \end{aligned}$$

Thus, $\text{Cov}(B_s, B_t) = s$. For $t < s$, by symmetry $\text{Cov}(B_s, B_t) = t$. In either case,

$$\text{Cov}(B_s, B_t) = \min\{s, t\}.$$