Autoregressive Moving Average (ARMA)

STA 3050A - Time Series and Forecasting

AUTOREGRESSIVE MOVING AVERAGE MODELS

In the previous sections, you have learnt autoregressive and moving average models which are used to model time series data. The autoregressive (AR) models are used when the current value of the time series variable depends on the past values of the series whereas the moving average models are used when the current value of the time series variable depends on the unpredictable shocks (residuals) in the previous periods.

But in real-life data, we also observe that the current value of the time series variable depends not only on its past values but also on the residuals in the previous periods. For example, the sale of a product of a company at a current time depends on the prior sales happening in the past time which plays a role of the AR component, and it also depends on the time-based campaigns launched by the company, such as distribution of coupons, buy one get one free, etc. will increase sales temporarily and such change in sales is captured by the moving average component.

Therefore, we need models that simultaneously use past data as a foundation for estimates, and can also quickly adjust to unpredictable shocks (residuals). In this section, we are going to talk about one such model, called autoregressive moving average (ARMA), which takes into account past values as well as past errors when constructing future estimates.

Autoregressive moving average (ARMA) models play a key role in the modelling of time series. An ARMA process consists of two models: an autoregressive (AR) model and a moving average (MA) model. In analysis, we tend to put the residuals at the end of the model equation, so that's why the "MA" part comes second.

As compared with the pure AR and MA models, ARMA models provide the most effective linear model of stationary time series since they are capable of modelling the unknown process with the minimum number of parameters. In this model, the impact of previous lags along with the residuals is considered for forecasting the future values of the time series.

We can define the ARMA model as follows:

Autoregressive moving average models are simply a combination of an AR model and an MA model.

Autoregressive Moving Average (ARMA) models are models in which the value of a variable in the current period is related to its own values in the previous period as well as values of the residual in the previous period.

Since ARMA is a combination of both autoregressive terms (p) and moving average (q) terms, therefore, we represent it as ARMA (p, q). It is also used for stationary time series.

Various Forms of ARMA Models

The ARMA model has various forms for different values of the parameters p and q of the model. We discuss some standard forms as follows:

ARMA(1,1) Models

ARMA(1,1) models are the models in which the value of a variable in the current period is related to its own value in the previous period as well as values of the residual in the previous period. It is a mixture of AR(1) and MA(1). If y_t and y_{t-1} are the values of a variable at time t and t –1, respectively and if ε_t and ε_{t-1} are the residuals at time t and t –1, respectively then the ARMA (1,1) model is expressed as follows:

$$y_t = \delta + \phi_1 y_{t-1} + \varepsilon_t + \theta_1 \varepsilon_{t-1}$$

As usual, the coefficients δ and ε_t denote the intercept/constant factor and error term at time t, respectively whereas the coefficients ϕ_1 and θ_1 represent AR and MA coefficients and represent the magnitude of the impact of past values and past error on the present value, respectively. After understanding the form of the ARMA (1,1) model, we now study the properties of the model.

Mean and Variance

We can find the mean and variance of an ARMA (1, 1) model as we have found in AR and MA models which are given as follows:

$$\text{Mean} = \frac{\delta}{1-\phi_1}$$

$$\text{Var}[y_t] = \frac{(1+\theta_1^2+\phi_1^2\theta_1)\sigma^2}{1-\phi_1^2} \geq 0 \text{ when } \phi_1^2 < 1$$

Autocovariance and Autocorrelation Functions

The autocovariance function of an ARMA(1, 1) model is given as follows:

$$\gamma_0 = \frac{(1+\theta_1^2+\phi_1^2\,\theta_1)\sigma^2}{1-\phi_1^2}$$

$$\gamma_1 = \frac{(\phi_1+\phi_1)(1+\phi_1\theta_1)\sigma^2}{1-\phi_1^2}$$

$$\gamma_k = \phi_1 \gamma_{k-1} \text{ for } k = 2,3,\ldots$$

Similarly, the autocorrelation function of an ARMA(1,1) model is given as follows:

$$\rho_1 = \frac{(\phi_1+\theta_1)(1+\phi_1\theta_1)}{1+\phi_1^2\theta_1+\theta_1^2}$$

$$\rho_k = \phi_1^{k-1}\rho_1 \text{ for } k = 2,3,\ldots$$

The autocorrelation function of an ARMA(1, 1) model exhibits exponential decay and/or sinusoid pattern towards zero. It does not cut off but gradually decreases as lag k increases.

Also, the autocorrelation function of an ARMA(1, 1) model displays the shape of an AR(1) process. The partial autocorrelation function of an ARMA(1, 1) model also gradually dies out (the same property as a moving average model) as k increases. It is relatively difficult to identify the order of the ARMA model.

Conditions for Stationarity and Invertibility

The stationarity of the ARMA (1, 1) is related to the AR component in the ARMA (1, 1) model. Therefore, stationarity conditions which are discussed for AR(1) are also for ARMA (1,1) model which is as follows:

$$|\phi_1| < 1 \Rightarrow -1 < \phi_1 < 1$$

Similarly to the stationarity conditions, the invertibility of an ARMA(1,1) model is related to the MA(1) component. Therefore, the invertibility conditions which are discussed for MA(1) are also for ARMA (1,1) model which is as follows:

$$|\theta_1| < 1 \Rightarrow -1 < \theta_1 < 1$$

ARMA (1, 2) Models

ARMA (1,2) models are the models in which the value of a variable in the current period is related to its own value in the previous period as well as the residuals of two previous periods. It is a mixture of AR(1) and MA(2) models. If y_t and y_{t-1} are the values of a variable at time t and t –1, respectively, and if ε_t, ε_{t-1}, and ε_{t-2} are the residuals at time t, t –1, and t – 2, respectively, then the ARMA (1,2) model is expressed as follows:

$$y_t = \delta + \phi_1 y_{t-1} + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2} + \varepsilon_t$$

The expression for mean, variance, autocovariance, and autocorrelation functions of ARMA(1, 2) model are more complicated, therefore, we are not giving the expression of these.

Conditions for Stationarity and Invertibility

The stationarity of the ARMA (1, 2) model is related to the AR component in the ARMA (1, 1) model. Therefore, stationarity conditions which are discussed for AR(1) are also for ARMA (1,2) model which is as follows:

$$|\phi_1| < 1 \Rightarrow -1 < \phi_1 < 1$$

Similarly, the invertibility conditions, of an ARMA(1,1) model is related to the MA(2) component which are as follows:

$$|\theta_2| < 1 \Rightarrow -1 < \theta_2 < 1$$

$$\theta_1 + \theta_2 < 1$$

$$\theta_2 - \theta_1 < 1$$

ARMA (p, q) Models

More generally, ARMA(p,q) models are the models in which the value of a variable in the

current period is related to its own values of the previous p period as well as the residuals of the previous q periods. It is a mixture of AR(p) and MA(q) models. If $y_t, y_{t-1}, \ldots, y_{t-p}$ are the value of a variable at time t, t-1, …, t-p, respectively and if $\varepsilon_t, \varepsilon_{t-1}, \ldots, \varepsilon_{t-q}$ are the residuals at time t, t-1, …, t-q, respectively, then the ARMA (p, q) model is expressed as follows:

$$y_t = \delta + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \ldots + \phi_p y_{t-p} + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2} + \ldots + \theta_q \varepsilon_{t-q} + \varepsilon_t$$

The expression for mean, variance, autocovariance, and autocorrelation functions of ARMA (p, q) models are more complicated, therefore, we are not giving the expression of these.

Autocorrelation and Partial Autocorrelation Functions

The autocorrelation function of an ARMA(p, q) model exhibits exponential decay and/or sinusoid pattern towards zero. It does not cut off but gradually decreases as lag k increases.

Also, the autocorrelation function of an ARMA(p, q) model displays the shape of an AR(p) process.

The partial autocorrelation function of an ARMA(p, q) model also gradually dies out (the same property as a moving average model) as k increases. It is relatively difficult to identify the order of the ARMA model.

Conditions for Stationarity and Invertibility of ARMA (p, q) model

When $p \geq 3$, the restrictions for stationarity are much more complicated. Similarly, when $q \geq 3$, the restrictions for invertibility become more complicated, therefore, we are not discussing them here.

Example 3:

Consider the time series model

$$y_t = 20 - 0.5 y_{t-1} + 0.7 \varepsilon_{t-1} + \varepsilon_t$$

Assuming that the variance of the white noise is 2.

1. Identify the model.
2. Check whether the model is stationary and invertible.
3. Calculate the autocorrelation function ρ_1, ρ_2, ρ_3.

Solution:

Since the variable in the current period is regressed against its previous value as well as previous residual, therefore, it is the ARMA model of order (1, 1). To check whether it is stationary and invertible, first of all, we find the parameters of the time series model. We now compare it with its standard form, that is,

$$y_t = \delta + \phi_1 y_{t-1} + \varepsilon_t + \theta_1 \varepsilon_{t-1}$$

We obtain

$$\delta = 20, \phi_1 = -0.5, \theta_1 = 0.7$$

The stationary constraints for ARMA(1, 1) is $-1 < \phi_1 < 1$. Since ϕ_1 lies between –1 and 1, therefore, the time series model ARMA(1, 1) is stationary. Similarly, the invertibility constraints for ARMA(1,1) is $-1 < \theta_1 < 1$. Since θ_1 lies between –1 and 1, therefore, the time series model ARMA(1,1) is invertible. We now calculate the autocovariance function of ARMA(1, 1) as

$$\rho_1 = \frac{\phi_1(1 + \phi_1\theta_1 + \theta_1)}{1 + 2\phi_1\theta_1 + \theta_1^2} = \frac{-0.5(1 - 0.5 \times 0.7 + 0.7)}{1 + 2 \times -0.5 \times 0.7 + 0.7^2} = \frac{-0.5(1 - 0.35 + 0.7)}{1 - 0.7 + 0.49} = \frac{-0.5 \times 1.35}{0.79} = -0.8557$$

$$\rho_2 = \phi_1\rho_1 = -0.5 \times -0.8557 = 0.4279$$

$$\rho_3 = \phi_1\rho_2 = -0.5 \times 0.4279 = -0.21395$$

Exercise 3

Consider the ARMA time series model

$$y_t = 27 + 0.8y_{t-1} + \varepsilon_t + 0.3\varepsilon_{t-1}$$

Assuming that the variance of the white noise is 1.5.

1. Is the process stationary and invertible?
2. Find ρ_1, ρ_2, ρ_3 for the process.
3. ARIMA ➔ Autoregressive Integrated Moving Average