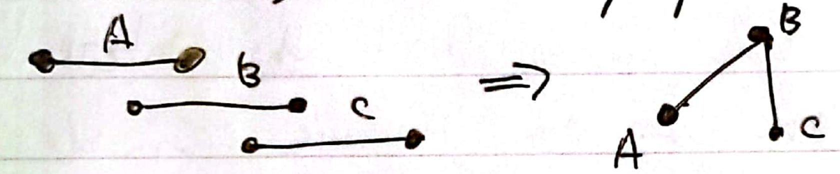
CS 401 HW#5

(1.) Yes, the interval scheduling problem $\leq_p$ Vertex Cover.

Explanation: We can transform the intervals to vertices and intersecting intervals will form an edge in the graph.



- Now, we need to find k nonoverlapping intervals. Notice that this corresponds to an independent set of size k in our graph. Also, the vertex cover is equal to $|V|-k$ where $|V|$ is the number of nodes and k is the size of the independent set.

- Thus, if the vertex cover finds a cover of size $|V|-k$, this implies an independent set (and thus nonoverlapping intervals) of size k.

This proves that Interval scheduling $\leq_p$ Vertex Cover because we turned the ISP to the Vertex Cov. problem.

(1b) Unknown, because it would resolve the question of whether P = NP.

As shown in part a, we related the interval scheduling problem to vertex cover by using the independent set. We can reduce the interval scheduling prob. to the independent set. So, if we can do it vice versa such that the NP-complete independent set problem can be reduced to the polynomial time interval scheduling problem, this would imply that P=NP.

* To map the independent set problem to interval scheduling, we can simply create an interval for every vertex in the graph. The edges in the graph will imply that there are overlaps. To solve the problem of choosing a subset of k independent vertices, you can also (ideally) solve it by choosing k non-overlapping intervals.

• If we can solve this in polynomial time, that would imply that P=NP.

(2.) We can show that the Diverse Subset is NP-complete by showing that it's in NP and by using a known NP-problem such that the (Independent set $\leq_p$ Diverse subset)

1. A solution is in NP: we can check by looping over the customers and for each customer, loop their bought products and if a product has already been bought, then the solution is invalid. Otherwise, add it to a set of bought products. If pass, solution is valid. The time is $O(C * P)$ so it's in polynomial.

(2.) Show that the independent set $\leq_p$ diverse subset:

1. Create a matrix A where rows is equal to the number of vertices (customers) and columns equal to the number of edges (product).

An edge in the graph of the independent set $\langle u, v \rangle$ shows $A[u][product] = 1$ & $A[v][product] = 1$ implying that both customers (u, v) bought the same product.

Correctness:

The diverse subset problem asks for a subset of customers such that no two customers share the same product. This is exactly the same as finding an independent set in the graph because if two vertices are connected by an edge, the corresponding customers in the Diverse Subset Problem share a product and cannot both be in the diverse set.

Therefore, finding a diverse set of size k corresponds to finding an independent set of size k.

<u>Polynomial Time Analysis</u>: The matrix construction involves checking each pair of vertices and edges, which takes $O(m \times n)$ where m = # of vertices and n = # of edges.

Thus, the reduction (indep. $\leq_p$ diverse) can be done in polynomial time.

By showing that the diverse subset is in NP and that the Independent Set is reducable to it, it's NP hard.

Thus, it's NP-complete.

3. We can show that Efficient Recruiting Problem (ERP) is NP-complete.

1. Show that ERP is in NP:
   Check whether there are at most k counselors.
   Check all the covered sports.
   time: $O(k)$ or $O(k \times n)$ where k = counselors, n = sports, which is polynomial.

2. Show that Set Cover $\leq_p$ ERP:

Set cover: $U = \{u_1, u_2, u_3 \ldots\} \rightarrow$ Sports $= \{u_1, u_2, u_3 \ldots\}$
$S_1 = \{u_1, u_2 \ldots\} \rightarrow$ Counselor1 $= \{u_1, u_2 \ldots\}$
$S_2 \ldots \rightarrow$ Counselor...

set cover → does there exist k or fewer subsets $S_1 \ldots$ such that their union is U?

reduced to → does there exist k or fewer counselors such that their union is Sports (where sports contain all the sports)

<u>Correctness</u>: If ERP finds a subset of k counselors covering all sports, it's equivalent to subsets covering all u_i element in U.

Polynomial time Analysis:

The entire conversion will just take polynomial time. This includes converting the universal set and all the collections of subsets. It takes $O(|c| \times |u|)$ worst case.

By showing that ERP is in NP and the Set cover problem is reducable to ERP then,

ERP is NP-complete.

4.) We can show that the Hitting Set problem (HSP) is NP-complete.

1. Show that HSP is in NP:
We can check that for every B_i, the intersection $H \cap B_i \neq \emptyset$.
It's in polynomial time as we're doing it for every B_i only. So, $O(k \cdot m)$ where $|H| = k$ and $m =$ number of subsets.

2. Show that the vertex cover $\leq_p$ HSP:
Given a graph $G = \langle V, E \rangle$ where V is the set of vertices and E is the set of edges,
we can transform/reduce this graph to an instance of the HSP:

$$A = \{\text{vertices in } G(V)\}$$

For every edge in E $(u, v) \Rightarrow B_i = \{u, v\}$

Correctness: We're able to map all the edges and vertices in G to HSP.
* If the HSP finds a Hitting Set of size k, then H forms a vertex cover in the original graph of size k because HSP loops through all $B_i \to B_n$ which are all the edges formed by the edge's vertex pairs.

Polynomial time Analysis:

the construction of the reduction is made in polynomial time.

1. Creating the set A is bounded by the # of vertices. $(|V|)$

2. Creating $B_i \Rightarrow B_m$ subsets are bounded by the number of edges.

Therefore, $O(|V| + m)$ is polynomial time.

∴ By showing that HSP is in NP and NP-Hard, we conclude that HSP is NP-complete.