CS 151: Mathematical Foundations of Computing
Homework Assignment 04
Fall 2023

Instructions

This assignment is due Sunday, November 05, at 11:59PM (Central Time).

This assignment must be submitted on *Gradescope*. Handwritten submissions are allowed as long as they are legible. Submissions typed in LaTeX or Word are preferred. Each answer must be clearly labeled (1, 2, etc.) and matched to the corresponding question on *Gradescope*. A 5-point penalty will be applied to submissions that do not follow these guidelines.

For more instructions on how to submit assignments on *Gradescope* see this guide.

Late submissions will be accepted within 0-12 hours after the deadline with a 5-point penalty and within 12-24 hours after the deadline with a 20-point penalty. No late submissions will be accepted more than 24 hours after the deadline.

This assignment is individual. Offering or receiving any kind of unauthorized or unacknowledged assistance (including searching for solutions online) is a violation of the University's academic integrity policies, will result in a grade of zero for the assignment, and will be subject to disciplinary action.

Part I: Induction (90 pt.)

(90 pt., 15 pt. each) Prove each of the following statements using *induction*, *strong induction*, or *structural induction*. For each proof, answer the following questions.

- **(3 pt.) Complete the basis step of the proof.**
- **(3 pt.) What is the inductive hypothesis?**
- **(3 pt.) What do you need to show in the inductive step of the proof?**
- **(6 pt.) Complete the inductive step of the proof.**

1. **Prove that**

$$\sum_{i=0}^{n} 3^i = \frac{3^{n+1} - 1}{2}$$

for any nonnegative integer n.

2. **Prove that $8^n - 5^n$ is divisible by 3 for any positive integer n.**

3. **Prove that $n! < n^n$ for any integer $n > 1$.**

Hint: The **factorial** of n, denoted by $n!$, is given by $n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot (n-1) \cdot n$.

4. Let $a_0, a_1, a_2, \ldots$ be the sequence defined by the following recurrence relation:
   - $a_0 = 0$
   - $a_i = 2a_{i-1} + 3^i + 5i$ for $i \geq 1$

   Prove that $a_n = 7 \cdot 2^n + 3^{n+1} - 5n - 10$ for any nonnegative integer n.

5. Let $b_0, b_1, b_2, \ldots$ be the sequence defined by the following recurrence relation:
   - $b_0 = 1$
   - $b_1 = 2$
   - $b_i = 5 \cdot b_{i-1} + 6 \cdot b_{i-2}$ for $i \geq 2$

   Prove that $b_n = \frac{1}{7}(4(-1)^n + 2^n \cdot 3^{n+1})$ for any nonnegative integer n.

6. Let S be the set of perfect binary trees, defined recursively as follows:
   - Basis: a single vertex v with no edges is a perfect binary tree T_0.
   - Recursive rule: if T_1 and T_2 are perfect binary trees of the same height, then a new perfect binary tree T can be constructed by taking T_1 and T_2, adding a new vertex v, and adding edges between v and the roots of T_1 and T_2.

   Prove that any perfect binary tree T has an odd number of vertices; that is, $n(T) = 2k + 1$ for some integer k, where $n(T)$ is the number of vertices of T.

Part II: Recursive Algorithms (10 pt.)

7. (10 pt.) Given the following recursive algorithm:

```
procedure doublePower(x, y)
Input: x: nonnegative integer, y: nonnegative integer
Output: x^(2^y): nonnegative integer
if y==0 then return x
else return (doublePower(x, y-1))^2
```

   Prove that `doublePower(x, y)` returns $x^{(2^y)}$ (i.e., the exponent of x is 2^y) for any nonnegative integers x and y using *induction*. Answer the following questions.
   - (2 pt.) Complete the basis step of the proof.
   - (2 pt.) What is the inductive hypothesis?
   - (2 pt.) What do you need to show in the inductive step of the proof?
   - (4 pt.) Complete the inductive step of the proof.