CS 151: Mathematical Foundations of Computing
Homework Assignment 03
Fall 2023

Instructions

This assignment is due Sunday, October 22, at 11:59PM (Central Time).

This assignment must be submitted on *Gradescope*. Handwritten submissions are allowed as long as they are legible. Submissions typed in LaTeX or Word are preferred. Each answer must be clearly labeled (1a, etc.) and matched to the corresponding question on *Gradescope*. A 5-point penalty will be applied to submissions that do not follow these guidelines.

For more instructions on how to submit assignments on *Gradescope* see [this guide].

Late submissions will be accepted within 0-12 hours after the deadline with a 5-point penalty and within 12-24 hours after the deadline with a 20-point penalty. No late submissions will be accepted more than 24 hours after the deadline.

This assignment is individual. Offering or receiving any kind of unauthorized or unacknowledged assistance (including searching for solutions online) is a violation of the University's academic integrity policies, will result in a grade of zero for the assignment, and will be subject to disciplinary action.

Part I: Sets (30 pt.)

1. **(10 pt., 2.5 pt. each) Let A, B, and C be sets such that $A \cap B \cap C \neq \emptyset$ (i.e., A, B, and C are not disjoint). Draw a *Venn diagram* for each of the following set operations.**

   a. $\overline{A \cap B \cap C}$

   b. $(A \cup C) \cap B$

   c. $(\overline{A} - B) \cup (\overline{A} - C)$

   d. $(A - C) \cup (B - A) \cup (C - B)$

2. **(10 pt., 2.5 pt. each) Let $A = \mathcal{P}(\{1, 2\})$, $B = \mathcal{P}(\{\emptyset, 1\})$, and $C = \mathcal{P}(\emptyset)$ where $\mathcal{P}(S)$ is the power set of S. Find the result of each of the following set operations.**

   a. $A \cap B$

   b. $A \cup B$

   c. $B - A$

   d. $A \times C$

3. **(10 pt., 5 pt. each) Find $\bigcup_{i=1}^{\infty} A_i$ and $\bigcap_{i=1}^{\infty} A_i$ where:**

   a. $A_i = \{-i, -i + 1, \ldots, -1, 0, 1, \ldots, i - 1, i\}$ **for every positive integer i.**

   b. $A_i = \{2i\}$ **for every positive integer i.**

Part II: Relations (60 pt.)

4. **(30 pt., 6 pt. each) For each of the following relations, determine whether the relation is:**
   - ***Reflexive*.**
   - ***Anti-reflexive*.**
   - ***Symmetric*.**
   - ***Anti-symmetric*.**
   - ***Transitive*.**
   - ***A partial order*.**
   - ***A strict order*.**
   - ***An equivalence relation*.**

   Justify all your answers.

   a. $\boldsymbol{R}$ **is a relation on the set of all people such that** $(\boldsymbol{a}, \boldsymbol{b}) \in \boldsymbol{R}$ **if and only if** $\boldsymbol{a}$ **is the older sibling of** $\boldsymbol{b}$**. Note:** Define **siblings** as two people who share at least one parent.

   b. $\boldsymbol{R}$ **is a relation on the power set of a set** $\boldsymbol{A}$ **such that** $(\boldsymbol{A}, \boldsymbol{B}) \in \boldsymbol{R}$ **if and only if** $\boldsymbol{A} \subset \boldsymbol{B}$ **(i.e.,** $\boldsymbol{A}$ **is a proper subset of** $\boldsymbol{B}$**).**

   c. $\boldsymbol{R}$ **is a relation on** $\mathbb{Z}$ **such that** $(\boldsymbol{a}, \boldsymbol{b}) \in \boldsymbol{R}$ **if and only if** $\boldsymbol{a} + \boldsymbol{2} = \boldsymbol{b}$**.**

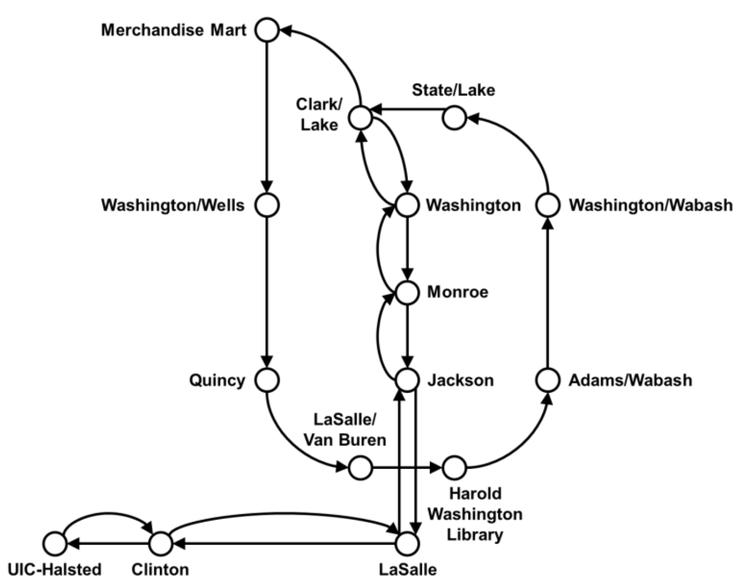
   d. $\boldsymbol{R}$ **is a relation on** $\mathbb{Z}$ **such that** $(\boldsymbol{a}, \boldsymbol{b}) \in \boldsymbol{R}$ **if and only if** $|\boldsymbol{a}| + |\boldsymbol{b}| > \boldsymbol{4}$**.**

   e. $\boldsymbol{R}$ **is a relation on** $\mathbb{Z}^+$ **such that** $(\boldsymbol{a}, \boldsymbol{b}) \in \boldsymbol{R}$ **if and only if** $\boldsymbol{a}$ **is divisible by** $\boldsymbol{b}$**. Note:** An integer a is **divisible** by an integer b with $b \neq 0$ if and only if there exists an integer k such that $a = bk$.

5. **(10 pt.) Given the following partial order, draw a *Hasse diagram*, find the *maximal* and *minimal* elements, and determine whether the partial order is a *total order*.**

   $\boldsymbol{R}$ **is a partial order on set** $\{\boldsymbol{2}, \boldsymbol{3}, \boldsymbol{4}, \boldsymbol{5}, \boldsymbol{7}, \boldsymbol{9}, \boldsymbol{15}, \boldsymbol{60}\}$ **such that** $(\boldsymbol{a}, \boldsymbol{b}) \in \boldsymbol{R}$ **if and only if** $\boldsymbol{a}$ **is divisible by** $\boldsymbol{b}$**. Note:** An integer a is **divisible** by an integer b with $b \neq 0$ if and only if there exists an integer k such that $a = bk$.

6. **(20 pt., 4 pt. each) Given the following directed graph** $\boldsymbol{G}$ **representing Chicago's 'L' system:**



Answer the following questions and <u>justify all your answers</u>:

a. **Is there a *walk* of length greater than 3 from UIC-Halsted to Jackson?**

b. **Is there a *path* of length greater than 3 from Jackson to Clark/Lake?**

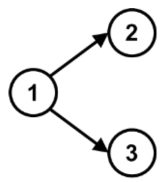
c. **Is there a *circuit* of length greater than 6 starting at Jackson?**

d. **Is there a *cycle* of length greater than 2 starting at Jackson?**

e. **Is there an edge from UIC-Halsted to itself in G^k for some <u>odd</u> integer k?**

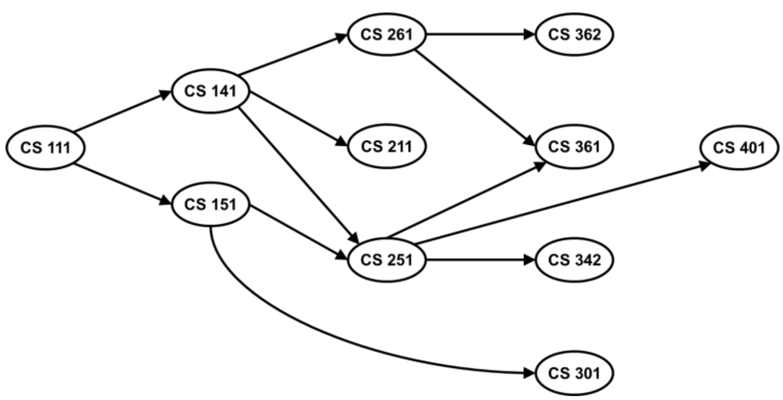
Additional Topics: Topological Sort (10 pt.)

A *strict order* relation can be represented by a **directed acyclic graph** (or **DAG**), which is a directed graph with *no cycles*. For example, the following directed graph is a **DAG**:



A **topological sort** of a DAG is an ordering of the vertices that is consistent with the edges of the graph. That is, if there is an edge (u, v) in the graph, then u should appear before v in the **topological sort**. For example, $1, 2, 3$ and $1, 3, 2$ are **topological sorts** of the DAG shown above, but $2, 1, 3$ is **<u>not</u>** a **topological sort** because 2 cannot be listed before 1.

7. **(10 pt.) Given the following DAG G representing UIC's Computer Science courses and prerequisites (note that edges implied by the transitive property are omitted):**



Answer the following questions:

a. **(4 pt.) Is the following ordering a *topological sort* of G? <u>Justify your answer</u>.**

CS 111, CS 141, CS 211, CS 261, CS 362, CS 361, CS 151, CS 251, CS 401, CS 342, CS 301.

b. **(4 pt.) Give two different *topological sorts* of G.**

c. **(2 pt.) If a student can take an unlimited number of courses per semester, what is the fewest number of semesters required to complete these courses? <u>Justify your answer</u>.**