Optimal Stopping Problem under Drift Uncertain

AMA4951 Capstone Project Presentation

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Outline

1. Introduction
   - Classic Black-Scholes Framework
   - Research object and motivation
   - Literature review
2. Problem Formulation
   - Financial markets and stock price
   - Stock loan optimal redeeming problem
3. Connection with PDE
   - Value function properties
   - Dynamic programming principle
   - Redeeming boundary under uncertain drift
4. Numerical methods
   - Finite difference method
   - Monte Carlo simulation

Outline

Assumpstions of Black-Scholes Framework

- Market movements are completely random.
- The market is frictionless (no transaction cost, highly liquid market).
- The drift rate is constant.
- The volatility is constant.

Problems from Black-Scholes framework

- The first one and the second assumptions are reasonable enough
- To address the problem of inconstant volatility, various models were proposed such as local volatility, stochastic volatility and so on (see [1] for more information about different volatility models).
- Surprisingly, there are few attempts of modeling the uncertainty of drift over the past research.

Importance of drift uncertain

- In practice, it's nearly impossible for investors to avoid estimating the current trend of a stock
- In the engineering area, when designing control systems, one must recognize the inevitable presence of uncertain drift
- Nowadays, people start to realize the importance of incomplete information and how they will affect the results when designing control systems.
- Various statistical techniques were developed to provide a reasonable estimate of the drifts.

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Stock loan

- A stock loan, is a special kind of loan between a borrower (usually an investor) and a lender (usually a bank), secured by the stock of the borrower.
- The borrower has the right to redeem anytime on or before the loan matures by repaying the lender the principal plus the pre-determined interest.
- Also, the borrower can choose not to repay the loan and the lender will take the stock.
- Essentially, a stock loan, can be regarded as an American option with changing strike price, which makes the problem different and harder than general American option problem.

Motivation of this project

- When comes to the optimal redeeming problem of stock loans with drift uncertain, the attempts were quite few.
- Prager and Zhang [8] derived closed-form analytical solution for European-style stock loans under mean-reverting and two-state regime switching stock price models apart from classic geometric Brownian motion.
- In the regime-switching model studied by Prager and Zhang [8], the current trend of the underlying stock is assumed to be known to the borrower although it will change over time. In reality, one need to admit the difficulty of obtaining a "reasonable" estimation of the current stock trend.

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Literature on optimal stopping problem under incomplete information

- Ekstr and Lu [5] studied the optimal selling problem for an agent under incomplete information of the asset with formulating the optimal stopping problem with drift uncertain and provided the optimal strategy for this asset selling problem.
- Martin [9] studied the optimal stopping strategy of the American put option under incomplete information and provided several numerical results to demonstrate the difference of approximation methods.

Literature on stock loan optimal redeeming problem under Black-Scholes Framework

- Xia and Zhou [12] originated the research on stock loans under classic Black-Scholes framework. The closed form solution was obtained by assuming the stock loans have infinite maturity date from pure probabilistic analysis.
- Dai and Xu [4] studied the optimal redeeming strategies under different dividends distributions and also provided some numerical examples.

Literature on stock loan optimal redeeming problem with variation

- Grasselli and Velez [6] investigates the stock loans pricing in an incomplete market by assuming market friction exists (e.g. hard for borrower to trade stocks).
- Chen et.al [2] addressed the inconstant risk-free rate issue and integrated stochastic interest rate model into stock loan pricing scheme also with ADI scheme.
- Wong and Wong [10] proposed a model for more general case by considering the stochastic volatility model.

Outline

Mathmatical reprentation of financial market

Suppose the financial market can be represented by a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, where

- $\mathbb{F} = \{\mathcal{F}_t\}_{t \geq 0}$ is a filtration satisfies usual conditions
- $\mathbb{P}$ is a probability measure.

Also, there exists a standard one-dimensional Brownian motion W in this filtered probability space.

Stock price process

Similar to Geometric Brownian motion, we have the following stock price process $(S_t)_{t\geq 0}$:

$$dS_t = (\mu - \delta)S_t dt + \sigma S_t dW_t$$

where volatility $\sigma > 0$ and dividend yield $\delta \geq 0$ are assumed to be known parameters.Without loss of generality, we can assume $\sigma = 1$ throughout the remaining of the report. Hence, the stock price process now reduces to

$$dS_t = (\mu - \delta)S_t dt + S_t dW_t \tag{1}$$

Drift uncertain

- We simply assume the drift μ is independent from Brownian motion W.
- $\mu - \delta$ only takes two possible values a and b which satisfies that $a > b$.
- The stock is said to be in the bull trend if $a > b$ and bear trend if $a < b$. Define the difference $\Delta = a - b$ which should be greater than 0.

Outline

Mathmarical representation of stock loan optimal redeeming problem

- The borrower can get K amount of cash at time 0 but need to use the stock as collateral.
- Anytime before the maturity, the borrower can repay the loan and get the stock back but the dividend gained will be left to the lender.
- The amount to repay at t should be $Ke^{\gamma t}$, where γ is the loan rate.
- the borrower needs to find the optimal time of redeeming the loan, which is to find the optimal stopping time to obtain:

$$\sup_{\tau \in \mathcal{T}_{[t,T]}} \mathbb{E}[e^{-r(\tau - t)}(S_\tau - Ke^{\gamma\tau})^+ | \mathcal{F}_t^S] \tag{2}$$

where $\mathcal{F}_t^S$ is the σ-algebra defined at t by the stock price process.

Transform stock price process under drift uncertain

Define the following probability:

$$\pi_t = \mathbb{P}(\mu - \delta = a | \mathcal{F}_t^S)$$

which yields the probability process $\pi = (\pi_t)_{t \geq 0}$.
Consider the log-price process $L = (\log S_t)_{t \geq 0}$. By Ito's lemma:

$$dL_t = \frac{1}{S_t} dS_t - \frac{1}{2} \frac{1}{S_t^2} S_t^2 dt = (\mu - \delta - \frac{1}{2}) dt + dW_t$$

From innovation representation, the process:

$$\bar{W}_t = L_t - \int_0^t \mathbb{E}[\mu - \delta - \frac{1}{2} | \mathcal{F}_\nu^L] d\nu = L_t - \int_0^t (\Delta \pi_\nu + b - \frac{1}{2}) d\nu$$

is a Brownian motion under the observable filtration $\mathbb{F}^L$.

Transform stock price process under drift uncertain (con't)

After rearranging terms and taking differentiation we find:

$$dL_t = (\Delta\pi_t + b - \frac{1}{2})dt + d\bar{W}_t$$

Again, by Ito's Lemma on $S_t = e^{L_t}$:

$$\begin{aligned} dS_t &= e^{L_t} dL_t + \frac{1}{2} e^{L_t} (dL_t)^2 \\ &= S_t[(\Delta\pi_t + b - \frac{1}{2})dt + d\bar{W}_t] + \frac{1}{2} S_t dt \\ &= (\Delta\pi_t + b) S_t dt + S_t d\bar{W}_t \end{aligned} \tag{3}$$

By applying general Bayes formula, we have:

$$d\pi_t = \Delta\pi_t(1 - \pi_t)d\bar{W}_t \tag{4}$$

Optimal stopping problem fromulation for stock loan

Hence, the optimal redeeming problem defined by 2 together with the underlying process defined by 3 and 4 can be reduced to determine:

$$V(s,\pi,t) = \sup_{\tau \in \mathcal{T}_{[t,T]}} \mathbb{E}[e^{-r(\tau - t)}(S_\tau - Ke^{\gamma\tau})^+ | S_t = s, \pi_t = \pi] \tag{5}$$

for (s,π,t) in the domain:

$$\mathcal{A} = (0,+\infty) \times (0,1) \times [0.T)$$

This is an optimal stopping problem of an observable Markov process $(S_t,\pi_t)_{t\geq 0}$, hence dynamic programming principle can be applied to obtain the corresponding HJB equation. We will see this later.

Problem formulation in discounted stock price process

It's more convenient to study the discounted stock price process $X_t = e^{-\gamma t} S_t$. By Ito's lemma, it follows:

$$\begin{aligned} dX_t &= -\gamma e^{-\gamma t} S_t dt + e^{-\gamma t} dS_t \\ &= -\gamma X_t dt + e^{-\gamma t} S_t (\Delta \pi_t + b) dt + e^{-\gamma t} S_t d\bar{W}_t \\ &= (\Delta \pi_t + b - \gamma) X_t dt + X_t d\bar{W}_t \end{aligned} \tag{6}$$

To adapt the problem in the above discounted stock price process, we introduce the following function:

$$u(x, \pi, t) = \sup_{\tau \in \mathcal{T}_{[t,T]}} \mathbb{E}[e^{(\gamma - r)(\tau - t)} (X_\tau - K)^+ | X_t = x, \pi_t = \pi] \tag{7}$$

Clearly, 7 has the following relationship with 5:

$$V(s, \pi, t) = e^{\gamma t} u(e^{-\gamma t} s, \pi, t), (s, \pi, t) \in \mathcal{A}$$

We call 7 as the value function which will be out main study object

Continuation region and redemption region

Define the continuation region as:

$$\mathcal{C} = \{(x, \pi, t) \in \mathcal{A} | u(x, \pi, t) > (x - K)^+\}$$

which implies that it's not optimal to redeem the loan as the expectation of future payoffs is still higher than the payoff of redeeming the loan.

Define the redeption region as:

$$\mathcal{R} = \{(x, \pi, t) \in \mathcal{A} | u(x, \pi, t) = (x - K)^+\}$$

By the general theory of optimal stopping, the optimal stopping strategy is given by the hitting time of redeeming region $\mathcal{R}$, which can be written as:

$$\tau^* = \inf\{t \in [0, T) | (X_t, \pi_t, t) \in \mathcal{R}\} \tag{8}$$

Continuation region and redemption region (con't)

Note that we can always choose the $\tau = T$ so that $u > 0$, then,

$$\mathcal{C} \supseteq \{(x, \pi, t) \in \mathcal{A} | x \leq K\}, \mathcal{R} \subseteq \{(x, \pi, t) \in \mathcal{A} | x > K\}$$

Clearly, the continuation region $\mathcal{C}$ always non-empty, however, the redemption region $\mathcal{R}$, may be empty or non-empty depends on parameters. This makes sense since the loan will not be redeemed in some cases such as the it's highly likely that $\mu - \delta = a$.

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Properties of value function

We will look at several properties of value function first:

Lemma

The upper bound of the value function is given by:

$$u(x, \pi, t) \leq xe^{(a-r)^{+}(T-t)}, \quad (x, \pi, t) \in \mathcal{A}$$

Lemma

The value function $u(x, \pi, t)$ is non-decreasing in π, non-increasing in t, non-decreasing and convex on x.

Implication from value function's properties

The monotonicity of the value function can be understood intuitively and provides meaningful financial information. From the payoff function of stock loan, we can regard it as an American call option with floating strike price. Therefore, we have

- for x, clearly, the value is high if current stock price is high.
- for t, since the payoff for exercise is $X_t - K$, then clearly, it's non-increasing on t.
- for π,with the probability of bull trend increasing, the value of stock loan will increase because again, it can be regarded as an American call option, the larger the uptrend probability, the higher the price.

Outline

Introduction of dynamic programming

To solve the optimal stopping problem, we may apply the dynamic programming scheme which is similar to pricing an option through stepping back through a tree [13].

Note that it is a simple case of dynamic programming when using the Binomial tree method to price an American option since the only control needed for every node is whether exercise the option. The idea behind is quite essential since it provide us a way to deal with optimal stopping problem in continuous time, where the evolution of the state need to be represented by a Stochastic Differential Equation (SDE).

Apply dynamic programming to our case

- We can discrete our time as $0 = t_0 \leq t_1 \leq t_2 \leq \ldots \leq t_N = T$ such that $\forall i \in [1, T], i \in \mathbb{N}$ we have $t_i - t_{i-1} = \Delta t$.
- In our case, let's assume at time t_i, the stock loan hasn't been redeemed before. Then the value function at time t_i is given by:

$$V_i(s,\pi) = \sup_{\tau \in \mathcal{T}_i} \mathbb{E}[e^{-r(\tau - t_i)}(S_\tau - Ke^{\gamma\tau})^+ | S_i = s, \pi_i = \pi]$$

- At time $t_N = T$, we have that $V_N(s,\pi) = (s - Ke^{\gamma T})^+, S_N = s$, $\pi_N = \pi$, then we move one step back and consider $t = t_{N-1}$. The borrower will only choose to redeem the loan if its current value is higher than the discounted expected payoff when not redeeming. Hence, we have the value of the stock loan in time t_{N-1} is:

$$V_i(s,\pi) = \sup_{\tau \in \mathcal{T}_i} \mathbb{E}[e^{-r(\tau - t_i)}(S_\tau - Ke^{\gamma\tau})^+ | S_i = s, \pi_i = \pi]$$

Applying dynamic programming to our case (con't)

Hence, the recursive system for the stock loan can be shown as follows:

$$\begin{cases} V_N(s,\pi) = (s - Ke^{\gamma T})^+, S_N = s, \pi_N = \pi \\ V_{i-1}(s,\pi) = \max\{(s - Ke^{\gamma t_{i-1}})^+, \\ \qquad \mathbb{E}[e^{-r(t_i - t_{i-1})} V_i(S_i, \pi_i) | S_{i-1} = s, \pi_{i-1} = \pi]\}, i = 1, 2, \ldots, N \end{cases}$$

We can absorb the discounting factor by define the following discounted payoff function and discounted value function of the stock loan:

$$h_i(s,\pi) = e^{-r(t_i - t_0)}(s - Ke^{\gamma t_i})^+$$

$$U_i(s,\pi) = e^{-r(t_i - t_0)} V_i(s,\pi)$$

Therefore, we have the following equations:

$$\begin{cases} U_N(s,\pi) = h_M(s,\pi), S_N = s, \pi_N = \pi \\ U_{i-1}(s,\pi) = \max\{h_{i-1}(s,\pi), \\ \qquad \mathbb{E}[V_i(S_i, \pi_i) | S_{i-1} = S, \pi_i = \pi]\}, i = 1, 2, \ldots, N \end{cases} \tag{9}$$

Continuation value

Also, we define the continuation value here for the stock loan, essentially, it is just the value of not redeeming the stock loan:

$$\begin{cases} C_N(s,\pi) = 0 \\ C_{i-1}(s,\pi) = \mathbb{E}[U_i(S_i,\pi_i)|S_i = s, \pi_1 = \pi], i = 1, 2, \ldots, N \end{cases} \tag{10}$$

Variational inequality

By applying dynamic programming scheme, we can obtain the variational inequality as below:

$$\begin{cases} \min\{-\mathcal{L}u, u-(x-K)^+\}=0, & (x,\pi,t)\in\mathcal{A}; \\ u(x,\pi,T)=(x-K)^+, \end{cases} \tag{11}$$

where

$$\begin{aligned}\mathcal{L}u = u_t &+ \frac{1}{2}x^2u_{xx} + \frac{1}{2}\Delta^2\pi^2(1-\pi)^2u_{\pi\pi} + \Delta\pi(1-\pi)xu_{x\pi} \\ &+ (\Delta\pi + b - \gamma)xu_x + (\gamma - r)u\end{aligned}$$

Variational inequality (con't)

The operator $\mathcal{L}$ in 11 is parabolic and which is degenerate in the whole domain $\mathcal{A}$. The regularity of $u(x, \pi, t)$ is not good enough to let 11 hold almost everywhere in $\mathcal{A}$. In contrast, it only holds in weak sense or viscosity sense. We will not discuss the weak solution or viscous solution for 11 in this report. The process for obtaining weak solution is long and tough, see [3] for more details.

Outline

Redeeming boundary

Define the following functions:

$$X(\pi, t) = \min\{x | u(x, \pi, t) = (x - K)^+, (x, \pi, t) \in \mathcal{A}\}$$

$$Y(\pi, t) = \max\{x | u(x, \pi, t) = (x - K)^+, (x, \pi, t) \in \mathcal{A}\}$$

Clearly, the continuation region and redemption region can be reduced as:

$$\mathcal{C} = \{(x, \pi, t) \in \mathcal{A} | x < X(\pi, t) \text{ or } x > Y(\pi, t)\},$$

$$\mathcal{R} = \{(x, \pi, t) \in \mathcal{A} | X(\pi, t) \leq x \leq Y(\pi, t)\}$$

By similar argument, we can represent the redemption region and continuation region by π. Define

$$\Pi(x, t) = \max\{\pi | u(x, \pi, t) = (x - K)^+, (x, \pi, t) \in \mathcal{A}\}$$

It is easy to see the following:

$$\mathcal{C} = \{(x, \pi, t) \in \mathcal{A} | \pi > \Pi(x, t)\},$$

$$\mathcal{R} = \{(x, \pi, t) \in \mathcal{A} | \pi \leq \Pi(x, t)\}$$

Case 1: $r > a$

In this case, we can see that the return of the stock is too low compared with the loan rate. Therefore, we have the following redemption region and continuation region:

$$\mathcal{C} = \{(x, \pi, t) \in \mathcal{A} | x < X(\pi, t)\}$$

$$\mathcal{R} = \{(x, \pi, t) \in \mathcal{A} | x \geq X(\pi, t)\}$$

Therefore, if the stock return is too low, then we need to wait until the stock price is high enough to redeem the stock loan.

Case 2: $b > \gamma > r$

In this case, we can see that the return of stock is much higher than the loan rate and discounting rate. Intuitively, we don't want to redeem the loan until maturity. From the property of $\mathcal{L}$, we can see that if $x \geq K$, then

$$
\begin{aligned}
-\mathcal{L}(x-K) &= -\mathcal{L}x + \mathcal{L}K \\
&= (r - (\Delta\pi + b))x + (\gamma - r)K \\
&< (r - b)x + (\gamma - r)K \\
&\leq (r - b)K + (\gamma - r)K \\
&\leq 0
\end{aligned}
$$

Then clearly, $\mathcal{R} = \emptyset$ and $\mathcal{C} = \mathcal{A}$. This implies that the it's never optimal to redeem the loan before maturity.

Outline

Basic principle of FDM

- Essentially, finite difference method is to discrete the space into grids and calculate the value at each point.
- The starting point's value is calculated by the boundary condition.
- From the variational inequality 11, the main part we need to deal with is $\mathcal{L}u$. We simply use the backward finite difference scheme.
- Basically, we want to know the value of u for every x and π in a given time point, then move to next time point.

Transform time variable

We introduce a new time variable $\tau = T - t$ for simplicity and rewrite 11 as

$$\begin{cases} \min\{-\bar{\mathcal{L}}u, u - (x-K)^+\} = 0, & (x, \pi, \tau) \in A = (0,\infty) \times (0,1) \times [0, T); \\ u(x, \pi, 0) = (x-K)^+, \end{cases} \tag{12}$$

Note that the time range has changed from $(0, T]$ to $[0, T)$ and therefore, we can start from time 0, which is more convenient.

Discrete space variables

We first discrete the space variables x and π. The first problem is that x is unbounded, which makes it hard to implement. The solution we propose is to give a reasonable upper bound of x, denoted as x^∞. Now we can discrete our bounded space into the following way:

$$0 = x_0 < x_1 < x_2 < \cdots < x_{N_1} = x^\infty, \quad x_j - x_{j-1} = h_1, \forall j \in [1, N_1], j \in \mathbb{N}$$

$$0 = \pi_0 < \pi_1 < \pi_2 < \cdots < \pi_{N_2} = 1, \quad \pi_k - \pi_{k-1} = h_2, \forall k \in [1, N_2], k \in \mathbb{N}$$

Backward approximation

$$u_\tau\big|_{x_j,\pi_k,\tau_n} \approx \frac{u_{jk}^{n+1} - u_{jk}^n}{h} \tag{13}$$

$$u_{xx}\big|_{x_j,\pi_k,\tau_n} \approx \frac{u_{j+1,k}^n - 2u_{jk}^n + u_{j-1,k}^n}{h_1^2} \tag{14}$$

$$u_{\pi\pi}\big|_{x_j,\pi_k,\tau_n} \approx \frac{u_{j,k+1}^n - 2u_{jk}^n + u_{j,k-1}^n}{h_2^2} \tag{15}$$

$$u_{x\pi}\big|_{x_j,\pi_k,\tau_n} \approx \frac{u_{j+1,k+1}^n - u_{j+1,k}^n - u_{jk}^n + u_{j,k-1}^n}{h_1 h_2} \tag{16}$$

$$u_x\big|_{x_j,\pi_k,\tau_n} \approx \frac{u_{j+1,k}^n - u_{jk}^n}{h_1} \tag{17}$$

Approximate variational inequality

Then the discrete version of $\bar{\mathcal{L}}u$ at (x_j, π_k, τ_n) is given by:

$$\bar{\mathcal{L}}u|_{x_j,\pi_k,\tau_n} = -u_\tau|_{x_j,\pi_k,\tau_n} + \frac{1}{2}x_j^2 u_{xx}|_{x_j,\pi_k,\tau_n} + \frac{1}{2}\Delta^2\pi_k^2(1-\pi_k)^2 u_{\pi\pi}|_{x_j,\pi_k,\tau_n} + \Delta\pi_k(1-\pi_k)u_{x\pi}|_{x_j,\pi_k,\tau_n} + (\Delta\pi_k + b_\gamma)x_j u_x|_{x_j,\pi_k,\tau_n} + (\gamma - r)u|_{x_j,\pi_k,\tau_n} \quad (18)$$

where the approximations are provided by 13 $\sim$ 17. Therefore, we obtain the equation links time $\tau_0 = 0$ with next time point τ_1:

$$\min\{u_{jk}^1 - (x_j - K)^+ - \frac{h}{2h_1^2}x_j^2((x_{j+1} - K)^+ - 2(x_j - K)^+ + (x_{j-1} - K)^+) - \frac{h}{h_1}(\Delta\pi_k + b_\gamma)x_j((x_{j+1} - K)^+ - (x_j - K)^+), h(u_{jk}^1 - (x_j - K)^+)\} = 0,$$
$$n = 0, \forall j \in [1, N_1 - 1], k \in [1, N_2 - 1], j, k \in \mathbb{N} \quad (19)$$

Now, we are able to compute all u_{jk}^1 and we can move forward to compute all u_{jk}^2 by the similar approach as 19. Just note that $u_{\pi\pi}|_{x_j,\pi_k,\tau_n}$ and $u_{x\pi}|_{x_j,\pi_k,\tau_n}$ can not be vanished and the recursive equation will be more complicated. Keep moving forward, we will obtain values of all u_{jk}^N, which is the numerical solution for $u(x,\pi,T)$.

Therefore, solving 11 can reduced to solve the following system:

$$\begin{cases} \min\{-\bar{\mathcal{L}}u|_{x_j,\pi_k,\tau_n}, u_{jk}^{n+1} - (x_j - K)^+\} = 0; \\ u(x,\pi,0) = (x-K)^+; \\ 0 = x_0 < x_1 < x_2 < \cdots < x_{N_1} = x^\infty,\ x_j - x_{j-1} = h_1, \forall j \in [1, N_1], j \in \mathbb{N}; \\ 0 = \pi_0 < \pi_1 < \pi_2 < \cdots < \pi_{N_2} = 1,\ \pi_k - \pi_{k-1} = h_2, \forall k \in [1, N_2], k \in \mathbb{N}; \\ 0 = \tau_0 < \tau_1 < \tau_2 < \cdots < \tau_N = T,\ t_n - t_{n-1} = h, \forall n \in [1, N], n \in \mathbb{N}, \end{cases} \tag{20}$$

Outline

Advantages of MC

By the nature of simulation, it is an ideal alternative to FDM in many perspectives.

- It allows the underlying follow different stochastic processes such as jump diffusions and also supports multiple factors.
- For American-style derivatives, it provides enough information for the pricing and optimal exercising.
- When comes to practical issue, it is a good fit for parallel programming to improve the efficiency.

General MC procedure

Step 1 Discrete time and generate stock price path according to the price distribution.

Step 2 Compute the payoff for every path in final time point.

Step 3 Calculate the price at time 0 as the expectation of discounted final payoff.

The difficulty for American-style product is to determine whether to exercise at each time point. Longstaff and Schwartz [7] proposed a regression based Monte Carlo method to price American options, namely least square approach. We will apply it to our stock loan model to get the estimation of u.

Generate stock price process

We have the following stock price process:

$$\begin{aligned} d\pi_t &= \Delta\pi_t(1-\pi_t)d\bar{W}_t \\ dS_t &= (\Delta\pi_t + b)S_t dt + S_t d\bar{W}_t \end{aligned} \tag{21}$$

Therefore, we can write it in discrete version as:

$$\begin{aligned} \pi_i &= \pi_{i-1} + \Delta\pi_{i-1}(1-\pi_{i-1})(\Delta\bar{W}_i) \\ S_i &= S_{i-1} + (\Delta\pi_{i-1} + b)S_{i-1}(\Delta t) + S_{i-1}(\Delta\bar{W}_i) \end{aligned} \tag{22}$$

where $\Delta t = t_i - t_{i-1}$ and $\Delta\bar{W}_i = \bar{W}_i - \bar{W}_{i-1}$ is the independent Brownian increment which follows normal distribution $\mathcal{N}(0, \Delta t)$, $i = 1, 2, \ldots, N$. The initial stock price can be obtained and denoted by $S_0 = s$. We need to estimate the initial probability of bull trend, namely $\pi_0 = \pi$.

Regression for continuation value

According to Longstaff and Schwartz [7], we want to approximate the continuation value by the linear combination of some basis function value. Therefore, our goal is to find the continuation value by regression, i.e.

$$C_i(s) = \sum_{j=1}^{J} \beta_{ij}\psi_j(s) = \beta_i^T \psi(s)$$

where β_i is a vector of regression coefficient depending on t_i and $\psi(s)$ is some basis function, can be as simple as trigonometric series. The regression coefficients are determined by least square method.

Regression for continuation value (con't)

According to Wu [11], we give the least square estimator of β_i as follows:

Theorem

The least square estimator is given by

$$\beta_i = (\mathbb{E}[\psi(S_i)\psi(S_i)^T])^{-1}\mathbb{E}[\psi(S_i)U_{i+1}(S_{i+1})] = B_\psi^{-1}B_{\psi V} \tag{23}$$

where $B_\psi = \mathbb{E}[\psi(S_i)\psi(S_i)^T]$ *is a* $J \times J$ *matrix and* $B_{\psi V} = \mathbb{E}[\psi(S_i)U_{i+1}(S_{i+1})]$ *is a* $J \times 1$ *vector.*

The proof is outlined here. Essentially we want to minimize

$$\mathbb{E}[(\psi(S_i)^T\beta_i - \mathbb{E}[U(S_{i+1})|S_i])^2]$$

By taking derivatives and setting it to zero, after rearranging terms, we should be able to derive the above result.

Regression for continuation value (con't)

Suppose we have M independent stock price paths denoted by $(S_1^m, S_2^m, \ldots, S_n^m), m = 1, 2, \ldots, M$. Then 23 can be approximated by

$$\begin{aligned}
\hat{\beta}_i &= \hat{B}_{\psi,i}^{-1} \hat{B}_{\psi V,i}, \\
\hat{B}_{\psi,i} &= \frac{1}{M} \sum_{m=1}^{M} \psi(S_i^m) \psi(S_i^m)^T, \\
\hat{B}_{\psi V,i} &= \frac{1}{M} \sum_{m=1}^{M} \psi(S_i^m) U_{i+1}(S_{i+1}^m)
\end{aligned} \tag{24}$$

Also, note that $U_{i+1}(S_{i+1}^m)$ need to be replaced by $\hat{U}_{i+1}(S_{i+1}^m) = \max\{h_{i+1}(S_{i+1}^m), \hat{C}_{i+1}(S_{i+1}^m)\}$ which is the estimated value of $U_{i+1}(S_{i+1}^m)$. Finally, the the continuation value is estimated by

$$\hat{C}_i(S_i) = \hat{\beta}_i^T \psi(S_i) \tag{25}$$

Summary of regression method

Step 1. Generate M independent stock price paths $(S_1^m, S_2^m, \ldots, S_n^m)$, $m = 1, 2, \ldots, M$

Step 2. In final time point t_N, set $\hat{U}_N(S_N^m) = h_N(S_N^m)$

Step 3. Apply backward induction, i.e. for $i = N-1, \ldots, 2, 1$:

i. Estimate the regression coefficient $\hat{\beta}_i$ which is given by 24.

ii. Calculate the continuation value for each path.

iii. Set the value of stock loan at time t_i as $\hat{U}_i(S_i^m) = \max\{h_i(S_i^m), \hat{C}_i(S_i^m)\}$ for each path.

Step 4. Store $\hat{\beta}_i$ for further use.

Pricing algorithm

Step 1. Load the estimated regression coefficients $\hat{\beta}_i$.

Step 2. Generate M independent stock price paths $(S_1^m, S_2^m, \ldots, S_n^m)$, $m = 1, 2, \ldots, M$. Note that this is a new set of paths which is independent from the one used in calculating $\hat{\beta}_i$.

Step 3. Work forward, for $i = 1, 2, \ldots, N-1$:

(a) Compute continuation value $\hat{C}_i(S_i^m) = \hat{\beta}_i^T \psi(S_i^m)$ for each $m = 1, 2, \ldots, M$.

(b) Compute the payoff of the stock loan $h_i(S_i^m)$.

Step 4. At the final time point t_N, set the continuation value to be 0.

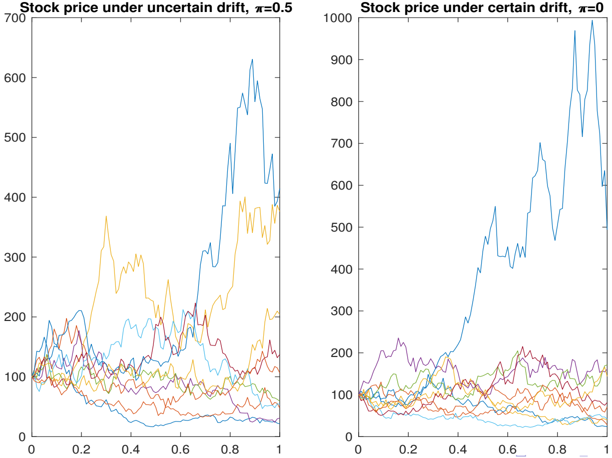
Step 5. For each path, the value of stock loan at time t_0 is given by the optimal stopping rule, i.e. $\hat{U}_0^m = h_{i^*}(S_{i*}^m)$ where the $i^* = \min\{i \in \{1, 2, \ldots, N\} : h_i(S_i^m) \geq \hat{C}_i(S_i^m)\}$.

Step 6. The price of the stock loan is computed by the average value of all paths, i.e. $\hat{U}_0 = \frac{1}{M}\sum_{m=1}^{M} \hat{U}_0^m$

Numerical results

stock price under certain and uncertain drift

Here is the graph for stock price paths under uncertain drift and certain drift with $S_0 = 100, a = 0.08, b = 0.02, T = 1, N = 100, M = 10$:



Numerical results

Stock loan price change under different S

We choose our basis function to be a set of Laguerre polynomials just as the settings in Longstaff and Schwartz's paper [7]. The Laguerre polynomials are defined as:

$$\begin{aligned} L_0(x) &= e^{-x/2} \\ L_1(x) &= e^{-x/2}(1-x) \\ L_2(x) &= e^{-x/2}(1-2x+\frac{x^2}{2}) \\ L_k(x) &= e^{-x/2}\frac{e^x}{k!}\frac{d^k}{dx^k}(x^k e^{-x}) \end{aligned} \tag{26}$$

Hence, the basis functions are chosen as

$$\psi_j(x) = L_k(x), j = k+1$$

Numerical results

For illustration purpose, we choose the J to be 5. For the Monte-Carlo simulation, we set $N = 2^6 = 64, M = 10^4$.
Other parameters are $T = 1, K = 30, r = 0.03$. Here is the case when $a = 0.1, b = 0.06, \gamma = 0.04$

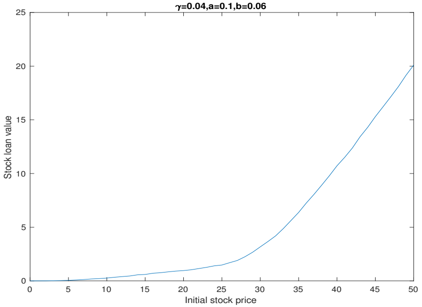


Figure: Stock loan value vs initial stock price with fixed initial estimation of $\pi = 0.5$

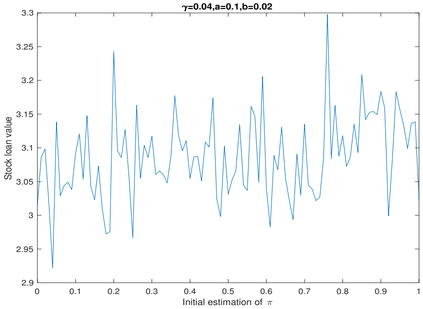


Figure: Stock loan value vs initial estimation of π with fixed initial stock price

Numerical method implication

Implied π

From previous discussion, we see that we need to estimate the initial probability π, however, given the market price of stock loan, we are more interested in whether it is possible to know the current π. Clearly, the above two numerical methods can also be used to obtain an estimation of π. Here we just present a simple algorithm using Monte Carlo simulation to get an estimation of π.

Suppose we have observed the stock loan price in the market for a specific (s, π, T) and denoted as $U^M(s, \pi, T)$. Then the estimation of π essentially is to solve the equation defined by the above. Since $\pi \in (0, 1)$, we can simply discrete the interval as a large enough points set. Then for every point, we check the stock loan price based on this point with actual price. Stop the process until the accuracy level is fulfilled.

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