1 Syntax

$e ::=$
$\quad | \; x \mid (e \;:\; e)$
$\quad | \; (\lambda \; b \; e) \mid (e \; e) \mid (\to \; t \; e)$
$\quad | \; (\text{Type} \; n \; e) \mid (\text{Level} \; n) \mid \text{lzero} \mid (\text{lsucc} \; e) \mid (\text{lmax} \; e \; e)$
$\quad | \; \text{Empty} \mid (\text{ind-Empty} \; e \; e \; e)$
$\quad | \; \text{Unit} \mid ()$
$\quad | \; (= \; e \; e \; e) \mid \text{refl} \mid (\text{ind-}= \; e \; e \; e \; e \; e)$
$\quad | \; (\text{W} \; t \; e) \mid (\text{w} \; e \; e) \mid (\text{ind-W} \; e \; e \; e \; e)$

$b ::= x \mid t$

$t ::= [x : e]$

2 Typing

$\boxed{\Gamma \vdash e \Rightarrow A}$ meaning e synthesizes type A under environment Γ
$\boxed{\Gamma \vdash e \Leftarrow A}$ meaning e checks against type A under environment Γ

2.1 Misc

$$\frac{x : A \in \Gamma}{\Gamma \vdash x \Rightarrow A} \textsc{Var} \qquad \frac{\Gamma \vdash A \Rightarrow (\text{Type} \; n \; l) \qquad \Gamma \vdash e \Leftarrow A}{\Gamma \vdash (e \;:\; A) \Rightarrow A} \textsc{Ann}$$

$$\frac{\Gamma \vdash e \Rightarrow A}{\Gamma \vdash e \Leftarrow A} \textsc{CheckSynth}$$

2.2 Functions

$$\frac{\Gamma \vdash b \Leftarrow A \qquad \Gamma, b_x : A \vdash e \Leftarrow B[b_x/x]}{\Gamma \vdash (\lambda \; b \; e) \Leftarrow (\to \; [x : A] \; B)} \lambda\text{-I-}\textsc{Check}$$

$$\frac{\Gamma \vdash b \Rightarrow A \qquad \Gamma, b_x : A \vdash e \Rightarrow B}{\Gamma \vdash (\lambda \; b \; e) \Rightarrow (\to \; [b_x : A] \; B)} \lambda\text{-I-}\textsc{Synth}$$

$$\frac{\Gamma \vdash f \Rightarrow (\to \; [x : A] \; B) \qquad \Gamma \vdash a \Leftarrow A}{\Gamma \vdash (f \; a) \Rightarrow B[a/x]} \lambda\text{-E}$$

$$\frac{\Gamma \vdash A \Rightarrow (\text{Type} \; n_A \; l_B) \qquad \Gamma, x : A \vdash B \Rightarrow (\text{Type} \; n_B \; l_B)}{\Gamma \vdash (\to \; [x : A] \; B) \Rightarrow \text{tmax}(n_A, n_B, l_A, l_B)} \to\text{-I}$$

2.3 Universes

$$\frac{\Gamma \vdash l \Leftarrow (\text{Level } n)}{\Gamma \vdash (\text{Type } n\ l) \Rightarrow (\text{Type } n\ (\text{lsucc } l))}\ \textsc{Type}$$

$$\frac{}{\Gamma \vdash (\text{Level } n) \Leftarrow (\text{Type } (n+1)\ l)}\ \textsc{Level} \qquad \frac{}{\Gamma \vdash \text{lzero} \Leftarrow (\text{Level } n)}\ \textsc{lzero}$$

$$\frac{\Gamma \vdash l \Leftarrow (\text{Level } n)}{\Gamma \vdash (\text{lsucc } l) \Leftarrow (\text{Level } n)}\ \textsc{lsucc}$$

$$\frac{\Gamma \vdash l_A \Leftarrow (\text{Level } n) \qquad \Gamma \vdash l_B \Leftarrow (\text{Level } n)}{\Gamma \vdash (\text{lmax } l_A\ l_B) \Leftarrow (\text{Level } n)}\ \textsc{lmax}$$

2.4 Base Types

$$\frac{}{\Gamma \vdash \text{Empty} \Leftarrow (\text{Type } n\ l)}\ \textsc{Empty}$$

$$\frac{\Gamma \vdash l \Rightarrow (\text{Level } n) \qquad \Gamma \vdash t \Leftarrow \text{Empty} \qquad \Gamma \vdash m \Leftarrow (\to [t : \text{Empty}]\ (\text{Type } n\ l))}{\Gamma \vdash (\text{ind-Empty } l\ t\ m) \Rightarrow (m\ t)}\ \textsc{Empty-E}$$

$$\frac{}{\Gamma \vdash \text{Unit} \Leftarrow (\text{Type } n\ l)}\ \textsc{Unit} \qquad \frac{}{\Gamma \vdash () \Rightarrow \text{Unit}}\ \textsc{Unit-I}$$

2.5 Propositional Equality

$$\frac{\Gamma \vdash A \Rightarrow (\text{Type } n\ l) \qquad \Gamma \vdash a \Leftarrow A \qquad \Gamma \vdash b \Leftarrow A}{\Gamma \vdash (=\ A\ a\ b) \Rightarrow (\text{Type } n\ l)}\ \textsc{Eq}$$

$$\frac{}{\Gamma \vdash \text{refl} \Leftarrow (=\ A\ a\ a)}\ \textsc{Eq-I}$$

$$\frac{\Gamma \vdash l \Rightarrow (\text{Level } n) \qquad \Gamma \vdash b \Rightarrow A \qquad \Gamma \vdash t \Rightarrow (=\ A\ a\ b) \qquad \Gamma \vdash m \Rightarrow (\to [b : A]\ (\to [t : (=\ A\ a\ b)]\ (\text{Type } n\ l)))}{\Gamma \vdash (\text{ind-=}\ l\ b\ t\ m\ r) \Rightarrow ((m\ b)\ t)}\ \textsc{Eq-E}$$

2.6 Inductive Data

$$\frac{\Gamma \vdash T \Rightarrow (\text{Type } n_T\ l_T) \qquad \Gamma, x : T \vdash B \Rightarrow (\text{Type } n_B\ l_B)}{\Gamma \vdash (\text{W } [x : T]\ B) \Rightarrow \text{tmax}(n_T, n_B, l_T, l_B)}\ \text{W}$$

$$\frac{\Gamma \vdash t \Leftarrow T \qquad \Gamma \vdash d \Leftarrow (\rightarrow [\_ : B[t/x]]\ (\text{W } [x : T]\ B))}{\Gamma \vdash (\text{w}\ t\ d) \Leftarrow (\text{W } [x : T]\ B)}\ \text{W-I}$$

$$\frac{\begin{array}{c}\Gamma \vdash l \Rightarrow (\text{Level } n) \qquad \Gamma \vdash t \Rightarrow (\text{W } [x : T]\ B) \\ \Gamma \vdash m \Leftarrow (\rightarrow [t : (\text{W } [x : T]\ B)]\ (\text{Type } n\ l)) \\ \Gamma \vdash e \Leftarrow \\ (\rightarrow [t : T]\ (\rightarrow [d : (\rightarrow [\_ : B[t/x]]\ (\text{W } [x : T]\ B))\ (\rightarrow [i : (\rightarrow [t : B[t/x]]\ (m\ (d\ t)))]\ (m\ (\text{w}\ t\ d)))))\end{array}}{\Gamma \vdash (\text{ind-W}\ l\ t\ m\ e) \Rightarrow (m\ t)}\ \text{W-E}$$

2.7 Bindings

$\boxed{\Gamma \vdash b \Rightarrow A}$ meaning b synthesizes type A under environment Γ
$\boxed{\Gamma \vdash b \Leftarrow A}$ meaning b checks against type A under environment Γ

$$\frac{\Gamma \vdash A \Rightarrow (\text{Type } n\ l)}{\Gamma \vdash x \Leftarrow A}\ \textsc{CheckUntyped} \qquad \frac{\Gamma \vdash A \Rightarrow (\text{Type } n\ l)}{\Gamma \vdash [x : A] \Leftarrow A}\ \textsc{CheckTyped}$$

$$\frac{\Gamma \vdash A \Rightarrow (\text{Type } n\ l)}{\Gamma \vdash [x : A] \Rightarrow A}\ \textsc{SynthTyped}$$

2.8 Type Maximum

$\boxed{tmax(n_A, n_B, l_A, l_B)}$ calculates a type representing the maximum of the two other types

$$tmax(n_A, n_B, l_A, l_B) = (\text{Type } n_A\ (\text{lmax } l_A\ l_B)) \text{ if } n_A = n_B$$
$$tmax(n_A, n_B, l_A, l_B) = (\text{Type } n_A\ l_A) \text{ if } n_A > n_B$$
$$tmax(n_A, n_B, l_A, l_B) = (\text{Type } n_B\ l_B) \text{ if } n_A < n_B$$