

1 Syntax

$e ::=$
 $| x \mid (e : e)$
 $| (\lambda b e) \mid (e e) \mid (\rightarrow t e)$
 $| (\text{Type } n e) \mid (\text{Level } n) \mid \text{lzero} \mid (\text{lsucc } e) \mid (\text{lmax } e e)$
 $| \text{Empty} \mid (\text{ind-Empty } e e e)$
 $| \text{Unit} \mid ()$
 $| (= e e e) \mid \text{refl} \mid (\text{ind-} = e e e e e)$
 $| (\text{W } t e) \mid (\text{w } e e) \mid (\text{ind-W } e e e e e)$

$b ::= x \mid t$

$t ::= [x : e]$

2 Typing

$\boxed{\Gamma \vdash e \Rightarrow A}$ meaning e synthesizes type A under environment Γ
 $\boxed{\Gamma \vdash e \Leftarrow A}$ meaning e checks against type A under environment Γ

2.1 Misc

$$\frac{x : A \in \Gamma}{\Gamma \vdash x \Rightarrow A} \text{VAR} \qquad \frac{\Gamma \vdash A \Rightarrow (\text{Type } n l) \quad \Gamma \vdash e \Leftarrow A}{\Gamma \vdash (e : A) \Rightarrow A} \text{ANN}$$

$$\frac{\Gamma \vdash e \Rightarrow A}{\Gamma \vdash e \Leftarrow A} \text{CHECKSYNTH}$$

2.2 Functions

$$\frac{\Gamma \vdash b \Leftarrow A \quad \Gamma, b_x : A \vdash e \Leftarrow B[b_x/x]}{\Gamma \vdash (\lambda b e) \Leftarrow (\rightarrow [x : A] B)} \lambda\text{-I-CHECK}$$

$$\frac{\Gamma \vdash b \Rightarrow A \quad \Gamma, b_x : A \vdash e \Rightarrow B}{\Gamma \vdash (\lambda b e) \Rightarrow (\rightarrow [b_x : A] B)} \lambda\text{-I-SYNTH}$$

$$\frac{\Gamma \vdash f \Rightarrow (\rightarrow [x : A] B) \quad \Gamma \vdash a \Leftarrow A}{\Gamma \vdash (f a) \Rightarrow B[a/x]} \lambda\text{-E}$$

$$\frac{\Gamma \vdash A \Rightarrow (\text{Type } n_A l_B) \quad \Gamma, x : A \vdash B \Rightarrow (\text{Type } n_B l_B)}{\Gamma \vdash (\rightarrow [x : A] B) \Rightarrow \text{tmax}(n_A, n_B, l_A, l_B)} \rightarrow\text{-I}$$

2.3 Universes

$$\begin{array}{c}
\frac{\Gamma \vdash l \Leftarrow (\text{Level } n)}{\Gamma \vdash (\text{Type } n \ l) \Rightarrow (\text{Type } n \ (\text{lsucc } l))} \text{TYPE} \\
\\
\frac{}{\Gamma \vdash (\text{Level } n) \Leftarrow (\text{Type } (n+1) \ l)} \text{LEVEL} \qquad \frac{}{\Gamma \vdash \text{lzero} \Leftarrow (\text{Level } n)} \text{LZERO} \\
\\
\frac{\Gamma \vdash l \Leftarrow (\text{Level } n)}{\Gamma \vdash (\text{lsucc } l) \Leftarrow (\text{Level } n)} \text{LSUCC} \\
\\
\frac{\Gamma \vdash l_A \Leftarrow (\text{Level } n) \quad \Gamma \vdash l_B \Leftarrow (\text{Level } n)}{\Gamma \vdash (\text{lmax } l_A \ l_B) \Leftarrow (\text{Level } n)} \text{LMAX}
\end{array}$$

2.4 Base Types

$$\begin{array}{c}
\frac{}{\Gamma \vdash \text{Empty} \Leftarrow (\text{Type } n \ l)} \text{EMPTY} \\
\\
\frac{\Gamma \vdash l \Rightarrow (\text{Level } n) \quad \Gamma \vdash t \Leftarrow \text{Empty} \quad \Gamma \vdash m \Leftarrow (\rightarrow [t : \text{Empty}] (\text{Type } n \ l))}{\Gamma \vdash (\text{ind-Empty } l \ t \ m) \Rightarrow (m \ t)} \text{EMPTY-E} \\
\\
\frac{}{\Gamma \vdash \text{Unit} \Leftarrow (\text{Type } n \ l)} \text{UNIT} \qquad \frac{}{\Gamma \vdash () \Rightarrow \text{Unit}} \text{UNIT-I}
\end{array}$$

2.5 Propositional Equality

$$\begin{array}{c}
\frac{\Gamma \vdash A \Rightarrow (\text{Type } n \ l) \quad \Gamma \vdash a \Leftarrow A \quad \Gamma \vdash b \Leftarrow A}{\Gamma \vdash (= A \ a \ b) \Rightarrow (\text{Type } n \ l)} \text{EQ} \\
\\
\frac{}{\Gamma \vdash \text{refl} \Leftarrow (= A \ a \ a)} \text{EQ-I} \\
\\
\frac{\Gamma \vdash l \Rightarrow (\text{Level } n) \quad \Gamma \vdash b \Rightarrow A \quad \Gamma \vdash t \Rightarrow (= A \ a \ b) \quad \Gamma \vdash m \Rightarrow (\rightarrow [b : A] (\rightarrow [t : (= A \ a \ b)] (\text{Type } n \ l)))}{\Gamma \vdash (\text{ind-} = \ l \ b \ t \ m \ r) \Rightarrow ((m \ b) \ t)} \text{EQ-E}
\end{array}$$

2.6 Inductive Data

$$\begin{array}{c}
\frac{\Gamma \vdash T \Rightarrow (\text{Type } n_T \ l_T) \quad \Gamma, x : T \vdash B \Rightarrow (\text{Type } n_B \ l_B)}{\Gamma \vdash (\text{W } [x : T] \ B) \Rightarrow \text{tmax}(n_T, n_B, l_T, l_B)} \text{ W} \\
\\
\frac{\Gamma \vdash t \Leftarrow T \quad \Gamma \vdash d \Leftarrow (\rightarrow [- : B[t/x]] (\text{W } [x : T] \ B))}{\Gamma \vdash (\text{w } t \ d) \Leftarrow (\text{W } [x : T] \ B)} \text{ W-I} \\
\\
\begin{array}{c}
\Gamma \vdash l \Rightarrow (\text{Level } n) \quad \Gamma \vdash t \Rightarrow (\text{W } [x : T] \ B) \\
\Gamma \vdash m \Leftarrow (\rightarrow [t : (\text{W } [x : T] \ B)] (\text{Type } n \ l)) \\
\Gamma \vdash e \Leftarrow \\
(\rightarrow [t : T] (\rightarrow [d : (\rightarrow [- : B[t/x]] (\text{W } [x : T] \ B))] (\rightarrow [i : (\rightarrow [t : B[t/x]] (m \ (d \ t)))] (m \ (\text{w } t \ d)))))
\end{array} \\
\hline
\Gamma \vdash (\text{ind-W } l \ t \ m \ e) \Rightarrow (m \ t) \text{ W-E}
\end{array}$$

2.7 Bindings

$$\begin{array}{l}
\boxed{\Gamma \vdash b \Rightarrow A} \text{ meaning } b \text{ synthesizes type } A \text{ under environment } \Gamma \\
\boxed{\Gamma \vdash b \Leftarrow A} \text{ meaning } b \text{ checks against type } A \text{ under environment } \Gamma
\end{array}$$

$$\begin{array}{c}
\frac{\Gamma \vdash A \Rightarrow (\text{Type } n \ l)}{\Gamma \vdash x \Leftarrow A} \text{ CHECKUNTYPED} \qquad \frac{\Gamma \vdash A \Rightarrow (\text{Type } n \ l)}{\Gamma \vdash [x : A] \Leftarrow A} \text{ CHECKTYPED} \\
\\
\frac{\Gamma \vdash A \Rightarrow (\text{Type } n \ l)}{\Gamma \vdash [x : A] \Rightarrow A} \text{ SYNTHTYPED}
\end{array}$$

2.8 Type Maximum

$\boxed{\text{tmax}(n_A, n_B, l_A, l_B)}$ calculates a type representing the maximum of the two other types

$$\begin{array}{l}
\text{tmax}(n_A, n_B, l_A, l_B) = (\text{Type } n_A \ (\text{lmax } l_A \ l_B)) \text{ if } n_A = n_B \\
\text{tmax}(n_A, n_B, l_A, l_B) = (\text{Type } n_A \ l_A) \text{ if } n_A > n_B \\
\text{tmax}(n_A, n_B, l_A, l_B) = (\text{Type } n_B \ l_B) \text{ if } n_A < n_B
\end{array}$$