Linear Algebra Assignment - 6

Q 1.1 → Prove that $X^T X$ is invertible if and only if the columns of X are linear independent.

(Forward implication)

To prove: columns of X are linear independent

Given: $X^T X$ is invertible

Proof: let's solve for below eqn

$Xu = 0$ where u coefficient matrix

~~$Xu = 0$~~

multiply X^T on both sides

$X^T(Xu) = X^T \cdot 0$

$(X^T X)u = 0$

but as $X^T X$ is invertible

$\Rightarrow (X^T X)u$ has only 1 soln which is trivial soln

$\Rightarrow Xu = 0$ ~~is 0~~ has soln when $u = 0$

let $c_1, c_2, \ldots c_n$ be columns of X

and $a_1, a_2, \ldots a_n$ be coefficients of u

$a_1 c_1 + a_2 c_2 + \ldots a_3 c_3 + \ldots a_n c_n = 0$

only when $a_1 = a_2 = a_3 = \ldots a_n = 0$

so acc to definition of linear independent, we can henceforth say that, columns of X are linear independent.

Hence, proved

(Backward implication)

To prove: $X^T X$ is invertible

Given: columns of X are linear independent

Proof:

$X^T X v = 0$, let's solve for this
multiply by v^T on both sides

$v^T X^T X v = 0$

$(Xv)^T (Xv) = 0$

$\Rightarrow (Xv) = 0$

$\Rightarrow X(v) = 0$, v lies in null space of x

columns of

but x is linear independent

$\Rightarrow$ zero vector is only vector in nullspace of x

$\Rightarrow v = 0$

$\Rightarrow X^T X v = 0$ has only 1 solⁿ that is $v = 0$

which is trivial solⁿ

so by theorem invertible matrix has only trivial solⁿ and vice-versa

$\therefore$ $X^T X$ is invertible

Hence, proved

Q1.2) Find the solution for β if the matrix X is decomposed as $X = QR$ using QR decomposition, where Q is an orthogonal matrix and R is an upper triangular matrix

To find: Solution for β when $X = QR$

Given: $Q \rightarrow$ orthogonal matrix and R is an upper triangular matrix.

$X\beta = Y$ for linear regression.

Soln:

$$Y = X\beta$$

$$(QR)\beta = Y$$

$Q^T \rightarrow$ transpose of Q

Multiply on both sides

$$Q^T(QR)\beta = Q^T Y$$

$$(Q^T Q)R\beta = Q^T Y$$

acc to property of orthogonal matrix

$$A^T A = A A^T = I$$

$$R\beta = Q^T Y$$

multiply R^{-1} on both sides

$$(R^{-1}R)\beta = R^{-1}Q^T Y$$

$$I\beta = (R^{-1}Q^T)Y$$

$$\beta = (R^{-1}Q^T)Y$$

Q1.3) Find the solution for β if the matrix X is decomposed as $X = U\Delta V^T$ using Singular Value Decomposition, where U and V are orthogonal matrices and Δ is a diagonal matrix.

To find: Solution for β when $X = U\Delta V^T$

Given: U, $V \rightarrow$ orthogonal matrices and Δ is a diagonal matrix.

$Y = X\beta$ for linear Regression

<u>Soln:</u> $Y = X\beta$

$$UDV^T\beta = Y$$

multiply by U^T on both sides

$$U^TUDV^T\beta = U^TY$$

acc to property of orthogonal matrix $AA^T = A^TA = I$

$$IDV^T\beta = U^TY$$

$$DV^T\beta = U^TY$$

$$D^{-1}DV^T\beta = D^{-1}U^TY$$

multiply by D^{-1} on both sides

$$IV^T\beta = D^{-1}U^TY$$

$$V^T\beta = D^{-1}U^TY$$

multiply by V on both sides

$$VV^T\beta = VD^{-1}U^TY$$

as V is orthogonal matrix

$$I\beta = VD^{-1}U^TY$$

$$\beta = VD^{-1}U^TY$$