$[c,\ d]$. We know that $[c,d]$ is a closed convex hull of it's extreme points c and d. Thus, the result is true for $n = 1$.

Now, assume that the result is true for the dimension $n-1$. We will show it for dimension n. Suppose K (of dimension n) be the closed convex hull of the extreme point of S. We claim that that $K = S$. Clearly $K \subseteq S$. Suppose $S \not\subseteq K$. Then there is a $\underline{y} \in S$ but $y \notin K$. But $\underline{y}$ is exterior to K, by Theorem 1.7 there exists $a \neq 0$ such that

$$\underline{a}^T \underline{y} < \inf_{\underline{x} \in K} (\underline{a}^T \underline{x}) \tag{1.1}$$

Let $s_0 = \inf_{\underline{x} \in S} \underline{a}^T \underline{x}$. Since the function $\underline{a}^T \underline{x}$ is continuous on compact set S, then the function $\underline{a}^T \underline{x}$ attains its minimum value at $\underline{x}_0 \in S$ with

$$s_0 = \inf_{\underline{x} \in S} \underline{a}^T \underline{x} = \min_{\underline{x} \in S} (\underline{a}^T \underline{x}) = \underline{a}^T \underline{x}. \tag{1.2}$$

It gives that

$$\underline{a}^T \underline{x}_0 \leq \underline{a}^T \underline{x} \ \ \forall\ \underline{x} \in S. \tag{1.3}$$

Then (1.1) and (1.2) implies that, the hyperplane $H = \{x : \underline{a}^T \underline{x} = s_0\}$ is a supporting hyperplane to S at $\underline{x}_0 \in S$. Using relation (1.2) and (1.3), we have

$$\underline{y} \in S \Rightarrow \underline{a}^T \underline{x}_0 \leq \underline{a}^T \underline{y} < \inf_{\underline{x} \in K} (\underline{a}^T \underline{x}).$$

Since $K \subseteq S$, $\underline{x}_0 \notin K$ and H is a supporting hyperplane to S at $\underline{x}_0$. Then the sets H and K are disjoint. Let $T = H \cap S$. Then T is a closed bounded subset of H and it is a space of dimension $(n-1)$. Since $\underline{x}_0 \in S$, $\underline{x}_0 \in H$ then $\underline{x}_0 \in T$. This means that T is a non-empty closed bounded subset of $\mathbb{R}^{n-1}$. Hence by induction hypothesis, T is a closed convex hull of extreme point of T, i.e. T contains extreme points. By using repeated use of this Theorem, we an prove all other extreme point of T are also the extreme point of S. Thus, we found $\underline{x}_0$ that lies in the convex hull of some extreme point of S and $\underline{x}_0 \notin K$. It is a contradiction to that K is a closed convex hull of the extreme points of S, so, we have $K = S$.

Definition 1.15 Let S and T be two non empty subset of $\mathbb{R}^n$ then a hyperplane H is said to be separate S and T if S is contained in one of the closed half spaces generated by H and T is contained in the other closed half space. The hyperplane H in this case is called a separating hyperplane.

Definition 1.16 A hyperplane H strictly separates S and T if S is contained in one of the open half spaces generated by H and T is contained in other half plane.

Theorem 1.11 If $S \subseteq \mathbb{R}^n$ is non empty convex set and $\underline{0} \notin S$, then there exists a hyperplane separating S and $\underline{0}$.

Proof. We will give proof in two different situations.

(a) Suppose $\underline{0}$ lies in an exterior S. Then by Theorem 1.7, there exists a vector $\underline{0} \neq \underline{a} \in \mathbb{R}^n$ such that $0 < \underline{a}^T \underline{x}$ for $\underline{x} \in S$. So, the hyperplane $H = \{\underline{x} : \underline{a}^T \underline{x} = c\}$ where $0 < c < \underline{a}^T \underline{x}$ separate S and $\underline{0}$.

(b) Suppose $\underline{0} \in \tilde{S}$, by Theorem, there is a supporting hyperplane $H = \{x : \underline{a}^T \underline{x} = 0\}$ to S at the $\underline{0}$ and it separates S and $\underline{0}$.

Theorem 1.12 Let S and T be two non empty disjoint convex sets in $\mathbb{R}^n$. Then there exists a hyperplane that separates S and T.

Proof. Clearly, $S - T$ is convex and $\underline{0} \notin S - T$, because $S \cap T = \phi$. So, there exists a vector $\underline{a}$ such that $\underline{a}^T \underline{x} \geq 0$ for all $\underline{x} \in S - T$. It means that for all $u \in S$ and $v \in T$, we have $\underline{a}^T (\underline{u} - \underline{v}) \geq 0$. So, there exist a number c satisfying.

$$\begin{aligned}
\underline{a}^T \underline{u} - \underline{a}^T \underline{v} &\geq 0 \\
\inf(\underline{a}^T \underline{u} - \underline{a}^T \underline{v}) &\geq 0 \\
\inf \underline{a}^T \underline{u} - \sup \underline{a}^T \underline{v} &\geq 0 \\
\inf \underline{a}^T \underline{v} \geq c &\geq \sup \underline{a}^T \underline{v}.
\end{aligned}$$

This implies that the hyperplane $H = \{x : \underline{a}^T \underline{x} = 0\}$ separate S and T.

Theorem 1.13 Let S be a non empty closed convex set in $\mathbb{R}^n$ not containing $\underline{0}$. Then there exists a hyperplane that strictly separates S and the $\underline{0}$.

Proof. Let S is closed set, so $\tilde{S} = S$ and $\underline{0} \notin S$ i.e. 0 is the exterior to S. Hence by Theorem 1.7, there exist $a \neq 0 \in \mathbb{R}^n$ such that $0 < \inf_{\underline{x} \in S} \underline{a}^T \underline{x}$. Now, we choose a real number c such that $0 < c < \inf_{\underline{x} \in S} \underline{a}^T \underline{x}$. Clearly the hyperplane $H = \{x : \underline{a}^T \underline{x} = c\}$ strictly separates 0 and S.

* Convex Polyhedron is the intersection of finite number of half spaces and its defined by its sides.

1.3 Convex polyhedron and polytope

Definition 1.17 The convex hull of a finite (non zero) number of points is called convex polytope spanned by these points.

Let $S = \{\underline{x}_1, \underline{x}_2, \cdots, \underline{x}_m\}$ where $\underline{x}_i \in \mathbb{R}^n$ then the convex polytope spanned by the points of S is the convex set

$$Co(S) = \left\{ \underline{x} : \underline{x} = \sum_{i=1}^{m} \lambda_i \underline{x}_i, \lambda_i \geq 0, \forall i, \ \sum_{i=1}^{m} \lambda_i = 1 \right\}.$$

Clearly a convex polytope is a non-empty convex set.

Theorem 1.14 The set of vertices of a convex polytope is a subset of the set of spanning points of the polytope.

Proof. Suppose V is the set of vertices of the convex polytope CO(S) spanned by the points of the set $S = \{\underline{x}_1, \underline{x}_2, \cdots, \underline{x}_m\}$. It is clear that the result is true when $m = 1$. Now, assume contrary that $V \not\subset S$. Then, there exist $\underline{x} \in V$ such that $x \notin S$. Since $\underline{x} \in V \Rightarrow \underline{x} \in CO(S)$. Therefore $x = \sum_{i=1}^{m} \mu_i \underline{x}_i$, Where $\mu_i \geq 0$, $i = 1, \cdots, m$ and $\sum_{i=1}^{m} \mu_i = 1$. Since $\underline{x} \notin S$, it follows that $\mu_i > 0$ and $\mu_i \neq 1$, $i = 1, 2, \cdots, m.$. Hence $x = \sum_{i=1}^{m} \mu_i \underline{x}_i$ implies that there exists μ_i $(0 < \mu_i < 1)$. Let it be μ_1

$$x = \mu_1 \underline{x}_1 + \sum_{i=2}^{m} \mu_i \underline{x}_i = \mu_1 \underline{x}_1 + (1 - \mu_1) \sum_{i=2}^{m} \frac{\mu_i}{(1 - \mu_1)} \underline{x} = \mu_1 \underline{x}_1 + (1 - \mu_1) \underline{x}_i,$$

where $y = \sum_{i=2}^{m} \frac{\mu i}{(1 - \mu_1)} \underline{x}_i$. Clearly $y = \sum_{i=2}^{m} \left(\frac{\mu_i}{(1 - \mu_1)} \right) \underline{x}_i, \underline{0} < \mu_1 < 1$. This implies that $y \in CO(S)$. Hence $\underline{x}$ is not a extreme (vertex) of $Co(S)$. It is a contradiction. Hence, we must have $V \subseteq S$.

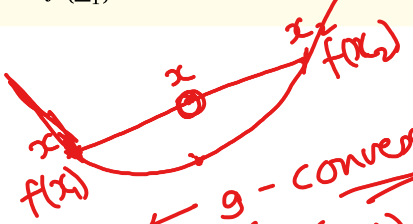
Theorem 1.15 Let $K = \{\underline{x} : A\underline{x} = \underline{b}, \underline{x} \geq 0\}$ be a non-empty polyhedral set. Then the set of extreme points of K is non empty and has a finite number of points.

1.4 Convex function

Definition 1.18 A function f defined on a set $T \subseteq \mathbb{R}^n$ is said to be convex at $\underline{x}_0 \in T$ if $\underline{x}_1 \in T$, $0 \leq \lambda \leq 1$, $\lambda \underline{x}_0 + (1 - \lambda) \underline{x}_1 \in T$, then

$$f(\lambda \underline{x}_0 + (1 - \lambda) \underline{x}_1) \leq \lambda f(\underline{x}_0) + (1 - \lambda) f(\underline{x}_1).$$

$[f(\alpha x_1 + (1-\alpha) x_2) \leq \alpha f(x_1) + (1-\alpha) f(x_2)]$



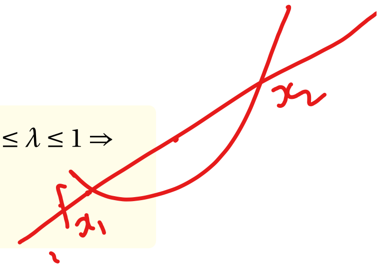
Jensen's Inequality $\rightarrow$ $E(g(x)) \geq g(E(x))$ $\leftarrow$ g - convex log x $\qquad$ $1/x$ $\qquad$ $-x^2$ $\qquad$ concave

x^2 $\qquad$ $|x|$ $\qquad$ e^x $\qquad$ convex

var ≥ 0 $\qquad$ $E(x^2) - (E(x))^2 \geq 0$ $\qquad$ $E(g(x)) \geq g(E(x))$

Definition 1.19 A function f is said to be convex on T if it is convex at every point of T.

Domain of f necessary to be a convex set. Therefore other way convex set is defined as follows.



Definition 1.20 If T is convex set then f is said to be convex on T if $\underline{x}_1, \underline{x}_2 \in T, 0 \leq \lambda \leq 1 \Rightarrow$

$$f\left(\lambda \underline{x}_0 + (1-\lambda)\underline{x}_1\right) \leq \lambda f\left(\underline{x}_0\right) + (1-\lambda) f\left(\underline{x}_1\right).$$

In geometrical point of view, a function $y = f(x)$ defined on a convex set T is convex if the chord joining, any two points on the graph of f lies on or above the graph.

Definition 1.21 A function f defined on a set $T \subseteq \mathbb{R}^n$ is said to be strictly convex at $\underline{x}_0 \in T$ if $\underline{x}_1 \in T, 0 < \lambda < 1, \underline{x}_0 \neq \underline{x}_1, \lambda \underline{x}_0 + (1-\lambda)\underline{x}_1 \in T$, then

$$f\left(\lambda \underline{x}_0 + (1-\lambda)\underline{x}_1\right) < \lambda f\left(\underline{x}_0\right) + (1-\lambda) f\left(\underline{x}_1\right).$$

Definition 1.22 A function f is said to be concave at $\underline{x}_0 \in T$ if $-f$ is convex at $\underline{x}_0 \in T$.

Example 1.12 Show that the linear function $f(\underline{x}) = \underline{c}^T \underline{x} + \underline{d}$ is both convex and convex on $\mathbb{R}^n$.

Solution. Clearly, $\mathbb{R}^n$ is a convex set. Let $\underline{x}_1, \underline{x}_2 \in \mathbb{R}^n$ and $0 \leq \lambda \leq 1$. Consider

$$\begin{aligned}
f\left(\lambda \underline{x}_1 + (1-\lambda)\underline{x}_2\right) &= \underline{c}^T\left(\lambda \underline{x}_1 + (1-\lambda)\underline{x}_2\right) + \underline{d} \\
&= \lambda \underline{c}^T \underline{x}_1 + \lambda \underline{d} + (1-\lambda)\underline{c}^T \underline{x}_2 + \underline{d} - \lambda \underline{d} \\
&= \lambda\left(\underline{c}^T \underline{x}_1 + \underline{d}\right) + (1-\lambda)\left(\underline{c}^T \underline{x}_2 + \underline{d}\right) \\
&= \lambda f(x_1) + (1-\lambda) f(x_2)
\end{aligned}$$

$\Rightarrow f$ is convex on $\mathbb{R}^n$. Similarly, we can show that $-f$ is convex function.

Example 1.13 Let f be a convex function on a convex set $T \subseteq \mathbb{R}^n$. Then for every $k \in \mathbb{R}$. Show that $T_k = \{\underline{x} : \underline{x} \in T,\ f(x) \leq k\}$ is convex set.

Solution. Let $\underline{x}_1, \underline{x}_2 \in T_K$ and $0 \le \lambda \le 1$. Then $\underline{x}_1$, $\underline{x}_2 \in T$. Since T is convex $\Rightarrow \lambda \underline{x}_1 + \left(1 - \lambda \underline{x}_2\right) \in T$ and, $f\left(\underline{x}_1\right) \le k$, $f\left(\underline{x}_2\right) \le k$. Given that f is convex on T, we have

$$\begin{aligned} f\left(\lambda \underline{x}_1 + (1-\lambda)\,\underline{x}_2\right) &\le f\left(\underline{x}_1\right) + (1-\lambda)\, f\left(\underline{x}_2\right) \\ &\le \lambda k x + (1-\lambda)\, k = k \\ \Rightarrow \lambda \underline{x}_1 + (1-\lambda)\,\underline{x}_2 &\in T_k. \end{aligned}$$

Theorem 1.16 Let f and g be convex function and let μ be a non negative number. Then μf and $f+g$ are convex functions. ~~Further~~ if $\mu > 0$ and f is strictly convex then μf is strictly convex.

Proof. Let f and g be convex function on convex set T. Let $\underline{x}_1$, $\underline{x}_2 \in T$, $0 \le \lambda \le 1$,

$$\begin{aligned} \mu f\left(\lambda \underline{x}_1 + (1-\lambda)\,\underline{x}_2\right) &= \mu\left(f\left(\lambda \underline{x}_1 + (1-\lambda)\,\underline{x}_2\right)\right) \\ &\le \mu\left(\lambda f\left(\underline{x}_1\right) + (1-\lambda)\, f\left(\underline{x}_2\right)\right) \\ &= \lambda \mu f\left(\underline{x}_1\right) + (1-\lambda)\,\mu f\left(\underline{x}_2\right) \\ &= \lambda\left(\mu f\right)\left(\underline{x}_1\right) + (1-\lambda)\left(\mu f\right)\left(\underline{x}_2\right). \end{aligned}$$

$\Rightarrow \mu f$ is a convex function. Now

$$\begin{aligned} \left(f+g\right)\left(\lambda \underline{x}_1 + (1-\lambda)\ \underline{x}_2\right) &= f\left(\lambda \underline{x}_1 + (1-\lambda)\ \underline{x}_2\right) + g\left(\lambda \underline{x}_1 + (1-\lambda)\ \underline{x}_2\right) \\ &\le \lambda f(\underline{x}_1) + (1-\lambda) f(\underline{x}_2) + \lambda g(\underline{x}_1) + (1-\lambda) g(\underline{x}_2) \\ &= \lambda\left(f\left(\underline{x}_1\right) + g\left(\underline{x}_1\right)\right) + (1-\lambda)\left(f\left(\underline{x}_2\right) + g\left(\underline{x}_2\right)\right) \\ &= \lambda\left(f+g\right)\left(\underline{x}_1\right) + (1-\lambda)\left(f+g\right)\left(\underline{x}_2\right). \end{aligned}$$

It shows that $f + g$ is a convex function.

If $\mu > 0$ and f is strictly convex, then

$$\begin{aligned} \left(\mu f\right)\left(\lambda \underline{x}_1 + (1-\lambda)\,\underline{x}_2\right) &= \mu\left(f\left(\lambda \underline{x}_1 + (1-\lambda)\,\underline{x}_2\right)\right) < \mu\left(\lambda f\left(\underline{x}_1\right) + (1-\lambda)\, f\left(\underline{x}_2\right)\right) \\ &= \lambda \mu f\left(\underline{x}_1\right) + (1-\lambda)\,\mu f\left(\underline{x}_2\right) = \lambda\left(\mu f\right)\left(\underline{x}_1\right) + (1-\lambda)\left(\mu f\right)\left(\underline{x}_2\right) \end{aligned}$$

Thus μf is strictly convex.

> **Theorem 1.17** Every linear combination $\sum_{i=1}^{k} \mu_i f_i$ of convex function f_i where $\mu_i \geq 0, \;\; i = 1, 2, \cdots, k$ is a convex function. Also, such a combination is strictly convex if at last one $\mu_i > 0, \;\; i = 1, 2, \cdots, k$ and the corresponding f_i strictly convex.

Proof. Theorem 1.16 follows proof.

> **Theorem 1.18** Let h be a non decreasing convex function on $\mathbb{R}$ and f be a convex function on a convex set $T \subseteq \mathbb{R}^n$. Then the composite function $h \circ f$ is convex on T.

Proof. Let $h : \mathbb{R} \to \mathbb{R}$ and $f : T \to \mathbb{R}$ be two convex functions. To show that $h \circ f$ ~~is~~ a convex on T. Let $\underline{x}_1, \underline{x}_2 \in T$and $0 \leq \lambda \leq 1$. Since f is convex on T

$$f\left(\lambda \underline{x}_1 + (1-\lambda)\, \underline{x}_2\right) \leq \lambda f\left(\underline{x}_1\right) + (1-\lambda)\, f\left(\underline{x}_2\right). \tag{1.4}$$

Since h is non decreasing, we have

$$h\left(f\left(\lambda \underline{x}_1 + (1-\lambda)\, \underline{x}_2\right)\right) \leq h\left(\lambda f\left(\underline{x}_1\right) + (1-\lambda)\, f\left(\underline{x}_2\right)\right). \tag{1.5}$$

Also h is convex function

$$h\left(\lambda f\left(\underline{x}_1\right) + (1-\lambda)\, f\left(\underline{x}_2\right)\right) \leq \lambda h\left(f\left(\underline{x}_1\right)\right) + (1-\lambda)\, h\left(f\left(\underline{x}_2\right)\right) \tag{1.6}$$

From (1.5) and (1.6)

$$h\left(f\left(\underline{x}_1 + (1-\lambda)\, \underline{x}_2\right)\right) \leq \lambda h\left(f\left(\underline{x}_1\right)\right) + (1-\lambda)\, h\left(f\left(\underline{x}_2\right)\right)$$

$\Rightarrow h \circ f$ is convex function on T.

> **Theorem 1.19** Let f be differentiable on an open convex set $T \subseteq \mathbb{R}^n$. Then f is convex on T iff
>
> $$f(\underline{x}_2) \geq f(\underline{x}_1) + \nabla f(\underline{x}_1)^T (\underline{x}_2 - \underline{x}_1), \tag{1.7}$$
>
> for all $\underline{x}_1, \underline{x}_2 \in T$. Further f is strictly convex on T if and on if the inequality is strict ($>$) for all $\underline{x}_1, \underline{x}_2 \in T, \underline{x}_1 \neq \underline{x}_2$.

Proof. Let f be differentiable on open convex set $T \subseteq \mathbb{R}^n$. Suppose f is convex on T. Let $\underline{x}_1, \underline{x}_2 \in T$ and $0 \leq \lambda \leq 1$,

$$f\left(\lambda \underline{x}_2 + (1-\lambda)\, \underline{x}_1\right) \leq \lambda f\left(\underline{x}_2\right) + (1-\lambda)\, f\left(\underline{x}_1\right).$$