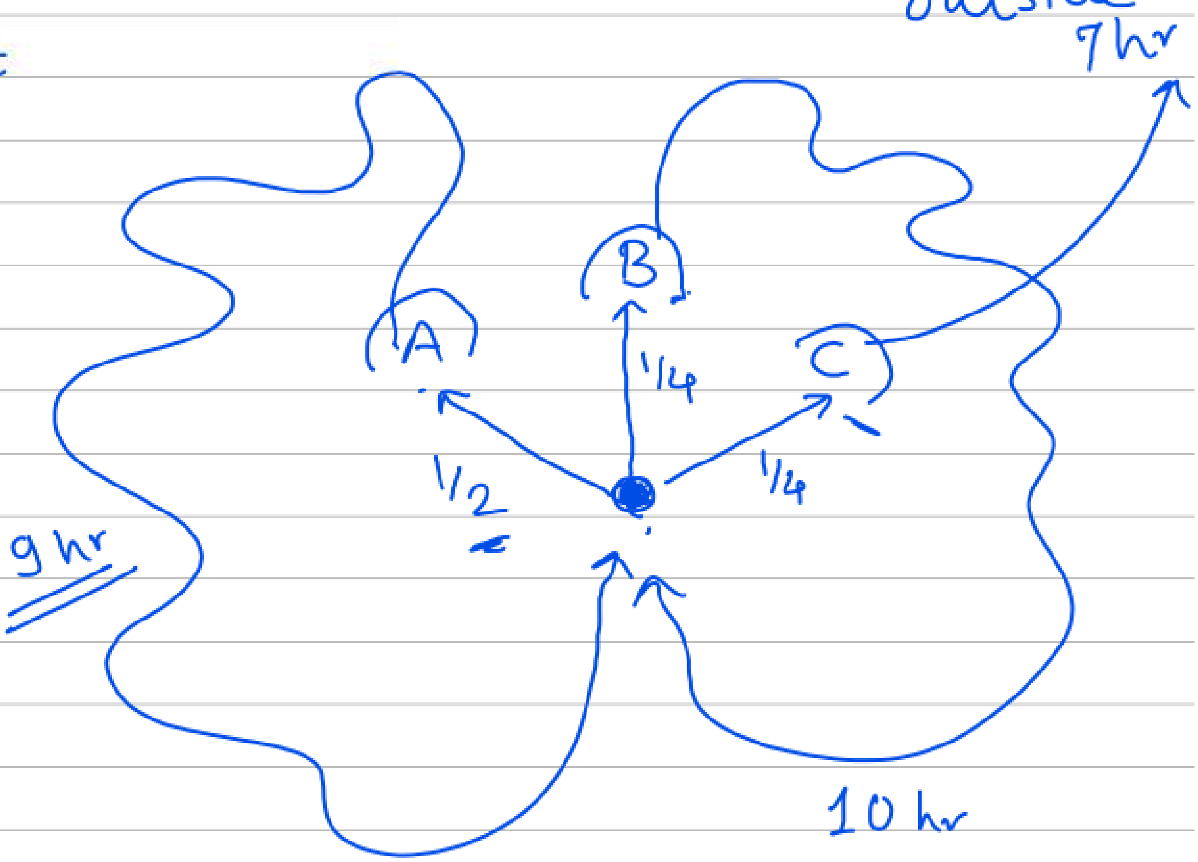
① Distributions Revision

Discrete

Continuous

Rat



Rat Cave

A 1/2 → 9 hr return

B 1/4 → 10 hr return

C 1/4 → 7hr leave

x

μ Assume

$E(X) = ?$

A 1/2 → $9 + \mu$ ✓

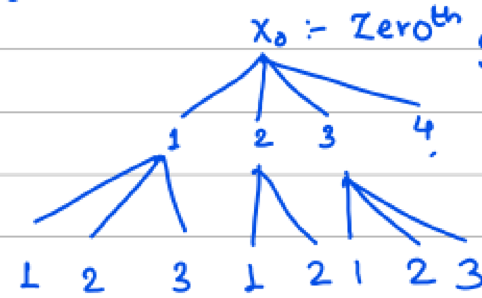
B 1/4 → $10 + \mu$ ✓

C 1/4 → 7

$$\mu = \frac{1}{2}(9+\mu) + \frac{1}{4}(10+\mu) + \mu$$

$=$

Offspring Distribution:-



x_0 :- Zeroth generation

$x_1 = 4$ 1^{st} generation size

$x_2 = 8$ 2^{nd} generation size

$x_n = 0$ & $x_m = 0 \ni m > n$

$x_{n+1} = 0$

Dinosaurs extints $\to x_n = 0$

$x_0 = 1$, $x_1 = 3$, $x_2 = 5$ $\quad \lim_{n\to\infty} x_n = 0$

Sum of expectations $\quad y = \sum_{i=1}^{n \leftarrow \text{fixed}} x_i \leftarrow \text{Random} \Rightarrow E(y) = n \cdot E(X) = \sum_{i=1}^{n} E(X_i)$

$V(X) = E(V(X/Y)) + V(E(X/Y))$

$E(X) = E_Y(E_{X/Y}(X/Y))$

Random sum of Random number $\quad V(y) = \sum_{i=1}^{n} V(X_i) + \sum_i \sum_{j, i\neq j} Cov(X_i, X_j)$

$y = \sum_{i=1}^{N \leftarrow \text{random}} X_i \leftarrow \text{random}$

↳ compound distributions

Y ① N ~ Bino → y Comp. Bino. distribution (K,p) $\quad E(Y)\ Kp \cdot E(X)$

Y ② ~ Pois ~ Comp. Poi. distribution $\quad E(Y)\ \lambda \cdot E(X)$

likewise Comp. Geometric, Negative Binomial

$E(Y) = E\left(\sum_{i=1}^{N} X_i\right) = E_N\left(E_{Y/N}\left(\sum_{i=1}^{N} X_i / N = n\right)\right)$ $\quad E(X_i) = \mu$

$V(X_i) = V(X)$ $\quad E(X_i) = E(X) = \mu$

$= E_N\left(\sum_{i=1}^{N} E(X_i)\right) = E_N(n \cdot E(X)) = E(X) \cdot E(N)$ ✓

$V(Y) = V\left(\sum_{i=1}^{N} X_i\right) = E(V(X/Y)) + V(E(X/Y))$

$= E\left(V\left(\sum_{i=1}^{N} X_i / N\right)\right) + V\left(E\left(\sum_{i=1}^{N} X_i / N\right)\right)$

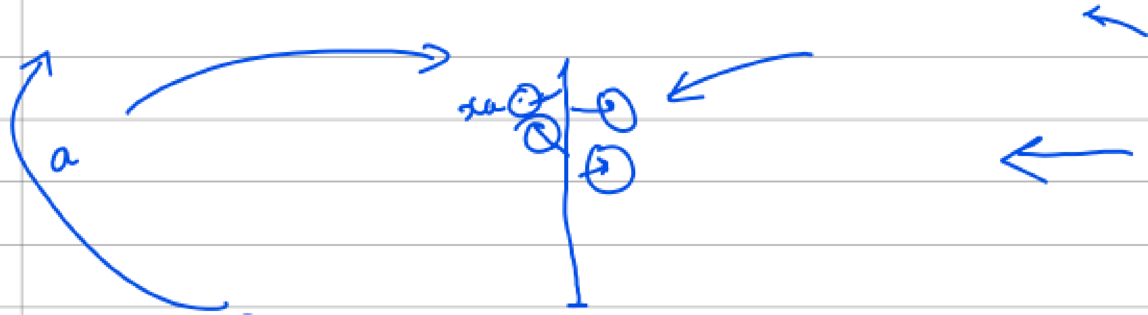
✓ $= E_N(N\, V(X)) + V_N(N\, E(X)) \Rightarrow E(N) V(X) + [E(X)]^2 V(N)$

Comp. Bino. $\Rightarrow V(Y) = kp \cdot V(X) + [E(X)]^2 (kpq)$ | $\Rightarrow$ Comp. Poisson $\Rightarrow V(Y) = \lambda [V(X) + [E(X)]^2]$

$\Rightarrow \lambda \cdot E(X^2)$

$V(X) = E(X^2) - E(X)^2$

Matching hat problem



$a \rightarrow x_a \Rightarrow$ —

$b \rightarrow x_b$

$n \rightarrow$ 0, 1, 2, 2, 3, ..., n

a_1	a_2		a_n
1	0		0

$\Rightarrow \sum_{i=1}^{n} a_i$ matching hats

$\rightarrow n$

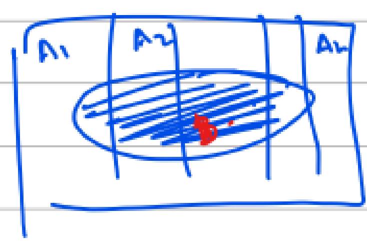
1	n	1/n	1
2	n−1	$\frac{1}{(n-1)} \times \frac{1}{n} + \frac{1}{(n-2)}\left(1-\frac{1}{n}\right)$	

Baye's Rule

$P(A/B) = \frac{P(A \cap B)}{P(B)}$

$P(A_i/B) = \frac{P(B/A_i)}{\sum_{i=1}^{n} P(B/A_i)}$

ΣA_i



$\cup A_i = \Omega$

mutually exclusive & exhaustive

e.g. Family 2 children

$X_i = 0$, if i is female $\frac{1}{2}$

$= 1$ if i is male $\frac{1}{2}$

$P\left[\sum_{i=1}^{2} X_i = 2 / \sum_{i=1}^{2} X_i \geq 1\right] = ?$

$\Omega = \{BB, BG, GB, GG\}$

$A = \{BB, BG, GB\}$ ✓

$B = \{BB\}$

$$P(B/A) = \frac{P(B \cap A)}{P(A)} = \frac{1/4}{3/4} = \frac{1}{3}$$

e.g. HIT

Maths $\rightarrow$ A $\rightarrow$ $1/2$

Stats $\rightarrow$ A $\rightarrow$ $1/3$

coin

P[Stats/A] = ?

$P[A/\text{stats}] = 1/3$ $P[A/\text{Maths}] = 1/2$

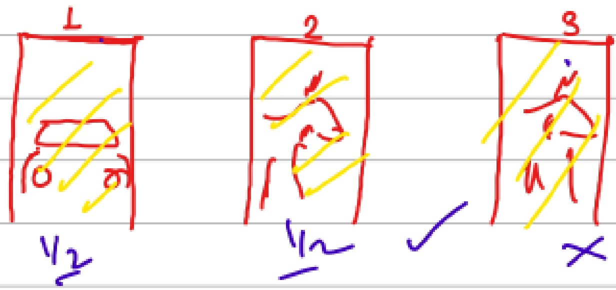
$P[\text{Stats}] = 1/2$ $P[\text{Maths}] = 1/2$

$$P[\text{Stats}/A] = \frac{P[\text{Stats}, A]}{P[\text{Stats}, A] + P[\text{Maths}, A]}$$

$$P[\text{Stats}/A] = \frac{1/3}{1/3 + 1/2} = \frac{2}{2+3} = \frac{2}{5}$$

Monty Hall Problem / Paradox ? Show 80

Monty Hall



R $\rightarrow$ 2 ✓

M $\rightarrow$ 3 open

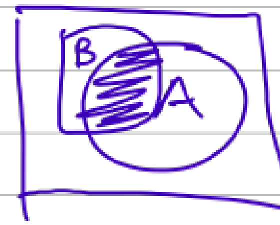
Paradox

1000	Papers	Letters	Original	Rashmi $\rightarrow$ 2	M open		
	W	1	1	2	3	2	N
	N	3	2	2	1/3	2	W
	W	3	3	2	1	2	N
	2/3						1/3

$E(X/Y)$

$= V(X/Y) \cdot E(X)$
$+ E(X/Y) \cdot V(X)$

$P(B/A) = \dfrac{P(B \cap A)}{P(A)}$



X, Y ind

x/y	1	2	3
prob	1/4	1/2	1/4

$S = x+y \Rightarrow$

y \ x	1	2	3
1	2	$\boxed{3}$	4
2	$\boxed{3}$	4	5
3	4	5	6

$E(X/S)$ $\quad \underline{S=3} \to (x=1)$ 1/4
$(x=2)$ 1/2
$(x=3)$ 1/4

		x	1/4	1/2	1/4
		Y	1	2	3
1/4	1		2	3	4
1/2	2		3	4	5
1/4	3		4	5	6

S	2	3	4
P(S)	1/16	$2 \times \frac{1}{2} \times \frac{1}{4}$ = 1/4	$\frac{1}{4}\times\frac{1}{4} + \frac{1}{2}\times\frac{1}{2} + \frac{1}{4}\times\frac{1}{4}$

$\frac{1}{16} + \frac{1}{4} + \frac{1}{16}$

$\frac{2}{16} + \frac{1}{4} = \frac{3}{8}$

5	6
1/8	1/16

	1/16	1/4	3/8	1/8	1/16
S =	2	3	4	5	6
x =	1	1, 2	1, 2, 3.	2, 3	3
x=1	1	1/2	1/6		
x=2		1/2	2/3		
			1/6		

$\dfrac{P(X=1, S=3)}{P(S=3)} = \dfrac{P(X=1, Y=2)}{1/4} = \dfrac{P(X=1)(P(Y=2))}{}$

$= \dfrac{1/4 \times 1/2}{1/4} = 1/2$

$\dfrac{P(X=2, S=4)}{P(S=4)} = \dfrac{1/2 \times 1/2}{3/8} = 2/3$

$\dfrac{P(X=1, S=4)}{P(S=4)} = \dfrac{1/4 \cdot 1/4}{3/8} = \dfrac{3}{8}$ $\quad \dfrac{1/16}{3/8}$

$$\frac{P(X=1, S=4)}{P(S=4)} = \frac{P(X=1).P(Y=3)}{3/8} \quad \frac{1/4 \times 1/4}{3/8} = 2 \quad \frac{1}{3}\,\frac{1}{2} = \frac{1}{6}$$

$$\frac{P(X=2, S=4)}{P(S=4)} = \frac{P(X=2).P(Y=2)}{P(S=4)} = \frac{\frac{1}{2}\times\frac{1}{2}}{3/8} = \frac{1/4}{3/8_2} = \frac{2}{3}$$

$$\frac{1}{6} + \frac{1}{3} = \frac{1+4}{6} = \frac{5}{6} \rightarrow P(X=3/S=4)\ 1/6$$

$E(X/S=2) = 1$

$E(X/S=3) = 1\times\frac{1}{2} + 2\times\frac{1}{2} = 1.5$

$E(\quad /S=4) =$ ______

$$E(X) = \cancel{E[E(X/Y)]} = E_Y[\underbrace{E_{X/Y}}(X/Y)]$$

MGF of compound distributions $\quad Y = \sum_{i=1}^{N} X_i$

$E(X_i) = E(X)$

$$M_Y(t) = E(e^{tY}) = E_N(E_{Y/N}(e^{tY}/N=n))$$

$$= E_N\left(E\left(e^{t\sum_{i=1}^{N} X_i}/N=n\right)\right)$$

X_i = i.i.d.

$$= E_N\left(\prod_{i=1}^{N} \underline{E(e^{tX_i})}\right)$$

$E(e^{tX}) = t$

$E(t^X)$

$e^{\log t}$

$$= E_N([\underline{E(e^{tX})}]^N)$$

$$M_Y(t) = E_N([\underline{M_X(t)}]^N) = P_N(M_X(t))$$

$$= E_N(e^{N \cdot \log M_X(t)}) = M_N(\log M_X(t))$$

Ballot Problem

n, m

$\underline{\underline{n > m}}$

	A		B
	1	>	0
	2	>	0, 1
	⋮	>	
		>	
	$\underline{n}$	>	$\underline{\underline{m}}$

$\boxed{\dfrac{n-m}{n+m}}$ ✓

Ballot Problem

$$P_{n,m} = \frac{n}{n+m} \cdot P_{n-1,m} + \frac{m}{n+m} P_{n,m-1}$$

$$= \frac{n}{n+m} \cdot \frac{(n-1)-m}{(n-1)+m} + \frac{m}{n+m} \frac{(n)-(m-1)}{n+m-1}$$

$$= \frac{n^2 - n - \cancel{mn} + \cancel{mn} - m^2 + m}{(n+m)(m+n-1)} = \frac{n^2 - n - m^2 + m}{(n+m)(m+n-1)}$$

$$= \frac{(n^2 - m^2) - (n-m)}{(n+m)(n+m-1)} = \frac{(n-m)(n+m) - (n-m)}{(n+m)(n+m-1)}$$

$$= \frac{(n-m)}{(n+m)} \frac{(\cancel{n+m-1})}{(\cancel{n+m-1})}$$

Excercise 1.8) $X_1 \sim Poi(\lambda_1)$ $X_2 \sim Poi(\lambda_2)$ $X_1 + X_2 \sim ?$

$Z = X_1 + X_2$

$$M_Z(t) = M_{X_1+X_2}(t) = M_{X_1}(t) \cdot M_{X_2}(t) = e^{\lambda_1(e^t - 1)} \cdot e^{\lambda_2(e^t-1)}$$

$$= e^{(\lambda_1+\lambda_2)(e^t-1)}$$

$Z \sim Poi(\lambda_1 + \lambda_2)$

Conditional distribution of $X_1 / X_1 + X_2$

$$P(X_1 = x / X_1 + X_2) = \frac{P(X_1 = x,\ X_1 + X_2 = z)}{P(X_1 + X_2 = z)} = \frac{P(X_1 = x, X_2 = z - x)}{P(z)}$$

$$= \frac{P(X_1 = x) \cdot P(X_2 = z - x)}{P(z)}$$

$$= \frac{\cancel{e^{-\lambda_1}} \lambda_1^{x}}{x!} \cdot \frac{\cancel{e^{-\lambda_2}} \lambda_2^{z-x}}{(z-x)!} \Bigg/ \frac{\cancel{e^{-(\lambda_1+\lambda_2)}} (\lambda_1+\lambda_2)^z}{z!}$$

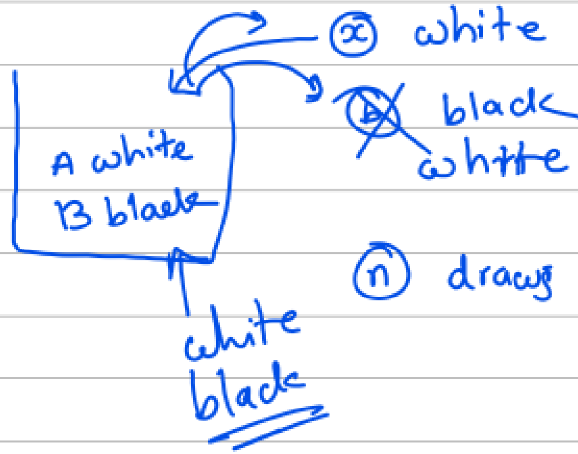
$$= \frac{z!}{x!(z-x)!} \left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^x \left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{z-x}$$

$$= {}^zC_x$$

$$X_1 / X_1+X_2 \sim \text{Bino}\left(z, \frac{\lambda_1}{\lambda_1+\lambda_2}\right)$$

1.19 excercise

Urn



a b $M_0 = a$

1 W|B W B W $\underline{\underline{M_1}} = a \times \frac{a}{\underline{\underline{a+b}}} + (a+1)\frac{b}{a+b} = \frac{a(a+b)}{(a+b)} + \frac{b}{(a+b)}$

M

$M_0 = \underline{a}$ white ball

$$M_1 = \underline{\underline{a}}\left(\frac{a}{a+b}\right) + (a+1)\left(\frac{b}{a+b}\right)$$

3+7

n+1 n M_n white

$$M_{n+1} = \underline{M_n}\left(\frac{?}{\underline{a+b}}\right) + (M_n+1)\left(\frac{(a+b)-?}{a+b}\right)$$

$$= M_n \cdot \left(\frac{M_n}{a+b}\right) + (M_n+1)\left(\frac{(a+b)-M_n}{a+b}\right) \qquad 1 - \frac{M_n}{a+b}$$

$$= \cancel{\frac{M_n^2}{(a+b)}} + (M_n+1) - \cancel{\frac{M_n^2}{(a+b)}} - \frac{M_n}{(a+b)}$$

① $\underline{\underline{M_{n+1}}} = 1 + M_n\left(1 - \frac{1}{a+b}\right)$ ② $M_0 = a$

$$M_n = a+b - b\left(1 - \frac{1}{a+b}\right)^n$$

$$M_1 = 1 + a\left(1 - \frac{1}{a+b}\right) = 1 + ax$$

$$M_2 = 1 + (1+ax)\cdot x = 1 + x + ax^2$$

$$M_3 = 1 + (1+x+ax^2)x = 1 + x + x^2 + ax^3$$

$$= 1 + x + x^2 + x^3 + (a-1)x^3$$

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \qquad = \left(\frac{1-x^3}{1-x}\right) + (a-1)x^3$$

$$\checkmark \quad \sum_{n=0}^{j} x^n = \frac{1-x^{j+1}}{1-x} \qquad M_n = \left(\frac{1-x^{n+1}}{1-x}\right) + (a-1)x^n$$

$$x = 1 - \frac{1}{a+b} \Rightarrow \frac{1}{1-x} = (a+b)$$

$$M_n = (a+b) - \frac{x^{n+1}}{1-x} + (a-1)x^n$$

$$= (a+b) - (a+b)x^{n+1} + (a-1)x^n$$

$$= (a+b) - x^n\left[(a+b)\cdot x - a + 1\right]$$

$$= (a+b) - \left(1 - \frac{1}{a+b}\right)^n \left[(a+b)\left(1 - \frac{1}{a+b}\right) - a + 1\right]$$

$$\cancel{a} + b - \cancel{1} - \cancel{a} + \cancel{1}$$

↑

Sheldon Ross

$$= (a+b) - b\left(1 - \frac{1}{a+b}\right)^n$$