* Counting Process

$X_n \longrightarrow \{X(t), t \in I\}$ — Statespace discrete, Time domain cont

$I = [a,b)$, $[0,\infty)$ etc.

$\{N(t), t \in I\}$ — no. of events occured during $(0,t]$

states discrete, time cont.

e.g. no. of customers arrived in bank / barber / Ticket coun

no. of deaths occurred due to Covid-19

① $N(t) \geq 0$

② $N(t)$ integer valued

③ $s \leq t \Rightarrow N(s) \leq N(t)$ — $10\text{-}3 \leq 10\text{-}5$; 20, 25

④ for $s<t$, $N(t)-N(s)$ increment $(s,t]$

* Counting Process with independent increments

for $s<t$, $N(t)-N(s)$ is increment,

independent increment if $N(s)$ & $N(t)-N(s)$ are indepen

Do you remember any stoc. process?

$$\underbrace{0 \quad\quad s}_{N(s)} \underbrace{\quad\quad t}_{N(t)-N(s)}$$

* Counting Process with stationary increments

$s \leq t$ $\quad N(t)-N(s)$ & $N(t+u)-N(s+u)$ if both have same dist^n

if dist^n of no. of events occured in an interval depends on length of interval only

$2 \leq 3$ $\quad N(3)-N(2)$ & $N(13)-N(12)$ if both have same dist^n

$(2,3] \to 1$ $\quad (12,13] \to 1$.

* Poisson Process :-

Def^n ① The counting process $\{N(t), t \geq 0\}$ is said to be a poisson process having rate λ, $\lambda > 0$ if

① $N(0) = 0$

② The process has indep. & stationary increments

$0,t$ $\quad$ $s, s+t$ $\quad \sim Poi(\lambda t)$

$N(t)$ $\quad$ $N(s+t) - N(s)$

③ The no. of events in any interval of length t, is poisson distributed with paramater λt

$(0,t)$ $\quad N(t) \sim Poi(\lambda t)$, $\quad (s, s+t]$, $\quad N(t+s) - N(s) \sim Poi(\lambda t)$

Def^n ②

Equivalence bet^n two def^{n}s

counting ✓

① $N(t) \geq 0$

② Integer value

③ $s \leq t$ $\quad N(s) \leq N(t)$

④ $(s,t]$, $\quad N(t) - N(s)$ incre

stoch ✓ poiss. process ✓

① $N(0)=0$

② incr. ind. & stationary

③ $\sim Poi(\lambda(\text{length of interval}))$

Poisson process example

$\{N(t), t \geq 0\}$ poiss $\longrightarrow$ $\lambda = 2/\text{day}$ $\quad$ ⑤ ?

$N(0)=0$ ✓ $\quad (0,5]$ $\quad N(5) = ?$

$\sim Poi(\lambda \cdot 5) = Poi(10)$

$\cancel{P(N(5))} = N(5) \sim Poi(10)$

$0, 1, 2, \ldots$

$E(N(5)) = 10$

Def^n ② Let $\{N(t), t \geq 0\}$ be a counting process is said to be Poisson process with rate λ, $\lambda > 0$ if

① $N(0) = 0$ ✓

② indep. & stationary increments. ✓

③ $P(N(h) = 1) = P_1(h) = \lambda h + o(h)$

$o(h) \to 0$ as $h \to 0$

h small

(4) $P(N(h) \geq 2) = o(h) = \sum_{k=2}^{\infty} P_k(h)$

equivalence

Def (1) $\rightarrow$ Def (2)

for first two conditions no need to prove

To prove Def (2) → (3) cond^n $\boxed{P_1(h) = \lambda h + o(h)}$

$P_n(t) = P[N(t) = n]$

$P_1(h) = P(N(h) = 1)$ $(0,h]$ $N(h) \sim Poi(\lambda h)$

$= e^{-\lambda h} \cdot \lambda h$

$= \lambda h \left[1 - \lambda h + \frac{(\lambda h)^2}{2!} - \frac{(\lambda h)^3}{3!} + \cdots\right]$ $o(h) = o(h) + o(h)$

$= \lambda h + o(h)$ $\rightarrow 0$

$\cancel{P_2(h)}$ $\sum_{k=2}^{\infty} P_k(h) = \sum_{k=2}^{\infty} \frac{e^{-\lambda h}(\lambda h)^k}{k!}$

$P[N(h) \geq 2]$

$= 1 - P_1(h) - P_0(h)$

$= 1 - e^{-\lambda h}\lambda h - e^{-\lambda h}$

$= 1 - (\lambda h + 1) \cdot e^{-\lambda h}$

$= 1 - (\lambda h + 1)\left[1 - \lambda h + \frac{(\lambda h)^2}{2} + \cdots\right]$

$= \cancel{1} - \lambda h\left[1 - \lambda h + \frac{(\lambda h)^2}{2!} + \cdots\right] - \left[\cancel{1} - \lambda h + \frac{(\lambda h)^2}{2!} + \cdots\right]$

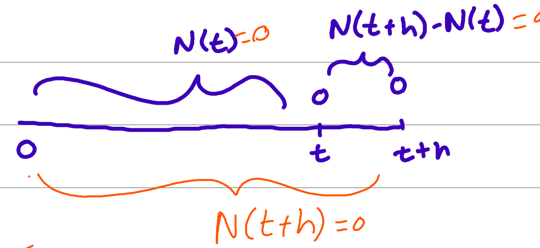
$= o(h)$

Def^n ② $\to$ Def^n ①

Def^n ②: ① $N(0)=0$ ② Stationary & indep. ③ $P_1(h) = \lambda h + o(h)$ ④ $\sum_{k=2}^{\infty} P_k(h) = o(h)$ $P(N(h)\geq 2)$

Def ①: ① ——— ② ——— ③ increment $\sim$ Poi(λ·length)

$$P_n(t) = P[N(t)=n]$$



$$P_0(t+h) = P[N(t+h)=0]$$

$$= P[N(t)=0, N(t+h)=0]$$

$$= P[\underbrace{N(t)=0}_{(0,t]}, \underbrace{N(t+h)-N(t)=0}_{(t,t+h]}]$$ ind. increment

$$= P[N(t)=0] \cdot P[N(t+h)-N(t)=0]$$

$$= P[N(t)=0] \cdot P_0(h)$$

$$= P[N(t)=0] \cdot [1 - P_1(h) - \sum_{k=2}^{\infty} P_k(h)]$$

$$P[N(t+h)=0] = P(N(t)=0] \cdot [1 - \lambda h + o(h)] \quad \cancel{+o(h)}$$

$$\to \lim_{h\to 0} \frac{P(N(t+h)=0) - P(N(t)=0)}{h} = \lim_{h\to 0} \frac{-\lambda \cancel{h} P(N(t)=0)}{\cancel{h}} + \frac{o(h)}{h}$$

$$P_0'(t) = -\lambda P_0(t)$$

$$\frac{1}{P_0(t)} \cdot P_0'(t) = -\lambda$$

$$\frac{d}{dt} \log P_0(t) = -\lambda$$

$$\int \frac{d}{dt} \log P_0(t) = \int_0^t -\lambda \, dt$$

$$\log P_0(t) = -\lambda t + C$$

$$P_0(t) = e^{-\lambda t} \cdot e^{c}$$

① $N(0)=0, \Rightarrow P_0(0) = 1 \Rightarrow P_0(0) = e^{-\lambda \cdot 0} e^{c} = e^{c} = 1$

$\Rightarrow c = 0$

$$P_0(t) = e^{-\lambda t} \checkmark$$

$\checkmark\ P_0(t) = e^{-\lambda t}$

$N(t) \sim Poi(\lambda t)$

Poisson $x = 0, 1, 2, \ldots$ 0, 1,

0 t t+h

n

$$P_n(t+h) = P[N(t+h)=n]$$

$= P[N(t+h)=n / N(t)=n]\ P[N(t)=n] +$
$P[N(t+h)=n / N(t)=n-1]\ P[N(t)=n-1] +$
$P[N(t+h)=n / N(t)\leq n-2]\ P[N(t)\leq n-2]$

0 0, 1 a N

0 t t+h

P_n

$= P_0(h) \cdot P_n(t) + P_1(h) \cdot P_{n-1}(t) + \sum_{k\geq 2}^{\infty} P_k(h) \cdot P_{n-k}(t)$

$$P_n(t+h) = [1-\lambda h] \cdot P_n(t) + \lambda h\, P_{n-1}(t) + o(h) \qquad +o(h)$$

$$\lim_{h\to 0} \frac{P_n(t+h) - P_n(t)}{h} = \lim_{h\to 0} \frac{-\lambda \cancel{h} \cdot P_n(t)}{\cancel{h}} + \frac{\lambda \cancel{h} P_{n-1}(t)}{\cancel{h}} + \frac{o(h)}{h}$$

$\downarrow 0$

$$P_n'(t) = -\lambda[P_n(t) - P_{n-1}(t)]$$

$$P_n'(t) + \lambda P_n(t) = \lambda P_{n-1}(t)$$

$$e^{\lambda t} P_n'(t) + \lambda e^{\lambda t} P_n(t) = \lambda e^{\lambda t} P_{n-1}(t)$$

$$\frac{d}{dt} e^{\lambda t} P_n(t)$$

$$\boxed{\frac{d}{dt}\left(e^{\lambda t} P_n(t)\right) = \lambda e^{\lambda t} P_{n-1}(t)} \quad \text{——— } (*)$$

$n = 1$

$$\frac{d}{dt}\left(e^{\lambda t} P_1(t)\right) = \lambda e^{\lambda t} P_0(t) = \lambda e^{\lambda t} e^{-\lambda t} = \lambda$$

$$\int_0^t \qquad\qquad \int_0^t \lambda$$

w.p. 1 $N(0)=0$

w.p. 0 $N(0)=1$

$$e^{\lambda t} P_1(t) = \lambda t + C$$

put $t=0$ $\Rightarrow$ $e^{\lambda \cdot 0} P_1(0) = 0 + C$

$\Rightarrow$ $0 = C$

$$e^{\lambda t} P_1(t) = \lambda t$$

$$\Rightarrow \quad P_1(t) = e^{-\lambda t} \lambda t$$

Abs. 2001, 5, 6, 9, 10, 14, 15, 16, 27, 33, 35, 39, 41, 43, 44, 45, 51, 54, 55

$P_0(t) = e^{-\lambda t}$ ✓ $\qquad$ $P_1(t) = e^{-\lambda t} \lambda t$ ✓

Assume it for $(n-1)$

$$P_{n-1}(t) = \frac{e^{-\lambda t} (\lambda t)^{n-1}}{(n-1)!} \quad \text{——— } (**)$$

from $(*)$

$$\frac{d}{dt} e^{\lambda t} P_n(t) = \lambda e^{\lambda t} P_{n-1}(t)$$

from $(**)$ $\quad \frac{d}{dt} e^{\lambda t} P_n(t) = \lambda \cancel{e^{\lambda t}} \frac{\cancel{e^{-\lambda t}} (\lambda t)^{n-1}}{(n-1)!} = \frac{\lambda^n t^{n-1}}{(n-1)!}$

$$\int_0^t \frac{d}{ds} e^{\lambda s} P_n(s)\, ds = \int_0^t \frac{\lambda^n s^{n-1}}{(n-1)!}\, ds$$

$$e^{\lambda t} P_n(t) = \left[\frac{\lambda^n}{(n-1)!}\,\frac{s^n}{n}\right]_0^t = \frac{(\lambda t)^n}{n!}$$

$$\boxed{P_n(t) = \frac{e^{-\lambda t}(\lambda t)^n}{n!}}$$

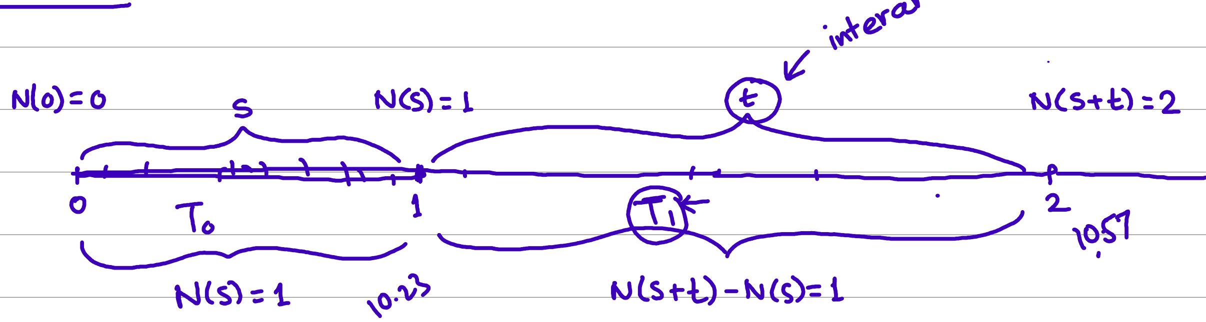
$$N(t) \sim Poi(\lambda t)$$

Inter event

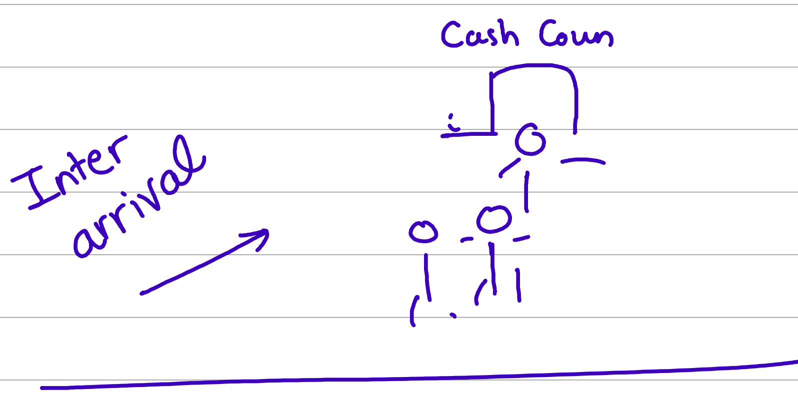
* Interarrival time between poisson events :-

$N(u)=1$ $s<u\leq s+t$

$N(u)=0$ $u<s$



Bank



$T_0 \rightarrow$
$\{ T_1 \rightarrow 2^{nd}$

T_0, T_1 interarrival

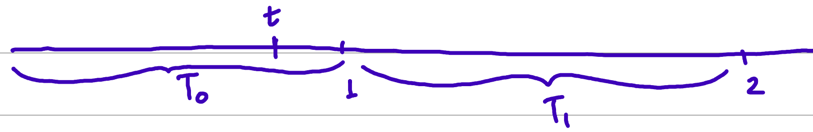
$N(T_0) = 1$

$N(T_0+T_1) = 2$

. . . .

T_0+T_1 arrival

- $N(t) \sim$ follows poisson process



$$F_{T_0}(t) = P[T_0 \le t] = 1 - P[T_0 > t]$$
$$= 1 - P[N(t) = 0]$$
$$= 1 - e^{-\lambda t}$$

$T_0 \sim \exp(\lambda)$

$$F_{T_1}(t) = P[T_1 \le t / T_0 = s] = 1 - P[T_1 > t / T_0 = s]$$

? no. of events =

$$= 1 - P[N(t) = 0]$$
$$\cancel{P[N(t+s) - N(s) = 0]} \text{ Same}$$
$$= 1 - e^{-\lambda t}$$

$T_1 \sim \exp(\lambda)$

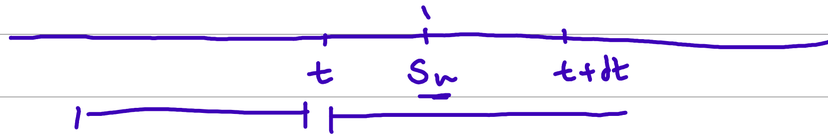
$\Rightarrow$ $T_0, T_1, \ldots$ $\exp(\lambda)$ $\qquad E(T_i) = 1/\lambda$

* Arrival Time

$S_n \Rightarrow$ Arrival time of n^{th} customer/event

$$S_n = \sum_{i=0}^{n-1} T_i$$

T_i's are i.i.d $\exp(\lambda)$, $N(t) \sim Poi(\lambda t)$



$$P(t < S_n < t+\delta t) = P(\underset{(0,t]}{N(t)=n-1},\ \underset{(t,t+\delta t]}{N(t+\delta t)-N(t)=1}) \quad \text{ind. incre.}$$

$$= P(N(t)=n-1) \cdot P(N(t+\delta t)-N(t)=1)$$

δt

$$F(t+\delta t) - F(t) = \frac{e^{-\lambda t}(\lambda t)^{n-1}}{(n-1)!} \cdot e^{-\lambda \delta t}\lambda \delta t$$

$$f(t) = \lim_{\delta t \to 0} \frac{F(t+\delta t) - F(t)}{\delta t} = \frac{e^{-\lambda t}(\lambda t)^{n-1}}{(n-1)!} \lim_{\delta t \to 0} \frac{e^{-\lambda \delta t}\lambda \cancel{\delta t}}{\cancel{\delta t}}$$

$$= \frac{e^{-\lambda t}}{(n-1)!}\lambda^n t^{n-1}$$

$$= \frac{\lambda^n}{\Gamma n} e^{-\lambda t} t^{n-1}$$

$$S_n \sim \text{Gamma}(n, \lambda)$$

* Conditional distribution of arrival time

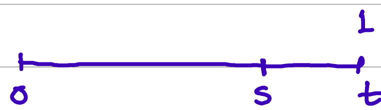
inter arrival ⇒ exp
arrival ⇒ gamma
condⁿ arrival ⇒ uniform

$$x_i \sim U(0,t) \qquad f(x) = \frac{1}{t} \quad 0<x<t$$

$$\boxed{f(x_{(1)}, x_{(2)}, \dots, x_{(n)}) = \frac{n!}{t^n}}$$

· $N(t) = 1$

S_n = arrival of n^{th} = $\sum_{i=0}^{n-1} T_i \Rightarrow S_1 = T_0$

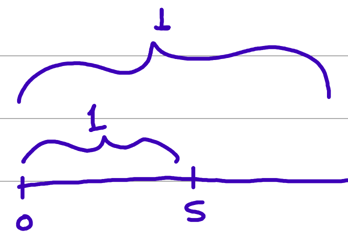


For $s \le t$

$$F(s) = P(S_1 < s \,/\, N(t) = 1)$$

$$= \frac{P[T_0 < s, N(t) = 1]}{P[N(t) = 1]}$$

$$= \frac{P[N(s) = 1, N(t) - N(s) = 0]}{P[N(t) = 1]}$$



ind. incr

$$= \frac{P[N(s) = 1] \cdot P[N(t) - N(s) = 0]}{P[N(t) = 1]}$$

$$= \frac{e^{-\lambda s} \lambda s \quad e^{-\lambda(t-s)}}{e^{-\lambda t} \lambda t}$$

$$= \frac{s}{t}$$

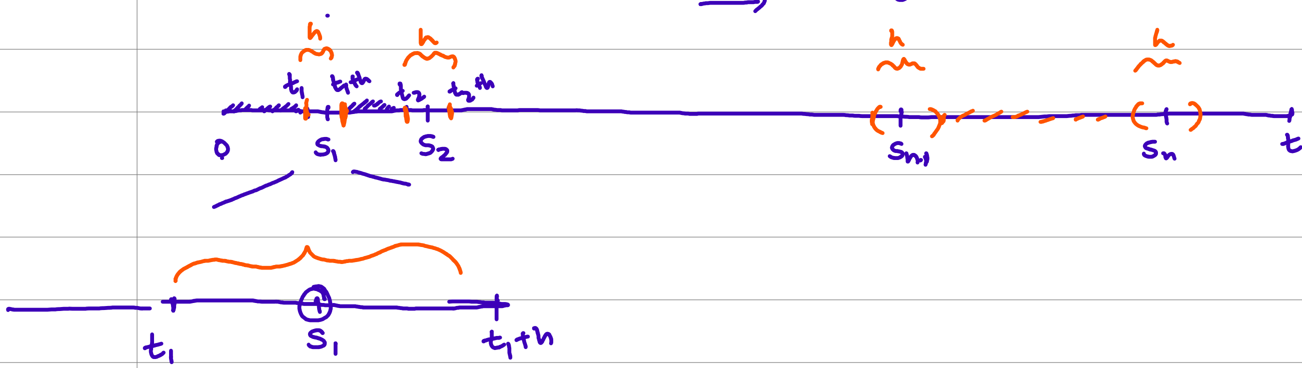
$$F(s) = \frac{s}{t}.$$

$\underline{\underline{U(0,t)}} \Rightarrow F(x) = \frac{x}{t}$

$\underline{\underline{S_1 / N(t) = 1}} \sim U(0,t)$ ✓

* Given that $N(t)=n$, n arrival times $S_1, S_2, \ldots, S_n$ have same distⁿ as of the order statistics of n indep. random variables from $U(0,t)$

$0\text{-}t \Rightarrow$ length $\underline{t-nh}$ 0



let t_i such that $0<t_1<S_1<t_1+h<t_2<\ldots < t_n<S_n<t_n+h<t$

$$P[t_i \le S_i \le t_i+h,\ i=1:n / N(t)=n]$$

$$= \frac{P[N(t_i+h)-N(t_i)=1 \ \forall\ i=1:n,\ \text{no event elsewhere in } (0,t)]}{P[N(t)=n]} \quad \overset{t-nh}{\uparrow}$$

$$= \frac{[\lambda h\, e^{-\lambda h}]^n \ e^{-\lambda(t-nh)}}{\frac{e^{-\lambda t}(\lambda t)^n}{n!}} \qquad P[N(t-nh)=0]$$

$$= \frac{n!\ \cancel{\lambda^n}\ \cancel{e^{-\lambda t}}}{\cancel{e^{-\lambda t}}(\cancel{\lambda} t)^n} \cdot h^n\ \cancel{e^{-\lambda nh}}\ \cancel{e^{+\lambda nh}}$$

$$= \frac{n!}{t^n} \quad h^n$$

$$\lim_{h\to 0} \frac{1}{h^n} P[t_i \le S_i \le t_i+h,\ i=1:n / N(t)=n] = \frac{n!}{t^n}$$

$$f(S_1, S_2, \ldots, S_n / N(t)=n) = \frac{n!}{t^n}$$

Conditional joint pdf of arrival times is equi to joint pdf of order stats $U(0,t)$

Poisson Process — Branching Process

① ①

②

③ $\sim Poi(\lambda t)$

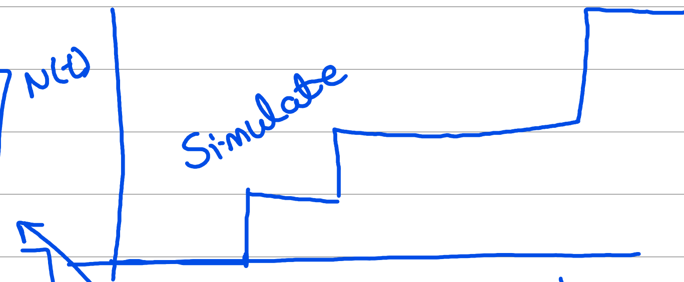
$N(t+s) - N(s) \sim Poi(\lambda t)$

Simulate Poisson ?

$N(t) = ?$ $t > 0$

✓ ② Interarrival → $\overset{?}{expo}(1/\lambda)$

Arrival times ~ Gamma$(n, 1/\lambda)$



③ Conditional distⁿ of arrival times

$S_1, S_2, \ldots, S_n / N(t) = n$

$\sim X_{(1)}, X_{(2)}, \ldots, X_{(n)} \sim U(0,t)$

(10) $\cdot U(0,7)$

order

$S_1, S_2 \ldots S_n / N(7) = 10$ (0,7)

Random Sample from

$\{N(t), t \geq 0\}$ (0 — 15)

$\lambda = 2/hr$ $t = 15$ hr

$N(t) = ?$ (30)

$N(22) = 0$

$N(37) = 1$

①	Arrival	Interarrival
①	✓23 min	23 min
②	55 min	32 min
③	✓75 min	20 min

$\lambda = 2/hr$ $t = 15$ hr

$T_0 \sim exp(\text{mean } 1/2)$

$T_1 \sim ex$

$T_2 \sim$

⋮

$T_{100} \sim$

$N(t)$

$(0, t]$

0 – 23	0
23 – 55	1
55 – 75	2

$N(t) = 0 \quad 0 \leq t < 23$

$\quad\quad 1 \quad 23 \leq t \leq 55$

Abs. 1, 6 to 10, 13, 15, 22, 23, 25, 27, 33 to 35, 37, 39, 42 to 45, 47, 50, 54 to 56

Poisson Process simulation ($\lambda =$)

① Expo ② Uni - Cond^n ③ Poi ~~X~~

$\lambda = 2$, 0-t, $t = 15$

Interarrival $\sim \exp(1/\lambda)$

$T_i \sim \exp(1/\lambda)$

simulate T_i, $n \geq \lambda t$

arr = Arrival time - cumsum

14.90 15 ~~15.23~~

$$N(t) = \begin{cases} 0 & 0 \leq t < S_1 \\ 1 & S_1 \leq t < S_2 \\ 2 & \\ 3 & S_n \\ & 14.90 \leq t \leq t = 15 \end{cases}$$

no. of arrivals 0-t

dataframe(c(0,arr), c(arr,t), 0:n)

$S_1, S_2 \ldots S_n / N(t) = n \sim X_{(1)}, X_{(2)}, \ldots, X_{(n)}$

where $X_{(i)}$ is order statistics of $U(0,t)$

$i = 1:n$

S ~~X~~ = sort (runif(n,0,t))

$N(t) =$ $0 \leq t \leq S$ 0

$S \leq t \leq t$ = n

$0, t$

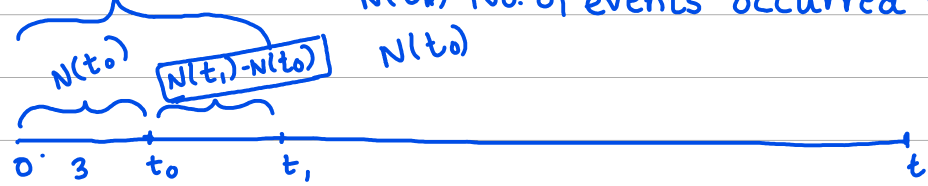
③ sample path $\quad t_0 \le t_1 \le t_2 \le \cdots \le t_n$

simulate $\quad N(t_0)\ N(t_1)\ N(t_2) \ldots\ N(t_n)$

$N(t_1)$ No. of events occurred in $(0, t_1]$

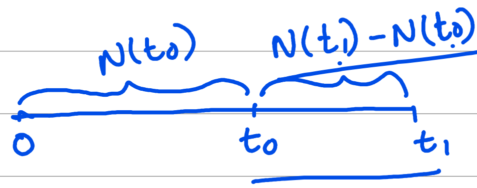
$N(t_0) \quad (0, t_0]$

① $N(0)=0$
② P
③



$N(t_0) \sim Poi(\lambda \cdot t_0) \qquad N(t_0) = 2$

$N(t_1) \sim Poi(\lambda t_1) \qquad N(t_1) / N(t_0)=2 \ \sim Poi($



① increment $\sim Poi(\lambda(t_1 - t_0))$

$t_0 \quad N(t_0) \sim Poi(\lambda t_0)$

$t_1 \quad \underline{N(t_0)} + \underline{N(t_1)-N(t_0)} \sim Poi(\lambda(t_1-t_0))$

$t_2 \quad N(t_1) + \underline{N(t_2)-N(t_1)} \sim Poi(\lambda(t_2-t_1))$

$\vdots$

Simulate the Poisson process at time points t with rate without simulating its interarrival time.

i) t = 1.5,2.2,3.8,7.5,8.8 & λ= 1 ii) t = 1.23,2.21,2.83,6.05,7.08,17.8 & λ= 1.5

time pts.	inte	λ.int.			
1.5	1.5	$\rightarrow$ Poi( )	N(1.5)		
2.2	2.2-1.5	Poi( )		+	= N(2.2)
3.8	3.8-2.2				
7.5	7.5-3.8				
8.8	8.8-7.5				

Let {Xn,n N} be a Bienayme-Galton-Watson Branching Process with offspring distribution given by P[= 0] = 0.2,P[= 1] = 0.3,P[= 2] = 0.2,P[= 3] = 0.3. If X0 = 2,then realize X1,X2,··· ,X5 where Xn denote size of n

th generation.

x1 = x2 x6

Branching $X_n = \sum_{i=1}^{X_{n-1}} Z_i$

$P(Z_i = z) = 0.2 \quad z = 0$

$0.3 \quad z = 1$

$0.2 \quad z = 2$

$0.3 \quad z = 3$

$X_0 = 2$

$x_1 = Z_1 + Z_2$

2

{ x_1 = sum(sample(x_0, 0:3, prob=c(0.2 0.3 0.2 0.3)

g (s) = 0.25 + 0.35s + 0.3s2 + 0.1s3

$P[Z=0] = 0.25$

1 0.35

2 0.3

3 0.1

Absent: 2006 9, 10, 33-35, 39, 43, 44, 45, 47, 55