Stochastic Processes

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Contents

Chapter 1

Background

1.1 Examples on Conditional Probability and Expectations

Example 1: A family has two children. What is the conditional probability that both are boys given that at least one of them is boy.

Solution:
S = {BB, GB, BG, GG}
A: Both are boys = {BB}
B: At least one of them is boy = {BG, GB}

P(A/B)=$\frac{(A \cap B)}{B}$ =$\frac{1/4}{3/4}$ =$\frac{1}{3}$

Example 2: Beve can either take a course in computer or chemistry. If she takes computer then she will receive an 'A' grade with probability $\frac{1}{2}$. If she takes chemistry then she will receive an 'A' grade with probability $\frac{1}{3}$. She decides to base her decision on the flip of coin. What is the probability that she will get an 'A' grade in chemistry.

Solution:
P(Chemistry)=$\frac{1}{2}$, P(Computer)=$\frac{1}{3}$
P(A/Chemistry)=$\frac{1}{3}$, P(A/Computer)=$\frac{1}{2}$
P(A in Chemistry)=P(A/Chemistry).P(Chemistry)=$\frac{1}{6}$
Also,
P(Chemistry/A)=$\frac{P(A/Chemistry).P(Chemistry)}{P(Chemistry).P(A/Chemistry)+P(A/Computer).P(Computer)}$

$$= \frac{2}{5}$$

Example 3: Let a ball be drawn from an Urn containing 4 balls numbered 1, 2, 3, 4. Let event E={1,2}, F={1,3}, G={1,4}. If all four events are assumed equally likely.
Solution:
E={1,2}, F={1,3}, G={1,4}

P(E ∩ F)=$\frac{1}{4}$
P(E).P(F)=$\frac{1}{2}.\frac{1}{2} = \frac{1}{4} \implies P(E\cap F) = P(E).P(F)$
$P(F\cap G) = P(F).P(G)$, $P(E\cap G) = P(E).P(G)$
∴ E, F, G are pairwise independent.
But, $P(E\cap F\cap G) = \frac{1}{4} \neq P(E).P(F).P(G)$
∴ E,F,G are not mutually exclusive.

Example 4: Three dies are thrown. What is the probability same number occurs on two of the three dies.
Solution:

1.2 Probability

An event is a subset of a sample space and is said to be occur is the outcome of experiment is an element of that subset for event E, sample space S,
Axiom 1: 0 ≤P(E)≤ 1
Axiom 2: P(S)=1
Axiom 3: For any sequence of events $E_1, E_2, ...$ that are mutually exclusive, that is, events for which $E_i \cap E_j = \phi$ when i ≠j (where ϕ is null set)
P($\cup_{i=1}^{\infty} E_i$) = P($\sum_{i=1}^{\infty} E_i$)

1. If E ⊂ F, P(E)≤P(F)
2. P(E^c)=1-P(E)
3. P($\cup_{i=1}^{n} E_i$) = P($\sum_{i=1}^{n} E_i$) when E_i are mutually exclusive.
4. P($\cup_{i=1}^{\infty} E_i$) ≤ P($\sum_{i=1}^{\infty} E_i$) ....(Boole's inequality)

Note: If {E_n, n≥ } is either an increasing or decreasing sequence of events then,

$$\lim_{n\to\infty} P(E_n) = P(\lim_{n\to\infty} E_n)$$

1.3 Borel-Cantelli Lemma:

Let $E_1, E_2, \cdots$ are independent events such that $\sum_{i=1}^{n} E_i < \infty$ then, P{An infinite number of E_i occur }=0

1.4 Converse to Borel-Cantelli Lemma:

If $E_1, E_2, \cdots$ are independent events such that $\sum_{i=1}^{n} E_i = \infty$ then, P{An infinite number of E_i occur }=∞ then P{An infinite number of E_n occur}=0.

Example 5:

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 0.5 & 0.5 & 0 \\ 0 & 0.5 & 0.5 \\ 0.5 & 0 & 0.5 \end{pmatrix} \end{array}$$

This is irreducible chain. **Example 6:**

$$\begin{array}{c} \\ H \\ S \\ O \end{array} \begin{array}{c} \begin{array}{ccc} H & S & O \end{array} \\ \begin{pmatrix} 0.8 & 0.15 & 0.05 \\ 0.5 & 0.3 & 0.2 \\ 0 & 0 & 1 \end{pmatrix} \end{array}$$

This is reducible chain. **Expected Value:**

- $E(x) = \int x f(x) dx$ if x is continuous
- $E(x) = \sum x P(x) dx$ if x is discrete
- $E(h(x)) = \int h(x) f(x) dx$
- $V(x) = E(x - E(x))^2 = E(x^2) - E[E(x)]^2$
- $Cov(X,Y) = E((X - E(X))(Y - E(Y))) = E(XY) - E(X) \cdot E(Y)$
- $E(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} E(X_i)$
- $V(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} V(X_i) + 2 \sum \sum_{i<j} Cov(X_i, X_j)$

1.5 The Matching Problem

Example 7: At a party of n people, put their hats in the centre of the room where the hats are mixed together. Each person that randomly selects one. We are interested in the mean and variance of X - The number that select their own hat.

Solution:

Solution:

Let's split X in $\{X_1 + X_2 + \cdots + X_n\}$

$$\begin{cases} 1 & \text{, if } i^{th} \text{ individual chooses his own hat} \\ 0 & \text{, otherwise} \end{cases}$$

E$(X_i) = \dfrac{1}{n}$,

V(X_i) = E(X_i^2)-[E$(X_i)]^2 = \frac{1}{n} - \frac{1}{n^2} = \frac{1}{n}(1-\frac{1}{n})$
Cov(X_i, X_j)=E$(X_i X_j)$-E(X_i)E(X_j)

$$X_j X_j = \begin{cases} 1 & ; X_i = 1, X_j = 1 \\ 0 & ; \text{Otherwise} \end{cases}$$

$\therefore E(X_i X_j) = \frac{1}{n(n-1)}$
$\therefore Cov(X_i, X_j) = \frac{1}{n(n-1)} - \frac{1}{n^2}$
E(X)=E($\sum_{i=1}^{n} X_i$)=n E(X_i)=1
V(X)=V($\sum_{i=1}^{n} X_i$)=n ■

1.6 Moment Generating Function

$$M_x(t) = E(e^{tx})$$

1. Binomial (n,p), 0<p<1

$$P(x) = \frac{x_!}{(n-x)!} p^x (1-p)^{n-x} x = 0, 1, ...n, 0 < p < 1$$

$M_x(t) = (pe^t + q)^n$
E(X)=np, V(X)=npq

2. Poisson(λ); λ>0

$$f(x) = \frac{e^{-\lambda}\lambda^x}{x!}, x = 0, 1, 2, 3, ..$$

$M_x(t) = e^{\lambda(e^t-1)}$
E(X)=V(X)=λ

3. Geometric (p)

$$f(x) = (1-p)^{x-1} p, x = 1, 2, ..., 0 < p < 1$$

$M_x(t) = pq^{(x-1)}$
E(X)=$\frac{1}{p}$, V(X)=$\frac{q}{p^2}$

4. Negative Binomial (r,p)

$$f(x) = \binom{x-1}{r-1} p^r (1-p)^{x-r}, x = r, r+1, r+2 \cdots$$

$$M_x(t) = (\frac{pe^t}{1-qe^t})^r$$

$$E(X) = \frac{r}{p}, V(X) = \frac{r(1-p)}{p^2}$$

5. Uniform (a,b)

$$f(x) = \frac{1}{b-a}, a < x < b$$

$$M_x(t) = \frac{e^{bt} - e^{at}}{t(b-a)}$$
$$E(X) = \frac{a+b}{2}$$
$$V(X) = \frac{(b-a)^2}{12}$$

6. Exponential (λ)

$$f(x) = \lambda e^{-\lambda x}, \lambda > 0$$
$$M_X(t) = \frac{\lambda}{\lambda - t}$$
$$E(X) = \frac{1}{\lambda}$$
$$V(X) = \frac{1}{\lambda^2}$$

7. Gamma (n, λ)

$$f(x) = \frac{\lambda e^{-\lambda x}(\lambda x)^{n-1}}{(n-1)!}; x \geq 0$$
$$M_x(t) = (\frac{\lambda}{\lambda - t})^n$$
$$E(X) = \frac{n}{\lambda}$$
$$V(X) = \frac{n}{\lambda^2}$$

8. Normal(μ,σ^2)

$$f(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}, -\infty < x < \infty$$
$$M_x(t) = e^{\mu t + \frac{1}{2}t^2\sigma^2}$$
$$E(X) = \mu$$
$$V(X) = \sigma^2$$

9. Beta(a,b) a>0, b>0

$$f(x) = \frac{\Gamma(a+b)x^{a-1}(1-x)^{b-1}}{\Gamma a \Gamma b}, 0 < x < 1$$

$$E(X) = \frac{a}{a+b}$$

$$V(X) = \frac{ab}{(a+b)^2(a+b+1)}$$

1.7 Ballot Problem

Example 8: In an election, candidate A receives 'n' votes and B receives 'm' votes, where $n > m$. Assuming that all orderings are equally likely show that the probability A is always ahead in the count of votes is $\frac{n-m}{n+m}$.

Solution:

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Chapter 2

Stochastic Process and Chapman Kolmogorov Equation

2.1 Stochastic Process

Definition 1 (State Space)**:** *The set of all possible values of single random variable X_n of a stochastic process is called state space.*

Definition 2 (Time Domain)**:** *The values 't' in the index set T is called time domain.*

Definition 3 (Stochastic Process)**:** *A stochastic process $\{x(t), t \in T\}$ is the collection of random variables i.e. for each t in the index set T, $x(t)$ is a random variable.*

We often indicate t as time t and call x(t), the state of the process at time t if the index set T is countable then x(t) is discrete time stochastic process and if T is continuous, we call it continuous time process.
Any realization of x(t) is called as sample path.

2.2 Markov Chains

(Discrete-Discrete)

Consider a stochastic process $\{X_n, n = 0, 1, 2, \cdots\}$ that takes on a finite or countable number of possible values. This set of possible values of the process will be denoted by the set of non-negative integers $\{0, 1, 2, \cdots\}$. If $X_n = i$ then the process is said to be in

State Space	**Time**(t)	**Example** $x(t)$
Discrete	Discrete	1. Number of guests on a particular day. 2. Number of students present on a particular day.
Continuous	Discrete	1. Temperature on a particular day. 2. Blood Pressure on a particular time.
Discrete	Continuous	Number of accidents in a particular time
Continuous	Continuous	Amount of rainfall in certain city at a particular time

Table 2.1: Stochastic Process

state 'i' at time n.

$$P_{ij} = P[X_{n+1} = j/X_n = i]; \quad P_{ij}^{(n)} = P[X_n = j/X_0 = i]$$

then X_n follows markov property and hence called as markov chain. Here, P_{ij} represents the probability that process will make transition to state j to i.

$$P_{ij} \geq 0, i, j \geq 0, \sum_{j=0}^{\infty} P_{ij} = 1, i = 0, 1, 2, \cdots$$

One Step Probability Matrix=

$$\begin{bmatrix} P_{00} & P_{01} & \cdots \\ P_{10} & P_{11} & \cdots \\ \vdots & \vdots & \end{bmatrix}$$

Example 9: Forecasting Weather

Suppose that the chance of raining tomorrow depends on previous weather condition only through weather or not it is raining today and not on past weather condition. Suppose also that if it rains today. Then it will rain tomorrow with probability α and if it does not rain today then it will rain tomorrow with probability β.

Solution:

$$\begin{array}{cc} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{pmatrix} 1-\beta & \beta \\ 1-\alpha & \alpha \end{pmatrix} \end{array}$$

$$P_{11}^{(2)} = P_{10}.P_{01} + P_{11}.P_{11}$$
$$P_{11}^{(2)} = (1-\alpha)\beta + \beta^2$$

2.3 Chapman Kolmogorov Equation

Statement: Let $P_{ij}^{(n)}$ denote n-step transition probability

$$P_{ij}^{(n)} = P\left\{X_{n+k} = j/X_k = i\right\}$$

The CK equation provide a method for computing these n-step transition probabilities.

$$P_{ij}^{(n+m)} = \sum_{k=0}^{\infty} P_{ik}^{(n)}.P_{kj}^{(m)} \quad \forall n, m \geq 0, i, j$$

Proof. Consider, $P_{ij}^{(n+m)} = P[X_{(n+m)} = j/X_0 = i]$

$$P_{ij}^{(n+m)} = \sum_{k \in S} P[X_{n+m} = j/X_n = k, X_0 = i].P[X_n = k/X_0 = i]$$

$$P_{ij}^{(n+m)} = \sum_{k \in S} P_{kj}^{(m)} P_{ik}^{n}$$

$$P^{(n+m)} = P^{(n)}.P^{(m)}$$

□

Example 10: Let $\alpha = \frac{2}{3}$ and $\beta = \frac{1}{3}$ in previous example then calculate probability that it will rain 4 days from two days given that its rain today.
Solution:

We have to calculate the probability that it will rain 4 days from today given that it is raining today. i.e. $P_{11}^{(4)}$

$$\text{P}= \begin{array}{c c} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{pmatrix} 1-\beta & \beta \\ 1-\alpha & \alpha \end{pmatrix} \end{array}$$

$$P = \begin{array}{c c} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{pmatrix} \end{array}$$

Now,

$$P^{(2)} = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix} \cdot \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

$$P^{(2)} = \begin{bmatrix} \frac{4}{9}+\frac{1}{9} & \frac{2}{9}+\frac{2}{9} \\ \frac{2}{9}+\frac{2}{9} & \frac{1}{9}+\frac{1}{9} \end{bmatrix}$$

$$P^{(2)} = \begin{bmatrix} \frac{5}{9} & \frac{4}{9} \\ \frac{4}{9} & \frac{5}{9} \end{bmatrix}$$

$$P^{(4)} = \begin{bmatrix} \frac{5}{9} & \frac{4}{9} \\ \frac{4}{9} & \frac{5}{9} \end{bmatrix} \cdot \begin{bmatrix} \frac{5}{9} & \frac{4}{9} \\ \frac{4}{9} & \frac{5}{9} \end{bmatrix}$$

$$P^{(4)} = \begin{bmatrix} \frac{25}{81}+\frac{16}{81} & \frac{20}{81}+\frac{20}{81} \\ \frac{20}{81}+\frac{20}{81} & \frac{16}{81}+\frac{25}{81} \end{bmatrix}$$

$$P^{(4)} = \begin{bmatrix} \frac{41}{81} & \frac{40}{81} \\ \frac{40}{81} & \frac{41}{81} \end{bmatrix}$$

$$\therefore P_{11}^{(4)} = \frac{41}{81}$$

Example 11:

A markov chain $\{X_n\}$ with S={0,1,2} has TPM P= $\begin{bmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{6} \\ 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$

if $P[X_0 = 0] = P[X_0 = 1] = \dfrac{1}{4}$. Find $E(X_3)$

Solution:

$$\alpha P^3 = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{bmatrix}. \begin{bmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{6} \\ 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}^3$$

$$= \frac{1}{4}.\frac{1}{6^3}.\begin{bmatrix} 1 & 1 & 2 \end{bmatrix}. \begin{bmatrix} 3 & 2 & 1 \\ 0 & 2 & 4 \\ 3 & 0 & 3 \end{bmatrix}^3$$

$$= \frac{1}{4}.\frac{1}{6^3}.\begin{bmatrix} 1 & 1 & 2 \end{bmatrix}. \begin{bmatrix} 3 & 2 & 1 \\ 0 & 2 & 4 \\ 3 & 0 & 3 \end{bmatrix} \begin{bmatrix} 12 & 10 & 14 \\ 12 & 4 & 20 \\ 18 & 6 & 12 \end{bmatrix}$$

$$= \frac{1}{4}.\frac{1}{6^3}.\begin{bmatrix} 9 & 4 & 11 \end{bmatrix}. \begin{bmatrix} 12 & 10 & 14 \\ 12 & 4 & 20 \\ 18 & 6 & 12 \end{bmatrix}$$

$$= \frac{1}{4}.\frac{1}{6^3}.\begin{bmatrix} 354 & 172 & 338 \end{bmatrix}$$

$$E(X_3) = \frac{[172 + 676]}{4 * 6^3}$$

$$E(X_3) = 0.9814$$

■

Example 12: Let the TPM of two state Markov Chain $P = \begin{bmatrix} p & 1-p \\ 1-p & p \end{bmatrix}$. Show that by mathematical induction.

$$P^{(n)} = \begin{bmatrix} \frac{1}{2} + \frac{1}{2}(2p-1)^n & \frac{1}{2} - \frac{1}{2}(2p-1)^n \\ \frac{1}{2} - \frac{1}{2}(2p-1)^n & \frac{1}{2} + \frac{1}{2}(2p-1)^n \end{bmatrix}$$

Solution:

Step 1: We will prove this for n=1

$$P^{(1)} = \begin{bmatrix} \frac{1}{2} + \frac{1}{2}(2p-1)^1 & \frac{1}{2} - \frac{1}{2}(2p-1)^1 \\ \frac{1}{2} - \frac{1}{2}(2p-1)^1 & \frac{1}{2} + \frac{1}{2}(2p-1)^1 \end{bmatrix}$$

$$P^{(1)} = \begin{bmatrix} p & 1-p \\ 1-p & p \end{bmatrix}$$

Step 2: We assume for n=k
i.e.

$$P^{(k)} = \begin{bmatrix} \frac{1}{2}+\frac{1}{2}(2p-1)^k & \frac{1}{2}-\frac{1}{2}(2p-1)^k \\ \frac{1}{2}-\frac{1}{2}(2p-1)^k & \frac{1}{2}+\frac{1}{2}(2p-1)^k \end{bmatrix}$$

Step 3: We will prove for n=k+1

$$\therefore P^{(k+1)} = P^k.P$$

$$= \begin{bmatrix} \frac{1}{2}+\frac{1}{2}(2p-1)^k & \frac{1}{2}-\frac{1}{2}(2p-1)^k \\ \frac{1}{2}-\frac{1}{2}(2p-1)^k & \frac{1}{2}+\frac{1}{2}(2p-1)^k \end{bmatrix} . \begin{bmatrix} p & 1-p \\ 1-p & p \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1+(2p-1)^k & 1-(2p-1)^k \\ 1-(2p-1)^k & 1+(2p-1)^k \end{bmatrix} . \begin{bmatrix} p & 1-p \\ 1-p & p \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} p+p(2p-1)^k+(1-(2p-1)^k)(1-p) & (1+(2p-1)^k)(1-p)+p-p(2p-1)^k \\ p-p(2p-1)^k+(1+(2p-1)^k)(1-p) & (1-(2p-1)^k)(1-p)+p+p(2p-1)^k \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1+(2p-1)^{k+1} & 1-(2p-1)^{k+1} \\ 1-(2p-1)^{k+1} & 1+(2p-1)^{k+1} \end{bmatrix}$$

$$P^{(k+1)} = \begin{bmatrix} \frac{1}{2}+\frac{1}{2}(2p-1)^{k+1} & \frac{1}{2}-\frac{1}{2}(2p-1)^{k+1} \\ \frac{1}{2}-\frac{1}{2}(2p-1)^{k+1} & \frac{1}{2}+\frac{1}{2}(2p-1)^{k+1} \end{bmatrix}$$

$\therefore$ By principle of mathematical induction, given statement is true. ■

Example 13: In previous example, suppose that it has rained neither yesterday nor the day before yesterday. What will be the probability that tomorrow will be rain.

Solution:

X_1:Day before yesterday, X_2:Yesterday, X_3:Today, X_4:Tomorrow
i.e. We have to find $P_{01}^{(2)}$.

$$P = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

$$P^{(2)} = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix} . \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

$$P^{(2)} = \begin{array}{c c} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{pmatrix} \frac{5}{9} & \frac{4}{9} \\ \frac{4}{9} & \frac{5}{9} \end{pmatrix} \end{array}$$

$$\therefore P_{01}^{(2)} = \frac{4}{9} = \text{Probability of raining tomorrow.}$$

■

2.4 Classification of States

Definition 4 (Accessible States)**:** *State j is accessible from state i, if $P_{ij}^{(n)}>0$ for some $n \geq 0$ i.e. starting from state i process will ever enter state j*

Definition 5 (Communicating States)**:** *Two states i and j that are accessible to each other are said to be communicating and written as i↔j*

If i and j are communicating

i) State i communicates with state i, $i \leftrightarrow i$

ii) $i \leftrightarrow j \Rightarrow j \leftrightarrow i$

iii) $i \leftrightarrow j, j \leftrightarrow k \Rightarrow i \leftrightarrow k$

Example 14: Consider, Markov Chain with TPM

$$\begin{array}{c c} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{3} & \frac{2}{3} \end{pmatrix} \end{array}$$

$$0 \leftrightarrow 0, 0 \leftrightarrow 1, 1 \leftrightarrow 1, 1 \leftrightarrow 2, 1 \rightarrow 2, 2 \rightarrow 1$$

$c(0) = \{0, 1, 2\}$ Reducible chain (recurrent)

Example 15:

$$\begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

$$2, 3, 1 - \text{transient state, 4-recurrent}$$

$c(1) = \{1, 2, 3\} = c(2) = c(3), c(4) = \{4\}$

For any state i, we let F_i denote the probability that, starting in state i, the process will ever re-enter state i. State i is said to be recurrent if F_i=1 and transient if F_i <1

$$P\left\{\text{Ever enter in j}\middle|\text{ start from i}\right\}$$
$$= P\left\{\cup_{n=0}^{\infty}\left\{X_n = j / X_0 = i\right\}\right\}$$
$$\leq \sum_{n=0}^{\infty} P\left\{\left\{X_n = j / X_0 = i\right\}\right\} = \sum_{n=0}^{\infty} P_{ij}^{(n)}$$

Example 16:

$$\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ 1 & & & & \\ 2 & & & & \\ 3 & & & & \\ 4 & & & & \end{array}$$

$$\begin{array}{cc} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

$$F_1 = f_1^1 + f_1^2 + f_1^3 + f_1^4 + \cdots$$
$$= \frac{1}{2} + \frac{1}{2} * \frac{1}{2} + \frac{1}{2} * \frac{1}{2} * \frac{1}{2} + 0 = \frac{7}{8} < 1$$
$$\therefore \text{State '1' is transient.}$$
$$F_2 = f_2^1 + f_2^2 + f_2^3 + f_2^4$$
$$= 0 + \frac{1}{4} + \frac{1}{8} * 2 + 2 * \frac{1}{16} < 1$$
$$\therefore \text{State '2' is transient.}$$
$$F_3 = f_3^1 + f_3^2 + f_3^3 + f_3^4 + \cdots$$
$$= 0 + 0 + (\frac{1}{2})^3 + (\frac{1}{2})^4 + 2 * (\frac{1}{2})^5 + 2 * (\frac{1}{2})^6 + \cdots + 2 * (\frac{1}{2})^n < \frac{1}{2}(\frac{1}{1-\frac{1}{2}}) = 1$$
$$\therefore \text{State '3' is transient.}$$

- Period: $d_{(i)}$=gcd$\left\{n \middle| P_{ii}^{(n)} > 0\right\}$
  $d_{(0)} = gcd\{1, 2, \cdots\} = 1$, here, 0 is aperiodic state.
  $d_{(1)} = gcd\{2, 4, 6 \cdots\} = 2$, here, '1' is not aperiodic state as period $\neq$1
  $d_{(3)} = gcd\{3, 4, \cdots\} = 1$, here '3' is aperiodic state.
- The state whose period is one is known as aperiodic state.
- If for state i, $P_{ii}^{(1)} > 0$ then it must be aperiodic state and all other states in its communicating class are aperiodic automatically.
- f_{ij}^n-denote probability that, starting with i, first transition into j occures at n time. (steps)
- In all finite state Markov Chain, at least one state must be recurrent.

Example 17: To prove: $i \leftrightarrow j \implies j \leftrightarrow i$

Proof.

$$i \leftrightarrow j, P_{ij}^{(n)} > 0 \text{ for some } n > 0$$
$$j \leftrightarrow i, P_{ji}^{(m)} > 0 \text{ for some m>0}$$
$$\Rightarrow j \leftrightarrow i$$

□

Example 18: To Prove: $i \leftrightarrow j, j \leftrightarrow k \implies i \leftrightarrow k$

Proof.

$$\begin{array}{ll} i \leftrightarrow j, P_{ij}^{(n)} > 0 & \text{for some } n > 0 \\ j \leftrightarrow k, P_{jk}^{(m)} > 0 & \text{for some } m > 0 \end{array}$$

$$\begin{array}{lll} P_{ik}^{(m+n)} = \sum_{l \in s} P_{il}^{(n)}.P_{lk}^{(m)} \geq 0 & \text{by C-K equation} & \Rightarrow i \to k \\ P_{ki}^{(m+n)} = \sum_{l \in s} P_{il}^{(m)}.P_{lk}^{(n)} + P_{ij}^{(n)}.P_{jk}^{(m)} \geq 0 & \text{by C-K equation} & \Rightarrow k \to i \end{array}$$

$$\therefore i \leftrightarrow j$$

□

Example 19:

$$d_{(i)} = gcd\{n \Big| P_{ii}^{(n)} > 0\}, P_{ii}^{(m)} > 0 \implies m = k \cdot d(i)$$

To Prove: if $i \leftrightarrow j \implies d(i) = d(j)$

Example 20: State i is recurrent if and only if $\sum P_{ii}^{(n)} = \infty$

Proof. State i is recurrent if with probability 1, a process starting at 'j' will eventually return. However, by Markovian property it follows that the process probabilistically restarts itself upon returning to i. Hence, with probability it will return again to i. Repeating to this argument, with probability 1, the number of visits to 'i' will be infinite and will thus have infinite exception.

On the other hand, suppose 'i' is transient then each time the process returns to i, there is positive probability (1-F_{ii}) that it will never again return. Hence, the number of visits is geometric with finite mean $\dfrac{1}{1-F_{ii}}$

By above argument, the state i is recurrent if and only if

$$E\left\{\text{Number of visits to i} \Big| X_0 = i\right\} = \infty$$

Let, $\begin{cases} 1, & \text{if } x_n \\ 0, & \text{otherwise} \end{cases}$

$$E\left\{\sum_{n=0}^{\infty} I_n / x_0 = i\right\} = \sum_{n=0}^{\infty} E\{I_n / x_0 = i\}$$

$$= \sum_{n=0}^{\infty} P_{ii}^n$$

$\implies$ 'i' recurrent if and only if $\sum_{n=0}^{\infty} P_{ii}^n = \infty$ □

Statement If i is recurrent and $i \leftrightarrow j$, then j is recurrent. (Recurrence is the class property).

Proof. Given that $i \leftrightarrow j$,

$$\therefore i \to j, \Rightarrow P_{ij}^{(s)} > 0$$

$$j \leftrightarrow i, \Rightarrow P_{ji}^{(t)} > 0$$

$$i \text{ is recurrent. } \implies \sum_{n=0}^{\infty} P_{ii}^{(n)} = \infty$$

□

Theorem 2.4.1. *Transience is a class property.*

Proof.

$$i \leftrightarrow j$$

$$i \to j \implies P_{ij}^{(s)} > 0$$

$$j \to i \implies P_{ji}^{(t)} > 0$$

Let j is transient $\implies \sum_{n=0}^{\infty} P_{ii}^{(n+s+t)} < \infty$

$$P_{jj}(n+s+t) \geq P_{ji}^{(t)}.P_{ii}^{(n)}.P_{ij}^{(s)}$$

$$\text{i.e.} P_{ji}^{(t)}.P_{ii}^{(n)}.P_{ij}^{(s)} \leq P_{jj}^{(n+s+t)}$$

$$P_{ji}^{(t)} \sum_{n=0}^{\infty} P_{ii}^{(n)} P_{ij}^{(s)} \leq P_{jj}^{(n+s+t)} < \infty$$

Since, j is transient $\therefore \sum_{n=0}^{\infty} P_{ii}^{(n)} < \infty$

$$\therefore \textbf{i is transient.}$$

□

- The markov chain is said to be irreducible if there is only one class that is, if all states communicates with each other.

2.5 Limit Theorem

Statement If state j is transient, then

$$\sum_{n=1}^{\infty} P_{ij}^{(n)} < \infty, \forall i$$

Meaning that, starting in i, the expected number of transitions into state j is finite.

$$\implies \sum P_{ij}^{(n)} \to 0 \text{ as } n \to \infty (\text{transient})$$

Let μ_{ij} denote the expected number of transitions needed to return to state j.

$$\mu_{ij} = \begin{cases} \infty & \text{, if j is transient} \\ \sum_{n=1}^{\infty} n f_{jj}^{(n)} & \text{, if j is recurrent} \end{cases}$$

- If state j is recurrent, then we say that it is positive recurrent if $\mu_{ij} < \infty$ and null recurrent if $\mu_{ij} = \infty$.
- if we let $\pi_j = \lim_{n\to\infty} P_{ij}^{n.d_{(j)}}$ it follows that recurrent state j is the non-null positive recurrent if $\pi > 0$& null recurrent if $\pi = 0$.
- A positive recurrent, aperiodic state is called as ergodic.

Example 21:

$$\begin{array}{cc} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{pmatrix} 0.8 & 0.2 \\ 0.2 & 0.8 \end{pmatrix} \end{array}$$

Solution:

$$\mu_{11} = 1.f_{11}^{(1)} + 2.f_{11}^{(2)} + 3.f_{11}^{(3)} + 4.f_{11}^{(4)} + \dots$$

$$= 0.8 + 2X0.2X0.2 + 3X0.2X0.8X0.2 + 4X0.2X0.8X0.8X0.2 + \dots$$

$$= 0.8 + (0.2)^2.[\sum_{n=2}^{\infty} n(0.8)^{n-2}(0.2)]$$

$$= 0.8 + 0.2[\sum_{n=2}^{\infty} n(0.8)^{n-2}(0.2)$$

$$= 0.8 + 0.2[\sum_{n=2}^{\infty} n(0.8)^{n-2}(0.2)(0.8)^{-2} < \infty$$

$\therefore$ State 1 is positive (non-null) recurrent and aperiodic.

$\implies$ The state 1 is ergodic.

■

Example 22:

$$\begin{array}{cc} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \end{array}$$

Solution:

$$
\begin{aligned}
\mu_{11} &= 1.f_{11}^{(1)} + 2.f_{11}^{(2)} + 3.f_{11}^{(3)} + \dots \\
&= 1.(0) + 2.(0) + 3.(1) + 0 \\
&= 3 < \infty
\end{aligned}
$$

$\therefore$ 1 is positive (non-null) recurrent.

■

2.6 Stochastic Process

Definition 6: *A probability distribution $\{P_j, j \geq 0\}$ is said to be stationary for the markov chain if*

$$P_j = \sum_{i=o}^{\infty} P_i.P_{ij}, j \geq 0$$

Example 23:

$$P^{(n+1)} = P^{(n)}.P$$

$$where, P = \begin{bmatrix} 0.8 & 0.2 \\ 0.2 & 0.8 \end{bmatrix}$$

Obtain stationary distribution.

Solution:

$$\underline{\Pi} = \underline{\Pi}.P$$

$$\begin{pmatrix} \underline{\pi}_1 & \underline{\pi}_2 \end{pmatrix} = \begin{pmatrix} \underline{\pi}_1 & \underline{\pi}_2 \end{pmatrix} \begin{bmatrix} 0.8 & 0.2 \\ 0.2 & 0.8 \end{bmatrix}$$

$$\begin{pmatrix} \underline{\pi}_1 & \underline{\pi}_2 \end{pmatrix} = \frac{1}{5}\begin{bmatrix} 4\underline{\pi}_1 + \underline{\pi}_2 & \underline{\pi}_1 + 4\underline{\pi}_2 \end{bmatrix}$$

$$\therefore \begin{pmatrix} 5\underline{\pi}_1 & 5\underline{\pi}_2 \end{pmatrix} = \begin{bmatrix} 4\pi_1 + \pi_2 & \pi_1 + 4\pi_2 \end{bmatrix}$$

$$\therefore 5\pi_1 = 4\pi_1 + \pi_2 \implies \pi_1 = \pi_2$$

$$\therefore 5\pi_2 = \pi_1 + 4\pi_2$$

$$\pi_1 + \pi_2 = 1$$

$$\therefore \pi_1 = \pi_2 = 0.5$$

■

Example 24:

$$P = \begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} 0 & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} \end{array}$$

Solution:

We have ,

$$\underline{\Pi} = \underline{\Pi}.P$$

$$\begin{pmatrix}\underline{\pi}_1 & \underline{\pi}_2\end{pmatrix}$$

$$\begin{bmatrix}\underline{\pi}_0 & \underline{\pi}_1 & \underline{\pi}_2\end{bmatrix} = \begin{bmatrix}\underline{\pi}_0 & \underline{\pi}_1 & \underline{\pi}_2\end{bmatrix}\begin{bmatrix}0 & \frac{2}{3} & \frac{1}{3}\\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4}\\ \frac{1}{2} & 0 & \frac{1}{2}\end{bmatrix}$$

$$\begin{bmatrix}\underline{\pi}_0 & \underline{\pi}_1 & \underline{\pi}_2\end{bmatrix} = \begin{bmatrix}\frac{\underline{\pi}_1+\underline{\pi}_2}{2} & \frac{2\underline{\pi}_0}{3}+\frac{\underline{\pi}_1}{4} & \frac{\underline{\pi}_0}{3}+\frac{\underline{\pi}_1}{4}+\frac{\underline{\pi}_2}{2}\end{bmatrix}$$

$$\therefore \frac{\underline{\pi}_1+\underline{\pi}_2}{2} = \underline{\pi}_0 \Rightarrow \underline{\pi}_1+\underline{\pi}_2 = 2\underline{\pi}_0$$

$$\therefore \frac{2\underline{\pi}_0}{3}+\frac{\underline{\pi}_1}{4} = \underline{\pi}_1 \Rightarrow 8\underline{\pi}_0+3\underline{\pi}_1 = 12\underline{\pi}_1 \Rightarrow 8\underline{\pi}_0 = 9\underline{\pi}_1 \Rightarrow \underline{\pi}_0 = \frac{9}{8}\underline{\pi}_1$$

$$\therefore \frac{\underline{\pi}_0}{3}+\frac{\underline{\pi}_1}{4}+\frac{\underline{\pi}_2}{2} = \underline{\pi}_2 \Rightarrow 4\underline{\pi}_0+3\underline{\pi}_1+6\underline{\pi}_2 = 12\underline{\pi}_2$$

Substituting $\underline{\pi}_0 = \frac{9}{8}\underline{\pi}_1$ in ,

$$\therefore \underline{\pi}_1+\underline{\pi}_2 = 2.\frac{9}{8}\underline{\pi}_1$$

$$\therefore \underline{\pi}_2 = \frac{9}{4}\underline{\pi}_1 - \underline{\pi}_1$$

$$\therefore \underline{\pi}_2 = \frac{5}{4}\underline{\pi}_1$$

We have,

$$\underline{\pi}_0+\underline{\pi}_1+\underline{\pi}_2 = 1$$

$$\therefore \frac{9}{8}\underline{\pi}_1+\underline{\pi}_1+\frac{5}{4}\underline{\pi}_1 = 1$$

$$\therefore \underline{\pi}_1\left(\frac{9}{4}+\frac{9}{8}\right) = 1$$

$$\therefore \underline{\pi}_1(\frac{27}{8}) = 1 \Rightarrow \underline{\pi}_1 = \frac{8}{27}$$

$$\therefore \underline{\pi}_0 = \frac{9}{8}\times\frac{8}{27} = \frac{1}{3} \Rightarrow \underline{\pi}_0 = \frac{1}{3}$$

$$\underline{\pi}_2 = \frac{5}{4}\times\frac{8}{27} = \frac{10}{27} \Rightarrow \underline{\pi}_2 = \frac{10}{27}$$

■

Example 25:

$$P = \frac{1}{6}\begin{bmatrix} 3 & 3 & 0 & 0 \\ 6 & 0 & 0 & 0 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & 4 & 2 \end{bmatrix}$$

Solution:

$$\underline{\pi} = \underline{\pi}.P$$

$$\underline{\pi} = \begin{pmatrix} \underline{\pi}_1 & \underline{\pi}_2 & \underline{\pi}_3 & \underline{\pi}_4 \end{pmatrix} \begin{bmatrix} 3 & 3 & 0 & 0 \\ 6 & 0 & 0 & 0 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & 4 & 2 \end{bmatrix}$$

$$\therefore 6\underline{\pi} = \begin{pmatrix} 3\underline{\pi}_1 + 6\underline{\pi}_2 & 3\underline{\pi}_1 & 4\underline{\pi}_3 + 4\underline{\pi}_4 & 2\underline{\pi}_3 + 2\underline{\pi}_4 \end{pmatrix}$$

$$\therefore 6\underline{\pi}_1 = 3\underline{\pi}_1 + 6\underline{\pi}_2$$

$$\therefore 6\underline{\pi}_2 = 3\underline{\pi}_1 \Rightarrow \underline{\pi}_2 = \frac{\underline{\pi}_1}{2}$$

$$\therefore 6\underline{\pi}_3 = 4\underline{\pi}_3 + 4\underline{\pi}_4 \Rightarrow 4\underline{\pi}_4 = 2\underline{\pi}_3 \Rightarrow \underline{\pi}_4 = \frac{\underline{\pi}_3}{2}$$

$$\therefore 6\underline{\pi}_4 = 2\underline{\pi}_3 + 2\underline{\pi}_4$$

We have,

$$\underline{\pi}_1 + \underline{\pi}_2 + \underline{\pi}_3 + \underline{\pi}_4 = 1$$

$$\frac{3}{2}\underline{\pi}_1 + \frac{3}{2}\underline{\pi}_3 = 1$$

$$\therefore \underline{\pi}_3 = \frac{2}{3} - \underline{\pi}_1$$

$$\therefore \underline{\pi}_4 = \frac{1}{3} - \frac{\underline{\pi}_1}{2}$$

Let

$$\underline{\pi}_1 = x$$

$$\therefore \underline{\pi}_2 = \frac{\underline{\pi}_1}{2} = \frac{x}{2}$$

$$\underline{\pi}_3 = \frac{2}{3} - \underline{\pi}_1 = \frac{2}{3} = x$$

$$\underline{\pi}_4 = \frac{1}{3} - \frac{\underline{\pi}_1}{2} = \frac{1}{3} - \frac{x}{2}$$

■

Theorem 2.6.1. *An irreducible aperiodic MC belongs to one of the following two classes:*

- *Either the states are all transient or all null recurrent.*
  In this case, $P_{ij}^{(n)} \to 0$ as $n \to \infty\ \forall i, j$ and there exists no stationary distribution.
- *Or else, all states are positive recurrent i.e $\underline{\pi}_j = \lim_{n\to\infty} P_{ij}^{(n)} > 0$,*
  In this case, $\{\underline{\pi}_j; j = 0,1,2,3,....\}$ is stationary distribution & $\exists$ no other stationary distribution.

Proof. We know that,

$$\sum_{j=0}^{M} P_{ij}^{(n)} \le \sum_{j=0}^{\infty} P_{ij}^{(n)} = 1\ \forall M$$

$$\lim_{n\to\infty} \sum_{j=0}^{M} P_{ij}^{(n)} = \sum_{j=0}^{M} \lim_{n\to\infty} P_{ij}^{(n)} = \sum_{j=0}^{M} \pi_j \le 1\ \forall M$$

$$\Rightarrow \sum_{j=0}^{\infty} \pi_j \le 1$$

by CK equations,
Now,

$$P_{ij}^{(n+1)} = \sum_{k=0}^{\infty} P_{ik}^{(n)}.P_{kj} \ge \sum_{k=0}^{M} P_{ik}^{(n)}.P_{kj}\ \forall M$$

$$\lim_{n\to\infty} P_{ij}^{(n+1)} \ge \sum_{k=0}^{M} (\lim_{n\to\infty} P_{ik}^{(n)}).P_{kj}\ \forall M$$

$$\pi_j \ge \sum_{k=0}^{\infty} \pi_k.P_{kj} \forall M$$

$$\Rightarrow \pi_j \ge \sum_{k=0}^{\infty} \pi_k.P_{kj} \tag{2.1}$$

$$\Rightarrow \sum_{j=0}^{\infty} \pi_j > \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \pi_k.P_{kj} = \sum_{k=0}^{\infty} \pi_k \sum_{j=0}^{\infty} P_{kj} = 1$$

$$\Rightarrow \sum_{j=0}^{\infty} \pi_j > \sum_{k=0}^{\infty} \pi_k$$

So our assumption is wrong in ,

$$\therefore \pi_j = \sum_{k=0}^{\infty} \pi_k.P_{kj}, j = 0,1,2,......$$

Putting $P_j = \frac{\pi_j}{\sum_{k=0}^{\infty}\pi_k}$, we see that $\{P_j, j = 0,1,2,.....\}$ is stationary distribution and hence at least one stationary distribution exists.
Now, let $\{P_j, j = 0,1,2,.....\}$ be any stationary distribution , then if $\{P_j = 0,1,2,.....\}$ be probability distribution of X_0.

$$P_j = P\{X_n = j\}$$

$$P_j = \sum_{i=0}^{\infty} P\{X_n = j|X_0 = i\}.P\{X_0 = i\}$$

$$P_j = \sum_{i=0}^{\infty} P_{ij}^{(n)}.P_i$$

$$\geq \sum_{i=0}^{M} P_{ij}^{(n)}.P_i, \forall M$$

$$\Rightarrow P_j = \sum_{i=0}^{M} P_{ij}^{(n)}.P_i, \forall M$$

As $n \to \infty$, $M \to \infty$,

$$P_j \geq \sum_{i=0}^{\infty} \lim_{n\to\infty} P_{ij}^{(n)}.P_i$$

$$P_j \geq \sum_{i=0}^{\infty} \pi_j.P_i = \pi_j \tag{2.2}$$

Now,

$$P_j = \sum_{i=0}^{M} P_{ij}^{(n)}.P_i + \sum_{i=M+1}^{\infty} P_{ij}^{(n)}.P_i$$

$$\leq \sum_{i=0}^{M} P_{ij}^{(n)}.P_i + \sum_{i=M+1}^{\infty} P_i \because \sum_{i=M+1}^{\infty} P_{ij}^{(n)} \leq 1$$

As $n \to \infty$,

$$P_j \leq \sum_{i=0}^{M} \pi_j.P_i + \sum_{i=M+1}^{\infty} P_i, \forall M$$

As $M \to \infty$,

$$P_j \leq \pi_j \tag{2.3}$$

From

$$P_j = \pi_j$$

If the states are transient or null recurrent and $\{P_j, j = 0,1,2,.....\}$ is stationary distribution then , $P_j = \sum_{i=0}^{\infty} P_{ij}^{(n)}.P_i$ and $P_{ij}^{(n)} \to 0$, which is clearly impossible. Thus, for case (i),no stationary distribution exists. □

2.7 Gambler's Ruin Problem

Consider a gambler who at each play of the game has probability p of winning 1 unit and $q = (1-p)$ of losing 1 unit. Assuming successive plays of the game are independent. What is the probability that starting with i units the gambler's fortune will reach N before reaching 0?

If we let x_n denote the players fortune at time n,then the process $\{x_n, n = 0, 1, 2, ...\}$ is markov chain with transition probabilities,

$$P_00 = P_NN = 1$$

$$p = P_{i.j+1} = 1 - Pi, j-1 = 1-q, i = 1, 2, ..., N-1$$

So MC has classes $\{0\}, \{N\}, \{1, 2, ..., N-1\}$ first two are recurrent and last one transient ,since each transient state is only visited finitely often,it follows that after some finite amount of time,the gambler will either attain her goal of N or go broke.

Let f_i be the probability that starting with i, $0 \leq i \leq N$,the gambler's fortune will eventually reaches N,

$$\begin{array}{c|cccccc} & 0 & 1 & 2 & ...... & N \\ \hline 0 & 1 & 0 & 0 & ...... & 0 \\ 1 & q & 0 & p & ...... & 0 \\ 2 & 0 & q & 0 & ...... & 0 \\ . & 0 & 0 & q & ...... & 0 \\ . & 0 & 0 & 0 & ....p & 0 \\ N & 0 & 0 & 0 & ....q & 1 \end{array}$$

$$f_i : P(i \to N)$$

$$f_i = p.f_{i+1} + q.f_{i-1}$$

$$(p+q).f_i = p.f_{i+1} + q.f_{i-1}$$

$$q(f_i - f_{i-1}) = p(f_{i+1} - f_i)$$

$$(f_{i+1} - f_i) = \frac{q}{p}(f_i - f_{i-1})$$

We know that, if gambler has 0 amount of money then, $f_0 = 0$

$$i = 1; f_2 - f_1 = \frac{q}{p} f_1$$

$$i = 2; f_3 - f_2 = \frac{q}{p}(f_2 - f_1) = \left(\frac{q}{p}\right)^2 f_1$$

$$\vdots$$

$$i = k; f_{k+1} - f_k = \left(\frac{q}{p}\right)^k f_1$$

Sum upto $i-1$ from 1,we get,

$$f_i - f_1 = \left[\frac{q}{p} + \left(\frac{q}{p}\right)^2 + ... + \left(\frac{q}{p}\right)^{i-1}\right] f_1$$

$$f_i = \left[1 + \frac{q}{p} + \left(\frac{q}{p}\right)^2 + ... + \left(\frac{q}{p}\right)^{i-1}\right] f_1$$

series converges if $\Rightarrow \frac{q}{p} < 1$,

$$f_i = \begin{cases} \left(\frac{1-\left(\frac{q}{p}\right)^i}{1-\frac{q}{p}}\right) f_1 & , \frac{q}{p} < 1 \\ i f_i & , \frac{q}{p} = 1 \end{cases}$$

$$f_N = 1 = \begin{cases} \left(\frac{1-\left(\frac{q}{p}\right)^N}{1-\frac{q}{p}}\right) f_1 & , \frac{q}{p} < 1 \\ N f_1 & , \frac{q}{p} = 1 \end{cases}$$

$$\Rightarrow f_1 = \begin{cases} \left(\frac{1-\frac{q}{p}}{1-\left(\frac{q}{p}\right)^N}\right) f_1 & , q < p \\ \frac{1}{N} & , q = p \end{cases}$$

Substituting f_1 in f_i,we get ,

$$\Rightarrow f_i = \begin{cases} \frac{1-\left(\frac{q}{p}\right)^i}{1-\frac{q}{p}} \times \frac{1-\frac{q}{p}}{1-\left(\frac{q}{p}\right)^N} & , q < p \\ \frac{i}{N} & , q = p = \frac{1}{2} \end{cases}$$

$$\Rightarrow f_i = \begin{cases} \frac{1-\left(\frac{q}{p}\right)^i}{1-\left(\frac{q}{p}\right)^N} & , q < p \\ \frac{i}{N} & , q = p = \frac{1}{2} \end{cases}$$

2.8 Random Walk

$$X_i = \begin{cases} +1 & , p \\ -1 & , q = 1-p \end{cases}$$

$$Y_n = \sum_{i=1}^{n} X_i$$

$\Rightarrow Y_n$ is called **Random Walk**.

$$Y_1 = X_1$$

$$Y_2 = X_1 + X_2 = Y_1 + X_2$$

$$\therefore Y_n = Y_{n-1} + X_n$$

If $X_i = \pm 1$ then $\{Y_n, n \geq 0\}$ is called as **Simple Random Walk**.
If $X_i = \pm 1$ & $p = \frac{1}{2} \Rightarrow \{Y_n, n \geq 0\}$ is called as **Simple Symmetric Walk**.
If $X_i = \pm C$ & $p = \frac{1}{2} \Rightarrow$ is called as **Symmetric Walk**.

2.8.1 Random walk with Absorbing Boundries

For a stochastic process with state space $\{0, 1, 2, ..., N\}$ if $P_{00} = 1 = P_{NN}$ then the stochastic process $\{X_n, n \geq 0\}$ is called as random walk with absorbing boundries.

- If $P_{01} = P_{N,N-1} = 1$ then $\{X_n\}$ is called as random walk with reflecting boundries.
- If $P_{01} = 1 - P_{00}$ & $P_{00} < 1$ then $\{X_n\}$ is called as random walk with absorbing boundries.

Chapter 3

Branching Process

3.1 Galton-Watson Branching Process

Consider the population consisting of individual able to produce offspring of the same kind. Suppose that each individual will by the end of its lifetime, have produced $'j'$ new offspring with prob $P_j, j \geq 0$,independent of the no. produced by any other individual. The no, of individuals initially present,denoted by X_0,is called as the $zero^{th}$ generation.All offspring of $zero^{th}$ generation constitute the first generation and their no. is denoted by X_1. In general, X_n is the size of n^{th} generation. The Markov chain $\{X_n\}$ is called **Branching Process**.

- Z_i offspring distribution with mean μ, σ^2.

$$X_n = \sum_{i=1}^{X_n - 1} Z_i$$

Where,
Z_i : No. of offsprings produced by each individual in $(n-1)^{th}$ generation.
μ : it denotes average no. of offspring per individual.

$$E(X_n) = E\left(\sum_{i=1}^{X_n-1} Z_i\right)$$

$$= E\left(E\left(\sum_{i=1}^{k} Z_i\right) | X_{n-1} = k\right)$$

$$= E_{X_n-1}\left(E_z\left(\sum_{i=0}^{k} Z+i\right)\right)$$

$$= E_{X_n-1}(k.\mu)$$

$$E(X_n) = \mu.E(X_{n-1})$$

$$= \mu.E(X_{n-1})......E(X_1)$$

$$= \mu.\mu.....\mu.\mu(1)(\because if X_0 = 1, E(X_0) = 1)$$

$$E(X_n) = \mu^n$$

Also,

$$V(X_n) = V\left(\sum_{i=0}^{X_{n-1}}\right)$$

$$= E\left(X_{n-1}.\sigma^2\right) + V_{Xn-1}.\left(\mu X_{n-1}\right)$$
$$= \sigma^2.E\left(X_{n-1}\right) + \mu^2 V X_{n-1}$$

$X_0 = 1$,

$$V(X_n) = \sigma^2\mu^(n-1) + \mu^2 v(X_{n-1})$$
$$V(X_1) = \sigma^2$$
$$V(X_2) = \sigma^2\mu + \mu^2\sigma^2 = \sigma^2(\mu+\mu^2)$$
$$V(X_3) = \sigma^2\mu^2 + \mu^2\sigma^2(\mu+\mu^2) = \sigma^2(\mu^2+\mu^3+\mu^4)$$
$$\therefore V(X_n) = \sigma^2[1+\mu+\mu^2+...+\mu^{n-1}].\mu^{n-1} \Rightarrow \sigma^2\mu^{n-1}.\left(\frac{1-\mu^n}{1-\mu}\right)$$

Let π_0 denote the probability that starting with single individual,population ever dies out.

$$\pi_0 = P\{Pop^n diesout\}$$
$$\pi_0 = \sum_{j=0}^{\infty}.P\{Pop^n diesout|X_i = j\}.P_j \Rightarrow \sum_{j=0}^{\infty}\pi_0^j.P_j$$
$$\therefore \pi_0 = P_z(\pi_0)$$

Example 26: $Z_i B(n=2, p=\frac{1}{3})$
Solution:

$$\therefore \pi_0 = P_2(\pi_0)$$
$$\pi_0 = \left(\frac{2}{3}+\frac{1}{3}\pi)0\right)^2$$
$$\therefore 9\pi_0 = 4 + \pi_0^2 + 4\pi_0$$
$$(\pi_0-4)(\pi_0-1) = 0$$
$$\therefore \pi_0 = 4, \pi_0 = 1$$
$$\therefore \text{Minimum value of} \pi_0 = 1$$

■

Example 27: Let $Z_i = \begin{cases} 0 & Prob. = 0.3 \\ 1 & Prob. = 0.3 \\ 2 & Prob. = 0.4 \end{cases}$

and n^{th} generation $X_n = \sum_{i=1}^{X_{n-1}} Z_i$
where, Z_i denotes the no. of offspring from i_{th} of $(n-1)^{th}$ generation.

Find the prob. of extinction.
Solution:

$$P_z(\pi_0) = E(\pi_0^{z_i}) = \sum_{i=1}^{3} \pi_0^{z_i}.P(Z_i = z_i)$$
$$= (\pi_0)^0(0.3) + (\pi_0)^1(0.3) + (\pi_0)^2(0.4)$$
$$\pi_0 = 0.3 + 0.3\pi_0 + 0.4\pi_0^2$$
$$\Rightarrow 0.4\pi_0^2 - 0.7\pi_0 + 0.3 = 0$$
$$\Rightarrow 4\pi_0^2 - 7\pi_0 + 3 = 0$$
$$\Rightarrow 4\pi_0^2 - 4\pi_0 + 3\pi_0 + 3 = 0$$
$$\Rightarrow (4\pi_0 - 3)(\pi_0 - 1) = 0$$
$$\therefore \pi_0 - \frac{3}{4} \& \pi_0 = 1$$
$$\therefore \text{Min. value of } \pi_0 = \frac{3}{4} = 0.75$$

■

Theorem 3.1.1. *Suppose that* $P_0 > 0$ & $P_1 + P_0 < 1$ *then,*

1. π_0 *is the smallest no. satisfying ,*

$$\pi_0 = \sum_{j=0}^{\infty} P_j$$

2. $\pi_0 = 1$ *if and only if* $\mu \leq \lim_{n\to\infty} P(X_n = 0) = \pi_0$

Proof. To show that π_0 is the smallest solution to $\pi_0 = \sum \pi_0^j.P_j$
Let $\pi \geq 0$, we have to show $\pi \geq \pi_0 = lim_{n\to\infty} P(X_n = 0)$, We will first show by induction that $\pi \geq P\{X_n = 0\}$ for all n.

$$\pi = \sum_{j=0}^{\infty} \pi^j.P_j \geq \pi^0 P_0 = P_0 = P\{X_1 = 0\}$$

and assume that $\pi \geq P\{X_n = 0\}$ then,

$$P\{X_{n+1} = 0\} = \sum_{j=0}^{\infty} P\{X_{n+1} = 0|X_1 = j\}.P_j$$
$$= \sum_{j=0}^{\infty} [P\{X_n = 0\}]^j.P_j$$
$$\leq \sum_{j=0}^{\infty} \pi^j.P_j = \pi$$

$$\Rightarrow P\{X_{n+1} = 0\} \leq \pi$$

$$\Rightarrow P\{X_n = 0\} \leq \pi, for all n$$

and letting $n \to \infty$, $\pi \geq \lim_n P\{X_n = 0\} = P$\{Population ever dies out\} $= \pi_0$
To prove 2), We define generating function,

$$\phi(s) = \sum_{j=0}^{\infty} s^j.P^j$$

Since, $P_0 + P_1 < 1$,

$$\phi(s) = \sum_{j=0}^{\infty} j(j+1)s^{j-2}P_j > 0$$

For all $s\epsilon(0,1)$. Here $\phi(s)$ is strictly convex function in open interval $(0,1)$. We will now distinguish two cases (a) and (b).
In (a) , $\phi(s) > s$ for all $s\epsilon(0,1)$ and , In (b) , $\phi(s) = s$ for some $s\epsilon(0,1)$.
It is geometrically clear that (a) represents the approximate picture when $\phi^1(1) \leq 1$ and (b) is approximate when $\phi^1(1) > 1$ .Thus, since $\phi(\pi_0) = \pi_0$
$\therefore \pi_0 = 1$ iff $\phi^1(1) \leq 1$
The result follows , since $\phi^1(1) = \sum j.P_j = \mu$ □

Example 28: $Z \sim B(2,P), X_c = 1, E(2) = 2P$
Solution:
We have,

$$s = \phi(s)$$

$$\therefore t = P_z(t)$$

$$t = (q+pt)^2$$

$$t = q^2 + P^2t^2 + 2pqt$$

$$\therefore \frac{p^2}{a}t^2 + \frac{2pq-1}{b}t + \frac{q^2}{c}$$

$$\therefore t = \frac{-(2pq-1) \pm \sqrt{1-4pq}}{2p^2}$$

■

Example 29: Calculate the probability that the population becomes extend for the 1^{st} time in third generation.
Solution:

$$Z_i \sim B(2,P)$$

$$X_n = \sum_{i=1}^{X_n-1} Z_i$$

$$X_0 = 1$$

$$X_1 = 1 = \sum_{i=1}^{1} Z_i = Z_i \sim B(2,P)$$

$$X_2 = \sum_{i=1}^{1} Z_i = Z_i \sim B(2, P)$$

$$\begin{aligned}\text{Required Prob.} &= P(X_2 = 0 | X_0 \neq 0, X_1 \neq 0)\\ &= \frac{P(X_2 = 0, X_0 \neq 0, X_1 \neq 0)}{P(X_0 \neq 0) P(X_1 \neq 0)}\\ &= \frac{q^2(1-q^2)(1-q^2)}{(1-q^2)(1-q^2)} = q^2\end{aligned}$$

■

3.2 Counting Process

A stochastic process $\{N(t), t \geq 0\}$ is said to be counting Process if $N(t)$ represents the total number of events that have occurred upto time t. Hence, a counting process$N(t)$ must satisfy,

1. $N(t) \geq 0$
2. $N(t)$ is integer valued
3. $if s < t, N(s) \leq N(t)$
4. For $s < t, N(t) - N(s)$ represents the no of events that have occurred in time $(s, t]$.

3.2.1 Counting Process with Independent Increments

If the no. of events that occur in disjoint time intervals are independent.
e.g For $0 < s < t$,$N(S)$&$N(t) - N(S)$ are independent.

3.2.2 Counting Process with Stationary Increments

If the distribution of no. of events that have occurred in any interval of time depends only on the length of time interval.
i.e. The process that has stationary increment. For any $s > 0$, no. of events occurred in $[t_1 + s, t_2 + s]$ has same distribution as $(t_1, t_2]$

3.3 Poisson Process

1. The counting process $\{N(t), t \geq 0\}$ is said to be a poisson process having rate $\lambda, \lambda > 0$,if
   (a) N(0) = 0
   (b) The process has independent increments and stationary.
   (c) The no. of events in any interval of length t is poisson distributed with mean λt

i.e. For all $s,\ t \geq 0$

$$P\{N(t+s)-N(s)=n\} = \frac{e^{(-\lambda t)}.(\lambda t)^n}{n!}$$

2. The counting process $\{N(t), t \geq 0\}$ is said to poisson process with rate $\lambda, \lambda > 0$,if,
   (a) N(0) = 0
   (b) The process has stationary and independent increment.
   (c) $P\{N(h)=1\} = \lambda h + 0(h) - (P_1(h))$
   (d) $P\{N(h) \geq 2\} = 0(h) - \sum_{k=2}^{\infty} P_k(h)$
   As $h \to \infty, 0(h) \to 0$, h is very short interval of time.

3.3.1 Proof of Equivalence of two definitions of Poisson Process :

$Def^n(2) \to Def^n(1)$
First two conditions are obivious.
So, we go to 3^{rd} condition , we define notation,

$$\begin{aligned}
P_n(t) &= P[N(t)=n] \\
P_0(t+h) &= P[N(t+h)=0] \\
&= P[N(t)=0, N(t+h)-N(t)=0] \\
&= P_0(t), P_0(h) \quad \ldots (\text{ indep-incre }) \\
&= P_0(t)[1-P_1(h)-P(N(h)\geq 2)] \\
&= P_0(t)-P_0(t).\lambda h + o(h) \\
\lim_{h\to 0} \frac{P_0(t+h)-P_0(t)}{h} &= \lim_{h\to 0}\left(\frac{-P_0(t).\lambda h}{h} + \frac{oh}{h}\right) \\
P_0'(t) &= -P_0(t).\lambda \\
\frac{P_0'(t)}{P_0(t)} &= -\lambda \\
\frac{d}{dt}\log(P_0(t)) &= -\lambda \quad \cdots \left(\frac{\partial}{\partial t}\log F(t) = \frac{f\prime(t)}{f(t)}\right) \\
\log_e P_0(t) &= -\lambda t + c \quad (\text{ (integrating w.r.t t })\\
P_0(t) &= e^{-\lambda t + c}
\end{aligned}$$

$$P(N(O)=0)=1, t=0 \Rightarrow P_0(0)=1=e^c \Rightarrow c=0$$

$$\therefore P_0(t) = e^{-\lambda t}$$

$$\therefore i.e. P[N(t)=0] = e^{-\lambda t}$$

$$
\begin{aligned}
\text{Now, } P_n[t+h] =& P[N(t+h)=n] \\
=& P[N(t)=n, N(t+h)-N(t)=0] \\
& + P[N(t)=n-1, N(t+h)-N(t)=1] \\
& + P[N(t)\le n-2, N(t+h)-N(t)\ge 2] \\
=& P_n(t)(1-\lambda h+o(h)) \\
& + P_{n-1}(t)(\lambda h+o(h)) \\
& + P_{n-2}(t)(O(h)) \\
=& P_n(t)-P_n(t).\lambda(h)+P_{n-1}(t)\lambda h+o(h)
\end{aligned}
$$

$$\therefore \lim_{h\to 0} P_n(t+h) = \lim_{h\to 0}[P_n(t)-P_n(t)-\lambda h+P_{n-1}(t)\lambda h+o(h)]$$

$$\therefore \lim_{h\to 0}\frac{P_n(t+h)-P_n(t)}{h} = \lim_{h\to 0}\frac{-\lambda\,[P_n(t)-P_{n-1}(t)]\,.h}{h}+\lim_{h\to 0}\frac{o(h)}{h}$$

$$\therefore P'_n(t) = -\lambda[P_n(t)-P_{n-1}(t)]+0$$

$$\therefore e^{\lambda t}P'_n(t)+\lambda e^{\lambda t}P_n(t) = e^{\lambda t}P_{n-1}(t)$$

$$\frac{d}{dt}e^{\lambda t}P_n(t) = e^{\lambda t}P_{n-1}(t)$$

Step -I : For n=1,

$$\frac{d}{dt}e^{\lambda t}P_{n=1}(t) = \lambda e^{\lambda t}P_0(t)$$

$$\frac{d}{dt}e^{\lambda t}P_1(t) = \lambda e^{\lambda t}e^{-\lambda t} = \lambda$$

$$\therefore e^{\lambda t}P_1(t) = \lambda t + c$$

We have, $P_1(0)=0 \Rightarrow c=0$

$$\therefore P_1(t) = e^{-\lambda t}.\lambda t$$

Step -II : Assume , for (n-1),

$$P_{n-1}(t) = \frac{e^{-\lambda t}(\lambda t)^{n-1}}{(n-1)!}$$

Step -III : Consider,

$$
\begin{aligned}
\frac{d}{dt}e^{\lambda t}P_n(t) &= \frac{\lambda.e^{\lambda t}.e^{-\lambda t}(\lambda t)^{n-1}}{(n-1)!} \\
&= \frac{\lambda^n.t^{(n-1)}}{(n-1)!} \\
e^{\lambda t}P_n(t) &= \frac{\lambda^n}{(n-1)!}.\frac{t^n}{n} \\
&= \frac{(\lambda t)^n}{n!} \\
P_n(t) &= \frac{e^{-\lambda t}(\lambda t)^n}{(n)!}
\end{aligned}
$$

$\therefore$ From principle of mathematical induction,
Given statement, $P_n(t) = \frac{e^{-\lambda t}.(\lambda t)^n}{n!}$ is true for all n.

$$\therefore P[N(t) = n] = \frac{e^{-\lambda t}.(\lambda t)^n}{n!}$$
$$\therefore N(t) \sim P(\lambda t)$$

$Def^n(1) \rightarrow Def^n(2):$
First two conditions are obivious.
We have to prove last ones.
Given that, $N(h) \sim P(\lambda h)$

$$\therefore P[N(h) = n] = \frac{e^{-\lambda h}.(\lambda h)^n}{n!}, \quad n = 0, 1, 2, ..$$

Put n=1,

$$\begin{aligned}
P[N(h) = 1] &= \frac{e^{-\lambda h}.(\lambda h)}{1!} \\
&= e^{-\lambda h}.\lambda h \\
&= \lambda h \left[1 - \lambda h + \frac{(\lambda h)^2}{2!} - ....\right] \\
&= \lambda h - (\lambda h)^2 + \frac{(\lambda h)^2}{2!} - .... \\
\therefore P[N(h) = 1] &= \lambda h + o(h)
\end{aligned}$$

where $\{o(h)$: Contains higher power of h which tends to 0 as $h \rightarrow 0\}$.
Also consider ,

$$\begin{aligned}
P[N(h) \geq 2] =& 1 - P[N(h) = 0] - P[N(h) = 1] \\
=& 1 - e^{\lambda h} - (\lambda h + o(h)) \\
=& 1 - \left(1 - (\lambda h) + \frac{(\lambda h)^2}{2!} - ....\right) - (\lambda h) - o(h) \\
=& \lambda h - \left[\frac{(\lambda h)^2}{2!} + \frac{(\lambda h)^3}{3!} - ...\right] - \lambda h - o(h) \\
=& o(h)
\end{aligned}$$

Hence proved.

3.4 Inter-Arrival Time Distribution :

Consider a Poisson process and let $'t_0'$ denote the time of 1^{st} event. Further ,for $n \geq 1$, let $'t_n'$ denote the time between $n\&(n+1)^{th}$ arrival. The sequence $t_n, n \geq 0$ is called as sequence of Inter-Arrival Time.

$$F_0(t) = P[T_0 \leq t] = 1 - P[T_0 > t]$$

$$= 1 - P[N(t) = 0]$$
$$= 1 - e^{\lambda t}$$
$$\therefore T_0 \sim Exp(\lambda)$$

$$F_1(t) = P(T_1 \leq t | T_0 = s)$$
$$= 1 - P(T_1 > t | T_0 = s)$$
$$= 1 - P[N(s+t) - N(s) = 0 | N(s) = 1]$$
$$= 1 - e^{-\lambda t}$$
$$\therefore T_1 \sim exp(\lambda)$$

$\Rightarrow T_0, T_1, T_2, \ldots$ are iid exponential r.v. having mean $\frac{1}{\lambda}$.
Arrival Time :, $S_n = \sum_{i=0}^{n-1} T_i$ where, $T_i \sim exp(\lambda)$ and $\{N(t), t \geq 0\}$ is poisson process.

$$P\{t < S_n < t + \partial t\} = P\{N(t) = n-1\}.P\{N(t+\partial t) - N(t) = 1\}$$
$$= \frac{e^{\lambda t}(\lambda t)^{n-1}}{(n-1)!}.\frac{e^{-\lambda \partial t}(\lambda \partial t)^1}{1!} + o(ot)$$

divide by ∂t and taking limit $\partial t \to 0$

$$= \frac{\lambda^n}{\gamma n}.e^{\lambda t}.t^{n-1}$$

This is the pdf of S_n, $\Rightarrow S_n \sim Gamma(\alpha, n)$.

3.4.1 Conditional Distribution of Arrival times :

$N(t) = 1$, Where $N(t)$ is poisson process with rate λ for $s \leq t$.

$$P\{x_1 < s | N(t) = 1\} = \frac{P\{x_1 < s, N(t) = 1\}}{P\{N(t) = 1\}}$$
$$= \frac{P\{N(s) = 1, N(t) - N(s) = 0\}}{e^{-\lambda t}.\lambda t}$$
$$= \frac{e^{\lambda s}.\lambda s.e^{-\lambda(t-s)}}{e^{-\lambda t}.\lambda t}$$
$$= \frac{s}{t}$$
$$\therefore x_1 < s | N(t) = 1 \sim U(0, t)$$

Theorem 3.4.1. *Given that $N(t) = n$, the n arrival times $s_1, s_2, \ldots, s_n$ have same distribution as the order statistics corresponding to n independent random variables uniformly distributed on the interval* $(0, t)$.

Proof. We shall compute conditional density function $s_1, s_2, ..., s_n$, given that $N(t) = n$. So, let $0 < t_1 < t_2 < ... < t_{n-1}$ & let h_i be small enough so that $t_i + h_i < t_{i+1}, \quad i = 1,2,...,n$. Now,

$$\therefore P t_i \le s_i \le t_i + h_i, i = 1,2,...,n | N(t) = n$$
$$= \frac{P\{N(t_i + h_i) - N(t_i) = 1, i = 1,2,..,n, \text{No. event elsewhere in}(0,t)\}}{P\{N(t) = n\}}$$
$$= \frac{\lambda h_1.e^{-\lambda h_1}.\lambda h_2.e^{-\lambda h_2}.....\lambda h_n.e^{-\lambda h_n}.e^{-\lambda(t-h_1-h_2-...-h_n)}}{\frac{e^{\lambda t}.(\lambda t)^n}{n!}}$$
$$= \frac{n!}{t^n}.h_1.h_2....h_n$$

Hence,

$$\therefore \frac{P\{t_i < s_i < t_i + h_i, i = 1,2,...,n | N(t) = n\}}{h_1.h_2....h_n} = \frac{n!}{t^n}$$

So, by letting the $h_i \to 0$, we obtain that conditional density of $s_1, s_2, ..., s_n$, given that $N(t) = n$,

$$f\{t_1, t_2, ..., t_n\} = \frac{n!}{t^n}, \quad 0 < t_1 < ... < t_n$$

□

Example 30: Let $N(t), t \ge 0$ be poisson process & $s < t$.
Solution:

$$P\{N(s) = k | N(t) = n\} = \frac{P\{N(s) = k, N(t) = n\}}{P(N(t) = n)}$$
$$= \frac{P\{N(s) = k, N(t) - N(s) = n - k\}}{P(N(t) = n)}$$
$$= \frac{\left[\frac{e^{-\lambda s}.(\lambda s)^k}{k!}\right].\left[\frac{e^{-lambda(t-s)}.(\lambda(t-s))^{n-k}}{(n-k)!}\right]}{\frac{e^{-\lambda t}.(\lambda t)^n}{n!}}$$
$$= \frac{n!}{k!.(n-k)!}.\frac{(\lambda s)^k.(\lambda t - \lambda s)^{n-k}}{(\lambda t)^{n+k-k}}$$
$$= \frac{n!}{k!.(n-k)!}.\left(\frac{\lambda s}{\lambda t}\right)^k.\left(1 - \frac{\lambda s}{\lambda t}\right)^{n-k}$$
$$N(s) = \frac{k}{N(t)} = n \sim B\left(n, \frac{\lambda s}{\lambda t}\right)$$

■

Example 31: Let $\{N(t), t \ge 0\}$ be a poisson process with rate λ . Calculate $E[N(t).N(t+s)]$. **Solution:**
We can write,

$$N(t + s) = N(t) + N(t + s) - N(t)$$

$$E[N(t).N(t+s)] = E[N(t).N(t+s)-N(t)]$$
$$= E[N(t)]^2.E[N(t).N(t+s)-N(t)]$$
$$= \lambda t + (\lambda t)^2 + \lambda t.(\lambda s)$$

For $s < t$,

$$\begin{aligned} Cov(N(s),N(t)) &= Cov(N(s),N(t)+N(s)-N(s)) \\ &= Cov(N(s),N(t)-N(s)+N(s)) \\ &= Cov(N(s),N(t)-N(s))) \\ &= 0+V(N(s)), \quad (\because N(s)\&N(t)-N(s) \quad \text{are independent.}) \end{aligned}$$
$\therefore cov(N(S),N(t)) = \lambda s$

■

If $N_i(t)$ represents the number of type i events that occur by time $t, i = 1,2.$ then $N_1(t)\&N_2(t)$ are independent poisson random variables having respective means $\lambda t_p \& \lambda t(1-p)$,where, $p = \frac{1}{t}\int_0^t p(s)ds$.

$$P[N_1(t)=n,N_2(t)=m] = \sum_{k=0}^{\infty} P\{N_1(t)=n,N_2(t)=m|N(t)=k\}$$
$$= P[N_1(t)=n,N_2(t)=m|N(t)=n+m].P(N(t)=n+m)$$

Now,consider an arbitrary event that occured in the interval $[0,t]$.if it had occured at time s, then probability that it would be a type-I event would be $P(s)$,Hence this event will have occured at same time uniformly distributed $(0,t)$.
It follows that the probability that it will be a type-I event is,

$$P = \frac{1}{t}\int_o^t P(s)ds$$

independently of the other events ,

$$P[N_1(t)=n,N_2(t)=m|N(t)=n+m]$$
$$= \binom{n+m}{m} p^n(1-p)^m$$
$$\Rightarrow P[N_1(t)=n,N_2(t)=m] = \frac{(n+m)!}{n!m!}.p^n.(1-p)^m.e^{-\lambda t}.\frac{(\lambda t)^{n+m}}{(n+m)!}$$
$$= e^{-\lambda tp}.\frac{(\lambda tp)^n}{n!}.e^{-\lambda t(1-p)}.\frac{(\lambda t(1-p))^m}{m!}$$

3.5 Compound Poisson Process :

A stochastic process $s(t), t \geq 0$ is said to be a compound poisson process , if if can be represented by, For $t \geq 0, s(t) = \sum_{i=0}^{N(t)} X_i$ where,
$\{N(t), t \geq\}$ is a poisson process and $\{X_i, i = 1, 2, ...\}$ is family of iid r.v that is independent of $N(t), t \geq 0$. Thus, if $s(t), t \geq 0$ is a compound poisson process then $s(t)$ is compound poisson r.v.

$$\begin{aligned}
\text{Let } S &= \sum_{i=1}^{N} x_i \\
\therefore E(S) &= E\left(\sum_{i=1}^{N} x_i\right) \\
&= E\left(E\left(\sum^{n} x_i / N = n\right)\right. \\
&= E\left(E\left(\sum_{i=1}^{n} x_i\right)\right) \\
&= E\left(\sum_{i=1}^{n} E(x_i)\right) \\
&= E(n \cdot E(x_i)) \\
&= E(N) \cdot E(x_i) = \lambda . E(x_i)
\end{aligned}$$

$$\begin{aligned}
V(S) &= V(E(S/N)) + E(V(S/N)) \\
&= V\left(E\left(\sum_{i=1}^{N} x_i / N = n\right)\right) + E\left(V\left(\sum_{i=1}^{N} x_i / N = n\right)\right) \\
&= V\left(E\left(\sum_{i=0}^{n} x_i\right)\right) + E\left(V\left(\sum_{i=1}^{n} x_i\right)\right) \\
&= V\left(\sum_{i=1}^{n} E(x_i)\right) + E\left(\sum_{i=1}^{n} \vee (x_i)\right) \\
&= V(n.E(x)) + E(n.V(x)) \\
&= (E(x_1))^2 V(N) + V(x_1) E(N). \\
&= (E(x_1))^2 \lambda + V(x_1) . \lambda \\
&= \lambda \left[(E(x_1))^2 + V(x_1)\right] \\
&= \lambda \cdot E\left(x_1^2\right)
\end{aligned}$$

Hence, for compound poisson process,

$$E(S(t)) = \lambda t . E(x)$$

$$V(S(t)) = \lambda t . E(x^2)$$

3.6 Continuous Time Markov Chain :

Consider a continuous stochastic process $\{X(t), t \geq 0\}$ taking on values in the set of non-negative integers.
We say that the process $\{X(t), t \geq 0\}$ is continuous time markov chain if for all $s, t \geq 0$ and non-negative integers $i, j, x(u), 0 \leq u \leq s$,

$$P\{X(t+s) = j|X(s) = i, \quad X(u) = x(u), 0 \leq u < s\}$$

$$= P\{X(t+s) = j|X(s) = i\}$$

3.7 Birth and Death Processes :

A Continuous Time Markov Chain (CTMC) with state space $0, 1, \ldots$ for which $q_{ij} = 0$ whenever $|i-j| > 1$ is called as Birth and Death process. The state of the process usually assumed to be the size of some population. The state increases by 1 if birth occurs and decreases by 1 if death occurs. Let $\lambda_i \& \mu_i$ be birth & death rates resp.

$$\lambda_i = q_{i,i+1}$$

$$\mu_i = q_{i,i-1}$$

$$P_i\lambda_i + P_i\mu_i = P_{i-1}\lambda_{i-1} + P_{i+1}\mu_{i+1}$$

$$\therefore i = 0 \Rightarrow P_0\lambda_0 = P_1\mu_1 \Rightarrow P_1 = \left(\frac{\lambda_0}{\mu_1}\right)P_0$$

$$\therefore i = 1 \Rightarrow P_1\lambda_1 + P_1\mu_1 = P_0\lambda_0 + P_2\mu_2 \Rightarrow P_2 = \left(\frac{\lambda_1}{\mu_2}\right)P_1$$

$$\therefore P_2 = \left(\frac{\lambda_0\lambda_1}{\mu_2\mu_1}\right)P_0$$

$$i = k-1 \Rightarrow P_k = \left(\frac{\lambda_0\lambda_1\ldots\lambda_{k-1}}{\mu_1\ldots\mu_k}\right)P_0$$

$$\therefore \sum_{k=0}^{\infty} P_K = 1$$

$$\therefore P_0\left[1 + \sum_{k=1}^{\infty}\left(\frac{\lambda_0\lambda_1\ldots\lambda_{k-1}}{\mu_1\ldots\mu_k}\right)\right] = 1$$

$$\Rightarrow P_0 = \left[1 + \sum_{k=1}^{\infty}\frac{\lambda_0\ldots\lambda_{k-1}}{\mu_1\ldots\mu_k}\right]^{-1}$$

$$\Rightarrow p_j = \frac{\lambda_0\ldots\lambda_{j-1}}{\mu_1\ldots\mu_j}\left[1 + \sum_{k=1}^{\infty}\frac{\lambda_0\ldots\lambda_{k-1}}{\mu_1\ldots\mu_k}\right]^{-1}$$

$If\lambda_i = \lambda$ of $\mu_i = \mu$, constants then,

$$P_j = \frac{\lambda^j}{\mu^j}\left[1+\sum_{k=1}^{\infty}\frac{\lambda^k}{\mu^k}\right]^{-1}$$

$$= \left(\frac{\lambda}{\mu}\right)^j\left[1+\left(\frac{\lambda}{\mu}\right)+\left(\frac{\lambda}{\mu}\right)^2+\cdots\right]^{-1}$$

If $r=\frac{\lambda}{\mu}<1$ then above series converges to $\frac{1}{1-r}$.

$$P_j = \left(\frac{\lambda}{\mu}\right)^j\left(\frac{1}{1-\frac{\lambda}{\mu}}\right)^{-1}$$

$$= \left(\frac{\lambda}{\mu}\right)^j\left(1-\frac{\lambda}{\mu}\right)$$

$$\therefore N \sim Geo\left(p=1-\frac{\lambda}{\mu}\right)$$

M|M| : If $\mu_1,\mu_2,\dots\mu_k$ are $\mu,2\mu,\dots,k\mu$ then $\mu_1.\mu_2.\dots.\mu_k = k!.\mu^k$

$$P_0 = \left[1+\sum_{k=1}^{\infty}\left(\frac{\lambda}{\mu}\right)^k.\frac{1}{k!}\right]^{-1}$$

$$= \left[\sum_{k=0}^{\infty}\left(\frac{\lambda}{\mu}\right)^k.\frac{1}{k!}\right]^{-1}$$

$$= e^{-\frac{\lambda}{\mu}}$$

$$\therefore P_j = \frac{\left(\frac{\lambda}{\mu}\right)^j}{j!}.e^{-\frac{\lambda}{\mu}}$$

$$\therefore N \sim Poi\left(\frac{\lambda}{\mu}\right)$$

$$P_j = \frac{\lambda_0\dots\lambda_{j-1}}{\mu_1\dots\mu_j}.P_0$$

$$P_0 = \left[1+\sum_{k=1}^{\infty}\frac{\lambda_0\dots\lambda_{k-1}}{\mu_1\dots\mu_k}\right]^{-1}$$

For M|M|1 : $P_j = \left(\frac{\lambda}{\mu}\right)^j\left(1-\frac{\lambda}{\mu}\right)$; $\lambda<\mu,\ j=0,1,2,\dots,$ $\lambda_i=\lambda,\mu_i=\mu$

For M|M|∞ : $P_j = \frac{e^{-\frac{\lambda}{\mu}}\left(\frac{\lambda}{\mu}\right)^j}{j!}$; $j=0,1,2,\dots,$ $\lambda_i=\lambda,\mu_i=\mu$

For M|M|S: $P_j = \frac{\lambda_0 ... \lambda_{j-1}}{\mu_1 ... \mu_j} . P_0 \quad ; \lambda_i = \lambda, \mu_i = \begin{cases} i.\mu & ; i \le s \\ s.\mu & ; i > s \end{cases}$

$$P_j = \begin{cases} \frac{\lambda^j}{\mu . 2\mu .... j\mu} = \frac{\lambda^j}{\mu^j} . \frac{1}{j!} . P_0 & ; \quad j \le s \\ \frac{\lambda^j}{\mu^j . s! s^{(j-s)}} . P_0 & ; j > s \end{cases}$$

Where , $P_0 = \left[1 + \sum_{k=1}^{s} \left(\frac{\lambda}{\mu} \right)^k . \frac{1}{k!} + \sum_{k=s+1}^{\infty} \left(\frac{\lambda}{\mu} \right)^k . \frac{1}{s! s^{k-s}} \right]$

Example 32: $\lambda_i = (i+1)\lambda \quad ; \mu_i = i.\mu$

Solution:

$$\therefore P_j = \frac{\lambda_0 . \lambda_1 \ldots \lambda_{j-1}}{\mu_1 . \mu_2 \ldots \mu_j} = \frac{1\lambda . 2\lambda \ldots j\lambda}{1\mu . 2\mu \ldots j\mu} = \frac{j! . \lambda^j}{j! \mu^j}$$

$$= \left(\frac{\lambda}{\mu} \right)^j . P_0 \quad ; P_0 = \left(1 - \frac{\lambda}{\mu} \right)$$

$\therefore$ for M|M|1 set-up,

$$P_0 = \left[1 + \sum_{j=1}^{\infty} \left(\frac{\lambda}{\mu} \right)^j \right]^{-1} = \left(1 - \frac{\lambda}{\mu} \right)$$

■

Example 33: We have a washing centre beside the highway and we have not place for parking . If 1 customer is arrive and our washing centre capacity is for 10 vehicles and washing center is already full and 11^{th} vehicle is come.

($\because$ with limited capacity)

$$\lambda_i = \begin{cases} \lambda & ; i < 10 \\ 0 & ; i \ge 10 \end{cases}$$

$$\mu_i = \mu$$

Also, As capacity is 10 ,$P_{11} = 0, \quad for\, j = 11$

3.8 Pure Birth Process :

A birth and death process is said to be pure birth process if $\mu_n = 0 \quad \forall n$

(i.e death rate = 0)

A simplest example of pure birth process is poisson process with rate $\lambda i = \lambda \quad \forall i$ (All pure birth processes are not poisson process).

Another example of pure birth process is Yule's process (linear birth rate process) where $\lambda_n = n\lambda$.

e.g : if 10 people have continuous disease then, 11^{th} person suffering From that disease by rate $10.\mu$

3.9 Kolmogorav Differential Equation :

$$lim_{t\to 0}\frac{1-P_{ii}(t)}{t}=\gamma_i \quad ,lim_{t\to 0}\frac{P_{ij}(t)}{t}=q_{ij}, i+j$$

$$\begin{cases} k=i-\text{Forward difference} \\ k=j-\text{Backward difference} \end{cases}$$

$$P_{ij}(t)=P[X_(t)=j|X_(0)=i]$$

Forward diff. equation CK eq^n is given as,

$$P_{ij}(t+h)=\sum_{k\epsilon s}P_{ik}(t).P_{kj}(h)$$

$$=P_{ij}(t).P_{jj}(h)+\sum_{k\epsilon s,k\neq j}P_{ik}(t).P_{kj}(h)$$

$$\therefore P_{ij}(t+h)-P_{ij}(t)=P_{ij}(t)(P_{jj}(h)-1)+\sum_{k\epsilon s}P_{ik}(t).P_{kj}(h)$$

$$\therefore lim_{h\to 0}\frac{P_{ij}(t+h)-P_{ij}(t)}{h}=lim_{h\to 0}\frac{P_{ij}(t)[P_{jj}(h)-1]}{h}+\sum_{k\epsilon s,k\neq j}P_{ik}(t).lim_{h\to 0}\frac{P_{kj}(h)}{h}$$

$$P'_{ij}(t)=-P_{ij}(t).\gamma_j+\sum_{k\epsilon s,k\neq j}P_{ik}(t).q_{kj}$$

Backward diff. equation CK eq^n is given as,

$$P_{ij}(t+h)=\sum_{k\epsilon s}P_{ik}(h).P_{kj}(t)$$

$$=P_{ij}(h).P_{jj}(t)+\sum_{k\epsilon s,k\neq i}P_{ik}(h).P_{kj}(t)$$

$$\therefore P_{ij}(t+h)-P_{ij}(t)=P_{ij}(t)(P_{ii}(h)-1)+\sum_{k\epsilon s,k\neq i}P_{ik}(h).P_{kj}(t)$$

$$\frac{P_{ij}(t+h)-P_{ij}(t)}{h}=\frac{P_{ij}(t)(P_{ii}(h)-1)}{h}+\sum_{k\epsilon s,k\neq i}P_{ik}(h).P_{kj}(t)$$

$$\therefore lim_{h\to 0}\frac{P_{ij}(t+h)-P_{ij}(t)}{h}=P_{ij}(t).lim_{h\to 0}\frac{[P_{ii}(h)-1]}{h}+\sum_{k\epsilon s,k\neq i}P_{kj}(t).lim_{h\to 0}\frac{P_{kj}(h)}{h}$$

$$P'_{ij}(t)=-P_{ij}(t).\gamma_i+\sum_{k\epsilon s,k\neq i}P_{ik}(t).q_{ik}$$

3.10 Two State Chain :

Consider a two state continuous time markov chain that spends exponential time with rate λ in state 0 before going to state 1 where it spends an exponential time with rate μ before returning to state 0 . The forward eq^n is,

$$P'_{00}(t)=\mu.P_{01}(t)-\lambda P_{00}(t)$$

$$P_{00}^{'}(t) = -(\lambda+\mu).P_{00}(t) + \mu$$
$$P_{00}^{'}(t) + (\lambda+\mu).P_{00}(t) = \mu$$
$$e^{(\lambda+\mu)t}\left[P_{00}^{'}(t) + (\lambda+\mu)P_{00}(t)\right] = \mu e^{(\lambda+\mu)t}$$
$$\frac{d}{dt}\left[e^{(\lambda+\mu)t}.P_{00}(t)\right] = \mu e^{(\lambda+\mu)t}$$
$$\Rightarrow e^{(\lambda+\mu)t}P_{00}(t) = \frac{\mu}{(\lambda+\mu)}.e^{(\lambda+\mu)t}$$

Since, $P_{oo}(0) = 1$

$$\Rightarrow \quad 1 = \frac{\mu}{\lambda+\mu} + c$$
$$\Rightarrow \quad C = \frac{\lambda+\mu-\mu}{\lambda+\mu} = \frac{\lambda}{\lambda+\mu}$$
$$\Rightarrow P_{00}(t) = \frac{\mu}{(\lambda+\mu)} + \frac{\lambda}{(\lambda+\mu)}e^{-(\lambda+\mu)t}$$

Similarly,

$$P_1(t) = \frac{\lambda}{(\lambda+\mu)} + \left(\frac{\mu}{\lambda+\mu}\right)e^{-(\lambda+\mu)t}$$

Machine Repairman Problem -

Example 34: Consider a shop having m no. of machines and 1 repairman. Amount of time is exp. with mean $1/\lambda$ and Repairing time is exp(mean = $1/\mu$). Find the prob. of n machines not in used.

Solution:

$$\mu_n = \mu$$
$$\lambda_n = \begin{cases} (m-n)\lambda & ,n = 1,2,\ldots m \\ 0 & ,n > m \end{cases}$$

we have to Find P_n =?(Prob. of n machines not in used)

$$P_0 = \left[1 + \sum_{k=1}^{\infty}\frac{\lambda_0.\lambda_1\ldots\lambda_{k-1}}{\mu_1\ldots\mu_k}\right]$$
$$= \left[1 + \sum_{j=1}^{m}\left(\frac{\lambda}{\mu}\right)^j m.(m-1).(m-j+1)\right]^{-1}$$
$$= \left[1 + \sum_{j=1}^{m}\left(\frac{\lambda}{\mu}\right)^j \frac{m!}{(m-j)!}\right]^{-1}$$
$$\therefore P_j = \left[\left(\frac{\lambda}{\mu}\right)^j \frac{m!}{(m-j)}\right]P_0$$

■

Chapter 4

Renewal Theory

A generalization of poisson process for which inter-arrival times are iid with an arbitrary distribution.Such a counting process is known as renewal process. Let $\{X_n, n = 1, 2, \dots\}$ be the sequence of non-negative independent r.vs with common distribution function F and to avoid trivialities .
Suppose, $F(0) = PX_n = 0 < 1$ X_n = time between $(n-1)^{st}$ to n^{th} arrival.
$\mu = E(X_n) = \int_0^\infty x.dF(x)$.
μ denotes mean time between successive events $X_n \geq 0, \quad F(0) < 1 \& 0 < \mu < \infty$,
$S_0 = 0, S_n = \sum_{i=1}^n x_i \quad , n \geq 1$. S_n denotes the time of n^{th} event.

$$N(t) = sup\{n | S_n \leq t\}$$

Such $N(t)$ is called as Renewable process.
$\therefore$ By strong law of large no.s

$$\frac{S_n}{n} = \frac{\sum_{i=1}^n x_i}{n} \to \mu$$

4.1 Distribution of N(t) :

The no. of renewals by time $t(N(t))$ is greater than or equal to n, iff the n^{th} renewal occurs before or at time t.

$$N(t) \geq n \Rightarrow S_n \leq t$$

$$\therefore P(N(t) = n) = P(N(t) \geq n) - P(N(t0 \geq n+1)$$

$$= P(S_n \leq t) - P(S_{n+1} \leq t)$$

$$= F_n(t) - F_{n+1}(t)$$

let $M(t)$ = Mean renewal function $= E(N(t)) = \sum_{n=0}^\infty n.P(N(t) = n)$
To Prove : $m(t) = \sum_{n=1}^\infty F_n(t)$

Proof. Consider,

$$\begin{aligned}
m(t) =& \sum^{\infty} n \cdot P(N(t) = n) \\
&= \sum_{n=0}^{\infty} n \cdot (F_n(t) - F_{n+1}(t)) \\
=& \sum_{n=0}^{\infty} n \cdot (F_n(t) - F_{n+1}(t)) \\
=& \sum_{n=1}^{\infty} nF_n(t) - \sum_{n=1}^{\infty} n \cdot F_{n+1}(t) \\
=& F_1(t) - F_2(t) \\
& + 2F_2(t) - 2F_3(t) \\
& + 3F_3(t) - 3F_4(t) \\
& + \vdots - \cdots \\
=& \sum_{n=1}^{\infty} F_n(t)
\end{aligned}$$

OR,
We define,

$$I_n = \begin{cases} 1 & ;\text{if } n^{th} \text{ renewal occurs in } (0, t] \\ 0 & ;else \end{cases}$$

$$\therefore N(t) = \sum_{n=1}^{\infty} I_n$$

$$E(N(t)) = \sum_{n=1}^{\infty} E(I_n)$$

$$= \sum_{n=1}^{\infty} P(I_n = 1)$$

$$= \sum_{n=1}^{\infty} P(S_n \leq t)$$

$$= \sum_{n=1}^{\infty} P(N(t) \geq t)$$

$$= \sum_{n=1}^{\infty} F_n(t)$$

□

To prove , with probability 1 ,

$$\frac{N(t)}{t} \to \frac{1}{\mu} \quad ,\text{as}, t \to \infty$$

Proof. We have ,

$$S_{N(t)} \leq t \leq S_{N(t)+1}$$

Divide by $N(t)$,

$$\frac{S_{N(t)}}{N(t)} \leq \frac{t}{N(t)} \leq \frac{S_{N(t)+1}}{N(t)+1}.\frac{N(t)+1}{N(t)}$$

$$\text{As}, t \to \infty, N(t) \to \infty$$

$$\therefore \frac{N(t)+1}{N(t)} \cong 1$$

$\therefore$ By SLLN , $\bar{x} \to \mu$ For $n \to \infty$

$$i.e \frac{\sum_{n=1}^{n} x_i}{n} \to \mu$$

$$\therefore \frac{\sum_{i=1}^{N(t)} x_i}{N(t)} \leq \frac{t}{N(t)} \leq \frac{\sum_{i=1}^{N(t)+1} x_i}{N(t)+1}$$

$$\mu \leq lim_{t\to\infty} \frac{t}{N(t)} \leq \mu$$

$$\therefore \frac{t}{N(t)} = \mu, \text{For} t \to \infty$$

$$\therefore \frac{N(t)}{t} = \frac{1}{\mu}, \text{For} t \to \infty$$

□

To prove, $E(S_{N(t)+1}) = \mu[m(t)+1]$ Now,

$$S_{N(t)+1} = \sum_{i=1}^{N(t)+1} x_i$$

$$\therefore E(S_{N(t)+1}) = E(\sum_{i=1}^{N(t)+1} x_i)$$

$=E(x).E(N(t)+1)$ ...(Compound poisson formula)

$=\mu[m(t)+1]$

Theorem 4.1.1. *Elementary Renewal Theorem :* $\frac{m(t)}{t} \to \frac{1}{\mu}$ *as*, $t \to \infty$

Proof. Suppose that $\mu < \infty$,

$$S_{N(t)+1} > t$$

$$E\left(S_{N(t)+1})\right) > t$$

$$\mu.(m(t)+1) > t$$

$$m(t)+1 > \frac{t}{\mu}$$

$$\frac{m(t)+1}{t} > \frac{1}{\mu}$$

$$\lim_{t\to\infty} \inf \frac{m(t)}{t} \geq \frac{1}{\mu}$$

lets fix some constant M and define new renewal prócess ,$\bar{x}_n, n = 1,2,\ldots$

$$\bar{X}_n = \begin{cases} X_n & \text{if } X_n \leq m \\ M & \text{if } X_n > M \end{cases}$$

Let ,

$$\bar{S_n} = \sum_{i=1}^{n} \bar{x}_i, \bar{N(t)} = \max\left\{n/\bar{S}_n \leq t\right\}$$

$$\therefore \bar{S}_{N(t)+1} \leq t + M$$

$$E\left(\bar{S}_{N(t)+1}\right) \leq t + M$$

$$\mu_m(\bar{m}(t)+1) \leq t + m$$

where, $\mu_m = E(\bar{x}_n)$

$$\lim_{t\to\infty} Sup \frac{\bar{m}(t)}{t} \leq \frac{1}{\mu_m}$$

Since,

$$\bar{S_n} \leq S_n \Rightarrow \bar{N(t)} \geq N(t) \quad \& \bar{m}(t) \geq m(t)$$

$$\Rightarrow \lim_{t\to\infty} Sup \frac{\bar{m}(t)}{t} \leq \frac{1}{\mu_m}$$

letting $M \to \infty$,

$$\lim_{t\to\infty} Sup \frac{\bar{m}(t)}{t} \leq \frac{1}{\mu}$$

$$\lim_{t\to\infty} \frac{\bar{m}(t)}{\leq} \frac{1}{\mu_m}$$

□

4.1.1 Renewal Reward Process :

$$R(t) = \sum_{i=1}^{N(t)} R_i$$

$N(t)$: No of Renewals.

Consider, a renewal process $\{N(t), t \geq 0\{$ having inter-arrival times $x_n, n \geq 1$, with distribution F & suppose that each time a renewal occurs, we receive a reward. We denote by R_n, the reward earned at time of n^{th} renewal. We shall assume that the $R_n, n \geq 1$ are iid. However, the reward may depend an x_n. So, we assume the pairs (x_n, R_n) are iid.

$$R(t) = \sum_{i=1}^{N(t)} R_i$$

then $R(t)$ represent the total reward earned by time t.
let, $E(R) = E(R_n), \quad E(X) = E(X_n)$

Theorem 4.1.2. *If* $E(R) < \infty \& E(X) < \infty$, *then*

1. *w.p. 1* , $\frac{R(t)}{t} \to \frac{E(R)}{E(x)}, \quad as, t \to \infty$
2. $\frac{E(R(t))}{t} \to \frac{E(R)}{E(x)}, \quad as, t \to \infty$

Proof. i) We have ,

$$R(t) = \sum_{i=1}^{N(t)} R_i$$

Divide by $N(t)$,

$$\frac{R(t)}{N(t)} = \frac{\sum_{i=0}^{N(t)} R_i}{N(t)} = E(R)$$

$$\therefore \frac{R(t)}{t} = \frac{R(t)}{N(t)} . \frac{N(t)}{t} \tag{4.1}$$

As $t \to \infty$, by SLLN ,

$$\lim_{t \to \infty} \frac{R(t)}{N(t)} \to E(R) \tag{4.2}$$

and as $t \to \infty$,

$$\frac{N(t)}{t} \to \frac{1}{\mu} = \frac{1}{E(x)} \tag{4.3}$$

$$\therefore \lim_{t \to \infty} \frac{R(t)}{t} \to \frac{E(R)}{E(X)}$$

ii) Let $N(t)+1$ is stopping time for seq^n $X_1, X_2, \ldots$, it is also stopping time for seq^n $R_1, R_2, \ldots$ So, by Wald's eq^n,

$$E(R(t)) = E\left(\sum_{i=1}^{N(t)} R_i\right) = E\left(\sum_{i=1}^{N(t)+1} R_i\right) - E\left(R_{N(t)+1}\right)$$

$$= (m(t)+1)\, E(R) - E\left(R_{N(t)+1}\right) - \mu\,(m(t)+1)$$

and So,

$$\frac{E(R(t))}{t} = \frac{m(t)+1}{t} . E(R) - \frac{E(R_{N(t)+1})}{t} . E(R)$$

$$= \frac{m(t)}{t} + \frac{1}{t}E(R) - \frac{E(R)}{t} \to 0$$

So, the result will follow from elementary renewal theorem, if we can show that, $\frac{E(R_{N(t)+1})}{t} \to 0 \quad \text{as} \quad t \to \infty$ So, let

$$g(t) = E\left(R_{N(t)+1}\right)$$

$$= E\left(R_{N(t)+1}|S_{N(t)} = 0\right).\bar{F}(t) + \int_0^t E\left(R_{N(t)+1}|S_{N(t)} = s\right).\bar{F}(t-s)dm(s)$$

However,

$$E\left(R_{N(t)+1}|S_{N(t)} = 0\right) = E\left(R_i|x_i > t\right)$$

$$E\left(R_{N(t)+1}|S_{N(t)} = s\right) = E\left(R_n|x_n > t-s\right)$$

and so ,

$$g(t) = E\left(R_1|X_1 > t\right).\bar{F}(t) + \int_0^t E\left(R_N|X_n > t-s\right).\bar{F}(t-s)dm(s)$$

$$h(t) = E\left(R_1|X_1 > t\right).\bar{F}(t) = \int_t^\infty E\left(R_1|X_1 = x\right).dF(x)$$

Since,

$$E(R_1) = \int_0^\infty E\left(|R_1| \quad |X_1 = x\right).dF(x) < \infty$$

$$\Rightarrow h(t) \to 0 \quad \text{as} \quad t \to \infty \quad \& \quad h(t) \le E(|R_1|) \quad \forall t$$

and thus , we can choose T so that,

$$|h(t)| < \epsilon \quad \text{whenever} \quad t \ge T$$

$$\frac{|g(t)|}{t} \le \frac{|h(t)|}{t} + \int_0^{t-T} \frac{|h(t-x)|dm(x)}{t} + \int_{t-T}^t \frac{|h(t-x)|dm(s)}{t}$$

$$\le \frac{\epsilon}{t} + \frac{\epsilon m(t-T)}{t} + E(R_1).\frac{[m(t) - m(t-T)}{t}$$

$$\to \frac{\epsilon}{E(x)} \quad \text{as} \quad t \to \infty$$

$\Rightarrow$ by elementary renewal theorem,

$$\frac{g(t)}{t} \to 0 \quad \text{as} \quad t \to 0$$

□

4.2 Regenerative Process :

Consider a stochastic process $\{X(t), \quad t \ge 0\}$ with state space having property that there exists time points at which the process (prababilistically) restarts itself. That is, suppose that with prob. $1, \exists$ a time s_1 such that continuation of the process beyond S_1 is a probalistic replica of the whole process starting at 0 . This property implies the existance of such $S_1, S_2, \ldots$ having same property as S_1. Such stochastic process is called as **Regenerative Process**.

4.2.1 Semi-Markov Chain :

A Semi - Markov process is one that changes states in accordance with Markov chain but takes a random amount of time bet^n changes. More specifically consider a stochastic process with states $0,1,2,\ldots$ which is such that whenever it enters state $i, \quad i \geq 0$.

- The next state, it will enter is state j with prob $P_{ij}, i, j \geq 0$.
- Given that the next state to be entered is state j, the until the transition from i to j occurs has $dist^n$ F_{ij}.

If we let $z(t)$ denote the state at time t , then $\{z(t), t \geq 0\}$ is called as **Semi-Markov Process**.

4.2.2 Time Reversible Markov Chain :

An irreducible positive recurrent MC is stationary if the initial state is choosen according to stationary probabilities.

Consider the stationary MC having transition prob P_{ij} of statianary prob π_i & suppose that starting at some time, we trace a seq^n of states going backwards in time. i.e , starting at time n , consider the seq^n of states $X_n, X_{n-1}, \ldots$ itself a MC with transition prob.

$$\begin{aligned} P^*_{ij} &= P\{X_m = j | X_{m+1} = i\} \\ &= \left[P\{X_{m+1} = i | X_m = j\}.P\{X_m = j\}\right] P\{X_{m+1} = i\} \\ &= \pi_j.\frac{P_{ij}}{\pi_i} \end{aligned}$$

If $P^*_{ij} = P_{ij} \quad \forall i, j$, then the MC is said to be time reversible .
The condition for time reversibility that ,

$$\pi_i.P_{ij} = \pi_j.P_{ji} \quad \forall i, j$$

Example 35: $\begin{array}{cc} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{pmatrix} 0.3 & 0.7 \\ 0.7 & 0.3 \end{pmatrix} \end{array}$

Solution:
we have, $\underline{\pi} = \underline{\pi}.P$

$$\begin{aligned} \begin{bmatrix} \pi_0 & \pi_1 \end{bmatrix} &= \begin{bmatrix} \pi_0 & \pi_1 \end{bmatrix} \begin{bmatrix} 0.3 & 0.7 \\ 0.7 & 0.3 \end{bmatrix} \\ &= \begin{bmatrix} 0.3\pi_0 + 0.7\pi_1 & 0.7\pi_0 + 0.3\pi_1 \end{bmatrix} \end{aligned}$$

$$\pi_0 = 0.3\pi_0 + 0.7\pi_1 \& \pi_1 = 0.7\pi_0 + 0.3\pi_1$$

$$\pi_1 = \pi_0 = 1/2$$

For time reversibility,

$$\begin{aligned} &\pi_1 P_{10} = \pi_0.P_{01} \\ &(0.5)(0.7) = (0.5)(0.7) \\ &3.5 = 3.5 \quad \therefore \text{ This stochastic process is time reversible.} \end{aligned}$$

■

Example 36: $\begin{array}{ccc} & 0 & 1 \\ 0 & \multicolumn{2}{c}{\multirow{2}{*}{}} \end{array}$ $\begin{array}{c|cc} & 0 & 1 \\ \hline 0 & 0.3 & 0.7 \\ 1 & 0.4 & 0.6 \end{array}$

Solution:
We have, $\underline{\pi} = \underline{\pi}.P$

$$\begin{bmatrix} \pi_0 & \pi_1 \end{bmatrix} = \begin{bmatrix} \pi_0 & \pi_1 \end{bmatrix} \begin{bmatrix} 0.3 & 0.7 \\ 0.4 & 0.6 \end{bmatrix}$$
$$\begin{bmatrix} \pi_0 & \pi_1 \end{bmatrix} = \begin{bmatrix} 0.3\pi_0 + 0.4\pi_1 & 0.7\pi_0 + 0.6\pi_1 \end{bmatrix}$$

$$\pi_0 = 0.3\pi_0 + 0.4\pi_1 \& \pi_1 = 0.7\pi_0 + 0.6\pi_1$$
$$\therefore 0.7\pi_0 = 0.4\pi_1 \quad \& 0.4\pi_1 = 0.7\pi_0$$

We know that ,

$$\pi_0 + \pi_1 = 1$$
$$\frac{7}{4}.\pi_0 + \pi_0 = 1$$
$$\therefore \pi_0 \left(\frac{11}{4}\right) = 1$$
$$\therefore \pi_0 = \frac{4}{11}$$
$$\therefore \pi_1 = \frac{7}{11}$$

For time reversibility,

$$\pi_0 P_{01} = \pi_1 P_{10}$$
$$\tfrac{4}{11}.(0.7) = \tfrac{7}{11}.(0.4)$$
$$2.8 = 2.8$$

■

4.3 Brownian Motion :

A stochastic process $\{x(t); t \geq 0\}$ is called a Weiner process or brownian motion process with drift coefficient μ and variance parameters / diffusion coefficient σ^2 if,

1. $X(t)$ has independent increment i.e. for disjoint intervals (s, t) of (u, v) where $s \leq t \leq u \leq v$ the random variables. $\{X(t) - X(s)\}$ and $\{X(v) - X(u)\}$ are independent.
2. Every increment $\{X(t) - X(s)\}$ is normally distributed with mean $\mu.(t - s)$ & variances $\sigma^2(t - s)$

Consider a particle performs random walk such that in small interval of time of duration Δt, the displacement of the particle to the right or left is also of small magnitude Δx, the total displacement $X(t)$ of the particle in time t being x.

$$P[z_i = \Delta x] = p\&$$

$$P[Z = -\Delta x] = q$$

$$X(t) = \sum_{i=1}^{n(t)} z_i \quad , n \cong n(t) \cong \frac{t}{\Delta t}$$

$$F(z_i) = (p-q)\Delta x$$

$$v(z_i) = 4pq(\Delta x)^2$$

$$E(x(t)) = nE(z_i) = n(p-q)\Delta x$$

$$\cong t(p-q)\frac{\Delta x}{\Delta t}$$

$$v(x(t)) = n.var(z_i) = \frac{t}{\Delta t} 4pq(\Delta x)^2$$

As $\Delta t \to 0 \quad , \Delta x \to 0$,

$$\frac{(\Delta x)^2}{\Delta t} \to \text{a limit} , (p-q) \to \text{multiple of } \Delta x$$

We may suppose $X(t)$ has mean value function μt and variance function $\sigma^2 t$, and suppose $\Delta x \to 0 \quad , \Delta t \to 0$

$$E(x(t)) \to \mu \quad \& \quad V(x(t)) \to \sigma^2$$

for $t = 1$, we have ,

$$\frac{(p-q).\Delta x}{\Delta t} \to \mu \quad \text{and} \quad \frac{4pq(\Delta x)^2}{\Delta t} \to \sigma^2$$

$$\Delta x = \sigma(\Delta t)^{\frac{1}{2}} \quad , P = \frac{1}{2}\left(1 + \frac{\mu(\Delta t)^{\frac{1}{2}}}{\sigma}\right)$$

Since, z_i are iid & $X(t) = \sum_{i=1}^{n(t)} z_i$ for large n is asympotically normal with mean μt & var $\sigma^2 t$.

Example 37: $\{X(t), t \geq 0\}$ is the Weiner Process. Show that $Y(t) = \sigma X(\frac{t}{\sigma^2}$ is Weiner process.

Solution:

1) Given that ,

$$\{X(t), t \leq 0\} \text{is the weiner process}$$

$$\therefore X(t) \sim N(\mu t, \sigma^2 t)$$

$$\therefore X(\frac{t}{\sigma^2}) \sim N\left(\frac{\mu t}{\sigma^2}, \frac{\sigma^2 t}{\sigma^4} = \frac{t}{\sigma^2}\right)$$

$$\therefore Y(t) = \sigma X\left(\frac{t}{\sigma^2}\right) \sim N\left(\frac{\mu t}{\sigma}, t\right)$$

$\therefore$ Drift Coeff. = $\frac{\mu}{\sigma}$ & Var Coeff. = 1

2) To show independence , Assume $s < t$,

$$Cov(X(t), x(s)) = E(X(t).X(s)) - E(X(t)).E(X(s))$$

$$= Cov\,(X(s), X(s) + X(t) - X(S))$$
$$= V(X(S)) + Cov(X(s), X(t) - X(s))$$
$$= V(X(s)) + 0$$
$$= \sigma^2.s$$

$\because Cov(X(s), X(t) - X(s)) = 0$, as for disjoint time intervals$s\&(t-s)$, $X(s)$, & $X(t) - X(s)$are independent. Parallelly, $Cov(Y(s), Y(t)) = s$ Consider, two disjoint time intervals $(0, s)\&(s, t)$,

$$\begin{aligned} Cov(Y(s), Y(t)) &= Cov\left(\sigma X\left(5/\sigma^2\right), \sigma X\left(t/\sigma^2\right) - \sigma X\left(s/\sigma^2\right)\right) \\ &= Cov\left(6X\left(s/\sigma^2\right), \sigma\left(x\left(t/\sigma^2\right) - X\left(s/\sigma^2\right)\right)\right) \\ &= \sigma^2 Cov\left(X\left(s/\sigma^2\right), X\left(t/\sigma^2\right) - X\left(\frac{s}{\sigma^2}\right)\right) \\ &= \sigma^2(0) \\ &= 0 \quad \because Cov(X(t) - X(s), X(5)) = 0 \end{aligned}$$

■

4.4 MLE :

Consider a time homogenous Markov chain with finite number of states and having transitian prob. matrix $P = \left(\left(P_{jk}\right)\right), j, k = 1, 2, \ldots, m$. Suppose that no. of observed direct transitions from state j to k is n_{jk} and total no. of observations are $N+1$

$$\text{Put} \sum_{k=1}^{m} n_{ik} = n_{j\cdot}, \quad \sum_{j=1}^{m} n_{jk} = n_{\cdot k}, \quad j, k = 1 : m$$

Here n_{jik} follows multinomial $dist^n$.

$$f\left(n_{jk}\right) = T\left(n_{jk}\right) \frac{\prod_j \left(n_{j\cdot}\right)!}{\prod_j \prod_k \left(n_{jk}\right)!} . \prod_j \prod_k {P_{jk}}^{n_{jk}}$$

$$L\left(P_{jk}\right) = C + \sum_j \sum_k n_{jk} \log P_{jk}$$

as $\sum P_{jk} = 1$

$$L\left(P_{jk}\right) = C + \sum_{j=1}^{m} \sum_{k=1}^{m-1} n_{ik} \log P_{jk} + \sum_{j=1}^{m} n_{jm} \log\left(1 - \sum_{k=1}^{m-1} P_{jk}\right).$$

Let r be specific state,

$$\frac{\delta L(P_{rk})}{\delta P_{rk}} = 0 \quad , k = 1, 2, \ldots, m-1$$

$$\Rightarrow \frac{n_{rk}}{P_{rk}} - \frac{n_{rm}}{1 - \sum_{k=1}^{m-1} P_{rk}} = 0 \quad ; k = 1, 2, \ldots, m-1$$

$\Rightarrow$ For specific values of r as s, k

$$\Rightarrow \frac{n_{rs}}{P_{rs}} = \frac{n_{rk}}{P_{rk}} = \frac{n_{rm}}{1 - sum_{k=1}^{m-1} P_{rk}}$$

$$\therefore 1 - \sum_{k=1}^{m-1} P_{rk} = \frac{n_{rm}}{n_{rs}}.P_{rs}$$

$$\Rightarrow P_{rk} = \frac{n_{rk}}{n_{rs}} P_{rs} \quad , k = 1, 2, \ldots, s, \ldots, m-1$$

$$\therefore \sum_k P_{rk} = 1 = \sum_k \frac{n_{rk}}{n_{rs}}.P_{rs}$$

Hence,

$$P_{rs} = \frac{n_{rs}}{\sum_{k=1}^{n} n_{rk}}$$

$$P_{jk} = \frac{n_{jk}}{\sum_{r=1}^{m} n_{jr}}$$

$$P_{jk} = \frac{n_{jk}}{n_{j\cdot}}$$

4.4.1 Hypothesis Testing :

Suppose that one wishes to test the H_0 that the observed realization comes from Markov chain with transition matrix P°

$$H_0 : P = P^{\circ}$$

So, for large N,

$$\sum_{k=1}^{m} \frac{n_{j\cdot}\left(p_{jk} - p_{jk}^{0}\right)^2}{p_{jk}^{0}}, \quad j = 1, 2, \ldots m$$

is distributed as χ^2_{m-1} with $m-1$ d.F ,
Here, P°_{jk} 's which are equal to zero are excluded d.F is reduced by no. of P°_{jk} 's equal to zero.
Let the likelihood ratio criterion for H_0

$$\lambda = \prod_j \prod_k \left(\frac{p_{jk}}{p_{jk}^{0}}\right)^{n_{jk}}$$

Under null hypothesis,

$$-2\log\lambda = 2\sum_j \sum_k n_{jk} \log \frac{n_{jk}}{(n_{j\cdot})\, p^{\circ}_{jk}}$$

has an asymptotic χ^2 $dist^n$ with $m(m-1)$ d.f.

4.5 Determination of Order of MC by MAICE :

The procedure for determination of order of MC by Akaike's Ibnformation Criterion (AIC) has been developed by,

$$AIC = (-2)\,log\left(Max^m \text{ likelihood}\right) + 2\left(\text{No. of independent parameters}\right)$$

Lets denote the transition probabilities for r order chain by $P_{ij},\ldots,kl \quad ,l = 1,2,\ldots,s$ Denote MLE by,

$$P_{ij\ldots kl} = \frac{n_{ij\ldots kl}}{n_{ij\ldots k\cdot}}$$

where,

$$n_{ij\ldots k} = \sum_{l=1}^{s} n_{ij\ldots kl}$$

The hypothesis tested is,

$$H_{r-1} : P_{ij\ldots kl} = P_{\cdot j\ldots kl} \quad ,i = 1,2,\ldots s \quad \left((r-1)\text{dependent}\right)$$

Vs

$$H_r = P_{ij\ldots kl} \text{ are different for some} i\epsilon(1,2,\ldots,s)$$

$$A_r = -2log\lambda_{r-1,1}$$
$$= 2\sum_{i,\ldots\lambda} n_{ij\ldots kl} log\left[\frac{n_{ij\ldots kl}(\sum_{i,\ldots,l} n_{ij\ldots kl}}{n_{ij\ldots k\cdot}(n_{\cdot j\ldots kl}}\right]$$

which is χ^2 variate with $S^{r-1}(s-1)^2$ d.f. **Example 38:**

4.6 Queuing Models (Using Birth and Death Process) :

Queue or waiting line is formed when units needed some service arrived at some service channel that offers such facility. The basic features that characterises a queuing system are,

1. Input
2. Service mechanism
3. Queue Discipline
4. No. of servers

$$a = \text{Inter-arrival time } dist^n$$
$$b = \text{Service time } dist^n$$
$$c = \text{No. of servers}$$
$$d = \text{Queue discipline}$$
$$e = \text{System capacity}$$
$$f = \text{Source capacity}$$

$a = M$: Markovian, $b = M$: Markovian

GI : General Inter-arrival, Er : Erlang

Er : Erlang, D : Deterministic

D : Deterministic, G : General $dist^n$ for Service time

4.6.1 Queue Discipline :

FCFS : First Come First Served .
LCFS : Last Come First Served.
SIRO : Service in Random Order.
GD : General Discipline.

λ_i **:** Arrival rate when there are i customers in system .
μ_i **:** Service rate when there are i customers in systems .

$$P_n = \left[\prod_{i=0}^{n} \frac{\lambda_{i-1}}{\mu_i}\right].P_0 \qquad , P_0 = \left[1 + \sum_{n=1}^{\infty}\prod_{i=1}^{n}\frac{\lambda_{i-1}}{\mu_i}\right].$$

L_s : Expected No. of customers in system .
L_q : Expected No. of customers in queue .
W_s: Expected waiting time in system for customer just arrived .
W_q : Expected waiting time in queue for customer just arrived .
$\bar{c}$: Expected no. of busy servers.

$$L_s = \sum_{n=0}^{\infty} n.P_n \qquad , L_q = \sum_{n=c+1}^{\infty}(n-c).P_n$$

Little's Formula:
$L = \lambda w$
$L_s = \lambda_{eff} W_s$
$L_q = \lambda_{eff}.W_q$
$\rho =$ traffic intensity $=$ Mean no of arrivals/mean service

M|M|1 : $\lambda_i = \lambda, \quad \mu_i = \mu, \quad i = 1,2,\ldots\infty$

$$\therefore P_n = \left(\frac{\lambda}{\mu}\right)^n\left(1-\frac{\lambda}{\mu}\right)$$

Where,

$$\text{Where } L_s = \sum_{n=0}^{\infty} n.P_n$$
$$= \frac{\frac{\lambda}{\mu}}{1-\frac{\lambda}{\mu}} = \frac{\lambda}{\mu-\lambda}$$
$$L_q = \sum_{n=2}^{\infty} (n-1)P_n$$
$$= \sum_{n=2}^{\infty} n.P_n - \sum_{n=2}^{\infty} P_n$$
$$= (L_s - P_1) - (1 - P_0 - P_1)$$
$$= L_s - 1 + P_0$$
$$= L_s - 1 + 1 - \frac{\lambda}{\mu}$$
$$\therefore L_q = L_s - \frac{\lambda}{\mu} \Rightarrow L_q = \frac{\lambda}{\mu-\lambda}$$
$$i.e \quad L_s = L_q + \frac{\lambda}{\mu}$$
$$W_s = \frac{L_s}{\lambda_{eff}} = \frac{\lambda}{\mu-\lambda}\frac{1}{\lambda} = \frac{1}{\mu-\lambda}$$
$$W_q = \frac{L_q}{\lambda_{eff}} = \frac{1}{\mu-\lambda} - \frac{1}{\mu} = W_s - \frac{1}{\mu}$$

M|M|∞ : $P_n = \frac{e^{\frac{\lambda}{\mu}}(\frac{\lambda}{\mu})^n}{n!}, \quad n = 0,1,2,\ldots$

$L_s = \frac{\lambda}{\mu}$

$L_q = 0$

$W_s = \frac{1}{\mu}$

M|M|C :GD|∞|∞

$\lambda_i = \lambda \quad i = 0,1,2,\ldots$

$$\mu_i = \begin{cases} i\mu & i = 1,2,\ldots c \\ c\mu & i = c+1,\ldots \end{cases}$$

$$P_n = \begin{cases} \left(\frac{\lambda}{\mu}\right)^n \frac{1}{n!} P_0 & ; n \le c \\ \left(\frac{\lambda}{\mu}\right)^n \frac{1}{c!.c^{n-c}}.P_0 & ; n \ge c \end{cases}$$

Where ,

$$P_0 = \left[1 + \sum_{n=1}^{c} \left(\frac{\lambda}{\mu}\right)^n \frac{1}{n!} + \sum \left(\frac{\lambda}{\mu}\right) \frac{1}{c!c^{n-c}}\right]^{-1}$$
$$\therefore \lambda_{eff} = \lambda - n.P_n$$

4.6.2 Strictly Stationary Process :

$$t_1 < t_2 < \ldots < t_n$$

$$(X(t_1), X(t_2), \ldots, X(t_n))$$

$$t_1 + s < t_2 + s < \ldots, X(t_n + s)$$

$$(X(t_1 + s), X(t_2 + s), \ldots, X(t_n + s))$$

If For arbitary $t_1, t_2, \ldots, t_n$ the joint $dist^{ns}$ of the vector r.vs $X(t_1), X(t_2), \ldots X(t_n)$ &$X(t_1 + s), X(t_2 + s), \ldots, X(t_n + s)$ are same for all $S > 0$ then stochastic process $\{X(t), t \in T\}$ is said to be stationary of order n, it is strictly stationary if it is stationary of order n ,for any n.

- Poisson process is strictly stationary process.

4.6.3 Weakly Stationary Process :

A stochastic process is called Co-variance stat. or weakly stationary if its mean function $M(t) = E(X(t))$ is independent of t and its covariance function, $Cov(X(t), X(s))$ is function of $s - t(t < s)$.

- Note that, a strictly stationary process will not necessarily be a weakly stationary process not will weakly stationary process be necessarily strictly stat. process.
- Standard brownian motion is weakly as well as strictly stationary process.
- strictly stationary process does not implies weakly Stationary.
- Poisson Process is counting process ,
  $\therefore T_i \sim^{iid} exp\left(\frac{1}{\lambda}\right)$

Chapter 5

Extra Examples from Question Bank

Example 39: Let $\{x_n, n \geq 0\}$ be two state markov chain with $P_{00} = 0.4$, $P_{01} = 0.6$, $P_{10} = 0.4$, $P_{11} = 0.6$ Show that the vector valued process $\{Y_n, n \geq 0$. Where $Y_n = (x_{n-1}, x_n)$ is again MC. Find TPM of process.

Solution:

Given that $\{x_n, n \geq 0\}$ is MC with TPM as, $\begin{array}{c c} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{pmatrix} 0.4 & 0.6 \\ 0.4 & 0.6 \end{pmatrix} \end{array}$

$$
\begin{aligned}
Y_n &= (X_{n-1}, X_n) \\
S &= \{(0,1), (0,0), (1,0), (1,1)\}
\end{aligned}
$$

TPM For $\{Y_n, n \geq 0\}$ is given by,

$$
\begin{array}{c c}
 & \begin{array}{cccc} (0,0) & (0,1) & (1,0) & (1,1) \end{array} \\
\begin{array}{c} (0,0) \\ (0,1) \\ (1,0) \\ (1,1) \end{array} &
\begin{pmatrix}
0.4 & 0.6 & 0 & 0 \\
0 & 0 & 0.4 & 0.6 \\
0.4 & 0.6 & 0 & 0 \\
0 & 0 & 0.4 & 0.6
\end{pmatrix}
\end{array}
$$

■

Example 40: A MC $\{x_n, n \geq 0\}$ with states $1,2\&3$ has the transition prob matrix $P = \frac{1}{4}\begin{bmatrix} 2 & 2 & 0 \\ 0 & 0 & 4 \\ 4 & 0 & 0 \end{bmatrix}$ If $P\{x_0 = 1\} = 0$ & $P\{x_0 = 2\} == P\{x_0 = 3\} = \frac{1}{2}$. Find EX_3

Solution:

To Find EX_3,

$$
\alpha.P^3 = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}^3
$$

$$= \begin{bmatrix} 0 & 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

$$= \frac{1}{64} \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 2 & 2 & 0 \\ 0 & 0 & 4 \\ 4 & 0 & 0 \end{bmatrix} \begin{bmatrix} 4 & 4 & 8 \\ 16 & 0 & 16 \\ 8 & 8 & 0 \end{bmatrix}$$

$$= \frac{1}{64} \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 40 & 8 & 48 \\ 32 & 32 & 0 \\ 16 & 16 & 32 \end{bmatrix}$$

$$= \frac{1}{64} \begin{bmatrix} 24 & 24 & 16 \end{bmatrix} = \begin{bmatrix} \frac{3}{8} & \frac{3}{8} & \frac{2}{8} \end{bmatrix}$$

i.e.

$$P\{X_3 = 1\} = \frac{3}{8} \quad , P\{X_3 = 2\} = \frac{3}{8} \quad , P\{X_3 = 3\} = \frac{2}{8}$$

$$\therefore EX_3 = 1\left(\frac{3}{8}\right) + 2\left(\frac{3}{8}\right) + 3\left(\frac{2}{8}\right)$$

$$EX_3 = \frac{15}{8}$$

■

Example 41: Define X_n as the state of the system after state chnge , so that $X_n = 0$, if the system running ; $X_n = 1$ if the system is under repair and $X_2 = 2$ if the system is idle . Assume that the matrix $P = \begin{bmatrix} 0 & 0.5 & 0.5 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$. Compute the matrix P_n for all possible n.

Solution:

$$P = \begin{bmatrix} 0 & 0.5 & 0.5 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$P^2 = \begin{bmatrix} 0 & 0.5 & 0.5 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} . \begin{bmatrix} 0 & 0.5 & 0.5 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.5 & 0.5 \\ 0 & 0.5 & 0.5 \end{bmatrix}$$

$$P^3 = \begin{bmatrix} 0 & 0.5 & 0.5 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} . \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.5 & 0.5 \\ 0 & 0.5 & 0.5 \end{bmatrix} = \begin{bmatrix} 0 & 0.5 & 0.5 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\therefore P^4 = P.P^3 = P.P = P^2$$

$$\therefore P^5 = P.P^4 = P.P^2 = P^3 = P$$

$$\therefore P^6 = P.P^5 = P.P^3 = P^4 = P^2$$
$$\therefore P^7 = P.P^6 = P.P^2 = P^3 = P$$

■

Example 42: Suppose that every man in a certain society has exactly 3 childern, which independent have prob. 5 of being boy and 0.5 of being girl. Suppose also that no. of males in n^{th} generation form branching process $\{x_n, n \geq 0\}$.. Find the prob. that male line of given man eventually become extinct.If given Man has 2 boys & 1 girl, what is the prob. that his male line will continue forever.

Solution:

Given that, Every man in society has 3 children which have prob 0.5 of being boy & 0.5 of being girl,

Let X_n : no. of males in n^{th} generation

Z_i : No of boy children to each male in $(n-1)^{th}$ generation.

$$X_n = \sum_{i=1}^{X_{n-1}} Z_i$$

$Z_i \sim B\left(n=3, p=0.5\right)$ Prob. that male line of given man become extinct $= \pi_0$

$$= P_Z(\pi_0) - \text{ PGF of Z}$$
$$= \sum_{j=0}^{3} \pi_0^j P(Z=j)$$
$$(q + p.\pi_0)^n$$
$$\pi_0 = (0.5 + 0.5\pi_0)^3$$
$$\pi_0 = \frac{1}{8}(1+\pi_0)^3$$

$$8\pi_0 = (1+\pi_0)^3$$
$$8\pi_0 = 1 + \pi_0^3 + 3\pi_0^2 + 3\pi_0$$

$$\pi_0^3 + 3\pi_0^2 - 5\pi_0 + 1 = 0$$

$$(\pi_0 - 1)\left(\pi_0^2 + 4\pi_0 - 1\right) = 0$$
$$\pi_0 = 1 \text{ OR } \pi_0^2 + 4\pi_0 - 1 = 0$$

$$\pi_0 = \frac{-4 \pm \sqrt{16-4}}{2} \therefore \pi_0 = \frac{-4 \pm 2\sqrt{3}}{2}$$
$$= -2 \pm \sqrt{3}$$
$$\pi_0 = 1 \quad OR \quad \Pi_0 = -2 - \sqrt{3} \quad OR \quad \pi_0 = -2 + \sqrt{3}$$

But prob. never be negative,

$$\therefore \pi_0 = 1$$

■

Example 43: Consider a game of 'ladder climbing'. There are 5 levels in game level 1 is the lowest (bottom) and level 5 is highest (top).A player starts at the bottom,Each time , a fair coin is tossed , if it turns up heads , the player moves up one rung,if tails , the player moves down to the very bottom . Once at the top level,the player moves to the very bottom if a tail turns up & stays at the top if head turns up.

1. Find the TPM.
2. Find Two-state TPM.
3. Find steady - state $dist^n$ of markov chain.

Solution:
i) For Transition prob. matrix,
$S = (1, 2, 3, 4, 5)$

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \end{pmatrix} \end{array}$$

ii) Two step transistion prob.matrix means P^2,

$$P^2 = P.P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \end{bmatrix}$$

$$P^2 = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} & 0 & 0 \\ \frac{1}{2} & \frac{1}{4} & 0 & \frac{1}{4} & 0 \\ \frac{1}{2} & \frac{1}{4} & 0 & 0 & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{4} & 0 & 0 & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{4} & 0 & 0 & \frac{1}{4} \end{bmatrix}$$

iii) To find steady state i.e stationary $dist^n$,
We have, $\pi = \pi.P$

$$\begin{bmatrix} \pi_1 & \pi_2 & \pi_3 & \pi_4 & \pi_s \end{bmatrix} = \frac{1}{2}[\pi_1\pi_2\pi_3\pi_4\pi_s]\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2\pi_1 & 2\pi_2 & 2\pi_3 & 2\pi_4 & 2\pi_5 \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{5}\pi_i\pi_1\pi_2\pi_3\pi_4 + \pi_5 \end{bmatrix}$$

$$\text{Since, } \pi_1+\pi_2+\pi_3+\pi_4+\pi_5 = 1$$

$$\therefore 2\pi_1 = \sum_{i=1}^{5}\pi_i = 1$$

$$\pi_1 = \frac{1}{2} \because 2\pi_2 = \pi_1$$

$$\pi_2 = \frac{\pi_1}{2} = \frac{1}{4}$$

$$\therefore 2\pi_3 = \pi_2$$

$$\therefore \pi_3 = \frac{\pi_2}{2} = \frac{1}{8}$$

$$\therefore 2\pi_4 = \pi_3$$

$$\pi_4 = \frac{\pi_3}{2} = \frac{1}{16}$$

$$2\pi_5 = \pi_4 + \pi_5$$

$$\pi_5 = \pi_4 = \frac{1}{16}$$

Hence ,

$$\pi_1 = \frac{1}{2}, \quad \pi_2 = \frac{1}{4}, \quad \pi_3 = \frac{1}{8}, \quad \pi_4 = \pi_5 = \frac{1}{16}$$

■

Example 44: Let us consider MC with SS , $S = \{1,2,3,4\}$ & TPM as ,

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{2}{5} & 0 & \frac{3}{5} & 0 \\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{bmatrix}$$

Classify the states into ,

1. Persistent
2. Transient
3. Periodic

Solution:
This is irreducible MC.
$c(1) = \{1,2,3,4\} = c(2) = c(3) = c(4)$ State 1 : $d(i) = \gcd\left(n / p_{ii}^{(n)} > 0\right)$ $d(1) = \gcd\{1,2,3,4,\ldots\}$
$d(1) = 1$ $\therefore$ state 1 is aperiodic.

$$\begin{aligned}
\mu_{11} &= \sum_n n.f_{11}^n \\
&= 1.f_{11}^{(1)} + 2.f_{11}^{(2)} + 3f_{11}^{(3)} + \ldots \\
&= 1.\left(\frac{1}{2}\right) + 2.\left(\frac{1}{2}\right)\left(\frac{2}{5}\right) + 3\left(\frac{1}{2}\right)\left(\frac{3}{5}\right)\left(\frac{1}{3}\right) + 4.\left(\frac{1}{2}\right)\left(\frac{3}{5}\right)\left(\frac{1}{3}\right).\left(\frac{2}{5}\right) + \ldots \\
&= \left(\frac{1}{2}\right)\left[1 + \frac{4}{5} + \frac{3}{5} + \frac{2}{5} + \ldots\right] \\
&= \frac{1}{10}[5+4+3+2+\ldots] < \infty
\end{aligned}$$

State 1 is recurrent (persistent) i.e positive recurrent. Hence , state 1 is ergodic.

State -2 : Using, "Periodicity of Recurrence is class property $\therefore$ state 2 is also aperiodic, positive recurrunt and hence ergodic.
Similarly, state 3&4 are also ergodic. ■

Example 45: An organization has N employee's where N is a large no. Each employee has one of three possible job classifications $\{a,b,c\}$ and changes classification independent according to MC with TPM $\begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0.2 & 0.6 & 0.2 \\ 0.1 & 0.4 & 0.5 \end{bmatrix}$
Over long period of time , what % of employee's is in each classification ?
Solution:

$$P = \begin{bmatrix} 0.7 & 0.2 & 0.1 \\ 0.2 & 0.6 & 0.2 \\ 0.1 & 0.4 & 0.5 \end{bmatrix}$$

Consider,

$$\pi = \pi.P$$

$$\begin{bmatrix} \pi_1 & \pi_2 & \pi_3 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} [\pi_1\pi_2\pi_3 \end{bmatrix} \begin{bmatrix} 7 & 2 & 1 \\ 2 & 6 & 2 \\ 1 & 4 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 10\pi_1 & 10\pi_2 & 10\pi_3 \end{bmatrix} = \begin{bmatrix} 7\pi_1 + 2\pi_2 + \pi_3 & 2\pi_1 + 6\pi_2 + 4\pi_3 & \pi_1 + 2\pi_2 + 5\pi_3 \end{bmatrix}$$

$$\begin{aligned}
&3\pi_1 - 2\pi_2 - \pi_3 = 0 \\
&4\pi_2 - 2\pi_1 - 4\pi_3 = 0 \\
&5\pi_3 - \pi_1 - 2\pi_2 = 0 \\
&\text{Also,} \quad \pi_1 + \Pi_2 + \pi_3 = 1
\end{aligned}$$

By simultaneous eq^n solving , we obtain values of $\pi_1 = \pi_2, \pi_3$ ■

Example 46: Calculate the prob. of extinction for branching processes with offspring $dist^n$ as,

1. $P_0 = \frac{1}{4}, P_1 = \frac{1}{2}, P_3 = \frac{1}{5}$
2. $P_0 = \frac{2}{3} \quad \& P_2 = \frac{1}{3}$

Solution:

$$P_0 = \frac{1}{4}, P_1 = \frac{1}{2}, P_3 = \frac{1}{4}$$

$$\begin{aligned}
\mu &= \sum_{j=0}^{3} j.P_j \\
&= 0.P_0 + 1.P_1 + 3.P_3 \\
&= 0 + 1\left(\frac{1}{2}\right) + 3\left(\frac{1}{4}\right) \\
&= \frac{5}{4} > 1
\end{aligned}$$

$\therefore$ Prob of exitinction is,

$$\begin{aligned}
\pi_0 &= \sum_j \pi_0^j.P(z = j) \\
&= \pi_0^0.P_0 + \pi_0^1.P_1 + \pi_0^3.P_3 \\
\pi_0 &= \frac{1}{4} + \pi_0.\frac{1}{2} + \pi_0^3\frac{1}{4} \\
4\pi_0 &= 1 + 2\pi_0 + \pi_0^3 \\
&= \pi_0^3 - 2\pi_0 + 1 = 0
\end{aligned}$$

$$\therefore (\pi_0 - 1)\left(\pi_0^2 + \pi_0 - 1\right) = 0$$

$$\therefore \pi_0 = 1 \text{ OR } \pi_0 = \frac{-1 \pm \sqrt{1+4}}{2}$$

$$\pi_0 = \frac{-1 \pm \sqrt{1+4}}{2}$$

$$\pi_0 = 1 \text{ OR } \pi_0 = \frac{-1 \pm \sqrt{5}}{2} = 0.618$$

ii) $P_0 = \frac{2}{3}, P_2 = \frac{1}{3}$

$$\begin{aligned}
\mu &= 0.\left(\frac{2}{3}\right) + 2\left(\frac{1}{3}\right) \\
&= \frac{2}{3} < 1 \\
\pi_0 &= 1
\end{aligned}$$

■

Example 47: Let $\{X_n, n > 0\}$ & be a MC with initial $dist^n$

$$P(x_0 = 0) = \frac{1}{2}, \quad P(x_0 = 1) = \frac{1}{2}, \quad P(x_0 = 2) = 0 \text{ \& TPM },$$

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 1 \end{bmatrix}$$

Obtain ,

1. $P(X_2 = 1)$
2. $P(X_4 = 2 | X_2 = 0)$
3. $P(X_3 = 1, \quad X_2 = 1, \quad X_1 = 1)$

Solution:
$\alpha = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$

1. To find $P(X_2 = 1)$
   Consider, $P(X_2 = 1)$

$$\alpha.P^2 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} \frac{5}{12} & \frac{5}{12} & \frac{1}{6} \\ \frac{5}{18} & \frac{5}{18} & \frac{4}{9} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.3472 & 0.3472 & 0.3055 \end{bmatrix} \therefore P(X_2 = 1) = 0.3472$$

2. $P(X_4 = 2 | X_2 = 0)$ i.e $P_{02}^{(2)}$
   $\therefore$ we first find P^2

$$P^2 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{5}{12} & \frac{5}{12} & \frac{1}{6} \\ \frac{15}{54} & \frac{15}{54} & \frac{4}{9} \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore P(X_4 = 2 | X_2 = 0) = P_{02}^{(2)} = \frac{1}{6}$$

3. $P(X_3 = 1, \quad X_2 = 1, \quad X_1 = 1) = P(X_3 = 1 | \quad X_2 = 1).P(X_2 = 1 | X_1 = 1).P(X_1 = 1)$

$$= P_{11}^{(1)}.P_{11}^{(1)}.P(X_1 = 1)$$

$$= \frac{1}{3}.\frac{1}{3}.P(X_1 = 1)$$

$$= \frac{1}{9}.P(X_1 = 1)$$

$$\alpha.P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} \frac{5}{12} & \frac{5}{12} & \frac{1}{6} \end{bmatrix}$$
$$\therefore P(X_1 = 1) = \frac{5}{12}$$
$$\therefore P(X_1 = 1, X_2 = 1, X_1 = 1) = \frac{1}{9}.\frac{5}{12} = \frac{5}{108}$$

■

Example 48: A job consist of 3 machines and 2 repairmen. The amount of time a machine works before breaking down is exponentially distributed with mean 8 then,

1. What is the avg. no.of machines not in use ?
2. What proportion of time are both repairemen busy ?

Solution:
This is $M|M|2$, $\lambda = 8,\quad \mu = 10$

$$\lambda_j = \begin{cases} (3-j)\lambda & ;\forall j = 0,1,2,3 \\ 0 & ;\forall j > 3 \end{cases} \qquad \mu_j = \begin{cases} j\mu & ;\forall j \le 2 \\ 2\mu & ;\forall j > 2 \end{cases}$$

Hence , $P_j = \begin{cases} \left(\frac{\lambda}{\mu}\right)^j \cdot \frac{3!}{j!(3-j)!}.P_0 & ; j \le 2 \\ \left(\frac{\lambda}{\mu}\right)^j \cdot \frac{3!}{2!2^{j-2}(3-j)!}.P_0 & ; j > 2 \end{cases}$

Where ,

$$P_0 = \left[1 + \sum_{j=1}^{2} \left(\frac{\lambda}{\mu}\right)^j 3c_j + \left(\frac{\lambda}{\mu}\right)^3 \frac{3!}{2!(2)0!}\right]^{-1}$$

$$P_0 = \left[1 + \left(\frac{\lambda}{\mu}\right)(3) + \left(\frac{\lambda}{\mu}\right)^2 (3) + \left(\frac{\lambda}{\mu}\right)^3 \frac{3}{2}\right]^{-1}$$
$$= \left[1 + \frac{12}{5} + \frac{48}{25} + \frac{192}{250}\right]^{-1}$$
$$= \left[\frac{250 + 600 + 480 + 192}{250}\right]^{-1}$$
$$= 0.16425$$

Avg. no. of machines not in use

$$
\begin{aligned}
&= \sum_{j=1}^{3} j.p_j \\
&= \Sigma^2 j \cdot p_j + 3 \cdot p_3 \\
&= 1 \cdot p_1 + 2 \cdot p_2 + 3p_3 \\
&= \left(\frac{4}{5}\right)\frac{31}{1!2!} + 2 \cdot \left(\frac{4}{5}\right)^2 \frac{3!}{2!1!} + 3 \cdot \left(\frac{4}{5}\right)^3 \frac{3}{2}\Bigg] \cdot p_0 \\
&= \left(\frac{4}{5} \times 3 + 6 \times \frac{16}{25} + \frac{64 \times 9}{250}\right] p_0 \\
&= 1.483 \\
&= \frac{1.5}{p(j \geqslant 2)} = \left[\left(\frac{4}{5}\right)^2 \times 3 + \left(\frac{4}{5}\right)^3 \frac{3}{2}\right] p_0 \\
&= 0.441504 \frac{3}{2}\Bigg) (0.1645)
\end{aligned}
$$

■