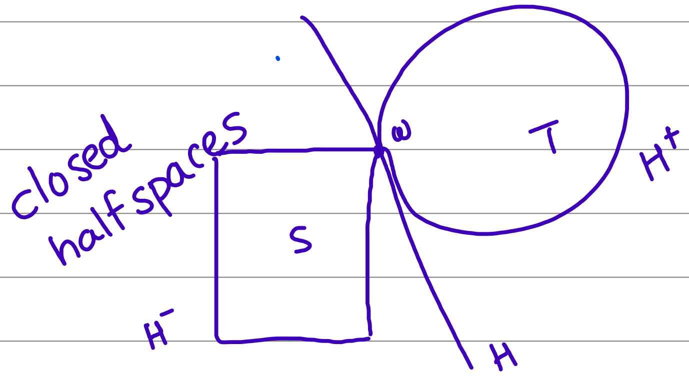
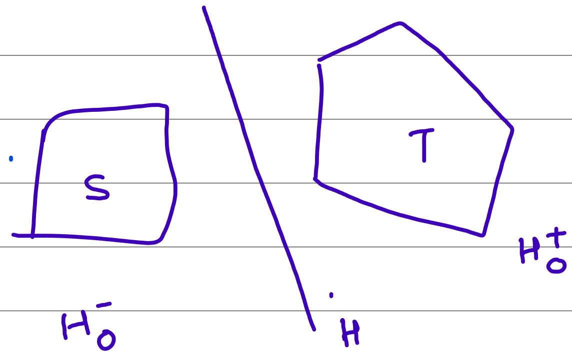
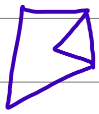
Separating hyperplane





open
half spaces



convex set

S convex

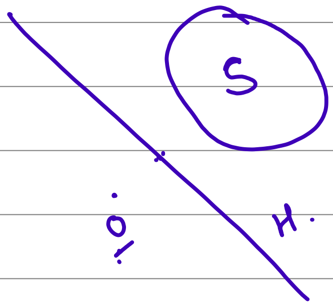
$\underline{0} \notin S$

$\tilde{S} = S$

$\underline{0} \notin \tilde{S}$

exterior pt.

$$\left[\, \underline{0} = \underline{a}'\underline{0} < \inf_{\underline{x} \in S} \underline{a}'\underline{x} \,\right] \Rightarrow \underline{a}'\underline{x} > 0 \quad \forall\ \underline{x} \in S \quad \text{—(1)}$$

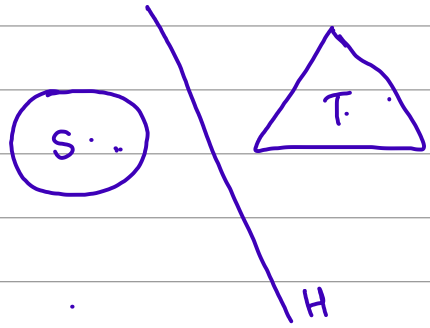


$$\text{if} \quad H = \{\underline{x} \mid \underline{a}'\underline{x} = c\} \quad \text{—(2)}$$

$$0 < c < \inf_{x \in S} \underline{a}'\underline{x}$$

$S \in H_0^+$

$\underline{0} \in H$

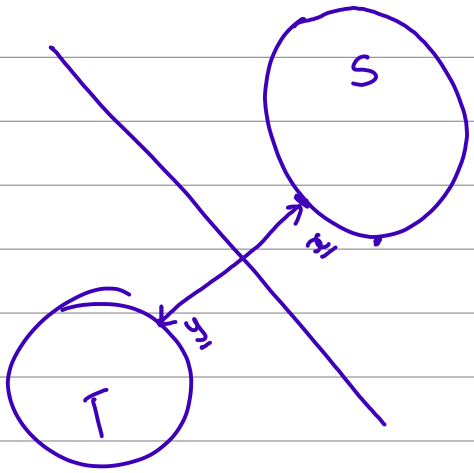


$\underline{0} \notin S-T$

$S-T = \{ \underline{z} \,/\, \underline{z} = \underline{x} - \underline{y},\ \underline{x} \in S,\ \underline{y} \in T \}$

$\underline{a}'\underline{z} > 0 \qquad \forall\ \underline{z} \in S-T$

$\underline{a}'\underline{x} - \underline{a}'\underline{y} > 0$



$\inf_{z \in S-T} \underline{a}'\underline{z} > 0$

$\inf_{\substack{x \in S \\ y \in T}} \underline{a}'\underline{x} - \underline{a}'\underline{y} > 0$

$\inf_{\underline{x} \in S} \underline{a}'\underline{x} > \sup_{y \in T} \underline{a}'\underline{y}.$

S, T

$0 \notin S-T$

$0 = \underline{a}'\underline{0} < \inf_{z \in S-T} \underline{a}'\underline{z}$

$\inf_{z \in S-T} a'z > 0$

$\inf_{x \in S} a'\underline{x} - \sup_{y \in T} a'y > 0$

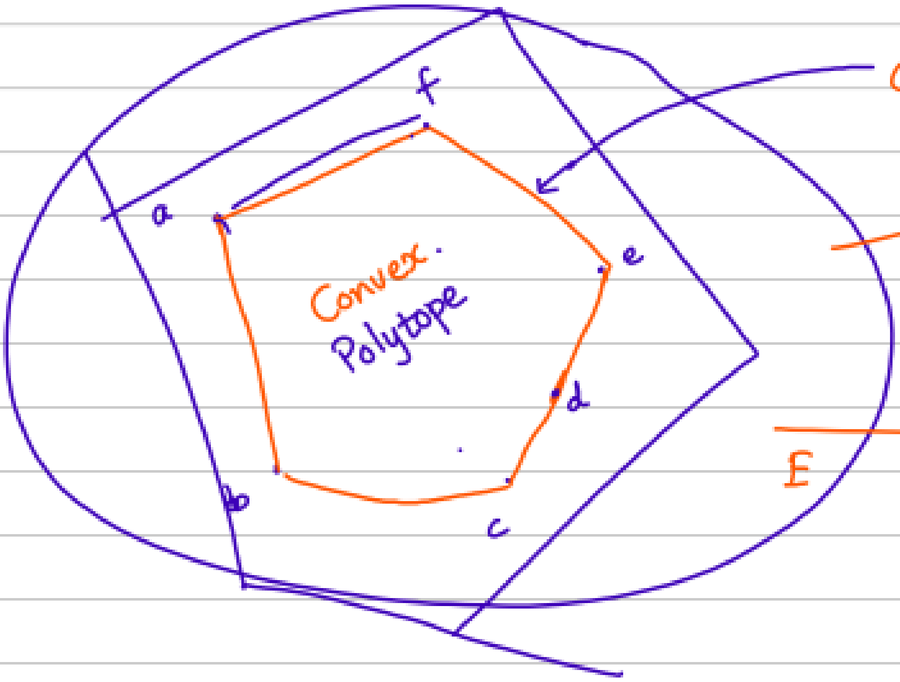
$\inf a'x > c > \sup a'y.$

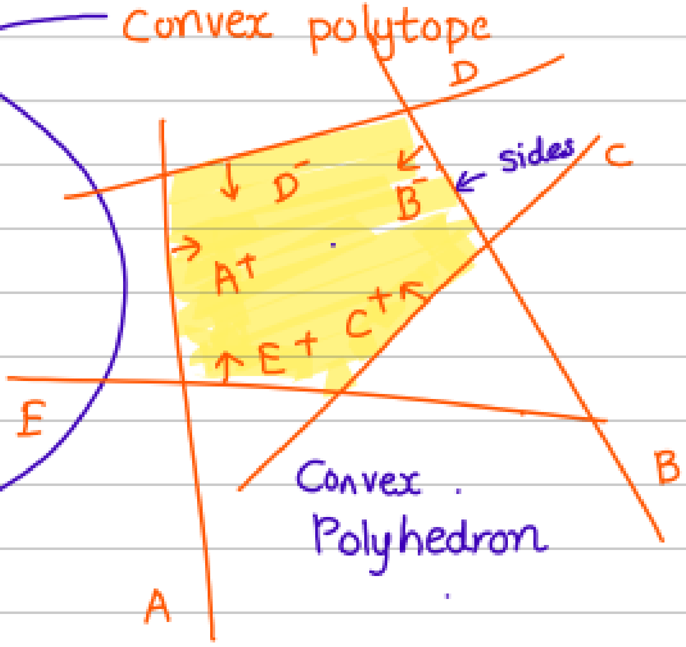
$\inf_{\underline{x} \in S} \underline{a}'\underline{x} > c > \sup_{\underline{y} \in T} \underline{a}'\underline{y}$

$H = \{ \underline{x} \,/\, \underline{a}'\underline{x} = c \}$ ✓

$S \in H_0^+ \qquad T \in H_0^-$

strictly separating.





Convex Polyhedron

$S = \{a, b, c, d, e, f\}$ $\quad$ $Co(S)$ = Convex hull / Polytope

$Co(S) = \{ z \mid z = \sum \alpha_i x_i, \ x_i \in S \}$ $\quad$ Vertices$(Co(S)) = \{a, b, c, e, f\} = V$

x

$u \in V \Rightarrow u \notin S$

$S = \{x_1, x_2, \ldots, x_m\}$ $\quad$ $Co(S) = \{ x \mid x = \sum_{i=1}^{m} \alpha_i x_i \}$

V = Vertices$(Co(S))$

$u \in V$ but $u \notin S$

$u \in Co(S)$

$\Rightarrow u = \sum_{i=1}^{m} \alpha_i x_i$ $\quad$ $\alpha_i \in (0,1)$ $\quad$ $\sum \alpha_i = 1$ $\quad$ $\alpha_i < 1$

x^*

$$u = \alpha_1 x_1 + \sum_{i=2}^{m} \alpha_i x_i = \alpha_1 x_1 + (1-\alpha_1) \boxed{\sum_{i=2}^{m} \frac{\alpha_i x_i}{(1-\alpha_1)}} = \alpha_1 x_1 + (1-\alpha_1) x^*$$

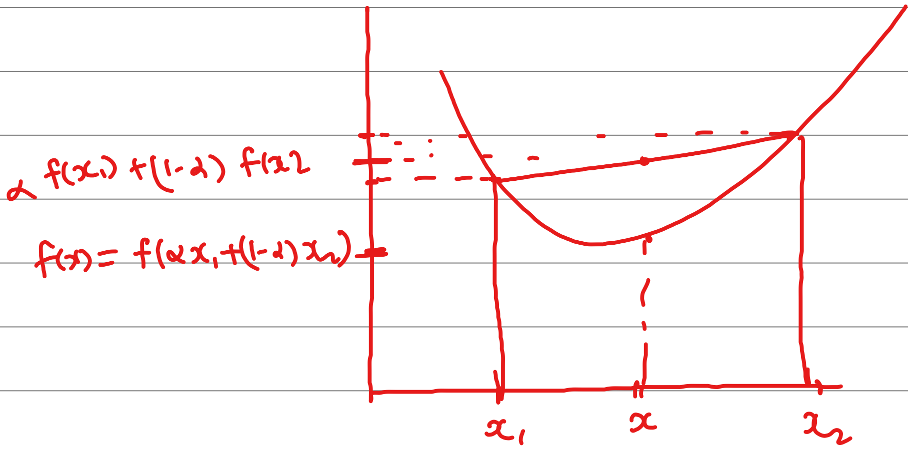
$A\underline{x} \leq \underline{b}$ $\quad a_1'\underline{x} = a_{11}x_1 + a_{12}x_2 \leq b_1$ $\quad H \quad H^+ \quad H^-$

$a_2'\underline{x} = a_{21}x_1 + a_{22}x_2 \leq b_2$ $\quad H \quad H^+ \quad H^-$

$\underline{x} \geq 0$ $\quad H^+$

$H = \{ \underline{x} \,/\, \underline{a}_1'\underline{x} = b_1 \}$

$\{ x : / Ax \geq b \}$ $\quad$ Polyhedron / Polyhedral



convex $\quad$ e.g x^2 $\qquad$ concave $\quad$ eg $-x^2$

$f(\underline{x}) = \underline{c}'\underline{x} + \underline{d}$

$\underline{x}_1, \underline{x}_2 \in S$

$f(\alpha \underline{x}_1 + (1-\alpha)\underline{x}_2) = \underline{c}'(\alpha \underline{x}_1 + (1-\alpha)\cdot \underline{x}_2) + \underline{d}$

$= \alpha \underline{c}'\underline{x}_1 + (1-\alpha)\underline{c}'\underline{x}_2 + \alpha \underline{d} + (1-\alpha)\underline{d}$

$= \alpha(\underline{c}'\underline{x}_1 + \underline{d}) + (1-\alpha)(\underline{c}'\underline{x}_2 + \underline{d})$

$= \alpha \cdot f(\underline{x}_1) + (1-\alpha) f(\underline{x}_2)$

$\leq \quad \Rightarrow$ convex

$\geq \quad \Rightarrow$ concave

f convex fun^c on T, $\Rightarrow$ $\underline{x}_1, \underline{x}_2 \in T$.

$f(\alpha \underline{x}_1 + (1-\alpha)\underline{x}_2) \leq \alpha f(x_1) + (1-\alpha) f(x_2)$

$T_K = \{\underline{x} \,/\, \underline{x} \in T,\ f(\underline{x}) \leq K\}$

let $\underline{x}_1, \underline{x}_2 \in T_K$

$\Rightarrow$ $\underline{x}_1, \underline{x}_2 \in T$ and $f(\underline{x}_1) \leq K$, $f(x_2) \leq K$

$\Rightarrow$ as T is convex, & $\underline{x}_1, x_2 \in T$ $\Rightarrow$ $\underline{x} = \alpha \underline{x}_1 + (1-\alpha) x_2 \in T$

$f(x) = f(\alpha x_1 + (1-\alpha) x_2)$

$\leq \alpha \underbrace{f(x_1)}_{\leq K} + (1-\alpha) \underbrace{f(x_2)}_{K}$ (as f is convex) $\alpha \in (0,1)$

$\leq K$

$\underline{x} \in T$ & $f(\underline{x}) \leq K$ $\Rightarrow$ $\underline{x} \in T_K$

$\Rightarrow$ T_K is convex set.

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f, g are convex functions

$\Big[$ f is convex fun^c $\Rightarrow$ $\underline{x}_1, \underline{x}_2 \in S$

$f(\alpha \underline{x}_1 + (1-\alpha)\underline{x}_2) \leq \alpha f(\underline{x}_1) + (1-\alpha) f(\underline{x}_2)$

$\mu \geq 0$

$(\mu \cdot f)(\underline{x}) = \mu \cdot f(\underline{x})$

$$\begin{aligned}
(\mu f)(\alpha \underline{x}_1 + (1-\alpha)\underline{x}_2) &= \mu \cdot f(\alpha \underline{x}_1 + (1-\alpha)\underline{x}_2) \\
&\leq \mu \cdot [\alpha f(\underline{x}_1) + (1-\alpha) f(\underline{x}_2)] \\
&\leq \alpha \cdot \mu \cdot f(x_1) + (1-\alpha)\, \mu \cdot f(x_2) \\
&\leq \alpha \cdot (\mu f)(x_1) + (1-\alpha)(\mu f)(x_2)
\end{aligned}$$

μf is also convex.

$(f+g)(\underline{x}) = f(\underline{x}) + g(\underline{x})$

$$\begin{aligned}
(f+g)(\alpha \underline{x}_1 + (1-\alpha)\underline{x}_2) &= f(\alpha \underline{x}_1 + (1-\alpha)\underline{x}_2) \\
&\quad + g(\alpha \underline{x}_1 + (1-\alpha)\underline{x}_2) \\
&\leq \alpha f(x_1) + (1-\alpha) f(x_2) \\
&\quad + \alpha g(x_1) + (1-\alpha) g(x_2) \\
&\leq \alpha (f(x_1) + g(x_1)) + \\
&\quad (1-\alpha)(f(x_2) + g(x_2)) \\
&\leq \alpha \cdot (f+g)(\underline{x}_1) + (1-\alpha)(f+g)(\underline{x}_2)
\end{aligned}$$

f & g conv