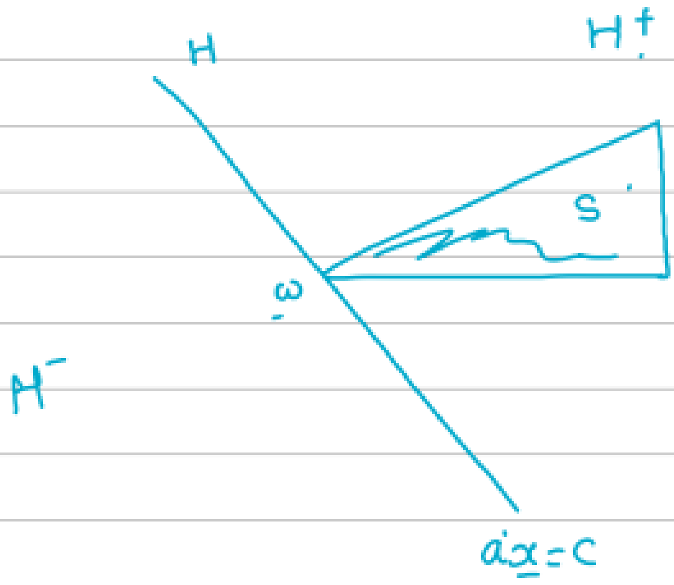


$S \in H$

$\forall\, \underline{x} \in S$, $\underline{a}'\underline{x} \geq c$

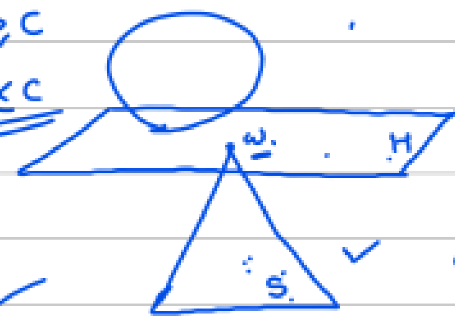
ω $\quad a'\omega = c$

$\omega \in S \cap H$

$\min_{\underline{x} \in S} \underline{a}'\underline{x} \Rightarrow \underline{a}'\underline{\omega}$

④

① Supporting hyperplane :- $-a'x \geq c$, $-a'x \leq c$



$\omega \in S \cap H$

$\underline{a}'\underline{x} = c$

$S \in H^-$

$\underline{x} \in S$

$\underline{a}'\underline{x} \leq c$

$H : \{ \underline{x} \,/\, \underline{a}'\underline{x} = c \}$ $\quad \boxed{a'\omega = c}$

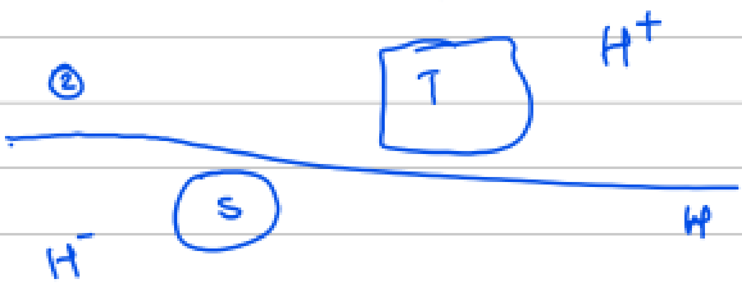
$\omega \in S \leftarrow$ boundary S.

$S \Rightarrow \underline{a}'\underline{x} \leq c$

$\underline{a}'\underline{\omega} = c$

Separating hyperplane

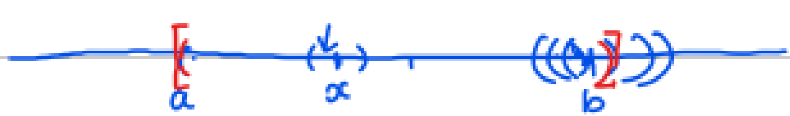
②

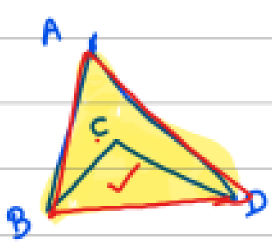


$S \in H^-$ & $T \in H^+$

$S' = (a,b) \cup \{a\} \cup \{b\} = [a,b]$

Closure. $S = (a,b)$





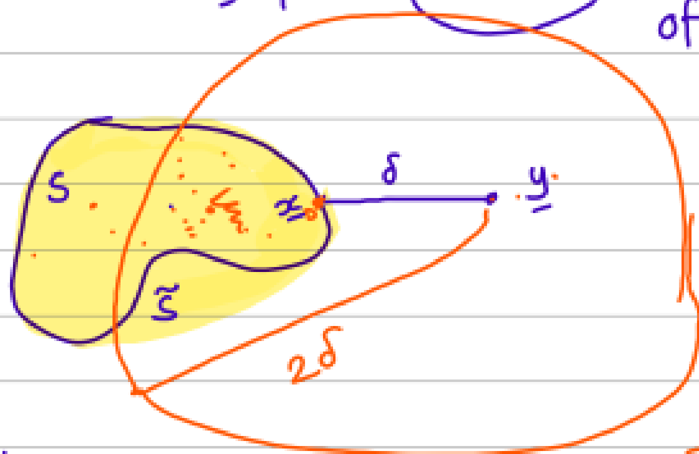
$|x - c| < \varepsilon$

$c - \varepsilon,\ c + \varepsilon$

$S \subseteq \mathbb{R}^n$ convex, y ~~boundary point~~ exterior to closure of S

$\underline{a} \neq \underline{0} \in \mathbb{R}^n$

$\Rightarrow \exists\ \underline{a} \in \mathbb{R}^n$, $\underline{a}'\underline{y} < \inf\limits_{x \in S} \underline{a}'x$



$\tilde{S}$ closure

let $\delta = \inf\limits_{x \in \tilde{S}} \|\underline{x} - \underline{y}\|$ $\qquad$ $\delta > 0$ as y is exterior to $\tilde{S}$

$y \notin \tilde{S}$

$$B_{2\delta} = \{ \underline{x} \mid \|\underline{x} - \underline{y}\| < 2\delta \}$$

$$\delta = \inf_{x \in \tilde{S}} \|\underline{x} - \underline{y}\| = \inf_{x \in \tilde{S} \cap B_{2\delta}} \|\underline{x} - \underline{y}\|$$

$\tilde{S} \cap B_{2\delta}$ is closed & bounded.

lets define $f : \tilde{S} \cap B_{2\delta} \to \mathbb{R}$, $f(\underline{x}) = \|\underline{x} - \underline{y}\|$

f contin____,

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by max^m min^m theo., f attains its extremum in that set

$\exists$ some $\underline{x}_0 \in \tilde{S} \cap B_{2\delta}$ $\ni$

$$\delta = \min_{\underline{x} \in \tilde{S} \cap B_{2\delta}} \|\underline{x} - \underline{y}\| = \|\underline{x}_0 - \underline{y}\|$$

$\Rightarrow \underline{x}_0$ is boundary pt. of $\tilde{S}$

$\underline{a}'\underline{y} \le \inf_{\underline{x}} \underline{a}'\underline{x}$

To show $\underline{a} = \underline{x}_0 - \underline{y}$

$\underline{x}, \underline{x}_0 \in \tilde{S}$, $\alpha \underline{x} + (1-\alpha)\underline{x}_0 \in \tilde{S}$

$$\|\alpha \underline{x} + (1-\alpha)\underline{x}_0 - \underline{y}\| \geq \|\underline{x}_0 - \underline{y}\|$$

$$\|(\underline{x}_0 - \underline{y}) + \alpha(\underline{x} - \underline{x}_0)\| \geq \|\underline{x}_0 - \underline{y}\|$$

$\|\underline{x}\| = \sqrt{\underline{x}'\underline{x}}$

$$\|(\underline{x}_0 - \underline{y}) + \alpha(\underline{x} - \underline{x}_0)\|^2 \geq \|\underline{x}_0 - \underline{y}\|$$

$$\Rightarrow \left((\underline{x}_0 - \underline{y}) + \alpha(\underline{x} - \underline{x}_0)\right)^T \left((\underline{x}_0 - \underline{y}) + \alpha(\underline{x} - \underline{x}_0)\right) \geq (\underline{x}_0 - \underline{y})^T(\underline{x}_0 - \underline{y})$$

$$\Rightarrow \left[(\underline{x}_0 - \underline{y})' + \alpha(\underline{x} - \underline{x}_0)'\right]\left[(\underline{x}_0 - \underline{y}) + \alpha(\underline{x} - \underline{x}_0)\right] \geq (\underline{x}_0 - \underline{y})'(\underline{x}_0 - \underline{y})$$

$$\Rightarrow \underline{(\underline{x}_0 - \underline{y})'(\underline{x}_0 - \underline{y})} + \alpha(\underline{x}_0 - \underline{y})'(\underline{x} - \underline{x}_0) + \alpha(\underline{x} - \underline{x}_0)'(\underline{x}_0 - \underline{y}) + \alpha^2(\underline{x} - \underline{x}_0)'(\underline{x} - \underline{x}_0)$$

$$\geq \underline{(\underline{x}_0 - \underline{y})'(\underline{x}_0 - \underline{y})}$$

$$2\alpha(\underline{x}_0 - \underline{y})'(\underline{x} - \underline{x}_0) + \alpha^2 |x - x_0|^2 \geq 0$$

$$\underline{2(\underline{x}_0 - \underline{y})'(x - x_0)} + \alpha\, \underline{|x - x_0|^2} \geq 0$$

let $\alpha \to 0$

$$(\underline{x}_0 - \underline{y})'(\underline{x} - \underline{x}_0) \geq 0$$

$$\Rightarrow (\underline{x}_0 - \underline{y})'\underline{x} \geq (\underline{x}_0 - \underline{y})'\underline{x}_0$$

$$\Rightarrow \underline{a}'\underline{x} \geq \underline{a}'(\underline{x}_0 - \underline{y} + \underline{y})$$

let $a = \underline{x}_0 - \underline{y}$

$$\geq a'(x_0 - y) + a'y$$

$\|z\| = \sqrt{z'z}$

$$a'x \geq (x_0-y)'(x_0-y) + a'y \qquad \forall x \in S$$

$$\geq \delta^2 + a'y \qquad \delta > 0$$

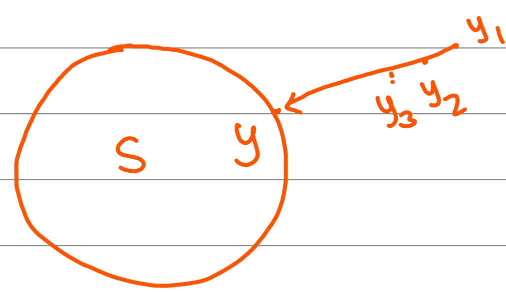
$$\inf_{x\in S} a'x \geq a'y$$

$$a'y < \inf_{x\in S} a'x$$

Theo. S convex, y boundary of S

To show $\exists\ H, \ni S \in H^+/H^-$, $y \in S\cap H$

→ [let y_n be seq^n of points exterior to closure of S
Assume $y_n \to y$]



$$\exists\ a_n \in \mathbb{R}^n, \qquad a_n'y_n \leq \inf_{x\in S} a_n'x \qquad \|a_n\| = 1$$

$$a_n'y_n - a_n'y + a_n'y \leq a_n'x \qquad \forall\ x \in S$$

for large n, $y_n \to y \Rightarrow a_n'y_n \to a'y$

$a'y \nless a'x$

$$a_n' y < a_n' x$$

$\|a_n\| \ni$ seq^n of a_n bounded.

Bolzano Weierstrauss Theo. for seq^ns.

$\exists$ convergent subseq^n $\{a_{n_k}\}$

Suppose it converges to $a_{n_k} \to a$

$$a_{n_k}' y < a_{n_k}' x$$

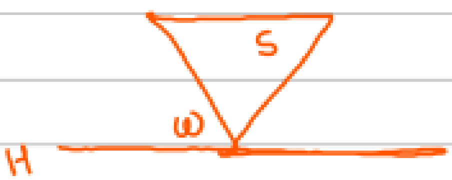
letting $k \to \infty$, $a_{n_k} \to a$

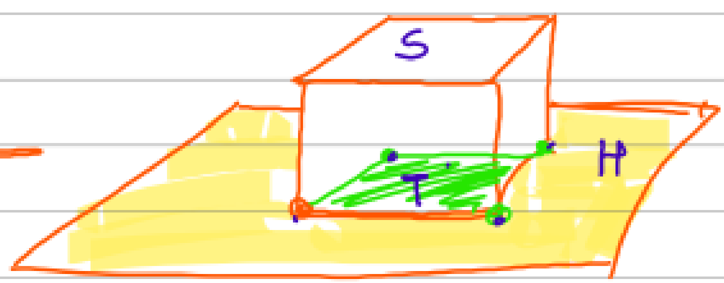
$$a'y = a_{n_k}' y < a_{n_k}' x = a'x$$

$\ni$ $a'y \le a'x$ $\forall x \in S$

$H = \{x \,/\, a'x = a'y\}$ is supporting hyperplane. at y

$$T = S \cap H = \{\omega\}$$





H Supportive Hyperplane , $T = S \cap H$
by method of contradiction

let $\underline{x}_0$ be extreme pt. of T but not of S.

let $\exists$ some $x_1, x_2 \in S \ni \underline{x}_0 = \alpha \underline{x}_1 + (1-\alpha)\underline{x}_2$

$H = \{ \underline{x} \mid \underline{a}'\underline{x} = c \}$ supportive hyperplane of S

let $S \in H^+$, $\underline{a}'\underline{x} \geq c \;\; \forall \; \underline{x} \in S$.

$\Rightarrow \underline{a}'\underline{x}_1 \geq c \; , \underline{a}'\underline{x}_2 \geq c$

$\underline{x}_0 \in T = S \cap H \qquad \Rightarrow a'\underline{x}_0 = c$

$\Rightarrow \underline{a}'(\alpha x_1 + (1-\alpha)\underline{x}_2) = c$

$\Rightarrow \alpha \underline{a}'\underline{x}_1 + (1-\alpha)\underline{a}'\underline{x}_2 = c$

$\alpha \in (0,1) , (1-\alpha) \in (0,1)$

$\Rightarrow \alpha . \underbrace{\quad\quad}_{\geq c} + (1-\alpha)\underbrace{(\quad\quad)}_{\geq c} = c$

it is only possible if $\underline{a}'\underline{x}_1 = c$ & $\underline{a}'\underline{x}_2 = c$

$\Rightarrow \underline{x}_1, \underline{x}_2 \in H$ & $\Rightarrow x_1, x_2 \in S \cap H$

$\Rightarrow \underline{x_1, x_2 \in T}$

$\therefore$ Which contradicts to our assumption that x_0 is extreme pt. of T.