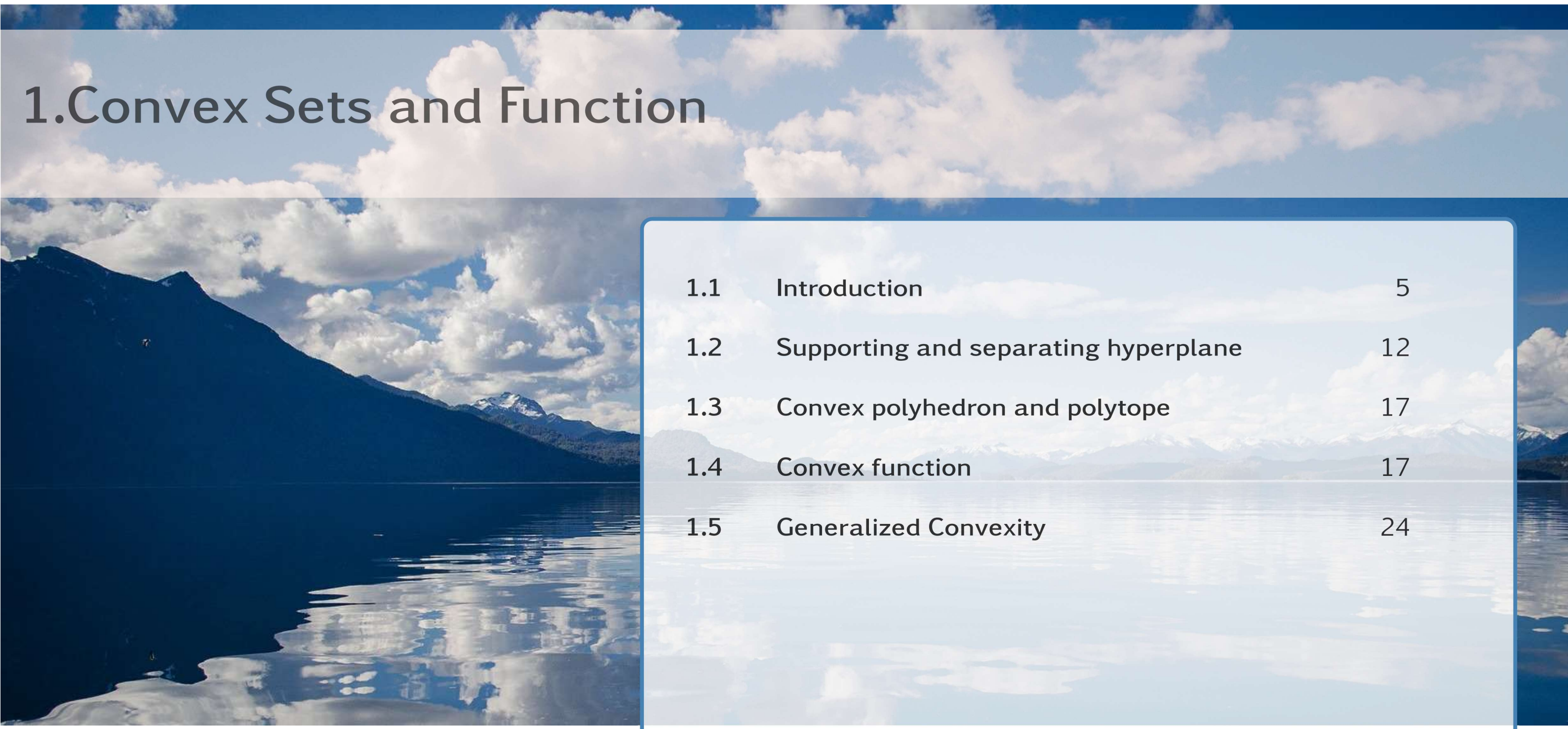

1.Convex Sets and Function

In this chapter, we have discussed the concept of a convex set and convex functions. This will help us to understand the important aspects of optimization techniques. These are useful for studying the optimum of function over the convex region, duality theory etc. Probably, we have to study the existence of solution of linear programming problems over the convex region. So it needs to study the properties of convex sets and convex functions.

1.1 Introduction

Definition 1.1 The line segment joining the points $\underline{x}_1, \underline{x}_2 \in \mathbb{R}^n$ is the set of points (vectors) given by

$$\left\{\underline{x} : \underline{x} = \lambda \underline{x}_1 + (1-\lambda)\,\underline{x}_2, 0 \le \lambda \le 1\right\}.$$

Definition 1.2 The line joining the points $\underline{x}_1$ and $\underline{x}_2$ is the set

$$\left\{\underline{x} : \underline{x} = \lambda \underline{x}_1 + (1-\lambda)\,\underline{x}_2, \lambda \in \mathbb{R}\right\}.$$

Definition 1.3 A vector $\underline{x} \in \mathbb{R}^n$ is called a linear combination of vector $\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_m$ in $\mathbb{R}^n$ if there exists numbers $\lambda_i, i = 1, 2, \cdots, m$ such that $\underline{x} = \sum_{i=1}^{m} \lambda_i \underline{x}_i$.

Definition 1.4 A vector $\underline{x} \in \mathbb{R}^n$ is called a convex combination of vector $\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_m$ if there exists number λ_i satisfying $\lambda_i \geq 0$ $(i = 1, 2. \cdots, m)$, $\sum_{i=1}^{m} \lambda_i = 1$ such that $\underline{x} = \sum_{i=1}^{m} \lambda_i \underline{x}_i$.

Definition 1.5 The set of all convex combination of vectors $\underline{x}_1, \underline{x}_2, \cdots, \underline{x}_m$ in $\mathbb{R}^n$ is the set of points

$$\left\{ \underline{x} : \underline{x} = \sum_{i=1}^{m} \lambda_i \underline{x}_i, \;\; \lambda_i \geq 0, \; i = 1, 2, \cdots, m \; \text{and} \sum_{i=1}^{m} \lambda_i = 1 \right\}.$$

Definition 1.6 A set $S \subseteq \mathbb{R}^n$ is called a convex set if $\underline{x}_1, \underline{x}_2 \in S$ then $\lambda \underline{x}_1 + (1 - \lambda) \underline{x}_2 \in S$ for all $0 \leq \lambda \leq 1$.

Remarks:

1. Empty set and singleton set are trivially convex sets.
2. A set S is convex if the line segment joining any two points of S lies in S.
3. Number of points in a convex set are zero, one or infinite.

x_0 $x_0 + \lambda d$

$\lambda \geq 0$

Definition 1.7 Given a point $\underline{x}_0 \in \mathbb{R}^n$ and a non zero vector $\underline{d} \in \mathbb{R}^n$ the set $\{\underline{x}_0 + \lambda \underline{d} : \lambda \geq 0\}$ is called a ray in $\mathbb{R}^n$.

The point $\underline{x}_0$ is called the vertex of the rays and the vector $\underline{d}$ is called the direction of the ray.

Definition 1.8 A set $S \subseteq \mathbb{R}^n$ is called a linear variety if $\underline{x}_1, \underline{x}_2 \in S$ then $\lambda \underline{x}_1 + (1 - \lambda) \underline{x}_2 \in S$ for all $\lambda \in R$.

Example 1.1

1. The straight line is a convex set.
2. The solid sphere is a convex set.
3. Triangle, square and rectangle are not convex sets.

4. Parabola, hyperbola, ellipse and circles are not convex sets.
5. Then intervals (0, 1), [0,1], (0,1],[0,1) are convex sets.
6. The set $\mathbb{R}^n$ is a convex set.
7. The set $\left\{(\underline{x}_1, \underline{x}_2) \in \mathbb{R}^2 : \underline{x}_1{}^2 + \underline{x}_2{}^2 \le 1\right\}$ it is a convex set.
8. The set $\mathbb{R}^2 - \{\underline{0}\}$ is not a convex set.

Definition 1.9 Let $c \in \mathbb{R}$, $\underline{a} \in \mathbb{R}^n$, $\underline{a} \neq \underline{0}$. Then the set $H = \{\underline{x} : \underline{a}^T \underline{x} = c\}$ is said to be a hyperplane in $\mathbb{R}^n$.

The nature of hyperplane in $\mathbb{R}$ is singleton set, in $\mathbb{R}^2$ it is a straight line and in $\mathbb{R}^3$ it is a plane.

Definition 1.10 Let $\underline{a} \in \mathbb{R}^n$. The sets $H_+ = \{\underline{x} \in \mathbb{R}^n | \underline{a}^T \underline{x} \ge c\}$, $H_- = \{\underline{x} \in \mathbb{R}^n | \underline{a}^T \underline{x} \le c\}$ are called the closed half spaces generated by the hyperplane H and the set H_+ is known as the positive closed half space. The set H_- is known as the negative closed half space. The sets $H_+^0 = \{\underline{x} : \underline{a}^T \underline{x} > c\}$ and $H_-^0 = \{\underline{x} : \underline{a}^T \underline{x} < c\}$ are called the positive and negative open half space generated by hyperplane H respectively. Clearly H_+, H_- are closed sets and H_+^0, H_-^0 are open sets.

Example 1.2 The hyperplane H is convex set.

Solution. Let $H = \{x : a^T x = c\}$ where $x, a \in \mathbb{R}^n$ and $c \in R$. Let $x, y \in H \Rightarrow a^T x = c$ and $a^T y = c$. To show that H is convex set. Let $0 \le \lambda \le 1$. Now

$$\begin{aligned} a^T(\lambda x + (1-\lambda) y) &= \lambda a^T x + (1-\lambda) a^T y \\ &= \lambda c + (1-\lambda) c \\ &= c. \end{aligned}$$

Hence, $\lambda x + (1-\lambda) y \in H$. Thus H is a convex set.

Example 1.3 The hyperplane H_+ and H_- are convex sets.

Solution. Let $H_+ = \{x : a^T x \ge c\}$ where $\underline{x}, \underline{a} \in \mathbb{R}^n$ and $c \in \mathbb{R}$. Let $\underline{x}, \underline{y} \in H_+ \Rightarrow \underline{a}^T \underline{x} \ge c$ and $\underline{a}^T \underline{y} \ge c$.

To show that H_+ is convex set. Let $0 \le \lambda \le 1$. Consider

$$\begin{aligned}\underline{a}^T\left(\lambda\underline{x} + (1-\lambda)\,\underline{y}\right) &= \lambda\underline{a}^T\underline{x} + (1-\lambda)\,\underline{a}^T\underline{y}\\ &\ge \lambda c + (1-\lambda)\,c\\ &= c.\end{aligned}$$

i.e. $\underline{a}^T\left(\lambda\underline{x} + (1-\lambda)\,\underline{y}\right) \ge c$. Hence, $\lambda\underline{x} + (1-\lambda)\,\underline{y} \in H_+$. Similarly we can show that H_- is convex set.

Example 1.4 The hyperplane H_+^0 and H_-^0 are convex sets.

Solution. Let $H_+^0 = \left\{\underline{x} : \underline{a}^T\underline{x} > c\right\}$, where $\underline{x}, \underline{a} \in \mathbb{R}^n$ and $c \in \mathbb{R}$. Let $\underline{x}, \underline{y} \in H_+ \Rightarrow \underline{a}^T\underline{x} > c$ and $\underline{a}^T\underline{y} > c$. To show that H_+^0 is convex set. Let $0 \le \lambda \le 1$. Consider

$$\begin{aligned}\underline{a}^T\left(\lambda\underline{x} + (1-\lambda)\,\underline{y}\right) &= \lambda\underline{a}^T\underline{x} + (1-\lambda)\,\underline{a}^T\underline{y}\\ &> \lambda c + (1-\lambda)\,c\\ &= c.\end{aligned}$$

i.e. $\underline{a}^T\left(\lambda\underline{x} + (1-\lambda)\,\underline{y}\right) > c$. Hence, $\lambda\underline{x} + (1-\lambda)\,\underline{y} \in H_+^0$. Similarly, we can show that H_-^0 is convex set.

Example 1.5 A ray in $\mathbb{R}^n$ is a convex set.

Solution. Let $\underline{x}_0 \in \mathbb{R}^n$, $\underline{d} \ne \underline{0} \in \mathbb{R}^n$ and $\lambda \ge 0$. Let $A = \left\{\underline{x} : \underline{x} = \underline{x}_0 + \lambda\underline{d},\ \ \lambda \ge 0\right\}$ is a ray in $\mathbb{R}^n$. Let $\underline{x}, y \in H$, then $\underline{x} = \underline{x}_0 + \lambda_1\underline{d}$, $\ y = \underline{x}_0 + \lambda_2\underline{d}$, for some $\lambda_1, \lambda_2 \ge 0$. Let $0 \le \mu \le 1$. Consider

$$\begin{aligned}\mu\underline{x} + (1-\mu)\,\underline{y} &= \mu\left(\underline{x}_0 + \lambda_1\underline{d}\right) + (1-\mu)\left(\underline{x}_0 + \lambda_2\underline{d}\right)\\ &= \mu\underline{x}_0 + \mu\lambda_1\underline{d} + \underline{x}_0 + \lambda_2\underline{d} - \mu\underline{x}_0 - \mu\lambda_2\underline{d}\\ &= \underline{x}_0 + \left(\mu\lambda_1 + \lambda_2 - \mu\lambda_2\right)\underline{d}\\ &= \underline{x}_0 + \lambda_3\underline{d}\end{aligned}$$

where $\lambda_3 = \mu\lambda_1 + (1-\mu)\lambda_2 \ge 0$ is real number, since $0 \le \lambda_1,\ \lambda_2 \le 1$ and $0 \le \mu \le 1$ are real numbers. Hence $\mu\underline{x} + (1-\mu)\,\underline{y} \in A$. Thus a ray in $\mathbb{R}^n$ is a convex set.

Example 1.6 Show that open ball $B\left(\underline{x}_0,\ r\right)$ is a convex set.