Remark. $Co(S)$ is a convex set. The convex hull $Co(S)$ is actually the smallest convex set in $\mathbb{R}^n$ containing S.

Theorem 1.6 The convex hull of the set S is the set of all convex combination of the point in S i.e.

$$Co(S) = \left\{ \underline{x} : \underline{x} = \sum_{i=1}^{k} \lambda_i \underline{x}_i, \underline{x}_i \in S, 0 \le \lambda_i \le 1, \sum_{i=1}^{k} \lambda_i = 1 \right\}.$$

1.2 Supporting and separating hyperplane

Definition 1.14 A hyperplane H containing a convex set $S \subseteq \mathbb{R}^n$ in one of its closed half spaced H_+ or H_- and a boundary point of S said to be supporting hyperplane of S, if the boundary point w of S lies in the supporting hyperplane H, then H is supporting hyperplane of S at w. Let w be a boundary point S, then $\underline{a}^T \underline{x} = c$ is a supporting hyperplane of S at w if $\underline{a}^T \underline{w} = c$ and either $\underline{a}^T \underline{x} \ge c$ for $\underline{x} \in S$ or $\underline{a}^T \underline{x} \le c$ for all $\underline{x} \in S$.

Theorem 1.7 If $S \subseteq \mathbb{R}^n$ be a convex set and y be a boundary point exterior to the closure of S, then there exist a vector $\underline{a} \neq 0 \in \mathbb{R}^n$ such that

$$\underline{a}^T \underline{y} < \inf_{\underline{x} \in S} \left(\underline{a}^T \underline{x}\right).$$

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$S \cup S' = \tilde{S}$

derived set
↳ collection of all limit pts. of S

Proof. Let $\tilde{S}$ be the closure of S. Define δ by

$$\delta = \inf_{\underline{x} \in S} ||\underline{x} - \underline{y}||.$$

Observing $\delta > 0$, since $\underline{y} \notin \tilde{S}$. Let $B_{2\delta}\left(\underline{y}\right) = \left\{\underline{x} : |\underline{x} - \underline{y}| < 2\delta\right\}$, then

$$\delta = \inf_{\underline{x} \in S} ||\underline{x} - \underline{y}|| = \inf_{\underline{x} \in \tilde{S} \cap B_{2\delta}} ||\underline{x} - \underline{y}||.$$

Define $f : \tilde{S} \cap B_{2\delta} \to R$ by $f\left(\underline{x}\right) = ||\underline{x} - \underline{y}||$. Then f is continuous function on closed set $\tilde{S} \cap B_{2\delta}$. Therefore f attains its minimum value on $\tilde{S} \cap B_{2\delta}$. Thus there exist a point $\underline{x}_0 \in \tilde{S} \cap B_{2\delta}$ such that

$$\delta = \min_{\underline{x} \in \tilde{S} \cap B_{2\delta}} ||\underline{x} - \underline{y}|| = ||\underline{x}_0 - \underline{y}||.$$

Clearly, $\underline{x}_0$ is a boundary point of $\tilde{S}$. We claim that $\underline{a} = \underline{x}_0 - \underline{y}$ will satisfies required condition. Let $\underline{x}, \underline{x}_0 \in \tilde{S}$. Then for $0 \le \lambda \le 1$, $\lambda \underline{x} + (1-\lambda)\,\underline{x}_0 \in \tilde{S}$ and therefore

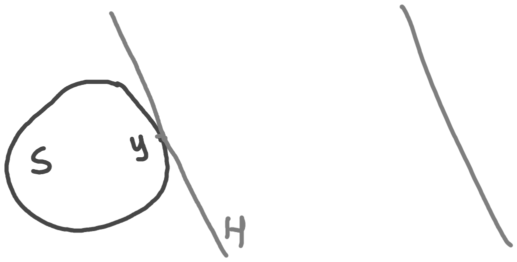
$$\begin{aligned}
|\lambda \underline{x} + (1-\lambda)\,\underline{x}_0 - \underline{y}| &\ge \left|\underline{x}_0 - \underline{y}\right| \\
\Rightarrow \left|\left(\underline{x}_0 - \underline{y}\right) + \lambda\,(x - \underline{x}_0)\right| &\ge \left|\underline{x}_0 - \underline{y}\right| \\
\Rightarrow \left|\left(\underline{x}_0 - \underline{y}\right) + \lambda\,(x - \underline{x}_0)\right|^2 &\ge \left|\underline{x}_0 - \underline{y}\right|^2
\end{aligned}$$

$$\begin{aligned}
&\Rightarrow \left[\left(\underline{x}_0 - \underline{y}\right) + \lambda\,(\underline{x} - \underline{x}_0)\right]^T \left[\left(\underline{x}_0 - \underline{y}\right) + \lambda\,(\underline{x} - \underline{x}_0)\right] \ge \left(\underline{x}_0 - \underline{y}\right)^T \left(\underline{x}_0 - \underline{y}\right) \\
&\Rightarrow \left[\left(\underline{x}_0 - \underline{y}\right)^T + \lambda(\underline{x} - \underline{x}_0)^T\right] \left[\left(\underline{x}_0 - \underline{y}\right) + \lambda\,(\underline{x} - \underline{x}_0)\right] \ge \left(\underline{x}_0 - \underline{y}\right)^T \left(\underline{x}_0 - \underline{y}\right) \\
&\Rightarrow \left(\underline{x}_0 - \underline{y}\right)\left(\underline{x}_0 - \underline{y}\right)^T + \lambda(\underline{x} - \underline{x}_0)^T\,(\underline{x} - \underline{x}_0) + \lambda(\underline{x} - \underline{x}_0)^T\left(\underline{x}_0 - \underline{y}\right) + \lambda^2\,(\underline{x} - \underline{x}_0)\left(\underline{x}_0 - \underline{y}\right) \\
&\qquad \ge \left(\underline{x}_0 - \underline{y}\right)^T \left(\underline{x}_0 - \underline{y}\right) \\
&\Rightarrow 2\lambda\left(\underline{x}_0 - \underline{y}\right)^T (\underline{x} - \underline{x}_0) + \lambda^2 |\underline{x} - \underline{x}_0|^2 \ge 0.
\end{aligned}$$

Letting $\lambda \to 0$, we have

$$\begin{aligned}
&(\underline{x}_0 - y)^T\,(\underline{x} - \underline{x}_0) \ge 0 \\
&\Rightarrow \left(\underline{x}_0 - \underline{y}\right)^T \underline{x} \ge \left(\underline{x}_0 - \underline{y}\right)^T \underline{x}_0 \\
&\Rightarrow \underline{a}^T \underline{x} \ge \left(\underline{x}_0 - \underline{y}\right)^T \left(\underline{x}_0 - \underline{y} + \underline{y}\right) \\
&\Rightarrow \underline{a}^T \underline{x} \ge \left(\underline{x}_0 - \underline{y}\right)^T \left(\underline{x}_0 - \underline{y}\right) + \left(\underline{x}_0 - \underline{y}\right)^T \underline{y} \\
&\Rightarrow \underline{a}^T \underline{x} \ge \left(\underline{x}_0 - \underline{y}\right)^T \underline{y} + \left|\underline{x}_0 - \underline{y}\right|^2 \\
&\Rightarrow \underline{a}^T \underline{x} \ge \left(\underline{x}_0 - \underline{y}\right)^T \underline{y} + \delta^2 \;\forall \underline{x} \in S \\
&\Rightarrow \underline{a}^T \underline{x} > \underline{a}^T \underline{y} \;\; \forall\; \underline{x} \in S,\; since\; \delta > 0 \\
&\Rightarrow \underline{a}^T \underline{y} < \inf_{\underline{x} \in S} \underline{a}^T \underline{x}.
\end{aligned}$$

Theorem 1.8 If $S \subseteq \mathbb{R}^n$ be a convex set and $\underline{y}$ be the boundary point of S, then there is a supporting hyperplane of S at $\underline{y}$.



$H = \{\underline{x} \,/\, \underline{a}'\underline{x} = c\}$

$y \in S \cap H$

$\underline{a}'\underline{y} = c$

$S \in H^+/H^-$

Proof. Let $S \subseteq \mathbb{R}^n$ and $\underline{y}$ be a boundary point of S. Let $\left\{\underline{y}_n\right\}$ be a sequence in exterior to $\tilde{S}$ converging to $\underline{y}$. Then by Theorem 1.7, there exists a sequence $\{\underline{a}_n\}$ of non zero vectors $\underline{a}_n$ such that $\underline{a}_n{}^T \underline{y}_n < \underline{a}_n{}^T \underline{x}$, $\forall\ \underline{x} \in S$. We can normalize $\{\underline{a}_n\}$ with $|\underline{a}_n| = 1$.

Thus $\underline{a}_n{}^T \underline{y} + \underline{a}_n{}^T \underline{y}_n - \underline{a}_n{}^T \underline{y} < \underline{a}_n{}^T \underline{x}$ $\forall x \in S$. But $\left\{\underline{y}_n\right\}$ converges to $\underline{y}$, so for large n, we have $\underline{a}_n{}^T \underline{y} < \underline{a}_n{}^T \underline{x}\ \forall\ x \in S$.

Since $\{\underline{a}_n\}$ is a bounded sequence, so there exists a sub sequence $\{\underline{a}_{nk}\}$ of $\{\underline{a}_n\}$ which is bounded such that $\underline{a}_{nk} \to a$ as $k \to \infty$. Now

$$\underline{a}^T \underline{y} = \lim_{k\in K} \underline{a}_n^T \underline{y} \leq \lim_{k\in K} \underline{a}_n^T \underline{x} = \underline{a}^T \underline{x}.$$

Thus, $\underline{a}^T \underline{y} \leq \underline{a}^T \underline{x}$ $\forall x \in S$. Hence the hyperplane $H = \left\{\underline{x} : \underline{a}^T \underline{x} = \underline{a}^T \underline{y}\right\}$ is supporting hyperplane to S at $\underline{y}$.

> Theorem 1.9 Let $S \subseteq \mathbb{R}^n$ be a convex set. H is a supporting hyperplane of S and $T = S \cap H$. Then every extreme point of T is an extreme point of S.

Proof. We will prove this by method of contradiction. Suppose $\underline{x}_0 \in T$ is an extreme point of T. If $\underline{x}_0$ is not an extreme point of S. Then there exists $\underline{x}_1, \underline{x}_2 \in S$ with $0 < \lambda < 1$ such that $\underline{x}_0 = \lambda \underline{x}_1 + (1-\lambda)\,\underline{x}_2$. Let $H = \{x : \underline{a}^T \underline{x} = c\}$ be a supporting hyperplane of S. We assume that S is contained in the negative closed half space H of S. Therefore, we have $\underline{a}^T \underline{x} \leq c$ for all $\underline{x} \in S$. Since, $\underline{x}_1, \underline{x}_2 \in S \Rightarrow \underline{a}^T \underline{x}_1 \leq c$ and $\underline{a}^T \underline{x}_2 \leq c$. Moreover $\underline{x}_0 \in T \Rightarrow \underline{x}_0 \in H \Rightarrow \underline{a}^T \underline{x}_0 = c$ and

$$\begin{aligned}
&\underline{a}^T \left(\lambda \underline{x}_1 + (1-\lambda)\,\underline{x}_2\right) = c \\
\Rightarrow\ & a^T \lambda \underline{x}_1 + \underline{a}^T (1-\lambda)\,\underline{x}_2 = c \\
\Rightarrow\ & \underline{a}^T \underline{x}_1 + \underline{a}^T \underline{x}_2 - \lambda \underline{a}^T \underline{x}_2 = c.
\end{aligned}$$

Since $0 < \lambda < 1$, we must have $\underline{a}^T \underline{x}_1 = c$ and $\underline{a}^T \underline{x}_2 = c$. It implies that $\underline{x}_1, \underline{x}_2 \in H \Rightarrow \underline{x}_1, \underline{x}_2 \in T$ and $\underline{x}_0 = \lambda \underline{x}_1 + (1-\lambda)\,\underline{x}_2$, $0 < \lambda < 1$. It shows that $\underline{x}_0$ is not extreme point of T. Clearly, it is a contradiction to our assumption. Hence, every extreme point of T is a extreme point of S.

> Theorem 1.10 Every closed bounded convex set in $\mathbb{R}^n$ in equal to the closed convex hull of the extreme point of S.

Proof. Suppose $S = \phi$, then there is no need to prove. So, we assume that $S \neq \phi$. We will give proof by induction on dimension on the spaces $\mathbb{R}^n$. For $n = 1$, S is a closed bounded interval