Branching Process
(Galton-Watson Branching process)

X_0

$\downarrow$

$X_1 = 2 \rightarrow Z_0$ - offsprings

A B

let $Z_0 = 2$

$X_2 = 2+3$ $Z_A = 2 + Z_B = 3$

C D E F G

$X_3 \Rightarrow Z_C + Z_D + Z_E + Z_F + Z_G$

Consider the pop^n consisting of individuals able to produce offspring of the same kind.

(2) Suppose that each individual have produced j offsprings with prob. P_j, $j \geq 0$, independent of each other.

$Z_i \sim$ i.i.d discrete

X_0 = size of $zero^{th}$ generation

X_n = size of n^{th} generation

$7 \Rightarrow Z_0$

let $X_n = 10$

$$X_{n+1} = \sum_{i=1}^{X_n} Z_i = \text{size of } (n+1)^{st} \text{ generation}$$

✓ $X_n = 3$

A B C

$\underline{\underline{X_{n+1}}}$ $Z_A + Z_B + Z_C$

$$X_{n+1} = \sum_{i=1}^{3 = X_n} Z_i \checkmark$$

$Z_0 = 0, 1, 2, \ldots$

$X_n = 0$

$X_{n+1} = ? \Rightarrow X_{n+1} = 0$

$$X_n = \sum_{i=1}^{X_{n-1}} Z_i$$

$$E(X_n) = E_{X_{n-1}}\left(E_Z\left(\sum_{i=1}^{X_{n-1}} Z_i / X_{n-1}=k\right)\right)$$

$$= E_{X_{n-1}}\left(\sum_{i=1}^{k} E(Z_i) / X_{n-1}=k\right)$$

$$= E_{X_{n-1}}\left(k\cdot E(Z) / X_{n-1}=k\right)$$

$$= \mu \cdot E_{X_{n-1}}(X_{n-1})$$

$\left\{ \text{as } Z_i \text{ are i.i.d}, \ E(Z)=\mu, \ V(Z)=\sigma^2 \right\}$

$$E(X_n) = \mu\cdot E(X_{n-1}) = \mu\cdot\mu\cdot E(X_{n-2})$$

$$E(X_{n-1}) = \mu\cdot E(X_{n-2})$$

$$E(X_n) = \mu\cdot\mu\cdot \ldots\ E(X_0) = \mu^n E(X_0)$$

Assume $E(X_0)=1$

$$\underline{E(X_n)=\mu^n}$$

$$V(X_n) = V\left(\sum_{i=1}^{X_{n-1}} Z_i\right)$$

$$= E_{X_{n-1}}\left(V_Z\left(\sum_{i=1}^{k} Z_i / X_{n-1}=k\right)\right) + V_{X_{n-1}}\left(E_Z\left(\sum_{i=1}^{k} Z_i / X_{n-1}=k\right)\right)$$

$$= E_{X_{n-1}}(k\cdot\sigma^2) + V_{X_{n-1}}(k\mu)$$

$$V(X_n) = \sigma^2\cdot E(X_{n-1}) + \mu^2 V(X_{n-1})$$

$X_0=1$ $$\underline{V(X_1)} = \sigma^2 \underbrace{E(X_0)}_{=1} + \mu^2\cdot \underbrace{V(X_0)}_{=0} = \sigma^2$$

$$V(X_2) = \sigma^2 \cdot E(X_1) + \mu^2 \cdot V(X_1)$$
$$= \sigma^2 \mu + \mu^2 \sigma^2$$
$$= \mu\sigma^2 (1 + \mu)$$

$$V(X_3) = \sigma^2 (E(X_2)) + \mu^2 \cdot V(X_2)$$
$$= \sigma^2 \mu^2 + \mu^2 \cdot \sigma^2 (\mu + \mu^2)$$
$$= \sigma^2 (\mu^2 + \mu^3 + \mu^4)$$
$$= \mu^2 \sigma^2 (1 + \mu + \mu^2)$$

$$V(X_n) = \mu^{n-1} \sigma^2 (1 + \mu + \mu^2 + \dots + \mu^{n-1})$$

↳ Geometric Series.

$\mu < 1$

$$= \mu^{n-1} \sigma^2 \left(\frac{1 - \mu^n}{1 - \mu} \right) \checkmark$$

* Probability of extinction → $X_n = 0$ for some $n \in \mathbb{N}$

starting from single individual $(X_0 = 1)$ but $X_n = 0$ ___

$$\pi_0 = \lim_{n \to \infty} P(X_n = 0) \quad ?$$

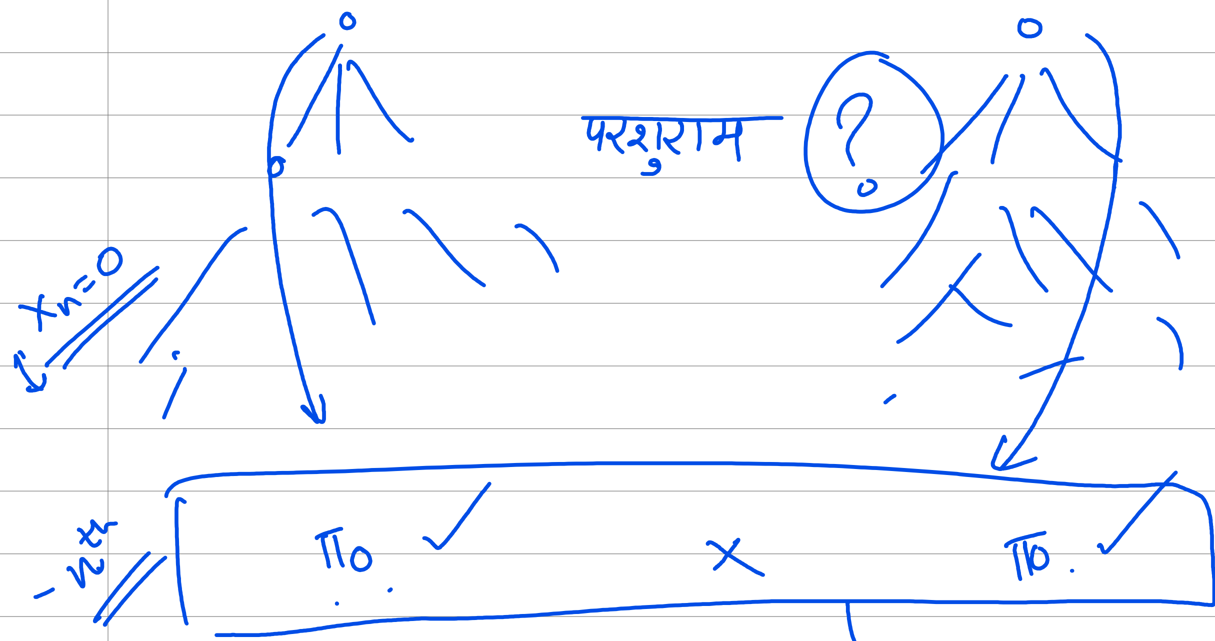
$X_0 = 1$

$$\pi_0 = P(\text{Pop}^n \text{ ever dies out}) \quad \leftarrow \text{single individual}$$

X_1

$$\pi_0 = \sum_{j=0}^{\infty} (\text{Pop}^n \text{ ever dies out} / X_1 = j) \cdot P_j$$

$$= \sum_{j=0}^{\infty} \pi_0^j \cdot P_j$$



$X_n = 0$

$X_{n+m} = 0 \quad \forall\, m \in \mathbb{N}$

PGF ?

$$P_X(t) = E(t^X) = \sum_x t^x P_x \quad ✓$$

$$\pi_0 = \sum_{j=0}^{\infty} \pi_0^j \cdot P_j = P_Z(\pi_0)$$

← PGF of Z

$$\pi_0 = P_Z(\pi_0)$$

by solving this eq^n we can get π_0 -

e.g. $Z_i \sim \text{Bino}(n=2,\ p=1/3)$

$\mu \Rightarrow n.p. = 2\times 1/3 < 1$

$\pi_0 = P_Z(\pi_0)$

✓ $t = P_Z(t) \Rightarrow t = \left(\frac{2}{3} + \frac{1}{3}.t\right)^2$

$9t = 4 + 4t + t^2$

$0 = 4 - 5t + t^2$

$0 \Rightarrow 4 - 4t - t + t^2$

$0 = 4(1-t) - t(1-t)$

$0 \Rightarrow (4-t)(1-t)$

$\underline{\underline{\pi_0 = \text{Prob. of}}}$ $\underline{\underline{0 \le \ \le 1}}$

$t=4$ or $t=1$

minⁿ value of π_0 obtained by solving this

$$Z = \begin{cases} 0 & 0.3 \\ 1 & 0.3 \\ 2 & 0.4 \end{cases}$$

$t = \sum t^j . p_j \quad = t^0 . 0.3 + t^1 . 0.3 + t^2 . 0.4$

$t = 0.3 + 0.3t + 0.4t^2$

$10t = 3 + 3t + 4t^2$

$4t^2 - 7t + 3 = 0$

$\Rightarrow 4t^2 - 4t - 3t + 3 = 0$

$\Rightarrow 4t(t-1) - 3(t-1) = 0$

$\Rightarrow t = \frac{3}{4}$ or $t = 1$ $\quad\Rightarrow \underline{\underline{\pi_0 = 3/4}}$

Absu 2001, 4, 6, 10, 15, 16, 22, 25, 30, 35, 38, 39, 43, 44, 45, 47, 50, 55

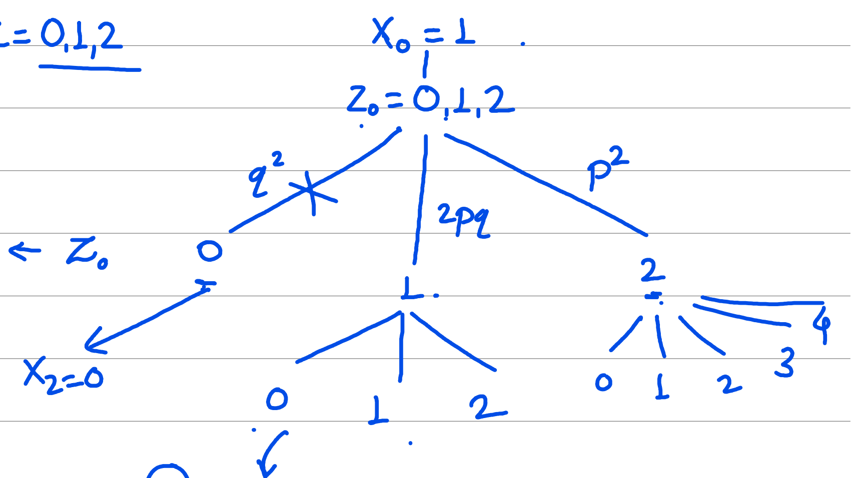
* Calculate the probability that the popⁿ become extint for the first time in 3^{rd} generation, consider $X_0 = 1$, $Z \sim B(2,p)$

→ So we have to find out $P(X_3 = 0 / X_2 \neq 0, X_1 \neq 0, X_0 = 1)$

$Z \sim Bino(2, p)$ $\quad Z = 0, 1, 2$

$$\boxed{\begin{aligned} P(Z=0) &= q^2 \\ P(Z=1) &= 2pq \\ P(Z=2) &= p^2 \end{aligned}}$$

$X_1 \leftarrow Z_0$



Hint? $X_1 = 2 \Rightarrow X_2 = \sum_{i=1}^{X_1} Z_i = \sum_{i=1}^{2} Z_i$ $\quad Z_i \sim B(2,p)$

✓ $X_2 \sim B(4, p)$

Obtain it?

$$\boxed{\begin{aligned} &✓ P(X_3 = 0 / X_2 \neq 0, X_1 \neq 0, X_0 = 1) \\ &= ? \end{aligned}}$$

$X_0 = 1$

$X_1 \sim Bin(2, p)$

$X_2 / X_1 \neq 0 \sim Bin(2 \times X_1, p)$ ✓

✓ $X_3 / X_2 \neq 0, X_1 \neq 0 \sim Bin(2 \times X_2, p)$

Random Walk

→ $X_n = X_0 + \sum_{i=1}^{n} Z_i$ $\qquad Z_i$ - i.i.d. $\quad X_n$ r.w.

$X_n = X_{n-1} + Z_n$ $\qquad Z_n = X_n - X_{n-1}$ $\quad$ indep. increments

i.i.d.