

Real Analysis

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Introduction to Real Analysis

1.1 The Algebraic Properties of \mathbb{R}

Algebraic Properties of \mathbb{R} On the set \mathbb{R} of real numbers there are two binary operations, denoted by $+$ and \cdot and called addition and multiplication, respectively. These operations satisfy the following properties :

- (A1) $a + b = b + a$ for all $a, b \in \mathbb{R}$ (commutative property of addition);
- (A2) $(a + b) + c = a + (b + c)$ for all $a, b, c \in \mathbb{R}$ (associative property of addition) ;
- (A3) there exists an element $0 \in \mathbb{R}$ such that $0 + a = a$ and $a + 0 = a$ for all $a \in \mathbb{R}$ (existence of a zero element) ;
- (A4) for each $a \in \mathbb{R}$ there exists an element $a \in \mathbb{R}$ such that $a + (-a) = 0$ and $(-a) + a = 0$ (existence of negative elements) ;
- (M1) $ab = ba$ for all $a, b \in \mathbb{R}$ (commutative property of multiplication) ;

- (M2) $(ab)c = a(bc)$ for all $a, b, c \in \mathbb{R}$ (associative property of multiplication) ;
- (M3) there exists an element $1 \in \mathbb{R}$ distinct from 0 such that $1a = a$ and $a1 = a$ for all $a \in \mathbb{R}$ (existence of a unit element) ;
- (M4) for each $a \in \mathbb{R} - \{0\}$ there exists an element $1/a \in \mathbb{R}$ such that $a(1/a) = (1/a)a = a$ and
- (D) $a(b+c) = (ab) + (ac)$ and $(b+c)a = (ba) + (ca)$ for all $a, b, c \in \mathbb{R}$ (distributive property of multiplication over addition).

1.2 The Order Properties of \mathbb{R}

There is a nonempty subset \mathbb{R}^+ of \mathbb{R} , called the set of positive real numbers, that satisfies the following properties :

1. If $a, b \in \mathbb{R}^+$, then $a + b \in \mathbb{R}^+$.
2. If $a, b \in \mathbb{R}^+$, then ab belongs to \mathbb{R}^+ .
3. If $a \in \mathbb{R}$, then exactly one of the following holds : $a \in \mathbb{R}^+$ OR $a = 0$ OR $(-a) \in \mathbb{R}^+$.

1. Let $a, b, c \in \mathbb{R}$
 if $a > b$ and $b > c$ then $a > c$
 Given that,
 $a > b$ and $b > c$
 $\therefore a - b > 0$ and $b - c > 0 \cdots$ i.e $(a - b), (b - c) \in \mathbb{R}^+$

$$\therefore (a - b) + (b - c) > 0 \dots (1^{st} \text{ order prop})$$

$$\therefore a - c > 0 \Rightarrow a > c$$

2. If $a > b$ then $a + c > b + c$

Given that,

$$a > b \text{ i.e. } a - b > 0$$

$$\therefore a - b \in \mathbb{R}^+$$

$$\therefore a + c - c - b > 0$$

$$\therefore (a + c) - (b + c) > 0$$

$$\therefore a + c > b + c$$

3. If $a > b$ and $c > 0$ then, $ca > cb$

Given that, $a > b$ & $c > 0 \therefore (a - b) > 0$ & $c > 0$

i.e $(a - b), c \in \mathbb{R}^+$

$$\therefore (a - b) \cdot c \in \mathbb{R}^+ \dots (2^{nd} \text{ order prop})$$

$$\therefore (a - b) \cdot c > 0 \Rightarrow a \cdot c - bc > 0 \Rightarrow ac > bc$$

4. If $a > b$ and $c < 0$ then, $ca < cb$

Given that, $a > b$ & $c < 0$

$$\therefore (a - b) \in \mathbb{R}^+ \text{ & } -c \in \mathbb{R}^+ \dots (3^{rd} \text{ order prop})$$

$$\therefore -c(a - b) > 0$$

$$\therefore -ca + cb > 0$$

$$\therefore cb > ca$$

$$\therefore ca < cb$$

1.3 Absolute Value and Real Line

Absolute value and Real line

Absolute value:-

$$|a| = \begin{cases} a & \text{if } a > 0 \\ -a & \text{if } a < 0 \\ 0 & \text{if } a = 0 \end{cases}$$

Theorem 1.3.1. For $a, b \in \mathbb{R}$

a) $|ab| = |a| \cdot |b| \forall a, b \in \mathbb{R}$

b) $|a|^2 = a^2 \forall a \in \mathbb{R}$

c) if $c > 0$ then $|a| \leq c$ iff $-c \leq a \leq c$

d) $-|a| \leq a \leq |a| \forall a \in \mathbb{R}$

Proof. a) $|ab| = |a| \cdot |b| \forall a, b \in \mathbb{R}$

- if $a = 0$ or $b = 0 \Rightarrow ab = 0 = |ab| = |a| \cdot |b|$
- if $a > 0$ or $b > 0 \Rightarrow ab > 0$
 $|ab| = a \cdot b = |a| \cdot |b|$

- if $a > 0$ or $b < 0 \Rightarrow ab < 0$
 $\therefore |ab| = -a \cdot b = (-a) \cdot b = a \cdot (-b) = |a| \cdot |b|$
- if $a < 0$ or $b > 0 \Rightarrow ab < 0$
 $\therefore |ab| = -ab = |a| \cdot |b|$
- if $a < 0$ or $b < 0 \Rightarrow ab > 0$
 $\therefore |ab| = ab = |a| \cdot |b|$
 - Hence proved -

b) $|a|^2 = a^2 \forall a \in \mathbb{R}$
 $\forall a \in \mathbb{R}, a^2 \in \mathbb{R}$ i.e $a^2 \geq 0$
 let $|a^2|^2 = |a| \cdot |a| = a \cdot a = a^2$, if $a > 0$
 $(-a) \cdot (-a) = a^2$, if $a < 0$
 Hence, $|a|^2 = a^2$

c) if $c > 0$ then $|a| \leq c$ iff $-c \leq a \leq c$
 Given that,
 $c > 0$ & $|a| \leq c$
 i) To show $-c \leq a \leq c$
 Now, $|a| = \max(a, -a) \leq c$
 $\Rightarrow a \leq c$ & $-a \leq c$
 $\Rightarrow a \leq c$ & $a \geq -c$

$$-c \leq a \leq c$$

ii) Given that, $-c \leq a \leq c$ & To show:- $|a| \leq c$

$$\Rightarrow a \leq c \text{ \& } -a \leq -c$$

$$\therefore |a| \leq c \dots (|a| = \max(a, -a))$$

d) For $a \neq 0 \in \mathbb{R}$, $|a| > 0 \dots |a| = \max(a, -a)$

Put $c = |a| > 0$ in c)

$$\therefore -c \leq a \leq c \Rightarrow -|a| \leq a \leq |a|$$



1.4 Triangular Inequality

Triangular Inequality:-

Theorem 1.4.1. If $a, b \in \mathbb{R}$ then $|a + b| \leq |a| + |b|$

Proof. if $a, b \in \mathbb{R}$ then

$$\begin{array}{rcl}
 & -|a| \leq a \leq |a| & \\
 + & -|b| \leq b \leq |b| & \\
 \hline
 & -(|a| + |b|) \leq a + b \leq (|a| + |b|) \dots (\text{Theorem:-1.1.1-d})) & \\
 \text{let } & |a| + |b| = c &
 \end{array}$$

$$\begin{aligned}
&\therefore -c \leq a + b \leq c \\
&\Rightarrow |a + b| \leq c \dots\dots (Th^m - 1.1.1 - c) \\
&\therefore |a + b| \leq |a| + |b|
\end{aligned}$$

□

Corollary 1.4.1.1. If $a, b \in \mathbb{R}$

$$a) ||a| - |b|| \leq |a - b|$$

$$b) |a - b| \leq |a| + |b|$$

Proof. a) We know that, $a, b \in \mathbb{R}$

$$a = a - b + b$$

$$\therefore |a| = |a - b + b| \leq |a - b| + |b|$$

$$\Rightarrow |a| - |b| \leq |a - b| \quad (1.1)$$

$$\text{Also, } b = b - a + a$$

$$|b| = |b - a + a| \leq |b - a| + |a|$$

$$\therefore |b| - |a| \leq |a - b| \quad (1.2)$$

from equation (1.1) & (1.2)

$$||a| - |b|| \leq |a - b|$$

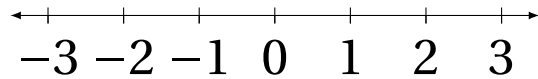
b)

$$\begin{aligned}
&\text{if } a, b, c \in \mathbb{R} \\
&\therefore |a + c| \leq |a| + |c| \\
&\text{Put } c = -b, |c| = |-b| = |b| \\
&\therefore |a + (-b)| \leq |a| + |-b| \\
&\therefore |a - b| \leq |a| + |b|
\end{aligned}$$

□

Corollary 1.4.1.2. *If $a_1, a_2 \dots a_n$ are any real no then $|a_1 + a_2 + \dots + a_n| \leq |a_1| + |a_2| + \dots + |a_n|$*

Definition 1.4.1 (Real line): *A convenient and Familiar interpretation of real no system is the real line.*



Definition 1.4.2 (ϵ -Neighbourhood:-): *let $a \in \mathbb{R}$ & $\epsilon > 0$, then ϵ - neighbourhood of a is the set $V_\epsilon(a) = \{x | x \in \mathbb{R}, |x - a| < \epsilon\} \dots 0 \leq |x - a| < \epsilon$*

$$\therefore V_\epsilon(a) = (a - \epsilon, a + \epsilon) \subseteq \mathbb{R}$$

Since $|x - a| < \epsilon \Rightarrow -\epsilon < x - a < \epsilon \Rightarrow a - \epsilon < x < a + \epsilon$

Definition 1.4.3 (Deleted- ϵ -Neighbourhood:-): $\delta_\epsilon(a) = v_\epsilon(a) - \{a\}$
 $= (a - \epsilon, a + \epsilon) - \{a\}$

$$i.e. 0 < |x - a| < \epsilon$$

Example 1:

If $a, b \in \mathbb{R}$. Show that $|a + b| = |a| + |b|$ if and only if $ab \geq 0$

Proof. i) Given that $ab \geq 0$, To prove- $|a + b| = |a| + |b|$
 if $ab \geq 0 \Rightarrow a \geq 0, b \geq 0$ or $a \leq 0, b \leq 0$

$$a + b \geq 0$$

$$\therefore |a| = a, |b| = b$$

$$\begin{aligned} |a + b| &= a + b \\ &= |a| + |b| \end{aligned}$$

$$a + b \leq 0$$

$$\therefore |a| = -a, |b| = -b$$

$$\begin{aligned} |a + b| &= -(a + b) \\ &= -a - b \\ &= |a| + |b| \end{aligned}$$

ii) Given that $|a + b| = |a| + |b|$, To prove $ab \geq 0$

$$|a + b|^2 = (|a| + |b|)^2$$

$$\therefore a^2 + b^2 + 2ab = |a|^2 + |b|^2 + 2 \cdot |a| \cdot |b|$$

$$\therefore 2ab = 2|a| \cdot |b| \dots (\because |a|^2 = a^2)$$

$$ab = |a| \cdot |b|$$

$$ab = |ab| \dots (\text{Theorem:- 1.1.1-a})$$

$$\therefore ab \geq 0$$



Example 2:

Show that if $a, b \in \mathbb{R}$ then

$$\text{i) } \max\{a, b\} = \frac{1}{2}(a + b + |a - b|), \min\{a, b\} = \frac{1}{2}(a + b - |a - b|)$$

$$\text{ii) } \min\{a, b, c\} = \min\{\min\{a, b\}, c\}$$

Proof. i) let $a > b \Rightarrow |a - b| = a - b$

$$\max(a, b) = a \tag{1.3}$$

Consider, *RHS*

$$\begin{aligned} &= \frac{1}{2}(a + b - |a - b|) \\ &= \frac{1}{2}(a + b + a - b) \dots \text{from (1.3)} (a - b) \geq 0 \\ &= a \\ &= \text{LHS} \end{aligned}$$

$$\text{let } \min(a, b) = b \tag{1.4}$$

$$\begin{aligned}
 \text{Consider, } RHS &= \frac{1}{2}(a + b - |a - b|) \\
 &= \frac{1}{2}(a + b - (a - b)) \\
 &= b \\
 &= \text{LHS}
 \end{aligned}$$

ii) Suppose, $a > b > c$

$$\text{LHS} = \min\{a, b, c\} = c$$

$$\text{RHS} = \min\{\min\{a, b\}, c\} = \min\{b, c\}$$

$$\text{RHS} = c$$

$$= \text{LHS}$$

$$\text{Hence, } \min\{a, b, c\} = \min\{\min\{a, b\}, c\}$$

□

Example 3:

If $x, y, z \in \mathbb{R}$ & $x \leq z$, Show that $x \leq y \leq z$ if and only if $|x - y| + |y - z| = |x - z|$

$$x \leq z \Rightarrow x - z \leq 0 \therefore |x - z| = z - x$$

Proof. i) Given that $x \leq y \leq z, x, y, z \in \mathbb{R}$

$$\therefore |x - y| = y - x \text{ \& } |y - z| = z - y$$

$$\text{To show } |x - y| + |y - z| = |x - z|$$

$$\begin{aligned}
\text{Consider, LHS} &= |x - y| + |y - z| \\
&= y - x + z - y \\
&= z - x \\
&= |x - z| \\
&= \text{RHS}
\end{aligned}$$

ii) Given that $|x - y| + |y - z| = |x - z|$

To show, $x \leq y \leq z$

let $a = (x - y)$, $b = (y - z)$

$$\therefore |(x - y) + (y - z)| = |x - y| + |y - z|$$

$$\Rightarrow (x - y)(y - z) \geq 0 \dots (\because \text{if } |a + b| = |a| + |b| \Leftrightarrow ab \geq 0)$$

$$\therefore a, b \geq 0$$

$$\text{i.e. } (x - y), (y - z) \geq 0$$

$$x \geq y, y \geq z$$

$$\therefore x \geq y \geq z$$

which is not possible Since $x \leq z$ -(given)

$$a, b \leq 0$$

$$(x - y), (y - z) \leq 0$$

$$\therefore x \leq y, y \leq z$$

$$\therefore x \leq y \leq z$$



Example 4:

If $a < x < b, a < y < b$. Show that $|x - y| < b - a$.

Proof. Given that,

$$a < x < b, a < y < b$$

$$0 < x - a < b - a \quad (1.5)$$

$$0 < y - a < b - a \quad (1.6)$$

multiplying by (-1) to (1.6) and add in (1.5)

$$\begin{array}{r} 0 \leq -a \leq b - a \\ + \quad -(b - a) \leq a - y \leq b - a \\ \hline -(b - a) \leq x - y \leq (b - a) \Rightarrow |x - y| < (b - a) \end{array}$$



Definition 1.4.4 (Upper bound): Let $S \neq \phi \subseteq \mathbb{R}$, the set s is said to be bounded above if $\exists a \in \mathbb{R} \xRightarrow{s.t} x \leq a \forall x \in S$ Each such 'a' is called as upper bound of S .

Definition 1.4.5 (Lower bound): Let $S \neq \phi \subseteq \mathbb{R}$. The set S is said to be bounded below if $\exists b \in \mathbb{R} \xRightarrow{s.t} x \geq b \forall x \in S$ Each such b is called as lower bound of S .

Definition 1.4.6 (Bounded Set): If both lower and upper bound exist.

Definition 1.4.7 (Unbounded set): *If set S is not bounded.*

Definition 1.4.8 (Supremum & Infimum): *Let S be a non-empty subset of \mathbb{R} if S is bounded above/below then a no u is said to be supremum/Infimum (least upper bound or greatest lower bound) of S if it satisfies the conditions:-*

- i) u is an upper(lower) bound of S .*
- ii) if v is any upper(lower) bound of S then $u \leq v$ ($u \geq v$).*

1.5 Completeness Property

Statement:-If set is bounded below then its infimum must be exists and if set is bounded above then its supremum must be exists this property is known as completeness property.

let $\mathbb{N} = 1, 2, \dots, 4$ bounded below

Unbounded = $\{\infty\}$ = Supremum

Lower bound = $\{\infty, \dots, -1, 0, 1\}$ = Infimum = 1

Example 5:

Let $A \subseteq B$ then Prove that,

I) $\inf A \geq \inf B$

II) $\sup A \leq \sup B$

Proof. I) Given that, $A \subseteq B, x \in A \Rightarrow x \in B$
 also, $\inf A = u$ and $\inf B = v \dots$ (assume)
 if u is $\inf A$ then, by definition,

i) u is lower bound, $x \geq u \forall x \in A$

ii) if u_1 is another lower bound, then $u_1 < u \forall u_1$ Assume that, $\inf B \geq \inf A$

Assume that, $\inf B \geq \inf A$

i.e $v \geq u$

i.e $x \geq v \geq u, \forall x \in B$

$\therefore x \geq v \geq u, \forall x \in A$

\Rightarrow if u is \inf , we can not have lower bound greater than u .

So, our assumption is wrong.

Hence, $u \geq v$ i.e $\inf A \geq \inf B$

II) let $\sup A = u$ and $\sup B = v$

if u is supremum of A then, by definition

i) u is upper bound of A i.e $x \leq u, \forall x \in A$

ii) if u_1 is any other upper bound then $u \leq u_1 \forall u_1$

Assume that, $\sup A \geq \sup B$

$u \geq v$

i.e $v \leq u$

$$x \leq v \leq u, \forall x \in B$$

$$x \leq v \leq u, \forall x \in A$$

\Rightarrow if u is sup of A then we can not have upper bound less than u . So assumption is wrong.

Hence, $u \leq v$ i.e $\sup A \leq \sup B$



Example 6:

$S = 1 - \frac{(-1)^n}{n}, n \in \mathbb{N}$. Find infimum & supremum

$$S = \{2, 1/2, 1 + 1/3, 1 - 1/4, 1 + 1/5, 1 - 1/6, \dots\}$$

$$\therefore \inf s = 1/2 \text{ of } \sup s = 2$$

Example 7:

$$S = \frac{(-1)^n}{n}, n \in \mathbb{N}$$

$$S = \{-1, 1/2, -1/3, 1/4, -1/5, \dots\}$$

$$LB = (-\infty, -1] \Rightarrow \inf = -1 \in S,$$

$$UB = [1/2, \infty) \Rightarrow \sup = 1/2 \in S$$

Example 8:

$$S = \{\frac{1}{m} - \frac{1}{n}, m, n \in \mathbb{N}\}$$

$$S = \{0, 1/2, -1/2, 1 - 1/3, -2/3, 1, -1, \dots\}$$

$$LB = (-\infty, -1] \Rightarrow \inf = -1 \notin S,$$

$$UB = (1, \infty) \Rightarrow \sup = 1 \notin S$$

Example 9:

$$S = \{\frac{n-1}{n}, n \in \mathbb{N}\} = \{1 - \frac{1}{n}, n \in \mathbb{N}\}$$

$$S = \{0, 1 - 1/2, 1 - 1/3, 1 - 1/4, \dots\}$$

$$LB = (-\infty, 0] \Rightarrow \inf S = 0 \in S,$$

$$UB = [1, \infty) \Rightarrow \sup S = 1 \notin S$$

Sets Operations

2.1 Set Operations

1. Union $A \cup B = \{x / x \in A \text{ or } x \in B\}$
2. Intersection $A \cap B = \{x / x \in A \& x \in B\}$
3. Complement $A^c = \{x / x \in A, x \in \Omega\}$
4. Substraction $A - B = A \setminus B = A \cap B^c = \{x / x \in A \text{ but } x \notin B\}$

Theorem 2.1.1. *if A, B, C are sets then,*

a) $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$

b) $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$

Proof. To Prove:-

i) $A \setminus (B \cup C) \subseteq (A \setminus B) \cap (A \setminus C)$

$$\text{ii) } (A \setminus B \cap A \setminus C) \subseteq A \setminus (B \cup C)$$

$$\text{i) } A \setminus (B \cup C) \subseteq (A \setminus B \cap A \setminus C)$$

$$\text{i) let } x \in A \setminus (B \cup C) \text{ i.e. } x \in A \cap (B \cup C)^C$$

$$\Rightarrow x \in A \text{ and } x \in (B \cup C)^C$$

$$\Rightarrow x \in A \text{ and } x \notin (B \cup C)$$

$$\Rightarrow x \in A \text{ and } (x \notin B \text{ and } x \notin C)$$

$$\Rightarrow (x \in A \& x \notin B) \text{ and } (x \in A \& x \notin C)$$

$$\Rightarrow x \in A \cap B^C \text{ and } x \in A \cap C^C$$

$$\Rightarrow x \in (A \setminus B) \cap (A \setminus C)$$

$$\therefore A \setminus (B \cup C) \subseteq (A \setminus B) \cap (A \setminus C) \quad (2.1)$$

$$\text{ii) } x \in (A \setminus B) \cap (A \setminus C)$$

$$\Rightarrow x \in A \setminus B \text{ and } x \in A \setminus C$$

$$\Rightarrow x \in (A \cap B^C) \text{ and } x \in (A \cap C^C)$$

$$\Rightarrow (x \in A \& x \notin B) \text{ and } (x \in A \& x \notin C)$$

$$\Rightarrow x \in A \text{ and } (x \notin B \text{ and } x \notin C)$$

$$\Rightarrow x \in A \text{ and } (x \in (B \cup C)^C)$$

$$\Rightarrow x \in A \cap (B \cup C)^C$$

$$A \setminus B \cap A \setminus C \subseteq A \setminus (B \cup C) \quad (2.2)$$

from (2.1) & (2.2)

$$A \setminus (B \cup C) = A \setminus B \cap A \setminus C$$

$$\text{ii) } A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$$

To Prove:-

$$\text{i) } A \setminus (B \cap C) \subseteq (A \setminus B) \cap (A \setminus C)$$

$$\text{ii) } A \setminus B \cup A \setminus C \subseteq A \setminus (B \cup C)$$

$$\begin{aligned} \text{i) let } x \in A \setminus (B \cap C) & \text{ i.e. } x \in A \cap (B \cap C)^C \\ \Rightarrow x \in A \text{ and } x \in (B \cap C)^C \\ \Rightarrow x \in A \text{ and } x \notin (B \cap C) \\ \Rightarrow x \in A \text{ and } (x \notin B \text{ or } x \notin C) \\ \Rightarrow (x \in A \& x \notin B) \text{ or } (x \in A \& x \notin C) \\ \Rightarrow x \in A \cap B^C \text{ or } x \in A \cap C^C \\ \Rightarrow x \in (A \setminus B) \cup (A \setminus C) \end{aligned}$$

$$\therefore A \setminus (B \cap C) \subseteq A \setminus B \cup A \setminus C \quad (2.3)$$

$$\begin{aligned} \text{ii) } x \in A \setminus B \cup A \setminus C \\ \Rightarrow x \in A \setminus B \text{ or } x \in A \setminus C \\ \Rightarrow x \in (A \cap B^C) \text{ or } x \in (A \cap C^C) \\ \Rightarrow (x \in A \& x \notin B) \text{ or } (x \in A \& x \notin C) \\ \Rightarrow x \in A \text{ and } (x \notin B \text{ or } x \notin C) \end{aligned}$$

$$\Rightarrow x \in A \text{ and } (x \in (B \cap C)^c)$$

$$\Rightarrow x \in A \cap (B \cap C)^c$$

$$\Rightarrow x \in A \setminus (B \cap C)$$

$$A \setminus B \cup A \setminus C \subseteq A \setminus (B \cap C) \quad (2.4)$$

from (2.3) & (2.4)

$$A \setminus (B \cap C) = A \setminus B \cup A \setminus C$$

-Hence Proved-



2.2 Distributive Law

Distributive Law:-

a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Proof. a) To Prove:-

i) $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$

let $x \in A \cup (B \cap C)$
 $\Rightarrow x \in A$ or $x \in B \cap C$
 $\Rightarrow x \in A$ or $x \in B$ and $x \in C$
 $\Rightarrow x \in A$ or $x \in B$ and $x \in A$ or $x \in C$
 $\Rightarrow x \in A \cup B$ and $x \in A \cup C$
 $\Rightarrow x \in (A \cup B) \cap (A \cup C)$

$$\therefore A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C) \quad (2.5)$$

ii) $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$

let $x \in (A \cup B) \cap (A \cup C)$
 $\Rightarrow x \in (A \cup B)$ and $x \in (A \cup C)$
 $\Rightarrow (x \in A$ or $x \in B)$ and $(x \in A$ or $x \in C)$
 $\Rightarrow x \in A$ or $(x \in B$ and $x \in C)$
 $\Rightarrow x \in A \cup (B \cap C)$

$$(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C) \quad (2.6)$$

from (2.5) & (2.6)

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

b) To Prove:-

$$\text{i) } A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$$

$$\text{let } x \in A \cap (B \cup C)$$

$$\Rightarrow x \in A \text{ and } x \in (B \cup C)$$

$$\Rightarrow x \in A \text{ and } (x \in B \text{ or } x \in C)$$

$$\Rightarrow (x \in A \text{ and } x \in B) \text{ or } (x \in A \text{ and } x \in C)$$

$$\Rightarrow x \in (A \cap B) \text{ or } (x \in A \cap C)$$

$$\Rightarrow x \in (A \cup B) \cup (A \cap C)$$

$$\therefore A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C) \quad (2.7)$$

$$\text{ii) } (A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$$

$$\text{let } x \in (A \cap B) \cup (A \cap C)$$

$$\Rightarrow x \in (A \cap B) \text{ or } x \in (A \cap C)$$

$$\Rightarrow (x \in A \text{ and } x \in B) \text{ or } (x \in A \text{ and } x \in C)$$

$$\Rightarrow x \in A \text{ and } (x \in B \text{ or } x \in C)$$

$$\Rightarrow x \in A \text{ and } (x \in B \cup C)$$

$$\Rightarrow x \in A \cap (B \cup C)$$

$$(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C) \quad (2.8)$$

from (2.7) & (2.8)

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$



Theorem 2.2.1. *If A & B are sets, Show that $A \subseteq B$ if and only if $A \cap B = A$*

Proof. i) Assume that $A \subseteq B$ to Prove that $A \cap B = A$

let $x \in A \Rightarrow x \in B$

$\therefore x \in B \dots (\because A \subseteq B)$

$\Rightarrow x \in A \cap B$

$$\therefore A \subseteq A \cap B \quad (2.9)$$

Also, by definition,

$$A \cap B \subseteq A \quad (2.10)$$

from (2.9) and (2.10)

$$A = A \cap B \quad (2.11)$$

ii) Assume that $A \cap B = A$, to prove $A \subseteq B$

We know that, $A \cap B \subseteq B$

$$\Rightarrow A \subseteq B \quad (2.12)$$

from (2.11) and (2.12)

$$A \subseteq B \text{ iff } A = A \cap B \quad (2.13)$$

-Hence Proved-



2.3 Basic Notations Theory

$$A \times B = \{ \langle \underline{x}, \underline{y} \rangle / x \in A, y \in B \} \quad \checkmark \quad \text{ordered}$$

Definition 2.3.1 (Cartesian Product): let A & B be two sets,
 $A = \langle 2, 3, 4 \rangle$ & $\langle 1, 5, 6 \rangle$ then cartesian prodct is given by

$$A = \{2, 3, 4\} \quad B = \{1, 5, 6\}$$

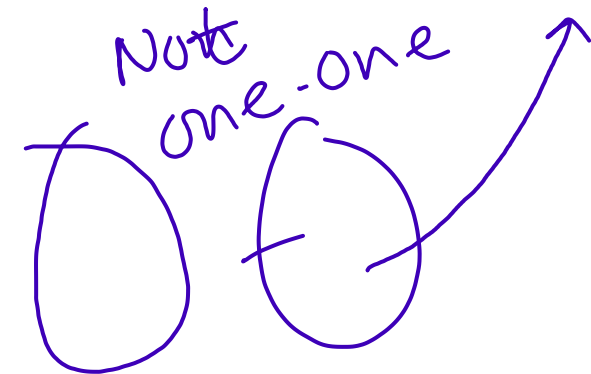
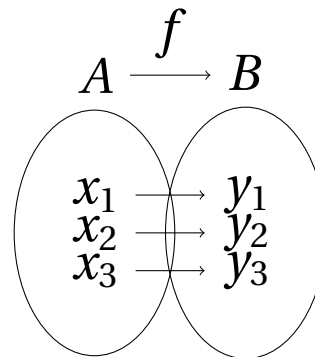
$$A \times B = \{ \langle 2, 1 \rangle, \langle 2, 5 \rangle, \langle 2, 6 \rangle, \langle 3, 5 \rangle, \langle 3, 6 \rangle, \langle 4, 1 \rangle, \langle 4, 5 \rangle, \langle 4, 6 \rangle \}$$

Definition 2.3.2 (Function): Let A & B be sets then a function from A to B is a set f of ordered pairs in $A \times B$ such that for each $a \in A$ then there exists a unique $b \in B$ with $\langle a, b \rangle \in f$.

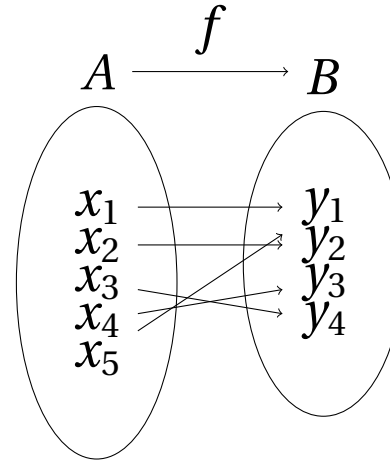
In other words if $\langle a, b \rangle \in f$ & $\langle a, b' \rangle \in f \Rightarrow \underline{b = b'}$.

Types of Function

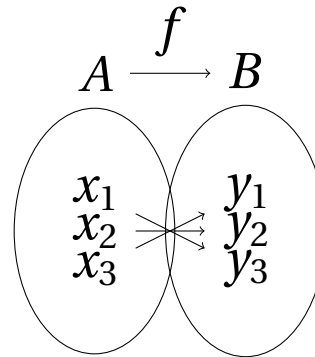
Definition 2.3.3 (One-One (Injective) Function): The Function f is said to be injective (or One-One) if whenever $x_1 \neq x_2$, then $\underline{f(x_1) \neq f(x_2)}$.



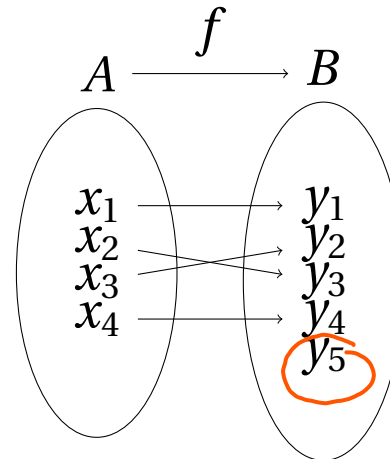
Definition 2.3.4 (Onto (Surjective) Function): The function f is said to be Surjective if $f(A) = B$ i.e if the range $R(f) = B$.



Definition 2.3.5 (One-One & Onto (Bijective) Function): *The Function f is both one-one and onto then it is said to be bijective.*



Definition 2.3.6 (Into Function): *If f is not onto then it is called as into function.*



Definition 2.3.7 (Composite Function): If $f : A \rightarrow B$ and $g : B \rightarrow C$ and if $R(f) \subseteq D(g) = B$ then the composite function $g \circ f$ is the function from $A \rightarrow C$

$g \circ f : A \rightarrow C$ is composite function if $g \circ f(x) = g(f(x)) \ x \in A$

Draw diagram Domain

Example 10:

$$f : \mathbb{R} \rightarrow \mathbb{R}, g : \mathbb{R} \rightarrow \mathbb{R}$$

$$f(x) = 2x, g(y) = 3y^2 - 1$$

Proof. Given that, $f(x) = 2x, g(y) = 3y^2 - 1$

$$\begin{aligned} g \circ f(x) &= g(f(x)) \\ &= g(2x) \\ &= 3(2x)^2 - 1 \\ &= 12x^2 - 1 \end{aligned}$$

$$\begin{aligned}
 f \circ g(y) &= f(g(y)) \\
 &= f(3y^2 - 1) \\
 &= 2(3y^2 - 1) \\
 &= 6y^2 - 2 \\
 \therefore g \circ f &\neq f \circ g
 \end{aligned}$$

□

Example 11:

Show that if $f : A \rightarrow B$ ~~then~~, E, F are subsets of A then,

a) $f(E \cup F) = f(E) \cup f(F)$ and

b) $f(E \cap F) \subseteq f(E) \cap f(F)$

Proof. a) $f : A \rightarrow B, E, F \subseteq A$

$$f(E) = \{y / y = f(x), x \in E \subseteq A\} \subseteq B$$

$$f(F) = \{y / y = f(x), x \in F \subseteq A\} \subseteq B$$

$$f(E \cup F) = \{y / y = f(x), x \in E \cup F\}$$

To Prove,

i) $f(E \cup F) \subseteq f(E) \cup f(F)$

ii) $f(E) \cup f(F) \subseteq f(E \cup F)$

let $y \in f(E \cup F)$

$$\Leftrightarrow y = f(x), x \in E \cup F$$

$$\Leftrightarrow y = f(x), x \in E \text{ or } x \in F$$

$$\Leftrightarrow y = f(x), x \in E \subseteq A \text{ or } y = f(x), x \in F \subseteq A$$

$$\Leftrightarrow y \in f(E) \text{ or } y \in f(F)$$

$$\Leftrightarrow y \in f(E) \cup f(F)$$

$$\begin{aligned} \therefore f(E \cup F) &\subseteq f(E) \cup f(F) & f(E) \cup f(F) &\subseteq f(E \cup F) \\ f(E \cup F) &= f(E) \cup f(F) \end{aligned}$$

To Prove,

$$\text{b) } f(E \cap F) \subseteq f(E) \cap f(F)$$

$$\text{let } y \in f(E \cap F)$$

$$\Rightarrow y = f(x), x \in E \cap F$$

$$\Rightarrow y = f(x), x \in E \text{ and } x \in F$$

$$\Rightarrow y = f(x), x \in E \text{ and } y = f(x), x \in F$$

$$\Rightarrow y \in f(E) \text{ and } y \in f(F)$$

$$\Rightarrow y \in f(E) \cap f(F)$$

$$\therefore f(E \cap F) \subseteq f(E) \cap f(F)$$



Example 12:

Example for $f(E) \cap f(F) \not\subseteq f(E \cap F)$?

let $f(x) = x^2$

$$E = \{1, 2\}, f(E) = \{1, 4\}$$

$$F = \{-2, 4\}, f(F) = \{4, 16\}$$

$$E \cap F = \{\emptyset\}, f(E) \cap f(F) = \{4\}$$

$$f(E \cap F) = \{\emptyset\}$$

$$f(E) \cap f(F) \not\subseteq f(E \cap F)$$

Example 13:

✓ Show that if $f : A \rightarrow B$ and G, H are subsets of B then,

a) $f^{-1}(G \cup H) = f^{-1}(G) \cup f^{-1}(H)$ and

b) $f^{-1}(G \cap H) = f^{-1}(G) \cap f^{-1}(H)$

Proof. a) $f : A \rightarrow B$

$$f^{-1}(G) = \{x / f(x) \in G\} \subseteq A$$

$$f^{-1}(H) = \{x / f(x) \in H\} \subseteq A$$

$$\text{let } x \in f^{-1}(G \cup H)$$

$$\Leftrightarrow f(x) \in G \cup H$$

$$\Leftrightarrow f(x) \in G \text{ or } f(x) \in H$$

$$\Leftrightarrow x \in f^{-1}(G) \text{ or } x \in f^{-1}(H)$$

$$\Leftrightarrow x \in f^{-1}(G) \cup f^{-1}(H)$$

$$\therefore f^{-1}(G \cup H) = f^{-1}(G) \cup f^{-1}(H)$$

-Hence Proved-

b) let $x \in f^{-1}(G \cap H)$

$$\Leftrightarrow f(x) \in G \cap H$$

$$\Leftrightarrow f(x) \in G \text{ and } f(x) \in H$$

$$\Leftrightarrow x \in f^{-1}(G) \text{ and } x \in f^{-1}(H)$$

$$\Leftrightarrow x \in f^{-1}(G) \cap f^{-1}(H)$$

$$\therefore f^{-1}(G \cap H) = f^{-1}(G) \cap f^{-1}(H)$$

-Hence Proved-



Example 14:

one-one
Show that if $f : A \rightarrow B$ is injective & $E \subseteq A$ then $f^{-1}(f(E)) = E$. Give an example to show that equality need not hold if f is not injective.

Proof. Given that, $f : A \rightarrow B$ is injective

i.e if $x \neq y \Rightarrow f(x) \neq f(y) \forall x, y \in A$

$$E = \{x / x \in E, f(x) \in B\} \subseteq A$$

$$f(E) = \{y / y = f(x) \in f(E), x \in A\} \subseteq B$$

To prove $f^{-1}(f(E)) = E$

$$\text{let } x \in f^{-1}(f(E))$$

$$\Rightarrow f(x) \in f(E)$$

$$\Rightarrow x \in E \dots (\because f \text{ is one-one function})$$

$$f^{-1}(f(E)) \subseteq E \quad (2.14)$$

Now, let $x \in E$

$$\Rightarrow f(x) \in f(E) \quad f^{-1}(f(E)) = \{x / f(x) \in f(E), x \in A\}$$

$$x \in f^{-1}(f(E))$$

$$E \subseteq f^{-1}(f(E)) \quad (2.15)$$

from (2.14) & (2.15)

$$f^{-1}(f(E)) = E$$

□

Example 15:

$$\text{let } f(x) = x^2$$

$$E = \{1, 2\} \Rightarrow f(E) = \{1, 4\}$$

$$E = \{1, 2\}$$

$$f(E) = \{1, 4\}$$

$$f^{-1}(f(E)) = \{(1, -2, -2)\}$$

$$f^{-1}(f(E)) \neq E$$

Example 16:

Show that if $f : A \rightarrow B$ is surjective and $E \subseteq A$ then $f(f^{-1}(H)) = H$. Give an example to show that equality need not hold if f is not surjective.

Proof. $f : A \rightarrow B, H \subseteq B$ and f is surjective i.e every element in B has inverse image in A

To prove: $f(f^{-1}(H)) = H$

let $y \in f(f^{-1}(H))$

$$\Rightarrow f(x) \in f(f^{-1}(H))$$

$$\Rightarrow x \in f^{-1}(H)$$

$$\Rightarrow y = f(x) \in H$$

$$\therefore f(f^{-1}(H)) \subseteq H \quad (2.16)$$

let $y \in H$ then

$\exists x \in A$ such that,

$$y = f(x) \in H \dots (\because f \text{ is onto})$$

$$\Rightarrow x \in f^{-1}(H)$$

$$\Rightarrow f(x) \in f(f^{-1}(H)) \dots (x \in E \Rightarrow f(x) \in f(E))$$

$$\Rightarrow y \in f(f^{-1}(H))$$

$$\therefore H \subseteq f(f^{-1}(H)) \quad (2.17)$$

from (2.16) & (2.17)

$$f(f^{-1}(H)) = H$$



Definition 2.3.8 (Finite & Infinite Sets): 1. The empty set ϕ is said to have zero elements

2. If $n \in \mathbb{N}$, a set S is said to have n elements if there exists a bijection from set $\mathbb{N} = \{1, 2, \dots, n\}$ on S

3. A set S is said to be finite if it is either empty or it has n elements for some $n \in \mathbb{N}$

4. A set S is said to be infinite if it is not finite

Theorem 2.3.1 (Uniqueness Theorem). If S is finite set, then the number of elements in S is unique number in \mathbb{N} .

The set \mathbb{N} of natural numbers is an infinite set.

Theorem 2.3.2. Suppose that S & T are sets and $T \subseteq S$

a) If S is finite Set, then T is a finite Set.

b) If T is an infinite set then S is an infinite Set.

Proof. a) $T \subseteq S$ and S is finite Set

- i) Suppose $S = \phi \Rightarrow T = \phi \Rightarrow T$ is finite
- ii) When $S \neq \phi$ then there are two possibilities.
 - 1) $T = \phi \Rightarrow T$ is a finite Set **or**
 - 2) $T \neq \phi$

We will prove this by method of mathematical induction.

- $\#(S) = 1$ and as $T \neq \phi \Rightarrow S = T$
Hence as S is finite $\Rightarrow T$ is finite
- Now assume that this statement is true for $\#(S) = k$
i.e $\#(S) = k \& T \subseteq S \Rightarrow T$ is finite set.
- Now, lets prove it for $\#(S) = k + 1$
As S is finite, it has bijection with N_{k+1}

$$S = \{f(1), f(2), \dots, f(k+1)\} \quad (2.18)$$

lets define, $S_1 = S - f(k+1)$

$\therefore \#(S)_1 = k$ and $T_1 = T - f(k+1)$

Now, if $f(k+1) \notin T \Rightarrow T_1 = T \subseteq S_1$

and as $\#(S)_1 = k \& T \subseteq S_1 \subseteq T$ is finite

if $f(k+1) \in T_1 \Rightarrow T_1 = T - f(k+1) \subseteq S_1$

$\therefore T_1 \subseteq S_1, \#(S_1) = k \Rightarrow T_1$ is finite $\Rightarrow T$ is finite.

-Hence Proved-

b) (b) is a contrapositive statement to (a).

Hence, if T is infinite $\Rightarrow S$ is also infinite.



Definition 2.3.9 (Countably Infinite): *A set is said to be denumerable or countably infinite if there exists bijection of \mathbb{N} onto S .*

Definition 2.3.10 (Countable Set): *A set S is said to be countable if it is either finite or denumerable.*

Definition 2.3.11 (Uncountable Set): *A set S is said to be uncountable if it is not countable.*

The following statements are equivalent

1. S is a countable set
2. \exists surjection of \mathbb{N} onto S
3. \exists injection of S into \mathbb{N}

Example

1. Set of even/odd numbers are denumerable
2. Set of all integers (denumerable)
3. The union of two disjoint denumerable sets is again denumerable
4. The sets $\mathbb{N}, \mathbb{N}^2, \mathbb{N}^n$ are denumerable

Theorem 2.3.3. Suppose that S & T are sets and $T \subseteq S$

a) If S is countable, then T is a countable set.

b) If T is an uncountable then S is an uncountable Set.

Theorem 2.3.4. The Set \mathbb{Q} of rational numbers is denumerable.

Proof. lets prove it for \mathbb{Q}^+ first.

$$\mathbb{Q} = \left\{ \frac{p}{q}, q \neq 0 \right\}, \mathbb{Q}^+ = \left\{ 1, \frac{1}{2}, \dots, \frac{2}{1}, \frac{2}{2}, \frac{2}{3} \right\}$$

We can map \mathbb{Q}^+ with \mathbb{N}^2 however, mapping will not be injection as

$$\frac{1}{1} = \frac{2}{2} = \frac{3}{3} \dots \text{ or } \frac{1}{2} = \frac{2}{4} = \frac{3}{6} \dots$$

To proceed $\mathbb{N}^2 = \mathbb{N} \times \mathbb{N}$ is countable.

lets define, $g : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{Q}^+$ is mapping of ordered pairs $\langle m, n \rangle$ into rational no $\frac{m}{n}$

$$\frac{1}{1} \frac{1}{2} \frac{1}{3} \frac{1}{4} \dots$$

$$\frac{2}{1} \frac{2}{2} \frac{2}{3} \frac{2}{4} \dots$$

$$\frac{3}{1} \frac{3}{2} \frac{3}{3} \frac{3}{4} \dots$$

\vdots
 \vdots

$\Rightarrow \mathbb{Q}^+$ is countable

Similarly, \mathbb{Q}^- is also countable

So, $\mathbb{Q} = \mathbb{Q}^+ \cup \mathbb{Q}^-$ is countable.... (\because Union of two disjoint denumerable sets is again denumerable)



- Countable union of countable sets again countable.

2.4 Archimedean Property

If $x \in \mathbb{R}$, then there exists $n_x \in \mathbb{N}$ subject to $x < n_x$.

Proof. By method of contradiction,

$$x \in \mathbb{R}, n_x < x \forall n_x \in \mathbb{N}$$

$\therefore x$ is upper bound for set \mathbb{N}

By completeness property, the set which has upper bound must have supremum (says)

$$n_x < u \quad n_x \in \mathbb{N}$$

$$n_{x+1} \leq u \quad \forall n_x$$

$$n_x \leq u - 1 \quad \forall n_x$$

$\therefore u - 1$ is also upper bound $< u$ (by definition)

But we know that, Supremum is the least upper bound i.e there exists no other upper bound which is less than u .

So our assumption is wrong

Hence, $x < n_x, x \in \mathbb{R}$ □

Corollary 2.4.0.1. If $S = \left\{ \frac{1}{n}, n \in \mathbb{N} \right\}$ then $\inf S = 0$

Proof. $S \neq \phi$ and 0 is lower bound of S .

\therefore By completeness Property, set S has infimum (v)

Let, $\varepsilon \in \mathbb{R}, \frac{1}{\varepsilon} > 0 \Rightarrow \frac{1}{\varepsilon} \in \mathbb{R}$

\therefore By archemidian property

$\exists n \in \mathbb{N}, 0 < \frac{1}{\varepsilon} < n \Rightarrow 0 < \frac{1}{n} < \varepsilon \Rightarrow 0 \text{ is } \inf(S)$ □

Corollary 2.4.0.2. If $t > 0$, $\exists n_t \in \mathbb{N} \Rightarrow 0 < \frac{1}{n_t} < t$

Proof. $t > 0, \frac{1}{t} > 0 \Rightarrow \frac{1}{t} \in \mathbb{R}$

\therefore By archemidian property, $\exists n \in \mathbb{N}$ subject to $\frac{1}{t} < n_t, \exists n_t \in \mathbb{N}$

$\Rightarrow 0 < \frac{1}{n_t} < t$ □

Corollary 2.4.0.3. If $y > 0$, $\exists n_y \in \mathbb{N} \quad \exists n_{y-1} \leq y < n_y$

Proof. Given that $y > 0$ i.e $y \in \mathbb{R}$

$y < n_y, \exists n_y \in \mathbb{N} \dots$ By archemidian property

$$E_y = \{n/y < n, n \in \mathbb{N}\}$$

$\Rightarrow y$ is lower Bound of E_y

\Rightarrow least element of E_y is $\inf(n_y)$

$$\Rightarrow n_{y-1} \leq y < n_y$$

□

Theorem 2.4.1 (Density Theorem). *If x & y are any real numbers with $x < y$, then \exists a rational numbers $r \in \mathbb{Q}$ such that $x < r < y$*

Proof. assume $x > 0, x \in \mathbb{R}$

Given, $x < y \Rightarrow y - x > 0, y - x \in \mathbb{R}$

$\exists n \in \mathbb{N}, \frac{1}{n} < y - x \dots$ (corollary 2.4.0.2)

$$1 < ny - nx$$

$$nx + 1 < ny \quad (2.19)$$

Also, $x > 0 \Rightarrow n_x > 0$ then $\exists m \in \mathbb{N}$ such that $m - 1 \leq n_x < m, \dots$ (corollary 2.4.0.3)

from (2.19)

$$n_x < m \leq n_{x+1} < n_y$$

$$\Rightarrow n_x < m < n_y$$

$$\Rightarrow x < \frac{m}{n} < y$$

$$\Rightarrow x < r < y, \text{ where } r = \frac{m}{n} = \text{rational number}$$

-Hence Proved-

□

Corollary 2.4.1.1. *If x and y are any real numbers with $x < y$ then \exists an irrational number $r \in \mathbb{Q}^c \ni x < r < y$*

Proof. By density theorem,

If $x < y$ then $\exists r_1 \in \mathbb{Q} \ni x < r_1 < y$. Here $x < y$

$$\therefore \sqrt{2}x < \sqrt{2}y$$

$$\sqrt{2}x < r_1 < \sqrt{2}y$$

$$x < \frac{r_1}{\sqrt{2}} < y$$

$$x < r < y \quad \text{where} \quad r = \frac{r_1}{\sqrt{2}} = \text{irrational number}$$

-Hence Proved-



Intervals:-

- $[a, b] = \{x / a \leq x \leq b\} = \text{Closed}$
- $(a, b) = \{x / a < x < b\} = \text{Open}$
- $[a, b) = \{x / a \leq x < b\} = \text{Half Closed- Half Open}$
- $(a, b] = \{x / a < x \leq b\} = \text{Half Closed- Half Open}$

Intersection:-

Finite :- $\bigcap_{i=1}^n \left[0, \frac{1}{n}\right]$

Arbitrary:-

$$\bigcap_{i=1}^{\infty} = \{0\}$$

$$\bigcap_{n=1}^{\infty} \left(0, \frac{1}{n}\right) = \phi$$

$$\bigcap_{n=1}^{\infty} (n, \infty) = \phi$$

$$\bigcup_{n=1}^{\infty} (-n, n) = \{-\infty, \infty\}$$

$$\bigcap_{n=1}^{\infty} (-n, n) = \{-1, 1\}$$

$$\bigcap_{n=1}^{\infty} \left[-1, 1 + \frac{1}{n}\right] = [-1, 1]$$

$$\bigcup_{n=1}^{\infty} \left[-1, 1 - \frac{1}{n}\right] = [-1, 1)$$

$$\bigcap_{n=1}^{\infty} [-n, n] = [-1, 1]$$

$$\bigcap_{n=1}^{\infty} [-n, n] = (-\infty, \infty)$$

Theorem 2.4.2. \mathbb{R} is uncountable.

Proof. Assume that \mathbb{R} is countable so does $(0, 1)$ is countable.

We can write one-one correspondence with \mathbb{N} as,

$$b_1 = 0.a_{11}a_{12}a_{13}\dots \neq C$$

$$b_2 = 0.a_{21}a_{22}a_{23}\dots \neq C$$

$$b_3 = 0.a_{31}a_{32}a_{33}\dots \neq C$$

:

:

:

$$b_i = 0.a_{i1}a_{i2}a_{i3}\dots a_{ii} \neq C$$

$$b_i = 0.C_1C_2C_3\dots \in (0, 1)$$

$$C_1 \neq a_{11}$$

$$C_2 \neq a_{22}$$

$$C_3 \neq a_{33}$$

:

:

$$C_i \neq a_{ii}$$

As $C_i \neq a_i$ there does not exist any $C_i \neq C$

- \Rightarrow Our counting Scheme is wrong.
- \Rightarrow Our assumption is wrong.
- $\Rightarrow (0, 1)$ must be uncountable .
- $\Rightarrow \mathbb{R}$ is uncountable.

□

2.5 Cauchy Schwartz Inequality

Let $a_i, b_i \in \mathbb{R} \forall i$ then

$$\left(\sum_{i=1}^n a_i b_i \right)^2 \leq \left(\sum_{i=1}^n a_i^2 \right) \left(\sum_{i=1}^n b_i^2 \right)$$

Proof. let $x \in \mathbb{R}$ then,

$a_i x + b_i \in \mathbb{R} \dots \because a_i, b_i \in \mathbb{R}$

$\therefore (a_i x + b_i)^2 \geq 0$

$a_i^2 x^2 + 2a_i x b_i + b_i^2 \geq 0$

$$\Rightarrow \left(\sum_{i=1}^n a_i^2 \right) x^2 + 2 \left(\sum_{i=1}^n a_i b_i \right) x + \sum_{i=1}^n b_i^2 \geq 0$$

$$Ax^2 + 2Bx + C \geq 0 \tag{2.20}$$

where,

$$A = \sum_{i=1}^n a_i^2, B = \sum_{i=1}^n a_i b_i, C = \sum_{i=1}^n b_i^2$$

let $x = \frac{-B}{A}$

\therefore from (2.20)

$$A \left(\frac{B}{A} \right)^2 + 2B \left(\frac{-B}{A} \right) + C \geq 0 \Rightarrow \frac{B^2}{A} - \frac{2B^2}{A} + C \geq 0$$

$$\Rightarrow \frac{-B^2}{A} + C \geq 0$$

$$\Rightarrow C \geq \frac{B^2}{A}$$

$$\Rightarrow A \cdot C \geq B^2$$

$$\Rightarrow B^2 \geq A \cdot C$$

$$\therefore \left(\sum_{i=1}^n a_i b_i \right)^2 \leq \left(\sum_{i=1}^n a_i^2 \right) \left(\sum_{i=1}^n b_i^2 \right)$$

-Hence Proved-



Note:-

Equality hold if a_i and b_i is equal to zero.

If $x = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$ then $Ax^2 + 2Bx + C = 0$

Elements of Point Set Topology

3.1 Terminology and Notations

Definition 3.1.1 (Member of a set): *If an element x is in a set A , we write $x \in A$ and say that x is a member of A , or that x belongs to A . If x is not in A , we write $x \notin A$*

Definition 3.1.2 (Subset): *If every element of a set A also belongs to a set B , we say that A is a subset of B and write $A \subseteq B$ or $B \supseteq A$*

Definition 3.1.3 (Proper Subset): *We say that a set A is a proper subset of a set B if $A \subset B$, but there is at least one element of B that is not in A .*

Definition 3.1.4 (Equal Sets): *Two sets A and B are said to be equal, and we write $A = B$, if they contain the same elements. i.e. $A \subseteq B$ and $B \supseteq A$.*

A set is normally defined by either listing its elements explicitly, or by specifying a property that determines the elements of the set.

- The set of natural numbers $\mathbb{N} := \{1, 2, 3, \dots\}$,
- The set of integers $\mathbb{Z} := \{0, 1, -1, 2, -2, 3, -3, \dots\}$,
- The set of rational numbers $\mathbb{Q} = \{m/n : m, n \in \mathbb{Z} \text{ and } n \neq 0\}$,
- The set of real numbers \mathbb{R}

Definition 3.1.5 (Open Set): A subset G of \mathbb{R} is open in \mathbb{R} if for each $x \in G$ there exists a neighbourhood \forall of x such that $v \subseteq G$.

Definition 3.1.6 (Closed set): A subset f of \mathbb{R} is closed in \mathbb{R} if the complement f^c is open in \mathbb{R}

G is open iff for $x \in G \exists \epsilon > 0$

$$x \in (x - \epsilon_x, x + \epsilon_x) \subseteq G$$

e.g. $(-\infty, \infty) = \mathbb{R}$ - open as well as closed

$(0, 1)$ - open

(a, b) - open

$[a, \infty)$ - not open but closed

$[a, b]$ - not open but closed

ϕ - open and closed

$[a, b)$ - neither open nor closed

$(a, b]$ - neither open nor closed

\mathbb{Q} - not closed not open

\mathbb{N} - closed but not open

\mathbb{I} - closed but not open

Definition 3.1.7 (Interior point): For some $x \in S$ if \exists open interval $I_x \ni x \in I_x \subseteq S$ then x is called interior point of set S .

Definition 3.1.8 (Interior of Set): Collection of all interior point is called interior of set (S_i).
example $S = \{[0, 1], [0, 1), (0, 1]\}$, $S_i(0, 1)$

Theorem 3.1.1. Finite union of open sets is open.

Proof. let A and B be two finite open sets.

Claim- $A \cup B$ is open set.

$\therefore A$ & B be two open set.

$\Rightarrow \forall x \in A, \exists I_x \subseteq A$ and $\forall x \in B \exists I_x \subseteq B$

let $x \in A \cup B$

$x \in A$ or $x \in B$

$\therefore x \in I_x \subseteq A$ or $x \in I_x \subseteq B$

$\Rightarrow x \in I_x \subseteq A \cup B$

$\Rightarrow A \cup B$ is open set. □

Theorem 3.1.2. Finite intersection of open set is open.

Proof. let A & B be two open sets.

claim- $A \cap B$ is open.

let $x \in A \cap B$

$\therefore x \in A \text{ or } x \in B$

$\Rightarrow \exists I_x \ni x \in I_x \subseteq A \text{ and } x \in I_x \subseteq B$

$\Rightarrow x \in I_x \subseteq A \cap B$

$\Rightarrow A \cap B$ is open set. □

Theorem 3.1.3. *Arbitrary union of open sets is open.*

Proof. let $\{A_i\}_{i=1}^{\infty}$ be collection of open sets.

claim- $\bigcup_{i=1}^{\infty} A_i$ is open set

let $x \in \bigcup_{i=1}^{\infty} A_i$

$\Rightarrow x \in A_j$, for some $j \in I$

$\Rightarrow x \in I_x \subseteq A_j$, for some $j \in I$

$\Rightarrow x \in I_x \subseteq A_j \subseteq \bigcup_{i=1}^{\infty} A_i$

$\therefore \bigcup_{i=1}^{\infty} A_i$ is open set. □

Theorem 3.1.4. *Arbitrary intersection of open sets may or may not be open set.*

Proof. Set $S_n = \left(1 - \frac{1}{n}, 1 + \frac{1}{n}\right)$

$\bigcap_{n=1}^{\infty} S_n = \{1\}$ which is not open set. □

Theorem 3.1.5. *Finite union of two closed set is closed.*

Proof. let A & B closed set.

Claim- $A \cup B$ is closed set.

Since, A^C & B^C are open sets.

$\Rightarrow A^C \cap B^C$ is open set.

$\Rightarrow (A \cup B)^C$ is open set

$\Rightarrow A \cup B$ is closed set

□

Theorem 3.1.6. *Finte intersection of two closed set is closed.*

Proof. let A & B two closed set.

$\Rightarrow A^C$ & B^C are two open sets.

$\Rightarrow A^C \cup B^C$ is again open set.

$\Rightarrow (A \cap B)^C$ is open set

$\Rightarrow A \cap B$ is closed set

□

Theorem 3.1.7. *Arbitrary union of closed sets may not be closed.*

Example 17:

[Counter example] $A_n = [0, n], \cup A_n [0, \infty)$ -closed

$$A_n = \left[0, 1 - \frac{1}{n}\right]$$

$$A_1 = \{0\}$$

$$A_2 = \left[0, \frac{1}{2}\right] \cup A_n[0, 1) \text{ -not closed}$$

$$A_3 = \left[0, 1 - \frac{1}{3}\right] \dots (\because (-\infty, 0) \cup [1, \infty) \text{ -not open})$$

Theorem 3.1.8. *Every open set is union of open intervals.*

Proof. Suppose $S = \{x_1, x_2, x_3\}$

let S be an open set, $S = \{x_1, x_2, x_3 \dots\} = \{x_i\}$

for each $x_i \in I_{x_i} \subseteq S$

$$\{x_i\} \subseteq I_{x_i} \subseteq S$$

$$S = \cup \{x_i\} \subseteq \cup_{i \in I} I_{x_i} \subseteq S$$

Hence, Every open set is union of open intervals. □

Theorem 3.1.9. *Interior of set is open set.*

Proof. Given that, Let S^i is interior.

S is open set.

Claim- $x \in S^i, \exists I_x \in S^i \ni x \in I_x \subseteq S^i$

let $x \in S^i$

$\Rightarrow x$ is interior point of S

$\Rightarrow x \in I_x \subseteq S^i$

let $y \in I_x \Rightarrow y \in S \Rightarrow y \in I_x \subseteq S$

$\Rightarrow y \in S^i, y \in I_x$

$\therefore y$ is also interior point of S

this is true for all $y \in I_x$

$\therefore I_x \subseteq S^i \Rightarrow x \in I_x \subseteq S^i$

$\Rightarrow S^i$ is open set. □

Theorem 3.1.10. *Interior of set is largest open subset of set.*

Proof. let $S \subseteq \mathbb{R}$, S^i is interior set of S .

Claim:- $S^i \subseteq S$ is largest open set.

We prove this by method of contradiction

Assume that, T is largest open subset of set S .

(S^i is not largest) i.e $S^i \subseteq T \subseteq S$

S^i is proper subset of T

Since, $S^i \in T$

\exists some $x \in T, x \notin S^i$

Now, $x \in T \subseteq S \Rightarrow x$ is interior point of S

This contradicts to our assumption that $x \notin S^i$

\therefore Our assumption is wrong.

Hence, Interior of set is largest open subset. □

Definition 3.1.9 (Limit point of set): *Let c be the limit point of set S if for any $\varepsilon > 0$, $\exists x \in S \ni$*

$0 < |x - c| < \varepsilon$

i.e $-\varepsilon < x - c < \varepsilon$

$$i.e\ c - \varepsilon < x < c + \varepsilon$$

$$i.e\ x \in \delta_\varepsilon(c)$$

$$\Rightarrow \#(\delta_\varepsilon \cap A) \neq 0$$

$$example- S = \left\{ \frac{1}{n}, n \in \mathbb{R} \right\}, 0 \text{ is limit point of } S.$$

Definition 3.1.10 (Derived Set): *The set of all limit points of Set S is called the derived set of S and denoted by S'*

$$S' = \{c / c \text{ is limit point of } S\}$$

Definition 3.1.11 (Closed Set): *The set S is said to be closed set if it contains all of its limit points (i.e $S' \subseteq S$)*

Definition 3.1.12 (Closure Set): $S = S \cup S'$

Example 18:

$$1. S = \left\{ \frac{1}{n}, n \in \mathbb{N} \right\}, S = \{0\} \notin S \text{ [Neither open nor closed]}$$

$$\bar{S} = \left\{ \frac{1}{n}, n \in \mathbb{N} \right\} \cup \{0\}$$

$$2. S = \mathbb{Q}, S' = \mathbb{R} \bar{S} = \mathbb{R}$$

$$3. S = \mathbb{I}, S' = \phi \bar{S} = \mathbb{I}$$

$$4. S = \mathbb{N}, S' = \phi \bar{S} = \mathbb{N}$$

Note:- If S is closed then $S = \bar{S}$

Theorem 3.1.11. *Let $S \subseteq T$ then $S' \subseteq T'$*

Proof. Let $c \in S'$

$\Rightarrow c$ is limit point of S

for any $\varepsilon > 0, \delta_\varepsilon(c) \cap S \neq \phi$

$\Rightarrow \delta_\varepsilon(c) \cap T \neq \phi$ as $S \subseteq T$

$\Rightarrow c$ is limit point of T

$\Rightarrow c \in T'$

$\therefore S' \subseteq T'$

□

Theorem 3.1.12. *Show that $(S \cup T)' = S' \cup T'$*

Proof. To prove, $(S \cup T)' = S' \cup T'$

i.e

$$a) (S \cup T)' \subseteq S' \cup T'$$

$$b) S' \cup T' \subseteq (S \cup T)'$$

- first we prove part b)

$$S \subseteq S \cup T \Rightarrow S' \subseteq (S \cup T)'$$

$$T \subseteq S \cup T \Rightarrow T' \subseteq (S \cup T)'$$

$$\Rightarrow S' \cup T' \subseteq (S \cup T)' \quad (3.1)$$

- a) let $c \in (S \cup T)'$
 - $\Rightarrow c$ is limit points of $S \cup T$
 - $\Rightarrow \exists S \cup T \ni x \in \delta_\epsilon(c)$
 - $\Rightarrow x \in S \ni x \in \delta_\epsilon(c)$ or $x \in T \ni x \in \delta_\epsilon(c)$
 - $\Rightarrow c$ is limit point of S or c is limit point of T
 - $\Rightarrow c \in S'$ or $c \in T'$
 - $\Rightarrow c \in S' \cup T'$

$$(S \cup T)' \subseteq S' \cup T' \quad (3.2)$$

from (3.1) and (3.2)

$$(S \cup T)' = S' \cup T'$$

□

Theorem 3.1.13. *Finite intersection of two closed set is closed.*

Proof. let S & T be two closed sets.

$$\therefore S' \subseteq S \text{ and } T' \subseteq T$$

Claim: $S \cap T$ is closed

$$\text{i.e. } (S \cap T)' \subseteq (S \cap T)$$

We know,

$$S \cap T \subseteq S \Rightarrow (S \cap T)' \subseteq S' \subseteq S$$

$$S \cap T \subseteq T \Rightarrow (S \cap T)' \subseteq T' \subseteq T$$

$$(S \cap T)' \subseteq (S \cap T)$$

$\therefore S \cap T$ is closed set. □

Theorem 3.1.14. let S & T be subsets of \mathbb{R} , $S' \cap T'$ may or may not be subset of $S \cap T'$

$$\text{Proof. } \therefore S' = [1, 2], T' = [2, 3]$$

$$(S \cap T) = \emptyset \text{ \& } S' \cap T' = \{2\}$$

$$\Rightarrow (S' \cap T')' = \emptyset$$

$$\therefore S' \cap T' \not\subseteq (S' \cap T')'$$

□

Definition 3.1.13 (Dense Set): A Subset $A \subseteq \mathbb{R}$ is said to be dense set in \mathbb{R} if every point of \mathbb{R} is point of A or limit point of \mathbb{R} or equivalently if closure of A is \mathbb{R}

$$\overline{A} = A' \cup A = \mathbb{R}$$

- A set A is said to be dense in itself if $\overline{A} = A$
- A set A is said to be nowhere dense relative to \mathbb{R} if no neighborhood of \mathbb{R} is contained in the closure of A
- A set is said to be perfect if it is identical with its derived set or equivalently a set which is closed and dense in itself.

Theorem 3.1.15. *Set is closed if and only if its complement is open.*

Proof. a) let S be closed set

To prove- S^c is open.

let $x \in S^c$

$\Rightarrow x$ is not limit point of S ($\overline{S} = S$)

for some $\varepsilon > 0$, $V_\varepsilon(x) \cap S = \phi$

$(x - \varepsilon, x + \varepsilon) \subseteq S^c$

$\therefore S^c$ is open.

b) let S^c is open set

To prove- S is closed set

By method of contradiction,

Assume that S is not closed.

$\therefore \exists$ some limit point of x of $S \ni x \notin S$

$\Rightarrow x \in S^c$

for some $\varepsilon > 0 \exists (x - \varepsilon, x + \varepsilon) \subseteq S^c \dots (\because S^c \text{ is open set})$

$\therefore \nu_\varepsilon(x) \cap S = \phi$

which is not possible as x is limit point

\Rightarrow Our Assumption is that $x \ni S$ is wrong

\Rightarrow All limit point of S are in S

\Rightarrow is closed set.

□

Theorem 3.1.16. *Derived set of set is closed.*

Proof. let $S \subseteq \mathbb{R}$, S' is derived set of S .

To prove- S' is closed i.e $(S')' \subseteq S' = S' /$

let $c \in S'' \Rightarrow c$ is limit point of S'

i.e every ε - neighborhood ν of c contains atleast one point x of $S' \ni x \neq c$

i.e $x \in S' \Rightarrow x$ is limit point of set S .

\therefore Every ε neighborhood ν of x contains atleast one point of S .

As $x \in \nu$, ν is also a ε neighborhood of x

$\therefore \nu$ contains atleast one point of S .

In this way, we can prove that, every ε neighborhood ν of c contains infinitely many points of S .

$\therefore C$ is limit point of set S .

Also $c \in S'$

As $c \in S'' \Rightarrow c \in S'$, $S'' \subseteq S' \Rightarrow S'$ is closed set when $S'' = \phi$

then $S'' \subseteq S' \Rightarrow S'$ is closed set. □

3.2 Compact Set

Definition 3.2.1 (Open Cover): Let A be a subset of \mathbb{R} . An open cover of A is an collection $G = \{G_\alpha\}$ of open sets in \mathbb{R} whose union contains A i.e

$$A \subseteq \bigcup_\alpha G_\alpha$$

Definition 3.2.2 (Subcover): if G' is subcollection of sets from G such that the union of sets in G' also contains A then G' is called a subcover of G

Definition 3.2.3 (Finite Subcover): A subset k of \mathbb{R} is said to be compact if every open cover of \mathbb{R} has finite subcover.

Example 19:

1. $S = (0, 1), G_i = \left(0, 1 - \frac{1}{i}\right)$
 $\cap G_i = (0, 1) \supsetneq (0, 1)$
 $\cap G_i = \left(0, 1 - \frac{1}{n}\right) \not\supseteq (0, 1)$
 $\therefore (0, 1)$ is not compact
2. \mathbb{N} is not compact

3.3 Heine Borel theorem

Theorem 3.3.1 (Heine Borel theorem). *The set k is compact set if and only if it is closed & bounded.*

Proof. Given that, k is compact set.

i.e Every open cover exists finite subcover.

claim- k is bounded & closed.

1. k is bounded

$$G_i = (-i, i), G = \mathbb{R}$$

$$\cup_{i=1}^n G_i = (-n, n), k \subseteq (-n, n)$$

$\therefore k$ is bounded

2. k is closed i.e k^c is open

let $x \in k^c$

$$G_n = \left(-\infty, x - \frac{1}{n}\right) \cup \left(x + \frac{1}{n}, \infty\right)$$

$$G_1 = (-\infty, x - 1) \cup (x + 1, \infty)$$

$$G_2 = \left(-\infty, x - \frac{1}{2}\right) \cup \left(x + \frac{1}{2}, \infty\right)$$

...

...

...

$$G_n = \left(-\infty, x - \frac{1}{n}\right) \cup \left(x + \frac{1}{n}, \infty\right)$$

$\therefore k$ is closed.

Hence, from a) and b),

k is compact if and only if it is closed and bounded.



Sequence and Series

Definition 4.0.1 (Sequence and Series): *A sequence of real numbers is function defined on the set \mathbb{N} whose range is contained in the set $\mathbb{R}(x : \mathbb{N} \rightarrow \mathbb{R})$*

Denoted by $x, (x_n), (x_n, n \in \mathbb{N})$

example $\frac{1}{n}, \frac{1}{n^2}, 2n, n^2 + 1, n^2 - n$

- *Constant Sequence-* $x_n = x, \forall n \in \mathbb{N}$
- *Increasing Sequence-* $x_n \leq x_{n+1}, \forall n \in \mathbb{N}$
- *Strictly increasing sequence-* $x_n < x_{n+1} \forall n \in \mathbb{N}$
- *Decreasing Sequence-* $x_n \geq x_{n+1}, \forall n \in \mathbb{N}$
- *Strictly Decreasing Sequence-* $x_n > x_{n+1}, \forall n \in \mathbb{N}$

Definition 4.0.2 (Fibonacci Sequence): $x_1, x_2, x_{n+2} = x_{n+1} + x_n$

- Limit of Sequence- A Sequence $(x_n) \in \mathbb{R}$ is said to be convergent to $x \in \mathbb{R}$ or x is said to be limit of (x_n) if for every $\varepsilon > 0 \exists k(\varepsilon) \in \mathbb{N}$ such that

$$|x_n - x| < \varepsilon, \forall n \geq k(\varepsilon)$$

If sequence has limit, we say that sequence is convergent. If it has no limit has no limit, we say that is divergent.

$$\lim(x_n) = x \text{ or } x_n \rightarrow x$$

- Oscillating Sequence: $(x_n) = (-1)^n, n \in \mathbb{N}$ - (non convergent)

$$(x_n) = \frac{(-1)^n}{n}, n \in \mathbb{N}$$

Definition 4.0.3 (Uniqueness of limit point): A sequence in \mathbb{R} have atmost limit point one.

let x_1 & x_2 be two limit points of x_n

\therefore for any $\varepsilon > 0 \forall, n \geq k_1(\varepsilon) \& |x_n - x_1| < \varepsilon$

$\exists k_1(\varepsilon) \in \mathbb{N} \ni |x_n - x_1| < \varepsilon, \forall n \geq k_1(\varepsilon)$

$\exists k_2(\varepsilon) \in \mathbb{N} \ni |x_n - x_2| < \varepsilon, \forall n \geq k_2(\varepsilon)$

$$k(\varepsilon) = \max(k_1(\varepsilon), k_2(\varepsilon))$$

$$\forall n \in \mathbb{N} \ni n \geq k(\varepsilon)$$

$$|x_1 - x_2|$$

$$= |x_1 - x_n + x_n - x_2|$$

$$\leq |x_n - x_1| + |x_n - x_2|$$

$$\leq \varepsilon + \varepsilon$$

$$\leq 2\varepsilon$$

As this statement is true for any $\varepsilon > 0$, $x_1 = x_2$

Hence, Sequence have atmost one limit point.

Definition 4.0.4 (Tail Sequence): If $\{x_1, x_2, \dots\}$ is sequence of real numbers and if m is given natural number then m – tail of x_n is sequence

$$x_m = \{x_{m+n} / x_{m+1}, x_{m+2} \dots\}$$

Theorem 4.0.1. Let x_n be sequence of real numbers and let $m \in \mathbb{N}$ then m – tail x_m of x_n converges if & only if x_n converges.

Proof. Let $x_n \rightarrow x$ i.e $\lim_{n \rightarrow \infty} x_n = x$

\Rightarrow for any $\varepsilon > 0$, $\exists k(\varepsilon) \in \mathbb{N}$

such that $|x_n - x| < \varepsilon$, $\forall n \geq k(\varepsilon)$

$\Rightarrow x - \varepsilon < x_n < x + \varepsilon$, $\forall n \geq k(\varepsilon)$

$\Rightarrow x - \varepsilon < x_k, x_{k+1}, \dots < x + \varepsilon$

let $y_n = x_{m+n}$, n

$\Rightarrow x - \varepsilon < y_{k-m}, y_{k+1-m}, \dots < x + \varepsilon$

$\Rightarrow x - \varepsilon < y_n < x + \varepsilon \forall n \geq k(\varepsilon) - m = k_1(\varepsilon)$

$\Rightarrow |y_n - x| < \varepsilon \forall n \geq k_1(\varepsilon)$

$y_n \rightarrow x$

-Hence proved-



Theorem 4.0.2. Let x_n be a sequence of real numbers and $x \in \mathbb{R}$ if a_n is sequence of positive real numbers with $\lim a_n = 0$ and if for some constant $c > 0$ and some $m \in \mathbb{N}$, we have $|x_n - x| \leq ca_n, \forall n \geq m$ then it follows that $\lim x_n = x$

Proof. Given that $\lim a_n = 0$

i.e $a_n \rightarrow 0$

\therefore by definition, for any $\varepsilon > 0, \frac{\varepsilon}{c} (\because c > 0)$

$\exists k(\varepsilon) \in \mathbb{N}$ such that

$$|a_n - 0| < \frac{\varepsilon}{c}$$

$$a_n < \frac{\varepsilon}{c} \dots (\because a_n > 0)$$

let $k_1(\varepsilon) = \max(m_1, k_1(\varepsilon))$

$\forall n \geq k_1(\varepsilon)$

$$|x_n - x|$$

$$\leq ca_n$$

$$\leq c(\varepsilon/c)$$

$$\leq \varepsilon, \forall n \geq k_1(\varepsilon)$$

$\therefore x_n \rightarrow x$

□

Definition 4.0.5 (Bounded Sequence): A Sequence of real numbers x_n is said to be bounded if $\exists m > 0$ such that $|x_n| \leq m, \forall n \in \mathbb{N}$

Theorem 4.0.3. The Convergent sequence of real numbers is bounded.

Proof. let $x_n \rightarrow x$

\therefore by definition, for any $\varepsilon > 0$, $\exists k(\varepsilon) \in \mathbb{N}$

such that $|x_n - x| < \varepsilon, \forall n \geq k(\varepsilon)$

$\therefore x - \varepsilon < x_n < x + \varepsilon, \forall n \geq k(\varepsilon)$

let

$$M = \max\{|x_1|, |x_2|, \dots, |x_k|, x + \varepsilon\}$$

$\therefore |x_n| \leq M, \forall n$

$\Rightarrow x_n$ is bounded.

-Hence Proved-

□

Theorem 4.0.4. a) Let x_n and y_n be sequence of real numbers that converges to x and y respectively and let $c \in \mathbb{R}$ then, the sequence $X + Y, X - Y, XY$ and CX converges to $x + y, x - y, xy$ and cx

b) If $x_n \rightarrow x$ and z_n is sequence of non-zero real numbers that converges to z and if $z \neq 0$ then

$$\frac{X}{Z} \rightarrow \frac{x}{z}$$

Proof. a) given that $x_n \rightarrow x$

\therefore by definition, for any $\varepsilon > 0, \frac{\varepsilon}{2} > 0$

$\exists k_1(\varepsilon) \in \mathbb{N}$ such that

$|x_n - x| < \varepsilon/2, \forall n \geq k_1(\varepsilon)$

also, $y_n \rightarrow y$

\therefore by definition, for any $\varepsilon > 0, \frac{\varepsilon}{2} > 0$

$\exists k_2(\varepsilon) \in \mathbb{N}$ such that

$$|y_n - y| < \varepsilon/2, \forall n \geq k_2(\varepsilon)$$

let $k(\varepsilon) = \max(k_1(\varepsilon), k_2(\varepsilon))$

$\therefore \forall n \geq k(\varepsilon)$

$$\begin{aligned} \text{i) } |(x_n + y_n) - (x + y)| &= |x_n - x + y_n - y| \\ &\leq |x_n - x| + |y_n - y| \dots (\text{triangular inequality}) \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \end{aligned}$$

$$\leq \varepsilon$$

$$\therefore (x_n + y_n) \rightarrow (x + y)$$

$$\begin{aligned} \text{ii) } |(x_n - y_n) - (x - y)| &= |x_n - x - y_n + y| \\ &\leq |x_n - x| + |y_n - y| \dots (\text{triangular inequality}) \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \end{aligned}$$

$$\leq \varepsilon$$

$$\therefore (x_n - y_n) \rightarrow (x - y)$$

iii) $x_n \rightarrow x$

\therefore by definition, for any $\varepsilon > 0, \frac{\varepsilon}{2M} > 0, \dots (\because M > 0)$

$\exists k_1(\varepsilon) \in \mathbb{N}$ such that

$$|x_n - x| < \varepsilon/2M, \forall n \geq k_1(\varepsilon)$$

also, $y_n \rightarrow y$

\therefore by definition, for any $\varepsilon > 0, \frac{\varepsilon}{2|x|} > 0, \dots (\because |x| > 0)$

$\exists k_2(\varepsilon) \in \mathbb{N}$ such that

$$|y_n - y| < \varepsilon/2|x|, \forall n \geq k_2(\varepsilon)$$

let $k(\varepsilon) = \max(k_1(\varepsilon), k_2(\varepsilon))$

$\therefore \forall n \geq k(\varepsilon)$

$$|(x_n y_n) - (xy)| = |x_n y_n - x y_n + x y_n - xy|$$

$$\leq |y_n||x_n - x| + |x_n||y_n - y| \dots \text{(triangular inequality)}$$

$$\leq M \frac{\varepsilon}{2M} + |x| \frac{\varepsilon}{2}$$

$$\leq \varepsilon$$

$$\therefore x_n y_n \rightarrow xy$$

iv) $x_n \rightarrow x$

\therefore by definition, for any $\varepsilon > 0, \frac{\varepsilon}{|c|} > 0, \dots (\because |c| > 0)$

$\exists k(\varepsilon) \in \mathbb{N}$ such that

$$|x_n - x| < \frac{\varepsilon}{|c|}, \forall n \geq k(\varepsilon)$$

$$|(cx_n - cx)| = |c| \cdot |x_n - x|$$

$$\leq |c| \cdot \frac{\varepsilon}{|c|}$$

$$\leq \varepsilon$$

$$\therefore cx_n \rightarrow cx$$

b) $x_n \rightarrow x$ and $z_n \rightarrow z$

\therefore by definition, for any $\varepsilon > 0, \varepsilon \cdot |z| \cdot m > 0$

$\exists k(\varepsilon) \in \mathbb{N}$ such that

$$|z_n - z| < \varepsilon \cdot |z| \cdot m, \forall n \geq k(\varepsilon)$$

$$\text{let } y_n = \frac{1}{z_n}$$

$$\text{consider, } |(y_n - x)| = \left| \frac{1}{z_n} - \frac{1}{z} \right|$$

$$= \frac{|z - z_n|}{|z_n \cdot z|}$$

$$\leq \frac{\varepsilon \cdot |z| \cdot m}{|z_n| \cdot |z|}$$

$$\leq \frac{\varepsilon \cdot m}{|z_n|}$$

$$\leq \varepsilon \dots (\because z_n \text{ is bounded } m < z_n < m)$$

$$\therefore \frac{1}{x_n} \rightarrow \frac{1}{z}$$

$$\therefore y_n \rightarrow y$$

we know that, $x_n y_n \rightarrow xy \dots (\because \text{if } x_n \rightarrow x \text{ \& } y_n \rightarrow y \text{ then } x_n y_n \rightarrow xy)$

$$\therefore \frac{x_n}{z_n} \rightarrow \frac{x}{y}$$

-Hence Proved-



Theorem 4.0.5. If $x_n \rightarrow x$ and if $x_n \geq 0, \forall n \in \mathbb{N}$ then $x = \lim x_n \geq 0$

Proof. Given that, $x_n \rightarrow x$

\therefore by definition, for any $\varepsilon > 0$

$\exists k(\varepsilon) \in \mathbb{N}$

such that $|x_n - x| < \varepsilon, \forall n \geq k(\varepsilon)$

we will prove this by method of contradiction.

let if possible $x < 0$

$\therefore -x > 0$

Assume, $0 < \varepsilon < -x$

$\therefore x - \varepsilon < 0$ and $x + \varepsilon < 0$ &

$\therefore x - \varepsilon < x_n < x + \varepsilon, \forall n \geq k(\varepsilon)$

$\therefore x_n < 0$

which contradicts to given statement that $x_n \geq 0$

\therefore Our assumption is wrong.

$$\therefore x = \lim x_n \geq 0$$

-Hence Proved-



Theorem 4.0.6. If $x_n \rightarrow x$, $y_n \rightarrow y$ are convergent sequence of real numbers and if $x_n \leq y_n, \forall n \in \mathbb{N}$ then $\lim x_n \leq \lim y_n$

Proof. Given that, $x_n \rightarrow x$, and $y_n \rightarrow y$ also, $x_n \leq y_n, \forall n$

$$\Rightarrow y_n - x_n \geq 0$$

$$\Rightarrow z_n \geq 0$$

Now, $y_n - x_n \rightarrow y - x$ (say z)

As, $z_n \geq 0, z_n \rightarrow z$

$\therefore z \geq 0 \dots$ (by above theorem)

$$\therefore y - x \geq 0$$

$$\therefore y \geq x$$

$$\therefore x \leq y$$

-Hence Proved



Theorem 4.0.7. If x_n is convergent to some $x \in \mathbb{R}$ and $a \leq x_n \leq b, \forall n$ then $a \leq x \leq b$

Proof. Given that, $x_n \rightarrow x$ and $a \leq x_n \leq b$

let $a_n = a$ & $b_n = b$

i.e $a_n \rightarrow a$ i.e $b_n \rightarrow b$

$$\therefore a_n \leq x_n \leq b_n$$

i.e $a_n \leq x_n$ & $x_n \leq b_n$

$\lim a_n \leq \lim x_n$ & $\lim x_n \leq b_n \dots$ (by above theorem)

$a \leq x$ and $x \leq b \therefore a \leq x \leq b$

-Hence Proved-



4.1 Squeeze Theorem

Theorem 4.1.1. Suppose x_n, y_n and z_n are sequence of real numbers $\ni x_n \leq y_n \leq z_n, \forall n \in \mathbb{N}$ and $\lim x_n = \lim z_n = w$ then y_n is convergent and $\lim x_n = \lim y_n = \lim z_n$.

Proof. Given that, $x_n \leq y_n \leq z_n, \forall n$

let, $\lim x_n = \lim z_n = w$

i.e $x_n \rightarrow w$ and $z_n \rightarrow w$

\therefore by definition, for any $\varepsilon > 0 \exists$

$k_1(\varepsilon) \in \mathbb{N}$ and $k_2(\varepsilon) \in \mathbb{N}$ such that

$|x_n - w| < \varepsilon, \forall n \geq k_1(\varepsilon)$ and $|z_n - w| < \varepsilon, \forall n \geq k_2(\varepsilon)$

$\therefore w - \varepsilon \leq x_n \leq w + \varepsilon$ and $w - \varepsilon \leq z_n \leq w + \varepsilon$

$\therefore w - \varepsilon \leq x_n \leq y_n$ and $y_n \leq z_n \leq w + \varepsilon$

$\therefore w - \varepsilon \leq x_n \leq y_n \leq z_n \leq w + \varepsilon$

i.e $w - \varepsilon \leq y_n \leq w + \varepsilon$

i.e $|y_n - w| < \varepsilon, \forall n \geq k(\varepsilon) = \max(k_1(\varepsilon), k_2(\varepsilon))$

$\therefore y_n \rightarrow w$

$\therefore \lim x_n = \lim y_n = \lim z_n = w$



Theorem 4.1.2. Given that, $x_n \rightarrow x$ then Show that,

$$a) |x_n| \rightarrow |x|$$

$$b) \sqrt{x_n} \rightarrow \sqrt{x}$$

Proof. Given that, $x_n \rightarrow x$

\therefore by definition, for any $\varepsilon > 0$, $\exists k(\varepsilon) \in \mathbb{N}$

such that $|x_n - x| < \varepsilon, \forall n \geq k(\varepsilon)$

consider,

$$||x_n| - |x||$$

$$\leq |x_n - x| \dots \text{(by corollary of triangular inequality)}$$

$$\leq \varepsilon$$

$$|x_n| \rightarrow |x|$$

Given that, $x_n \rightarrow x$

\therefore by definition, for any $\varepsilon > 0$, $\sqrt{x} > 0$, $\frac{\varepsilon}{\sqrt{x}} > 0$, $\varepsilon\sqrt{x} > 0$

$\exists k(\varepsilon) \in \mathbb{N}$ such that

$$|x_n - x| \leq \varepsilon\sqrt{x}, \forall k(\varepsilon) \in \mathbb{N}$$

As, $\sqrt{x} > 0$

$$\therefore 0 < \sqrt{x} < \sqrt{x_n} + \sqrt{x}$$

$$\frac{1}{\sqrt{x}} > \frac{1}{\sqrt{x_n} + \sqrt{x}} \quad (4.1)$$

$$\begin{aligned}
& |\sqrt{x_n} - \sqrt{x}| \\
&= \frac{|\sqrt{x_n} - \sqrt{x}| \cdot |\sqrt{x_n} + \sqrt{x}|}{|\sqrt{x_n} - \sqrt{x}|} \\
&= \frac{|x_n - x|}{\sqrt{x_n} + \sqrt{x}} \\
&\leq \frac{|x_n - x|}{\sqrt{x}} \dots \text{(from 4.1)} \\
&\leq \frac{\varepsilon \cdot \sqrt{x}}{\sqrt{x}} \\
&\leq \varepsilon \\
&\therefore \sqrt{x_n} \rightarrow \sqrt{x}
\end{aligned}$$

□

4.2 Monotone Sequence

- *Monotone decreasing:* $x_n \geq x_{n+1}, \forall n$
- *Monotone increasing:* $x_n \leq x_{n+1}, \forall n$

x_n is called as monotone if it is increasing or decreasing.

Theorem 4.2.1 (Monotone Convergence theorem). *A monotone sequence of real numbers is convergent if and only if*

a) If x_n is bounded increasing sequence

$$\lim(x_n) = \sup\{x_n, n \in \mathbb{N}\}$$

b) If x_n is bounded decreasing sequence

$$\lim(x_n) = \inf\{x_n, n \in \mathbb{N}\}$$

Proof. We know that, Convergent sequence must be bounded.
Conversely, let x_n be monotone bounded sequence.

a) Assume x_n is increasing and bounded.

As x_n is bounded $M \in \mathbb{R}, |x_n| \leq M, \forall n$

let, $S = \{x_n, \forall n \in \mathbb{N}\}$

M upper bound of S

\therefore By completeness property, $\exists x^* \in \mathbb{R}$

$$\ni x^* = \sup\{x_n, n \in \mathbb{N}\}$$

$$\therefore x_n \leq x^* \forall n$$

for any $\varepsilon > 0$ $x^* - \varepsilon$ is not supremum of S

$$\therefore x^* - \varepsilon < x_k \leq x^*, \text{ for some } k$$

$$\Rightarrow x^* - \varepsilon < x_k \leq x_{k+1} \leq x_{k+2} \leq \dots \leq x^*$$

$$\therefore x^* - \varepsilon < x_n < x^*, \forall n \geq k(\varepsilon)$$

$$\therefore x^* - \varepsilon < x_n < x^* + \varepsilon$$

$$\therefore x^* = \lim x_n$$

i.e x_n is convergent sequence.

b) Assume x_n is decreasing and bounded.

As x_n is bounded $M \in \mathbb{R}, |x_n| \leq M, \forall n$

let, $S = \{x_n, \forall n \in \mathbb{N}\}$

$-M$ lower bound of S

\therefore By completeness property, $\exists x^* \in \mathbb{R}$

$\ni x^* = \inf\{x_n, n \in \mathbb{N}\}$

$\therefore x_n \geq x^* \forall \mathbb{N}$

for any $\varepsilon > 0$ $x^* + \varepsilon$ is not lower bound of S

$\therefore x^* < x_k < x^* + \varepsilon$, for some k

$\Rightarrow x^* < \dots \leq x_{k+2} \leq x_{k+1} \leq x_k < x^* + \varepsilon$

$\therefore x^* < x_n < x^* + \varepsilon$

$\therefore x^* - \varepsilon < x_n < x^* + \varepsilon$

$\therefore x^* = \lim x_n$

i.e x_n is convergent sequence.

□

Theorem 4.2.2. If x_n converges to x then any subsequences x_{n_k} of x_n also converges to x .

Proof. for any $\varepsilon > 0 \exists k(\varepsilon) \in \mathbb{N}$ such that,

$|x_n - x| < \varepsilon, \forall n \geq k(\varepsilon)$

let subsequence $x_{n_k} = \{x_{n_1}, x_{n_2}, x_{n_3}, \dots\}$

As $x_n \rightarrow x \Rightarrow x - \varepsilon < x_n < x + \varepsilon$

Also, $n_k \geq n \geq k(\varepsilon)$

$$\rightarrow x - \varepsilon < x_{n_k} < x + \varepsilon, \forall n_k \geq k(\varepsilon)$$

$$\therefore x_{n_k} \rightarrow x$$

□

Theorem 4.2.3 (Monotone Subsequence theorem). *If x_n is sequence of real numbers then there is subsequence of x_n that is monotone.*

Proof. We will say that m^{th} term x_m is a peak if $x_m \geq x_n \forall n \geq m$.

Note that, In a decreasing sequence, every term is peak while in increasing sequence, no term is peak.

Case-1:-

x_n has infinitely many peaks. In this case, we list the peaks by,

$$x_{m_1} \geq x_{m_2} \geq x_{m_3} \dots \geq x_{m_k}, \dots$$

\therefore subsequence x_{m_k} is decreasing subsequence of x_n .

Case-2:-

x_n has finitely number of peaks.

let these peaks be denoted by,

$$x_{m_1}, x_{m_2}, x_{m_3} \dots x_{m_r}$$

let $S_1 = m_r + 1$ be the first index beyond the last peak since x_{S_1} is not peak $\exists S_2 > S_1$

$\ni x_{S_1} < x_{S_2}$ since x_{S_2} is not peak $\exists S_3 > S_2$

$\ni x_{S_2} < x_{S_3}$ continuing this way, we obtain an increasing sequence.

□

Theorem 4.2.4 (Bozano- Weistress theorem). *A bounded sequence of real numbers has convergent subsequence.*

Proof. Let x_n be bounded sequence.

\therefore by monotone subsequence theorem,

$\exists x_{n_k}$ subsequence of x_n that is monotone.

As x_n is bounded x_{n_k} is also bounded

\therefore by monotone convergence theorem,

x_{n_k} is monotone and bounded so convergent. □

4.3 Cauchy Sequence

Definition 4.3.1 (Cauchy Sequence): *A sequence of real numbers is said to be cauchy if for every $\varepsilon > 0$, $\exists H(\varepsilon) \in \mathbb{N} \ni |X_n - x_m| < \varepsilon, \forall n, m \geq H(\varepsilon)$*

Theorem 4.3.1. *Every convergent sequence is cauchy.*

Proof. let $x_n \rightarrow x$

for any $\frac{\varepsilon}{2} > 0$, $\exists k(\varepsilon) \in \mathbb{N}$

$\ni |X_n - x| < \frac{\varepsilon}{2}, \forall n \geq k(\varepsilon)$ let, $k_1, k_2 \in \mathbb{N}$ such that $\forall k_1, k_2 \geq k(\varepsilon)$

$$|X_{k_1} - x_{k_2}|$$

$$\leq |X_{k_1} - x| + |X_{k_2} - x|$$

$$\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

$$\leq \varepsilon$$

Hence, every convergent sequence is cauchy. □

Theorem 4.3.2. *A cauchy sequence of real numbers is bounded.*

Proof. let x_n be cauchy sequence and
let $\varepsilon = 1$ if $H = H(1)$ and $n \geq H$ then $n \geq H$.

$$M = \sup\{|x_1|, |x_2|, |x_3|, \dots, |x_{H-1}|, |x_H| + 1\}$$

then it follows that $|x_n| \leq M \forall n$

\therefore cauchy sequence of real numbers is bounded. □

Definition 4.3.2 (Cauchy convergence criterion): *A Sequence of real numbers is convergent if and only if it is cauchy sequence.*

Definition 4.3.3 (Contractive Sequence): *We say that the sequence x_n of real numbers is contractive sequence if there exists a constant c , $0 < c < 1$ such that,*

$$|x_{n+2} - x_{n+1}| \leq c \cdot |x_{n+1} - x_n|, \forall n$$

Theorem 4.3.3. *Contractive sequence is cauchy sequence.*

Proof. let x_n is contractive sequence

$\therefore \exists c, 0 < c < 1$ such that

$$|x_{n+2} - x_{n+1}| \leq c \cdot |x_{n+1} - x_n|, \forall n$$

for $\varepsilon > 0$ choose $k(\varepsilon) \in \mathbb{N} \ni$ for $m > n$

$$|x_m - x_n|$$

$$= |x_m - x_{m-1} + x_{m-1} - x_{m-2} + \dots + x_{n+1} - x_n|$$

$$\leq |x_m - x_{m-1}| + |x_{m-1} - x_{m-2}| + \dots + |x_{n+1} - x_n|$$

$$\leq c|x_{m-1} - x_{m-2}| + c|x_{m-2} - x_{m-3}| + \dots + c|x_n - x_{n-1}|$$

$$\leq c^2|x_{m-2} - x_{m-3}| + c^2|x_{m-3} - x_{m-4}| + \dots + c|x_n - x_{n-1}|$$

$$\begin{aligned} &\leq (c^{m-n} + c^{m-n-1} + \dots + c)|x_n - x_{n-1}| \\ &\leq \frac{c(1 - c^{m-n})}{1 - c}|x_n - x_{n-1}| \end{aligned}$$

$$\leq \varepsilon \quad \because \frac{c(1 - c^{m-n})}{(1 - c)} < 1$$

$\therefore x_n$ is cauchy sequence. □

Divergent Sequence Let x_n be sequence of real numbers

a) $x_n \rightarrow +\infty$ and $\lim x_n = +\infty$

if every $\alpha \in \mathbb{R}$ there exists a natural number $k(\alpha)$ such that if $n \geq k(\alpha)$, then $x_n > \alpha$.

b) $x_n \rightarrow -\infty$ and $\lim x_n = -\infty$

if every $\beta \in \mathbb{R}$ there exists a natural number $k(\beta)$ such that if $n \geq k(\beta)$, then $x_n < \beta$.

We say that x_n is properly divergent if $\lim x_n = +\infty$ or $-\infty$

4.4 Infinite Series

Definition 4.4.1 (Infinite Series): If x_n is sequence in \mathbb{R} , then the infinite series generated by x_n is sequence S_n

$$S_1 = x_1$$

$$S_2 = x_1 + x_2$$

.

•
•

$$S_n = x_1 + x_2 + \dots x_n$$

Denoted by $\sum x_n$ or $\sum_{n=1}^{\infty} x_n$

Example 20:

$$1. \sum_{n=0}^{\infty} r_n = 1 + r + r^2 + \dots$$

$$2. \sum_{n=1}^{\infty} (-1)^n = (-1) + 1 + (-1) + \dots$$

$$3. \sum \frac{1}{n(n+1)} = \frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4}$$

Theorem 4.4.1 (The n^{th} term test). *if $\sum x_n$ converges then $\lim x_n = 0$*

Proof. By definition $\sum x_n$ converges if S_n converges,

$$\text{Since } S_n = \sum_{i=1}^n x_i$$

$$\therefore x_n = S_n - S_{n-1}$$

$$\therefore \lim x_n = \lim S_n - \lim S_{n-1} = 0$$

□

Definition 4.4.2 (Cauchy Criterion for Series): *The series $\sum x_n$ converges if and only if $\forall \varepsilon > 0$,*

$\exists M(\varepsilon) \in \mathbb{N} \ni \text{if } m > n \geq M(\varepsilon) \text{ then}$

$$|S_m - S_n| = |x_{n+1} + x_{n+2} + \dots + x_m| < \varepsilon$$

Theorem 4.4.2. *let x_n be a sequence of non-negative real numbers then the series $\sum x_n$ converges if and only if the sequence S_k of partial sum is bounded.*

$$\sum x_n = \lim S_k = \sup\{S_k : k \in \mathbb{N}\}$$

Theorem 4.4.3. *Show that, $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$*

Proof. Suppose,

$$S_{n+1} = 1 + r + \dots + r^n$$

$$S_n = 1 + r + \dots + r^{n-1}$$

$$rS_n = (r + r^2 + \dots + r^n)$$

$$\therefore S_{n+1} - rS_n = 1$$

$$\therefore \lim_{n \rightarrow \infty} (S_{n+1} - rS_n) = 1 \lim_{n \rightarrow \infty} 1 = 1$$

$$\therefore (S - rS) = 1 \dots \text{where } S = \sum_{n=0}^{\infty} r^n$$

$$S(1 - r) = 1$$

$$S = \frac{1}{(1 - r)}$$

□

Theorem 4.4.4. *The p Series $\sum \frac{1}{n^p}$ converges when $p > 1$*

Proof. if $k_1 = 2 - 1 = 1, S_{k_1} = 1$

$$k_1 = 2^2 - 1 = 3, 2^p < 3^p$$

$$S_{k_2} = \frac{1}{1^p} + \left(\frac{1}{2^p} + \frac{1}{3^p} \right) < \frac{1}{1^p} + \frac{2}{2^p} = 1 + \frac{1}{2^{p-1}}$$

further, if $k_3 = 2^3 - 1$ then

$$S_{k_3} < S_{k_2} + \frac{1}{4^p} < 1 + \frac{1}{2^{p-1}} \frac{1}{4^{p-1}}$$

finally, let $r = \frac{1}{2^{p-1}}$ Since $p > 1$

Using mathematical induction

we can show that if $k_j = 2^j - 1$

$$0 < S_{k_j} < 1 + r + r^2 + \dots + r^{j-1} < \frac{1}{1-r}$$

\Rightarrow The p-series converges if $p > 1$

□

The alternating harmonic series

$$\sum \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots + \frac{(-1)^{n+1}}{n} + \dots$$

is convergent

$$\text{let } S_{2n} = \left(\frac{1}{1} - \frac{1}{2} \right) + \left(\frac{1}{3} - \frac{1}{4} \right) + \dots + \left(\frac{1}{2n-1} - \frac{1}{2n} \right)$$

$$S_{2n+1} = 1 - \left(\frac{1}{2} - \frac{1}{3} \right) - \left(\frac{1}{4} - \frac{1}{5} \right) - \dots - \left(\frac{1}{2n} - \frac{1}{2n+1} \right)$$

Since $0 < S_{2n} < S_{2n} + \frac{1}{2n+1} = S_{2n+1} \leq 1$

S_{2n} and S_{2n+1} both bounded and monotone, so by monotone convergence theorem, must be convergent and to same point.

$\sum \frac{(-1)^{n+1}}{n}$ must be convergent.

Theorem 4.4.5 (The Comparison Test). *Let x_n and y_n be real sequence and for some $k \in \mathbb{N}$ $0 \leq x_n \leq y_n, \forall n \geq k$*

a) *Convergent of $\sum y_n \Rightarrow$ Convergence of $\sum x_n$*

b) *Divergence of $\sum x_n \Rightarrow$ divergence of $\sum y_n$*

Proof. a) Suppose $\sum y_n$ is convergent,
 i.e for any $\varepsilon > 0, \exists M(\varepsilon) \in \mathbb{N} \ni m > n \geq M(\varepsilon)$
 $|y_{n+1} + \dots + y_m| < \varepsilon$
 if $m > \text{Sup}(k, M(\varepsilon))$
 $0 \leq x_{n+1} + \dots + x_m \leq y_{n+1} + \dots + y_m < \varepsilon$
 $\Rightarrow \sum x_n$ converges.

b) This statement is contrapositive to a)

□

Theorem 4.4.6 (Limit Comparison Test). *Suppose x_n and y_n are strictly positive sequence and Suppose following limit exists*

$$r = \lim \left(\frac{x_n}{y_n} \right)$$

a) If $r \neq 0$ then $\sum x_n$ convergent iff $\sum y_n$ convergent.

b) If $r = 0$ then if, $\sum y_n$ convergent then $\sum x_n$ convergent.

Proof. a) Given $r = \lim \frac{x_n}{y_n}$

\therefore by definition, For any $\varepsilon > 0$, $\exists, k(\varepsilon) \in \mathbb{N}$

such that $\left| \frac{x_n}{y_n} - r \right| < \varepsilon, \forall n \geq k(\varepsilon)$

As $r \neq 0, \Rightarrow r > 0 \Rightarrow \varepsilon \frac{r}{2}$

$$r - \varepsilon < \frac{x_n}{y_n} < r + \varepsilon$$

$$\left(\frac{r}{2} \right) y_n < x_n < \left(\frac{3r}{2} \right) y_n$$

$$\therefore \left(\frac{r}{2} \right) y_n < x_n$$

\Rightarrow if x_n converges then $\sum y_n$ also converges. ... (by comparison test)

$$\therefore x_n < \left(\frac{3r}{2} \right) Y_n$$

\Rightarrow If $\sum y_n$ converges then x_n also converges. ... (by comparison test)

$$r = 0 \text{ i.e } \lim \left(\frac{x_n}{y_n} \right) = 0$$

\therefore by definition, For any $\varepsilon > 0$, \exists , $k(\varepsilon) \in \mathbb{N}$ such that

$$\left| \frac{x_n}{y_n} - 0 \right| < \varepsilon$$

$$\left| \frac{x_n}{y_n} \right| < \varepsilon$$

$$\frac{x_n}{y_n} < \varepsilon$$

$$0 < x_n < \varepsilon y_n$$

\therefore By comparison test,
 $\sum x_n$ converges if $\sum y_n$ converges. □

Definition 4.4.3 (Absolute Convergence): *let x_n be sequence in \mathbb{R} . We say that $\sum x_n$ is absolutely convergent if the series $\sum |x_n|$ is convergent . A series is said to be conditionally convergent if it is convergent but not absolutely convergent.*

Example 21:

$\sum \frac{(-1)^n}{n}$ is convergent but $\sum \left| \frac{(-1)^n}{n} \right| = \sum \frac{1}{n}$ is not convergent
 $\therefore \sum \frac{(-1)^n}{n}$ is conditionally convergent.

Theorem 4.4.7. *If a series is absolutely convergent then it is convergent.*

Proof. $\sum |x_n|$ is convergent

\therefore for any $\varepsilon > 0$ $M(\varepsilon) \in \mathbb{N}$

$$||x_{n+1}| + |x_{n+2}| + \dots + |x_m|| < \varepsilon \forall m > n > M(\varepsilon)$$

$$|x_{n+1} + x_{n+2} + \dots + x_m| \leq \varepsilon$$

$$|x_{n+1}| + |x_{n+2}| + \dots + |x_m| \leq \varepsilon \forall m > n > M(\varepsilon)$$

$\Rightarrow \sum x_n$ is convergent. □

Theorem 4.4.8 (Limit Comparison Test- II-). *Suppose x_n and y_n are non-zero real sequence and Suppose that following limit exists in \mathbb{R}*

$$r = \lim \left(\frac{x_n}{y_n} \right)$$

a) *If $r \neq 0$ then $\sum x_n$ absolutely convergent iff $\sum y_n$ is absolutely convergent.*

b) *If $r = 0$ and $\sum y_n$ is absolutely convergent then $\sum x_n$ absolutely convergent.*

Theorem 4.4.9 (Root test). *Let x_n be sequence in \mathbb{R} . Suppose that the limit $r = \lim |x_n|^{\frac{1}{n}}$ exists in \mathbb{R} then $\sum x_n$ is absolutely convergent when $r < 1$ and is divergent when $r > 1$.*

Proof. $r < 1$, $r = \lim |x_n|^{\frac{1}{n}}$, $\exists r_1, r_1 \in (r, 1)$

$$|x_n|^{\frac{1}{n}} \leq r_1$$

$$\therefore |x_n| \leq \sum r_1^n$$

by comparison test,

$|x_n| < (r_1)^n$ it is convergent

$|x_n| < (r_1)^n$ it is absolutely convergent. □

Theorem 4.4.10 (Ratio Test). *Let x_n be non-zero sequence in \mathbb{R} . Suppose $r = \lim \left| \frac{x_{n+1}}{x_n} \right|$ exists then $\sum x_n$ is absolutely convergent when $r < 1$ and divergent when $r > 1$*

Proof. $r < 1$, $r_1 \in (r, 1)$

$$\left| \frac{x_{n+1}}{x_n} \right| \leq r_1, \forall n > k(\varepsilon)$$

$$|x_{n+1}| \leq r_1 |x_n|$$

$$|x_{n+1}| \leq r_1 |x_n| < r_1 \cdot r_1 |x_{n-1}| < \dots < r_1^n |x_1|$$

$$\therefore |x_{n+1}| < r_1^n \cdot c$$

$$\therefore \sum |x_{n+1}| < \sum r_1^n \cdot c$$

\therefore by comparison test,

$\sum x_n$ is absolutely convergent □

4.5 Establish the converges/divergence of series

Example 22:

$$\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)}$$

$$\sum_{n=1}^{\infty} = \frac{1}{2.3} + \frac{1}{3.4} + \frac{1}{4.5} + \dots$$

The series is converges to zero

or $(n+1)(n+2) > n.n$

$$\therefore \frac{1}{(n+1)(n+2)} < \frac{1}{n^2}$$

$$\therefore 0 < \frac{1}{(n+1)(n+2)} < \frac{1}{n^2}$$

by comparison test,

$$\sum \frac{1}{(n+1)(n+2)} \text{ is convergent.}$$

Example 23:

$$2^{(\frac{-1}{n})}$$

$$\lim_{n \rightarrow \infty} 2^{(\frac{-1}{n})} = 1 \neq 0$$

\therefore by n^{th} term test

$2^{(\frac{-1}{n})}$ is divergent

Example 24:

$$\frac{n}{2^n}$$

Applying ratio test

$$\left| \frac{x_{n+1}}{x_n} \right| = \left| \frac{(n+1)/2^{(n+1)}}{n/2^n} \right| = \left| \frac{n+1}{n} \right| \cdot \frac{1}{2}$$

$$\lim_{n \rightarrow \infty} \left| \frac{x_{n+1}}{x_n} \right| = \frac{1}{2} < 1$$

$\therefore \frac{\sum n}{2^n}$ is convergent.

Definition 4.5.1 (Integral test): *Let f be a positive decreasing function on $\{t, t > 1\}$ then the*

series $\sum_{k=1}^{\infty} f(k)$ converges if and only if the improper integral

$$\int_1^{\infty} f(t) dt = \lim_{b \rightarrow \infty} \int_1^b f(t) dt$$

exists. In the case of convergence, the partial sum

$S_n = \sum_{k=1}^n f(k)$ and sum $S = \sum_{k=1}^{\infty} f(k)$ satisfy the estimates

$$\int_{n+1}^{\infty} f(t) dt \leq S - S_n \leq \int_1^{\infty} f(t) dt$$

Example 25:

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{1}{n^p} \\ &= \int_1^{\infty} \frac{1}{t^p} dt, \quad x_n \frac{1}{n^p} \\ &= \left[\frac{t^{-p+1}}{-p+1} \right]_1^{\infty} \\ &= \frac{1}{1-p} \left[\frac{1}{t^{p-1}} \right]_1^{\infty} \end{aligned}$$

$$\frac{1}{p-1}, p > 1$$

$\therefore \sum \frac{1}{n^p}$ is convergent

Definition 4.5.2 (Raabies Test): Let x_n be non-zero sequence in \mathbb{R} and let

$a = \lim n \left(1 - \left| \frac{x_{n+1}}{x_n} \right| \right)$ whenever this limit exists then $\sum x_n$ absolutely convergent when $a > 1$ and is not absolutely convergent when $a < 1$

Example 26:

$$x_n = \frac{1}{n^p}$$

$$\left| \frac{x_{n+1}}{x_n} \right| = \left| \frac{\frac{1}{(n+1)^p}}{\frac{1}{n^p}} \right| = \left| \frac{n^p}{(n+1)^p} \right| = \left| \frac{1}{1 + \frac{1}{n}} \right|$$

$$\therefore \lim n \left(1 - \left| \frac{x_{n+1}}{x_n} \right| \right) = \lim n \left(1 - \left| \left(\frac{1}{1 + \frac{1}{n}} \right)^p \right| \right)$$

$$= \lim \left(\frac{\left(1 + \frac{1}{n}\right)^p - 1}{\frac{1}{n} \left(1 + \frac{1}{n}\right)^p} \right)$$

$$= \lim \left(\frac{p \left(1 + \frac{1}{n}\right)^{p+1} \left(-\frac{1}{n^2}\right)}{\frac{1}{n} \left(1 + \frac{1}{n}\right)^p} \right)$$

$$= \lim \left(\frac{-\frac{p}{n} \left(1 + \frac{1}{n}\right)}{\left(1 + \frac{1}{n}\right)^p} \right)$$

$$= \lim_{n \rightarrow \infty} \left(\frac{-p \left(1 + \frac{1}{n}\right)}{n} \right)$$

$$= \lim_{n \rightarrow \infty} p \left(-\frac{1}{n} - \frac{1}{n^2} \right)$$

$$= p$$

Example 27:

$$x_n = \frac{1}{n(n+1)}$$

$$\left| \frac{x_{n+1}}{x_n} \right| = \left| \frac{\frac{1}{(n+1)(n+2)}}{\frac{1}{n(n+1)}} \right| = \left| \frac{n}{n+2} \right| = \left| \frac{1}{1 + \frac{2}{n}} \right|$$

$$\therefore \lim \left| \frac{x_{n+1}}{x_n} \right| = \left| \frac{1}{1 + \frac{2}{n}} \right| = 1$$

\therefore Ratio test fails ($\because r = 1$)

we know $n(n+1) > n.n$

$$\frac{1}{n(n+1)} < \frac{1}{n^2}$$

$$\therefore \frac{1}{n(n+1)} < \frac{1}{n^2} \quad (0 < x_n < y_n)$$

by comparison test

As $\sum \frac{1}{n^2}$ is convergent, $\sum \frac{1}{n(n+1)}$ is also convergent.

Example 28:

$$\frac{n!}{n^n}$$

Using raabies test, we have,

$$\text{consider, } \left| \frac{x_{n+1}}{x_n} \right| = \frac{1}{\left(1 + \frac{1}{n}\right)^n}$$

$$\Rightarrow \left(1 - \frac{1}{\left(1 + \frac{1}{n}\right)^n}\right)$$

$$= \frac{\left(1 + \frac{1}{n}\right)^n - 1}{\left(1 + \frac{1}{n}\right)^n}$$

$$= n \left(1 - \frac{1}{\left(1 + \frac{1}{n}\right)^n}\right)$$

$$\text{as } n \rightarrow \infty, \quad r = \lim n \left(1 - \left| \frac{x_{n+1}}{x_n} \right| \right) = 0 < 1$$

$\therefore \sum x_n = \sum \frac{n!}{n^n}$ is not absolutely convergent. i.e divergent.

Example 29:

$$\frac{n^2}{\sqrt{n+1}} = \left| \frac{x_{n+1}}{x_n} \right| = \frac{(n+1)^2}{\sqrt{n+2}} \cdot \frac{\sqrt{n+1}}{n^2}$$

$$= \left(1 + \frac{1}{n}\right)^2 \frac{\sqrt{1 + \frac{1}{n}}}{1 - \frac{2}{n}}$$

$$\Rightarrow n \left(1 - \left| \frac{x_{n+1}}{x_n} \right| \right)$$

$$= \frac{\sqrt{1 + \frac{2}{n}} - \left(1 + \frac{1}{n}\right)^2 \sqrt{1 + \frac{1}{n}}}{\sqrt{\frac{1}{n^2} + \frac{2}{n^3}}}$$

$\therefore \sum \frac{n^2}{\sqrt{n+1}}$ is not absolutely convergent. i.e divergent.

4.6 Test for Non-Absolute Convergence

Definition 4.6.1 (Alternative Series): A sequence of non-zero real numbers is said to be alternating if the terms $(-1)^{(n+1)} x_n$, $n \in \mathbb{N}$ are all positive (or all negative) real numbers. If the sequence x_n is alternating, we say that the series $\sum x_n$ is alternating series.

Theorem 4.6.1 (Alternating Series test). Let z_n be decreasing sequence with strictly positive numbers with $\lim z_n = 0$ then the alternating series $\sum (-1)^{n+1} z_n$ is convergent.

Proof. Given that z_n decreasing sequence and let $S_n = \sum (-1)^{n+1} z_n$

We have

$$S_{2n} = (z_1 - z_2) + (z_3 - z_4) + \dots + (z_{2n-1} - z_{2n})$$

and Since $(z_k - z_{k+1}) \geq 0$, it follows that S_{2n} is increasing sequence

$$S_{2n} = z_1 - (z_2 - z_3) + \dots - (z_{n-2} - z_{n-1}) - z_{2n}$$

$$\therefore S_{2n} \leq z_1$$

\therefore bounded by MCT, S_{2n} must be convergent to some number $c \in \mathbb{R}$.

We have to show that entire $S_n \rightarrow c$ if $\varepsilon > 0$, let $k \in \mathbb{N}$. if $n \geq k$

$$|S_{2n} - c| \leq \frac{\varepsilon}{2} \text{ and } z_{2n+1} \leq \frac{\varepsilon}{2}$$

$$|S_{2n+1} - c| = |S_{2n} + z_{n+1} - c|$$

$$\leq |S_{2n} - c| + |z_{n+1}|$$

$$\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

$$S_n \rightarrow c$$

$S_n = \sum (-1)^{n+1} z_n$ is convergent. □

Lemma 4.6.2 (Abels Lemma). $x_n, y_n \in \mathbb{R}$ $S_n = \sum_{i=1}^n y_i$ with $S_0 = 0$ if $m > n$ then,

$$\sum_{k=n+1}^m x_k y_k = (x_m S_m - x_{n+1} S_n) + \sum_{k=n+1}^{m-1} (x_k - x_{k+1}) S_k$$

Proof. $y_k = S_k - S_{k-1} \quad \left(\because S_k = \sum_{i=1}^k y_i \& S_{k-1} = \sum_{i=1}^{k-1} y_i \right)$

$$x_k y_k = x_k S_k - x_k S_{k-1}$$

$$\sum_{k=n+1}^m x_k y_k$$

$$= \sum_{k=n+1}^{m-1} (x_k S_k - x_k S_{k-1})$$

$$= x_{n+1} \cdot S_{n+1} - x_{n+1} S_n + x_{n+2} S_{n+2} - x_{n+2} S_{n+1} + \dots + x_m S_m - x_m S_{m-1}$$

$$= (x_m S_m - x_{n+1} S_n) + \sum_{k=n+1}^{m-1} (x_k - x_{k+1}) S_k$$

□

Theorem 4.6.3 (Dirichlet's Test). If x_n is decreasing 0, if $S_n = \sum y_i$ is bounded then $x_n y_n$ is convergent.

Proof. Let $S_n \leq B$, $\forall n \in \mathbb{N}$. if $m > n$, by abels lemma and $x_k - x_{k+1} > 0$ (as x_n is decreasing)

Consider,

$$\begin{aligned} & \left| \sum_{k=n+1}^m x_k y_k \right| \\ &= \left| (x_m S_m - x_{n+1} S_n) + \sum_{k=n+1}^{m-1} (x_k - x_{k+1}) S_k \right| \\ &\leq |(x_m S_m - x_{n+1} S_n)| + \sum_{k=n+1}^{m-1} (x_k - x_{k+1}) |S_k| \end{aligned}$$

Suppose,

$$S_m, S_n, S_k = B$$

$$\begin{aligned} &\leq |x_m - x_{n+1}| B + B \sum_{k=n+1}^{m-1} (x_k - x_{k+1}) |S_k| \\ &\leq \frac{\varepsilon}{2B} B + B \frac{\varepsilon}{2B} \end{aligned}$$

$$\leq \varepsilon$$

$\therefore \sum x_n y_n$ is convergent. □

Theorem 4.6.4 (Abel's Test). *If x_n convergent monotone sequence and y_n is convergent then the series $x_n y_n$ also convergent.*

Proof. Let x_n is decreasing x

$u_n = x_n - x$ decreasing 0

$\sum u_n y_n$ is convergent by diricblets test

$$\sum_n x_n y_n$$

$$= \sum_n (x + u_n) y_n$$

$$= x \sum_n y_n + \sum_n u_n y_n$$

$\sum_n x_n y_n$ is convergent sequence. □

Example 30:

$\sum a_n$ convergent then

1. $\sum b_n = \frac{a_n}{n}$ is convergent sequence.
2. $\sum n^{1/n} a_n$ is divergent sequence.
3. $\sum a_n \sin n$ is divergent sequence.
4. $\sum \frac{\sqrt{a_n}}{n}$ is convergent sequence.
5. $\sum \sqrt{a_n}$ is divergent sequence.

Function and Continuity

Definition 5.0.1 (Cluster Point): Let $A \subseteq \mathbb{R}$. A point $c \in \mathbb{R}$ is cluster point of A if every $\delta > 0 \exists$ at least one point $x \in A$, $x \neq c \ni |x - c| < \delta$

Theorem 5.0.1. The number $c \in \mathbb{R}$, is cluster point of $A \subseteq \mathbb{R}$ if and only if \exists sequence a_n in A such that $\lim(a_n) = c$ and $a_n \neq c$, $\forall n$

Proof. If c is cluster point of A then for any $n \in \mathbb{N}$ the $\frac{1}{n}$ neighbourhood $v_{1/n}(c)$ contains at least one point a_n in A distinct from c , then $a_n \in A$, $a_n \neq c$ & $|a_n - c| < \frac{1}{n} \Rightarrow \lim a_n = c$ conversly, if \exists a sequence a_n in $A \setminus \{c\}$ with $\lim(a_n) = c$, then for any $\delta > 0$, $\exists k$ such that if $n \geq k$, then $a_n \in v_\delta(c)$. Therefor, δ neighbourhood $v_\delta(c)$ contains the point a_n , $\forall n \geq k$ which belong to A and are distinct from c . \square

Definition 5.0.2 (Limit of Function): Let $A \subseteq \mathbb{R}$ and let $c \in \mathbb{R}$ be the cluster point of A . for a function $f : A \rightarrow \mathbb{R}$ a real number L is said to be limit of f at c if, given any $\varepsilon > 0$, $\exists \delta > 0, \ni$

$x \in A$ and $0 < |x - c| < \delta$ then $|f(x) - L| < \varepsilon$ then we say f converges to L at c .

Theorem 5.0.2. If $f : A \rightarrow \mathbb{R}$ and if c is a cluster point of A , then f can have only one limit at c .

Proof. We will prove this by method of contradiction.

Let L and L' be limits of f at c

For any $\varepsilon > 0$, $\exists \delta \left(\frac{\varepsilon}{2}\right) > 0 \ni x \in A$ and $0 < |x - c| < \delta \left(\frac{\varepsilon}{2}\right)$

$$\Rightarrow |f(x) - L| < \frac{\varepsilon}{2}$$

Also, $\exists \delta' \left(\frac{\varepsilon}{2}\right) > 0 \ni x \in A$ and $|x - c| < \delta' \left(\frac{\varepsilon}{2}\right)$

$$\Rightarrow |f(x) - L'| < \frac{\varepsilon}{2}$$

$$|L - L'|$$

$$= |L - f(x) + f(x) - L'|$$

$$\leq |L - f(x)| + |f(x) - L'|$$

$$\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

$$\leq \varepsilon$$

Since, $\varepsilon > 0$ is arbitrary, $L = L'$

□

Theorem 5.0.3 (Sequential Criterion). *Let $f : A \rightarrow \mathbb{R}$ and let c be a cluster point of A then the following are equivalent.*

1. $\lim_{x \rightarrow c} f(x) = L$
2. *for every x_n in A , $x_n \rightarrow c$, $x_n \neq c$, $\forall n \in \mathbb{N} \Rightarrow f(x_n) \rightarrow L$.*

Definition 5.0.3 (Divergence Criterion): *Let $A \subseteq \mathbb{R}$ let $f : A \rightarrow \mathbb{R}$ and let $c \in \mathbb{R}$ be cluster point of A .*

- a) *If $L \in \mathbb{R}$ then f does not have limit L at c iff \exists sequence x_n in A with $x_n \neq c$, $\forall n \in \mathbb{N}$ such that sequence x_n converges to c . but the sequence $f(x_n)$ does not converges to L*
- b) *the function does not have a limit L at c iff $\exists x_n$ in A with $x_n \neq c$, $\forall n \in \mathbb{N}$ such that the sequence x_n converges to c but the sequence $f(x_n)$ does not converges in \mathbb{R}*

$f(x) =$

$$f(x) = \begin{cases} +1 & \text{if } x > 0 \\ -0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$$

Theorem 5.0.4 (Limit Theorem). *Let $A \subseteq \mathbb{R}$. and $c \in \mathbb{R}$, be cluster point of A we say that f is bounded on neighbourhood of c if \exists a δ neighbourhood of $v_\delta(c)$ of c and constant $M > 0 \ni |f(x)| \leq M \quad \forall x \in A \cap V_\delta(c)$*

Theorem 5.0.5. *If $A \subseteq \mathbb{R}$ and $f : A \rightarrow \mathbb{R}$ has limit at $c \in \mathbb{R}$ then f is bounded on some neighbourhood of c*

Proof. If $L = \lim_{x \rightarrow c} f$ then for $\varepsilon = 1, \exists \delta_c < 0$

Such that $0 < |x - c| < \delta \Rightarrow |f(x) - L| < 1$

$$|f(x)| - |L| \leq |f(x) - L| < 1$$

if $x \in A \cap V_\delta(c), \quad x \neq c$ then,

$$|f(x)| < |L| + 1$$

if $c \notin A$, Take $M = |L| + 1$

while if $c \in A$, Take $M = \sup\{|f(x)|, |L| + 1\}$

$$\therefore |f(x)| \leq M$$

\therefore by limit theorem

$\therefore f$ is bounded on neighbourhood of c . □

Definition 5.0.4: *Let $A \subseteq \mathbb{R}$ and let f & g be function defined on A to \mathbb{R} . We define the sum $f + g$, the difference $f - g$ and the product $f.g$ on $A \rightarrow \mathbb{R}$ to be function from A to \mathbb{R} given by,*

$$(f + g)(x) = f(x) + g(x)$$

$$(f - g)(x) = f(x) - g(x)$$

$$(f.g)(x) = f(x).g(x)$$

Further if $b \in \mathbb{R}$

$$(bf)(x) = b.f(x)$$

finally, if $h(x) \neq 0$,

$$\left(\frac{f}{h}\right)(x) = \frac{f(x)}{h(x)}$$

Theorem 5.0.6. Let $A \subseteq \mathbb{R}$ let f & g be function on $A \rightarrow \mathbb{R}$ and let $c \in \mathbb{R}$ be a cluster point of $A \rightarrow \mathbb{R}$ & let

1. If $\lim_{x \rightarrow c} f = L$ & $\lim_{x \rightarrow c} g = M$ then

$$\lim_{x \rightarrow c} (f \pm g) = L \pm M$$

$$\lim_{x \rightarrow c} (f \cdot g) = L \cdot M$$

$$\lim_{x \rightarrow c} (b \cdot f) = b \cdot L$$

2. $\lim_{x \rightarrow c} \left(\frac{f}{h}\right) = \frac{L}{H}$
 where, $h(x) \neq 0$ and $\lim_{x \rightarrow c} h(x) = H \neq 0$

Theorem 5.0.7. Let $A \subseteq \mathbb{R}$ let $f : A \rightarrow \mathbb{R}$ and let $c \in \mathbb{R}$ be the cluster point of A . if $a \leq f(x) \leq b \forall x \in A, x \neq c$ and if $\lim_{x \rightarrow c} f$ exists then $a \leq \lim_{x \rightarrow c} f \leq b$

Proof. Given, $f : A \rightarrow \mathbb{R}$ and c is cluster point of A .

let $x_n \in A$ such that $x_n \rightarrow c$

$$\therefore f(x_n) \rightarrow L = \lim_{x \rightarrow c} f(x_n) = \lim_{x \rightarrow c} f(x)$$

Also,

$$a \leq f(x) \leq b$$

$$a \leq f(x_n) \leq b$$

$$a \leq \lim_{x \rightarrow c} f(x_n) \leq b$$

$$a \leq \lim_{x \rightarrow c} f(x) \leq b$$

$$a \leq L \leq b$$

□

Theorem 5.0.8 (Squeeze Theorem). *Let $A \subseteq \mathbb{R}$ let $f, g, h : \rightarrow \mathbb{R}$ & $c \in \mathbb{R}$ be a cluster point of A . If $f(x) \leq g(x) \leq h(x)$, $\forall x \in A, x \neq c$ & $\lim_{x \rightarrow c} f = \lim_{x \rightarrow c} h$ then, $\lim_{x \rightarrow c} g = L$.*

Proof. Given, $f, g, h : \rightarrow \mathbb{R}$ & c is cluster point of A $x_n \in A \Rightarrow x_n \rightarrow c$

$$f(x_n) \rightarrow L = \lim_{x \rightarrow c} f(x_n) = \lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} h(x) = \lim_{x \rightarrow c} h(x_n)$$

i.e $f(x_n) \rightarrow L$ & $h(x_n) \rightarrow L$

Also,

$$f(x) \leq g(x) \leq h(x)$$

$$f(x_n) \leq g(x_n) \leq h(x_n)$$

$$\lim_{x_n \rightarrow c} f(x_n) \leq \lim_{x_n \rightarrow c} g(x_n) \leq \lim_{x_n \rightarrow c} h(x_n)$$

$$\therefore L \leq \lim_{x \rightarrow c} g(x_n) \leq L$$

$$\lim_{x \rightarrow c} g(x_n) = L$$

$$\text{i.e } g(x_n) \rightarrow L$$

$$\text{i.e } \lim_{x \rightarrow c} g = L$$

-Hence Proved-



Definition 5.0.5: Let $A \in \mathbb{R}$ & let $f: A \rightarrow \mathbb{R}$

1. If $c \in \mathbb{R}$ is a cluster point of the set $A \cap (c, \infty) = \{x \in A, x > c\}$ then we say that $L \in \mathbb{R}$ is right hand limit of f at c

$$\lim_{x \rightarrow c^+} f(x) = L \text{ If given any } \varepsilon > 0 \quad \exists \quad \delta(\varepsilon) > 0 \quad \ni \quad \forall x \in A \text{ with } 0 < x - c < \delta \text{ then } |f(x) - L| < \varepsilon$$

2. If $c \in \mathbb{R}$ is a cluster point of the set $A \cap (-\infty, 0) = \{x \in A, x < c\}$ then we say that $L \in \mathbb{R}$ is left hand limit of f at c

$$\lim_{x \rightarrow c^-} f(x) = L \text{ If given any } \varepsilon > 0 \quad \exists \quad \delta(\varepsilon) > 0 \quad \ni \quad \forall x \in A \text{ with } 0 < -x + c < \delta \text{ then } |f(x) - L| < \varepsilon$$

5.1 Continuous Function

Definition 5.1.1 (Continuous Function): Let $A \subseteq \mathbb{R}$ let $f: A \rightarrow \mathbb{R}$ & let $c \in A$ we say that f is continuous at c if given any $\varepsilon > 0$, $\exists \quad \delta(\varepsilon) > 0 \quad \ni$ if x is any point of A satisfying $|x - c| < \delta$, then $|f(x) - f(c)| < \varepsilon$ if f fails to be continuous at c then we say that f is discontinuous at c

Theorem 5.1.1. A function $f : A \rightarrow \mathbb{R}$ is continuous at point $c \in A$ if and only if given any $\varepsilon > 0$, $\nu_\varepsilon(f(c))$ of $f(c) \ni$ of c such that if x is any point of $A \cap \nu_\delta(c)$ then $f(x) \in \nu_\varepsilon(f(c))$ i.e $A \cap \nu_\delta(c) \subseteq \nu_\varepsilon(f(c))$

Proof. $\therefore \lim_{x \rightarrow c} = L$

i.e any $\varepsilon > 0$, $\exists \delta(\varepsilon) > 0 \ni$

$|x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon$

and $\lim f(x) = f(c)$

for any $\varepsilon > 0$, $\exists \delta(\varepsilon) > 0 \ni$

$|x - c| < \delta, \Rightarrow |f(x) - f(c)| < \varepsilon$

$\therefore x \in A \cap \nu_\delta(c) \Rightarrow f(x) \in \nu_\varepsilon(f(c)), \quad \forall x$

$\therefore f(A \cap \nu_\delta(c)) \subseteq \nu_\varepsilon(f(c)) \dots (\because \text{if } x \in A \Rightarrow x \in B \text{ then } A \subseteq B)$ □

Definition 5.1.2 (Combination of Continuous function): Let $A \subseteq \mathbb{R}$. Let f & g be function on A to \mathbb{R} , let $b \in \mathbb{R}$, Suppose that $c \in A$ & that f & g are continuous at c

a) then $f + g, f - g, f \cdot g$ and $b \cdot f$ are continuous at c

b) if $h : A \rightarrow \mathbb{R}$ is continuous at $c \in A$ & if $h(x) \neq 0, \quad x \in A$, then $\left(\frac{f}{h}\right)$ is also continuous at c

Definition 5.1.3 (Continuous Point): Let $A \subseteq \mathbb{R}$ & $f : A \rightarrow \mathbb{R}$. if $B \subseteq A$ we say that f is continuous on set B if f is continuous at every point of B

Example 31:

Continuous

- $f(x) = x, \quad x \in \mathbb{R}$
- $f(x) = x^2, \quad x \in \mathbb{R}$
- $f(x) = \frac{1}{x}, \quad x \in \mathbb{R}^+, \{0\}$
- $f(x) = \text{Polynomial function} \quad x \in \mathbb{R}$
- $f(x) = \text{Rational function}$
- $f(x) = \text{Trigonometric function}$
- $f(x) = \sqrt{f}, \quad x \in \mathbb{R}$

Example 32:

Discontinuous

- $\psi(x) = \frac{1}{x}, \quad x = 0$
- $\psi(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \in \mathbb{Q}^c \\ \text{discount everywhere} \end{cases}$

$$\bullet \sin(x) = \begin{cases} 1, & x > 0 \\ 0, & x < 0 \\ 0, & x = 0 \end{cases}$$

• $\psi(x) = [x]$ = greatest integer function discount at integer

Theorem 5.1.2. Let $A \subseteq \mathbb{R}$. Let $f : A \rightarrow \mathbb{R}$ & let $|f|$ be defined by $|f|x = |f(x)| \quad \forall x \in A$

1. If f is continuous at point $c \in A$ then $|f|$ is continuous at c
2. If f is continuous on A then $|f|$ is continuous on A .

Theorem 5.1.3. Let $A, B \subseteq \mathbb{R}$ & let $f : A \rightarrow \mathbb{R}$ & $g : B \rightarrow \mathbb{R}$ be function such that $f(A) \subseteq B$ if f is continuous at point $c \in A$ and g is continuous at $b = f(c) \in B$ then the composition $g \circ f : A \rightarrow \mathbb{R}$ is continuous at c .

Proof. Let W be ε -neighbourhood of $g(b)$. since g is continuous at b there is a δ -neighbourhood of ν of $b = f(c)$ such that if $y \in B \cap \nu$ then $g(y) \in W$. Since f is also continuous at c , there is a ν -neighbourhood ν of $c \ni x \in A \cap U$ then $f(x) \in \nu$

Since $f(A) \subseteq B$, it follows that if $x \in A \cap U$ then $f(x) \in B \cap \nu$ so that $g \circ f(x) = g(f(x)) \in W$ But, Since W is an arbitrary ε -neighbourhood of $g(b)$ this implies $g \circ f$ is continuous at c . \square

5.2 Continuous function on Interval

Definition 5.2.1 (Bounded Function): A function $f : A \rightarrow \mathbb{R}$ is said to be bounded on A if \exists a constant $M > 0$ such that $|f(x)| \leq M$ for all $x \in A$

Theorem 5.2.1 (Boundedness Theorem-). Let $I = [a, b]$ be a closed bounded interval and let $f : I \rightarrow \mathbb{R}$ be continuous on I then f is bounded on I .

Proof. Suppose f is bounded on I .

then, for any $n \in \mathbb{N}$, $\exists x_n \in I \ni |f(x_n)| > n$.

Since, I is bounded, sequence x_n is bounded.

\therefore By Bolzano weiestress theorem,

\exists subsequence x_{n_k} that converges to some x

Since, I is closed, elements of sequence $x_{n_k} \in I \Rightarrow x \in I$.

then, f is continuous at x so that $f(x_{n_k})$ converges to $f(x)$.

$\Rightarrow |f(x_{n_k})| > n_k > k \quad \forall k \in \mathbb{N}$

\therefore Our assumption is wrong.

Hence, f must be bounded. □

Definition 5.2.2 (Absolute Extremum): Let $A \subseteq \mathbb{R}$ and let $f : A \rightarrow \mathbb{R}$. We say that f has an absolute maximum on A if there is $x^* \in A$ such that

$$f(x^*) \geq f(x), \quad \forall x \in A$$

We say that f has absolute minimum on A if there is $x^* \in A$ such that

$$f(x^*) \leq f(x), \quad \forall x \in A$$

Theorem 5.2.2 (Maximum-Minimum Theorem). *Let $I = [a, b]$ be closed bounded interval and let $f : I \rightarrow \mathbb{R}$ be continuous on I then f has an absolute maximum and absolute minimum on I .*

Proof. $f(I) = \{f(x); x \in I\}$

I is a closed bounded and f is continuous on I then $f(x)$ is also bounded $\subseteq \mathbb{R}$

\therefore By completeness property, it has supremum and infimum

$$\therefore S^* = \sup\{f(I)\}, \quad S_* = \inf\{f(I)\}$$

claim- To show, $\exists x^*, x_* \in I$

$$\exists S^* = f(x^*) = \text{absolute maximum}$$

$$S_* = f(x_*) = \text{absolute minimum}$$

$$S_* = \inf\{f(I)\}$$

if $n \in \mathbb{N}$ then $S^* - \frac{1}{n}$ is not upper bound

$$\therefore S^* - \frac{1}{n} < f(x_n) < S^*, \quad \forall n \in \mathbb{N}$$

Since, I is bounded x_n is bounded By Bolzano Weierstrass theorem,

$$\exists x_{n_k} \text{ subsequence of } x_n \text{ and } x_{n_k} \rightarrow \text{some } x^*$$

Also, As I is closed and $x_{n_k} \in I \Rightarrow x^*$ must be in I

$$\Rightarrow f \text{ is continuous at } x^*, \lim f(x_{n_k}) = f(x^*)$$

$$S^* - \frac{1}{n} < f(x_{n_r}) \leq S^*, \quad \forall r \in \mathbb{N}$$

\therefore by squeeze theorem

$$\lim f(x_{n_r}) = S^*$$

$$\therefore S^* = f(x^*) \text{ i.e } f(x^*) \geq f(x), \quad \forall x$$

$\therefore x^*$ is absolute maximum

Similarly, we show x_* is absolute minimum □

Theorem 5.2.3 (Location of Root). *Let $I = [a, b]$ & let $f : I \rightarrow \mathbb{R}$ be continuous on I . If $f(a) < 0 < f(b)$ or $f(b) < 0 < f(a)$, then $\exists c \in (a, b) \ni f(c) = 0$.*

Proof. Assume that $f(a) < 0 < f(b)$

Let $I_1 = [a_1, b_1]$ where, $a_1 = a, b_1 = b$

let $P_1 = \frac{a+b}{2}$ if $f(P_1) = 0$ then $c = P_1$

if $P_1 \neq 0$, then wither $f(P_1) > 0$ or $f(P_1) < 0$

if $f(P_1) > 0$ then $a_2 = a_1, b_2 = P_1$ and if $f(P_1) < 0$

$a_2 = P_1, b_2 = b_1$ thus, we get $I = [a_2, b_2] \in I_1$

continuing this bisectins, we obtain intervals I_1, I_2, \dots, I_k

In this process, we terminate by locating a point $P_n \ni f(P_n) = 0$

if process does not terminate, we obtain nested sequence of bounded interval

$$I_n = [a_n, b_n]$$

$$\ni f(a_n) < 0 \text{ \& } f(b_n) > 0$$

$$\text{\& length of interval } b_n - a_n = \frac{(b-a)}{2^{n-1}}$$

$$\Rightarrow \exists \text{ a point } c \in I_n \quad \forall n \in \mathbb{N}$$

$$a_n \leq c \leq b_n, \quad \forall n \in \mathbb{N}$$

$$\Rightarrow 0 \leq c - a_n \leq b_n - a_n$$

$$\Rightarrow 0 \leq c - a_n \leq \frac{(b - a)}{2^{n-1}}$$

$$\Rightarrow \lim f(a_n) = \lim f(b_n) = f(c)$$

$$\Rightarrow 0 \leq b_n - c \leq b_n - a_n$$

$$\Rightarrow 0 \leq b_n - c \leq \frac{(b - a)}{2^{n-1}}$$

□

Theorem 5.2.4 (Bolzano's Intermediate Theorem). *Let I be an interval and let $f : I \rightarrow \mathbb{R}$ be continuous on I if $a, b \in I$ and if $k \in \mathbb{R}$ satisfies $f(a) < k < f(b)$ then a point $c \in I$ between a & b $\ni f(c) = k$.*

Proof. 1. Assume that, $a < b, a, b \in I, f$ continuous on I

Define $g(x) = f(x) - k$

As $f(x)$ is continuous, $g(x)$ is also continuous on I

Also, $f(a) < k < f(b)$

$f(a) - k < 0 < f(b) - k$

$g(a) < 0 < g(b)$

\therefore by location of root theorem

$\exists c \ni g(c) = 0$

i.e $f(c) - k = 0$

$$\therefore f(c) = k$$

2. Assume that, $a > b, a, b \in I, f$ continuous on I

Define $h(x) = k - f(x)$

As $f(x)$ is continuous, $h(x)$ is also continuous on I

Also, $f(a) < k < f(b)$

$$k - f(a) < 0 < k - f(b)$$

$$h(a) < 0 < h(b)$$

\therefore by location of root theorem

$$\exists c \ni h(c) = 0$$

$$\text{i.e } k - f(c) = 0$$

$$\therefore f(c) = k$$

□

Corollary 5.2.4.1. Let $I = [a, b]$ be a closed bounded interval. Let $f : I \rightarrow \mathbb{R}$ be continuous on I if $k \in \mathbb{R}$ is any number satisfying $\inf f(I) \leq k \leq \sup f(I)$ then \exists a number $c \in I \ni f(c) = k$

Proof. Given that, I is a closed bounded interval and $f : I \rightarrow \mathbb{R}$ is continuous on I

\therefore By maximum- minimum theorem,

$$\exists x^*, x_* \in I \text{ such that } f(x^*) = \sup\{f(I)\}$$

$$f(x_*) = \inf\{f(I)\}$$

Also, Given that, $\text{Inf } f(I) \leq k \leq \text{Sup } f(I)$

i.e $f(x^*) \leq k \leq f(x_*)$

\therefore by Bolzano intermediate theorem,

$\exists c \in I \quad \ni \quad f(c) = k$

-Hence Proved-

□

Theorem 5.2.5. Let I be closed bounded interval and let $f : I \rightarrow \mathbb{R}$ be continuous on I then, the set $f(I) = \{f(x) : x \in I\}$ be closed bounded interval.

Proof. let ,

$m = \text{Inf } \{f(I)\}$

$M = \text{Sup } \{f(I)\}$

by maximum - minimum theorem, $m, M \in f(I)$

$f(I) \subseteq [m, M]$

if $k \in [m, M]$

\therefore by bolzano-itermediate theorem

$\exists c \in I, \quad f(c) = k$

Hence, $k \in f(I)$

$\Rightarrow [m, M] \subseteq f(I)$

$\therefore, f(I)$ is the interval $[m, M]$

□

5.3 Continuity

Definition 5.3.1 (Uniform Continuous): Let $A \subseteq \mathbb{R}$ & let $f : A \rightarrow \mathbb{R}$. We say that f is uniformly continuous on A if for each $\varepsilon > 0$, $\exists \delta(\varepsilon) > 0$ \ni if $x, y \in A$ are any numbers satisfying $|x - y| < \delta(\varepsilon) \Rightarrow |f(x) - f(y)| < \varepsilon$

Definition 5.3.2 (Non- Uniform Continuity): Let $A \subseteq \mathbb{R}$ & let $f : A \rightarrow \mathbb{R}$ then following statements are equivalent.

- i) f is not uniformly continuous on A .
- ii) $\exists \varepsilon_0 > 0$ \ni for every $\delta > 0$ there are points x_δ, y_δ in A such that,
 $|x_\delta - y_\delta| < \delta$ and $|f(x_\delta) - f(y_\delta)| \geq \varepsilon_0$
- iii) $\exists \varepsilon_0 > 0$ and two sequence x_n & y_n in A such that $\lim x_n - y_n = 0$ and $|f(x_n) - f(y_n)| \geq \varepsilon_0$, $\forall n \in \mathbb{N}$

Theorem 5.3.1 (Uniform Continuity Theorem). Let I be closed bounded interval and let $f : I \rightarrow \mathbb{R}$ be continuous on I then f is uniform continuous on I .

Proof. If f is not uniform continuous on I then,

$\exists \varepsilon_0 > 0$ and two sequence $x_n, y_n \in I$

$|x_n - y_n| < \frac{1}{n}$ & $|f(x_n) - f(y_n)| \geq \varepsilon_0$

Since I is bounded x_n, y_n are bounded.

\exists subsequenece x_{n_k} of x_n that converges to some elements $z \in I$ (as I closed) as

$$|x_n - y_n| < \frac{1}{n} \quad \forall n$$

Subsequence y_{n_k} of y_n also converges to z

$$|y_{n_k} - z|$$

$$= |y_{n_k} - x_{n_k} + x_{n_k} - z|$$

$$\leq |y_{n_k} - x_{n_k}| + |x_{n_k} - z|$$

$\therefore y_{n_k}$ is also converges to z

Now if f is continuous at z both $f(x_{n_k})$ and $f(y_{n_k})$ must converges $f(z)$

But this not possible as $|f(x_n) - f(y_n)| \geq \varepsilon_0$

\therefore Our assumption is wrong. □

Definition 5.3.3 (Lipschitz Function): Let $A \subseteq \mathbb{R}$ and $f : A \rightarrow \mathbb{R}$ if there exists a constant $k > 0$ such that

$$|f(x) - f(u)| < k|x - u| \quad \forall x, u \in A \text{ then } f \text{ is said to be a Lipschitz function on } A$$

Theorem 5.3.2. Lipschitz function is an uniformly continuous function always.

Proof. for Lipschitz function

$$|f(x) - f(u)| < k|x - u|$$

$$\text{Now, } |x - u| < \frac{\varepsilon}{k} = \delta, \quad | < \frac{\varepsilon}{k} > 0 \text{ as } k > 0$$

$$|f(x) - f(u)| < k \cdot \frac{\varepsilon}{k}$$

$$< \varepsilon$$

$$|x - u| < \delta \Rightarrow |f(x) - f(u)| < \varepsilon$$

Lipschitz function is always uniformly continuous function. □

Theorem 5.3.3. *If $f : A \rightarrow \mathbb{R}$ is uniformly continuous on subset A of \mathbb{R} and if x_n is a cauchy sequence in A , then $f(x_n)$ is cauchy sequence in \mathbb{R} .*

Proof. let x_n is a cauchy sequence in A and let $\varepsilon > 0$ choose $x, y \in A$, $\delta > 0$

$$|x - u| < \delta \Rightarrow |f(x) - f(u)| < \varepsilon$$

Since, x_n is a cauchy sequence $\exists H(\delta)$

$$|x_n - x_m| < \delta, \quad \forall n, m \geq H(\delta)$$

(as f is uniformly continuous)

$$|f(x_n) - f(x_m)| < \varepsilon$$

Therefore, the sequence $f(x_n)$ is cauchy sequence. □

Theorem 5.3.4 (Continuous Extension Theorem). *A function f is uniformly continuous on (a, b) iff it can be defined at the end points a & b such that the extended function is continuous on $[a, b]$.*

Proof. Assume that function f is continuous on $[a, b]$

\therefore by defination,

for any $\varepsilon > 0$, $\frac{\varepsilon}{2} > 0$, $\exists \delta(\varepsilon) > 0$ such that,

$$|x - c| < \delta \Rightarrow |f(x) - f(c)| < \varepsilon$$

let x_1 & $x_2 \in [a, b]$

by defination,

for any $\varepsilon > 0$, $\frac{\varepsilon}{2} > 0$, $\exists \delta(\varepsilon) > 0$ such that,

$$|x_1 - c| < \frac{\delta}{2} \Rightarrow |f(x_1) - f(c)| < \frac{\varepsilon}{2}$$

and,

for any $\varepsilon > 0$, $\frac{\varepsilon}{2} > 0$, $\exists \delta(\varepsilon) > 0$ such that,

$$|x_2 - c| < \frac{\delta}{2} \Rightarrow |f(x_2) - f(c)| < \frac{\varepsilon}{2}$$

consider,

$$|x_1 - x_2|$$

$$= |x_1 - c + c - x_2|$$

$$\leq |x_1 - c| + |x_2 - c|$$

$$\leq \frac{\delta}{2} + \frac{\delta}{2}$$

$$\leq \delta$$

and,

$$|f(x_1) - f(x_2)|$$

$$= |f(x_1) - f(c) + f(c) - f(x_2)|$$

$$\leq |f(x_1) - f(c)| + |f(x_2) - f(c)|$$

$$\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

$$\leq \varepsilon$$

$$\therefore |x_1 - x_2| \leq \delta \Rightarrow |f(x_1) - f(x_2)| \leq \varepsilon$$

$\therefore f$ is uniformly continuous on (a, b)

Conversely, Suppose f is uniformly continuous on (a, b) . Let's define $f(a)$ & $f(b)$

Let's x_n be sequence in (a, b) $\ni \lim x_n = a$

$\Rightarrow x_n$ is cauchy sequence and as f is uniformly continuous on (a, b) and $x_n \in (a, b)$

\therefore by sequential criteria, $\lim f(x_n) = L$ exists if y_n is any other sequence in (a, b) that converges to a then

$$\lim x_n - y_n = a - a = 0$$

$$\lim f(y_n) = \lim (f(y_n) - f(x_n) + f(x_n)) = L$$

So we define, $L = f(a)$

then f is continuous at a

Similarly, we can find some $M = f(b)$ and we can say that f is continuous on extended $[a, b]$ □

Definition 5.3.4 (Step Function): $I \subseteq \mathbb{R}$ be an interval and let $S: I \rightarrow \mathbb{R}$ then S is called a step function if it has only a finite number of distinct values.

5.4 Continuity And Gauges

Definition 5.4.1 (Partition): A partition of an interval $I = [a, b]$ is collection $P = \{I_1, I_2, \dots, I_n\}$ of non-over-lapping closed intervals whose union is $[a, b]$. We generally denote $I_i = [x_{i-1}, x_i]$ where $a = x_0 < \dots < x_{i-1} < x_i < \dots < x_n = b$

The points x_i ($i = 0, 1, 2, \dots, n$) are called the partition points of p . If a point t_i has been chosen from each interval I_i , for $(i = 0, 1, 2, \dots, n)$ then the points t_i are called tags and set of ordered pairs $\dot{p} = \{(I_1, t_1), (I_2, t_2), \dots, (I_n, t_n)\}$ is called as tagged partition of I

Definition 5.4.2: A gauge on I is a strictly positive function defined on I . if δ is a gauge on I , then a tagged partition \dot{p} is said to be δ -fine if

$$t_i \in I_i \subseteq [t_i - \delta(t_i), t_i + \delta(t_i)]$$

If a partition p of $I = [a, b]$ is a δ -fine & $x \in I$, then \exists a tag t_i in p such that $|x - t_i| \leq \delta(t_i)$

Alternative proof of Boundedness Theorem

Proof. Since f is continuous on I , then for each $t \in I$ \exists $\delta(t) > 0$ \ni if $x \in I$ and

$$|x - t| < \delta(t) \text{ then } |f(x) - f(t)| < 1$$

Thus, δ -gauge on I let $\{(I_i, t_i)\}_{i=1}^n$ be δ -fine partition on I and let

$$k = \max\{|f(t_i)| \mid i = 1, 2, \dots, n\}$$

Given any $x \in I$ \exists i with $|x - t_i| \leq \delta(t_i)$

$$|f(x)|$$

$$= |f(x) - f(t_i) + f(t_i)|$$

$$\leq 1 + k$$

Since $x \in I$ is arbitrary, f is bounded. □

Definition 5.4.3 (Monotone and Inverse Function): If $A \subseteq \mathbb{R}$, then a function $f : A \rightarrow \mathbb{R}$ is said to be increasing on A if whenever $x_1, x_2 \in A$ and $x_1 < x_2$ then $f(x_1) \leq f(x_2)$

if $x_1, x_2 \in A$ and $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$ then f is called strictly increasing function.

Similarly, for decreasing function,

$x_1 < x_2$ then $f(x_1) \geq f(x_2)$ and strictly decreasing function

$$x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$$

Theorem 5.4.1. Let $I \subseteq \mathbb{R}$ be an interval and let $f : I \rightarrow \mathbb{R}$ be increasing on I . Suppose $c \in I$ is not endpoint of I then ,

$$1. \lim_{x \rightarrow c^-} = \sup\{f(x) / x \in I, x < c\}$$

$$2. \lim_{x \rightarrow c^+} = \sup\{f(x) / x \in I, x > c\}$$

Proof. 1. Let $x \in I$ & $x < c \Rightarrow f(x) < f(c)$

So, for set $\{f(x) / x \in I, x > c\}$, $f(c)$ is upper bound, So by completeness property,

\exists Supremum, say L .

if $\varepsilon > 0$, then $L - \varepsilon$ is not upper bound

Hence, $\exists y_\varepsilon \in I, y_\varepsilon < c$

$$\ni L - \varepsilon < f(y_\varepsilon) \leq L$$

Since, f is increasing, if $\delta_\varepsilon = c - y_\varepsilon$ and if

$0 < c - y < \delta_c$ then

$$y_\varepsilon < y < c$$

So that, $t - \varepsilon < f(y_\varepsilon) \leq f(y) \leq L$

$$\Rightarrow |f(y) - L| < \varepsilon \text{ when } 0 < c - y < \delta_c$$

Similarly we can prove (ii)

□

Theorem 5.4.2 (Continuous Inverse Function). *Let $I \subseteq \mathbb{R}$ be an interval and let $f : I \rightarrow \mathbb{R}$ be strictly monotone and continuous on I then the function g – inverse to f is strictly monotone and continuous on $I = f(I)$*

Proof. Let f is strictly increasing

Since f is continuous on I

By preservation of interval theorem,

$J = f(I)$ is also an interval. Also,

$f : I \rightarrow \mathbb{R}$ is strictly monotone and injective on I , therefore, inverse function $g : J \rightarrow \mathbb{R}$ exists if

$y_1, y_2 \in J$ with $y_1 < y_2$ then

$$y_1 = f(x_1), \quad y_2 = f(x_2) \text{ for some } x_1, x_2 \in I$$

$\Rightarrow x_1 < x_2$ as function is increasing

$$\Rightarrow x_1 = g(y_1) < g(y_2) = x_2$$

Since, y_1, y_2 arbitrary elements of J with

$y_1 < y_2$, we conclude that g is strictly increasing on J .

Now, we have to show that g is continuous on J .

As $g(J) = I$ is an interval.

Indeed, if g is discontinuous at a point $c \in J$, then the jump at c is non-zero so that $\lim_{y \rightarrow c^-} g <$

$\lim_{y \rightarrow c^+} g$

if we choose any number $x \neq g(c)$ satisfying $\lim_{y \rightarrow c^-} g < x < \lim_{y \rightarrow c^+} g$

then, $x \neq g(y)$, for any $y \in J$

Hence, $x \notin I$ which contradicts to our given condition that I is interval.

\therefore The inverse function g is continuous on J . □

Differentiation

6.1 Derivative

Definition 6.1.1 (Derivative): Let $I \subseteq \mathbb{R}$ be an interval. let $f : I \rightarrow \mathbb{R}$ and let $c \in I$. We say that a real number L is derivative of f at c if given any $\varepsilon > 0 \quad \exists \quad \delta(\varepsilon) > 0 \quad \ni$ if $x \in I$ satisfies

$$0 < |x - c| < \delta \Rightarrow \left| \frac{f(x) - f(c)}{x - c} - L \right| < \varepsilon$$

We say, f is differentiable at c .

$$f'(c) = \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c}$$

Theorem 6.1.1. If $f : I \rightarrow \mathbb{R}$ has a derivative at $c \in I$, then f is continuous at c .

Proof. $\forall \quad x \in I, x \neq c$

$$f(x) - f(c) = \frac{f(x) - f(c)}{x - c} (x - c)$$

$$\begin{aligned}
& \lim_{x \rightarrow c} f(x) - f(c) \\
&= \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c} (x - c) \\
&= f'(c) \cdot 0 \\
&= 0 \\
& \lim_{x \rightarrow c} f(x) = f(c)
\end{aligned}$$

$\therefore f$ is continuous at point c

if $f : I \rightarrow \mathbb{R}$ is continuous at point c then f may or may not be derivable at c . □

Example 33:

$f(x) = |x|$ is continuous at 0 but not differentiable at 0.

Theorem 6.1.2. *Let $I \subseteq \mathbb{R}$ be an interval, let $c \in I$ & let $f : I \rightarrow \mathbb{R}$ & $g : I \rightarrow \mathbb{R}$ be function that are differentiable at c then*

a) $(\alpha f)'(c) = \alpha f'(c), \quad \alpha \in \mathbb{R}$

b) $(f + g)'(c) = f'(c) + g'(c)$

c) $(fg)'(c) = f'(c)g(c) + f(c)g'(c)$

$$d) \left(\frac{f}{g} \right)'(c) = \frac{f'(c)g(c) - f(c)g'(c)}{(g(c))^2} \quad (g(c) \neq 0)$$

$$\begin{aligned} \text{Proof. a) } (\alpha f)'(c) &= \lim_{x \rightarrow c} \frac{(\alpha f)(x) - (\alpha f)(c)}{x - c} \\ &= \lim_{x \rightarrow c} \frac{\alpha f(x) - \alpha f(c)}{x - c} \\ &= \alpha \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c} \end{aligned}$$

$$\underline{(\alpha f)'(c) = \alpha f'(c)}$$

$$b) \lim_{x \rightarrow c} \frac{(f + g)(x) - (f + g)(c)}{x - c} = (f + g)'(c)$$

$$\therefore (f + g)'(c)$$

$$\begin{aligned} &= \lim_{x \rightarrow c} \frac{(f(x) + g(x)) - (f(c) + g(c))}{x - c} \\ &= \lim_{x \rightarrow c} \frac{(f(x) - f(c)) - (g(x) - g(c))}{x - c} \end{aligned}$$

$$= \lim_{x \rightarrow c} \left\{ \frac{f(x) - f(c)}{x - c} \right\} + \lim_{x \rightarrow c} \left\{ \frac{g(x) - g(c)}{x - c} \right\}$$

$$\underline{(f + g)'(c) = f'(c) + g'(c)}$$

c) Let $h(x) = fg(x)$

$$h'(c) = \lim_{x \rightarrow c} \frac{h(x) - h(c)}{x - c}$$

$$(fg)'(c)$$

$$= \lim_{x \rightarrow c} \frac{fg(x) - fg(c)}{x - c}$$

$$= \lim_{x \rightarrow c} \frac{f(x).g(x) - f(c).g(c)}{x - c}$$

$$= \lim_{x \rightarrow c} \frac{f(x).g(x) - f(c).g(x) + f(c).g(x) - f(c).g(c)}{x - c}$$

$$= \lim_{x \rightarrow c} \frac{(f(x) - f(c))g(x) + f(c).(g(x) - g(c))}{x - c}$$

$$= \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c} \cdot \lim_{x \rightarrow c} g(x) + \lim_{x \rightarrow c} \frac{g(x) - g(c)}{x - c} \cdot f(c)$$

$$\underline{(fg)'(c) = f'(c)g(c) + f(c)g'(c)}$$

d) let $h = \frac{f}{g}$

$$\therefore h'(c) = \lim_{x \rightarrow c} \frac{h(x) - h(c)}{x - c}$$

$$\therefore \left(\frac{f}{g} \right)'(c)$$

$$= \lim_{x \rightarrow c} \frac{\left(\frac{f}{g} \right)(x) - \left(\frac{f}{g} \right)(c)}{x - c}$$

$$= \lim_{x \rightarrow c} \frac{\frac{f(x)}{g(x)} - \frac{f(c)}{g(c)}}{x - c}$$

$$= \lim_{x \rightarrow c} \frac{f(x).g(c) - f(c).g(x)}{g(x).g(c)(x - c)}$$

$$= \lim_{x \rightarrow c} \frac{f(x).g(c) - f(c).g(c) + f(c).g(c) - f(c).g(x)}{g(x).g(c)(x - c)}$$

$$\begin{aligned}
&= \lim_{x \rightarrow c} \frac{(f(x) - f(c)) \cdot g(c) + f(c) \cdot (g(x) - g(c))}{g(x) \cdot g(c)(x - c)} \\
&= \lim_{x \rightarrow c} \left(\frac{1}{g(x) \cdot g(c)} \right) \left[\lim_{x \rightarrow c} \left(\frac{f(x) - f(c)}{x - c} \right) \cdot g(c) - \lim_{x \rightarrow c} \left(\frac{g(x) - g(c)}{x - c} \right) f(c) \right] \\
&= \frac{1}{(g(c))^2} [f'(c) \cdot g(c) - g'(c) f(c)] \\
&\underline{\left(\frac{f}{g} \right)'(c) = \frac{f'(c)g(c) - f(c)g'(c)}{(g(c))^2} \quad (g(c) \neq 0)}
\end{aligned}$$

□

Theorem 6.1.3. Let f be defined on an interval I containing point c . Then f is differentiable at c iff \exists a function ψ on I that is continuous at c and satisfies $f(x) - f(c) = \psi(x)(x - c)$ $x \in I$. In this case, $\psi(c) = f'(c)$.

Proof. If $f'(c)$ exists we can define,

$$\psi(x) = \begin{cases} \frac{f(x) - f(c)}{x - c} & \text{for } x \neq c, x \in I \\ f'(c) & \text{for } x = c \end{cases}$$

$$\lim_{x \rightarrow c} \psi(x) = f'(c)$$

Now, assume that ψ function is continuous at c and satisfies

$$f(x) - f(c) = \psi(x).(x - c)$$

$$\lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c} = \lim_{x \rightarrow c} \psi(x) = \psi(c) \text{ exists}$$

$\therefore f$ is differentiable at c and $\psi(c) = f'(c)$ □

6.2 Chain Rule

Theorem 6.2.1 (Chain Rule). *Let I, J be intervals in \mathbb{R} . Let $g : I \rightarrow \mathbb{R}$ & $f : J \rightarrow \mathbb{R}$ be function $\ni f(J) \subseteq I$ and let $c \in J$*

If f is differentiable at c & g is differentiable at $f(c)$ then the composite function $g \circ f$ is differentiable at c and $(g \circ f)'(c) = g'(f(c)).f'(c)$

Proof. Given that f is differentiable at c

$\therefore \exists$ function ψ on $J \ni$

$$f(x) - f(c) = \psi(x).(x - c) \text{ \& } f'(c) = \psi(c)$$

Also, g is differentiable at $f(c)$

\exists function ψ on $I \ni$

$$g(f(x)) - g(f(c)) = \psi(f(x)).(f(x) - f(c)) \text{ \& } g'(f(c)) = \psi(f(c))$$

Consider,

$$g \circ f(x) - g \circ f(c)$$

$$= g(f(x)) - g(f(c))$$

$$= \psi(f(x)).(f(x) - f(c))$$

$$= \psi(f(x)).(\psi(x).(x - c))$$

$$= [\psi(f(x)).\psi(x)].(x - c)$$

$\therefore g \circ f$ is differentiable at c

$$\text{Also, } \lim_{x \rightarrow c} \frac{g \circ f(x) - g \circ f(c)}{(x - c)}$$

$$= \lim_{x \rightarrow c} [\psi(f(x)).\psi(x)]$$

$$= \psi(f(c)).\psi(c)$$

$$(g \circ f)'(c) = g'(f(c)).f'(c)$$

□

Definition 6.2.1 (Inverse Function): Let I be an interval in \mathbb{R} and $f : I \rightarrow \mathbb{R}$ strictly monotone and continuous on I . let $J = f(I)$ and let $g : J \rightarrow \mathbb{R}$ be strictly monotone and continuous function inverse to f .

Theorem 6.2.2. If f is differentiable at c , $c \in I$ & $f'(c) \neq 0$ then g is differentiable at $d = f(c)$ &

$$g'(d) = \frac{1}{f'(c)} = \frac{1}{f'(g(d))}$$

Proof. Given that, f is differentiable at $c \in I$

$\therefore \exists \psi$ on I continuous at $c \ni$

$$f(x) - f(c) = \psi(x).(x - c) \text{ \& } \psi(c) = f'(c)$$

Since $f'(c) \neq 0 \Rightarrow \psi(c) \neq 0$

\exists neighbourhood of c , $\nu = (c - \delta, c + \delta)$

$$\ni \psi(x) \neq 0 \quad \forall x \in \nu \cap I$$

If $U = f(\nu \cap I)$ then inverse function g satisfies

$$f(g(y)) = y, \quad \forall y \in U$$

$$y - d = f(g(y)) - f(c) = \psi(g(y)) \cdot (g(y) - g(d))$$

$$\text{since, } \psi(g(y)) \neq 0, \quad \forall y \in U$$

$$g'(y) - g(d) = \frac{1}{\psi(g(y))}(y - d)$$

Since, $\psi(g(y))$ is continuous at d

$\therefore g'(d)$ exists and

$$g'(d) = \frac{1}{\psi(g(d))} = \frac{1}{\psi(c)} = \frac{1}{f'(c)}$$

□

Theorem 6.2.3. *Prove that if $f : \mathbb{R} \rightarrow \mathbb{R}$ is an even function [i.e $f(-x) = f(x) \forall x$] and has derivative at every point, then the derivative f' is an odd function. Also, prove that if $g : \mathbb{R} \rightarrow \mathbb{R}$ is a differentiable odd function, then g' is even function.*

Proof. a) Given that f is even function

$$f(x) = f(-x) \forall x$$

Also, f is differentiable at c

$$\therefore f'(c) = \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c} \text{ exists}$$

To prove, f' is odd function

$$\text{i.e } f'(-c) = -f'(c)$$

consider,

$$f'(-c)$$

$$= \lim_{x \rightarrow -c} \frac{f(x) - f(-c)}{x + c}$$

$$= \lim_{-x \rightarrow c} \frac{f(x) - f(c)}{-x + c}$$

$$= \lim_{-x \rightarrow c} \frac{f(x) - f(c)}{-(x - c)}$$

$$= -f'(c)$$

$\therefore f'$ is odd function.

b) Given that g is odd function

$$g(x) = g(-x) \forall x$$

Also, g is differentiable at c

$$\therefore g'(c) = \lim_{x \rightarrow c} \frac{g(x) - g(c)}{x - c} \text{ exists}$$

To prove, g' is even function

$$\text{i.e } g'(-c) = -g'(c)$$

consider,

$$g'(-c)$$

$$= \lim_{x \rightarrow -c} \frac{g(x) - g(-c)}{x + c}$$

$$= \lim_{-x \rightarrow c} \frac{-g(x) + g(c)}{-x + c}$$

$$= \lim_{x \rightarrow c} \frac{g(x) - g(c)}{(x - c)}$$

$$= g'(c)$$

$\therefore g'$ is even function.

□

Theorem 6.2.4 (Interior Extremum). *Let c be an interior point of the interval I at which $f : I \rightarrow \mathbb{R}$ has a relative extremum. If derivative of f at c exists, then $f'(c) = 0$*

Proof. Let f has relative maximum at c

if $f'(c) > 0$, $\exists V_\varepsilon(c) \subseteq I$

$$\frac{f(x) - f(c)}{x - c} > 0, \quad \forall x \in v_\varepsilon(c), x \neq c$$

if $x \in v, x > c$

$$f(x) - f(c) = (x - c) \cdot \frac{f(x) - f(c)}{x - c} > 0$$

$$\therefore f(x) > f(c) \quad \forall x > 0, \quad x \in v_\varepsilon(c)$$

but, f has relative maximum at c .

So, our assumption is wrong that $f'(c) > 0$

Similarly, we can show that $f'(c) < 0$

$$\therefore f'(c) = 0$$

□

Corollary 6.2.4.1. *Let $f : I \rightarrow \mathbb{R}$ be continuous on an interval I and suppose that f has relative*

extremum at an interior at c of I then either the derivative of f at c does not exist or it is equal to 0

Theorem 6.2.5 (Rolle's theorem). *If a function f defined on $[a, b]$ is*

- 1. Continuous on $[a, b]$*
- 2. derivable on (a, b)*
- 3. $f(a) = f(b)$*

then $\exists c \in \mathbb{R}, c \in (a, b) \ni f'(c) = 0$

Proof. Since, f is continuous $[a, b] \Rightarrow f$ is bounded

\therefore by maximum- minimum theorem,

If $m = \inf\{f(I)\}$ and $M = \sup\{f(I)\}$ then $\exists c, d \in (a, b)$

$f(c) = m$ & $f(d) = M$

there are two possibilities $m = M$ or $m \neq M$

If $m = M$

$\Rightarrow \inf\{f(I)\} = \sup\{f(I)\} \Rightarrow f$ is continuous

$\Rightarrow f'(c) = 0, \quad \forall c \in (a, b)$

If $m \neq M$

$\Rightarrow f(c) = m \neq f(a) \Rightarrow c \neq a$

$\Rightarrow f(c) = m \neq f(b) \Rightarrow c \neq b$

$\Rightarrow c$ lies in (a, b)

Now, we have to show $f'(c) = 0$

IF $f'(c) < 0$, $\exists (c, c + \delta), \delta_1 > 0$ for every x of which $f(x) < f(c) = m$ which contradicts to our assumption that infimum attains at c .

Simillarly, $f'(c) > 0$ is not possible

$\therefore f'(c) = 0$

□

Theorem 6.2.6 (Langrange's Mean Value theorem). *If a function f defined on $[a, b]$*

i) Continuous on $[a, b]$

ii) differentiable on (a, b)

then $\exists c \in (a, b) \ni f'(c) = \frac{f(b) - f(a)}{b - a}$

Proof. Let us define function ψ on $[a, b]$ such that

$\psi(x) = f(x) - Ax$, where A is constant.

As $f(x)$ is continuous on $[a, b]$ and differentiable on (a, b) ,

$\psi(x)$ is also continuous on $[a, b]$ and differentiable on (a, b)

Assume, $\psi(a) = \psi(b)$

$$f(a) - A.a = f(b) - A.b$$

$$f(b) - f(a) = A(b - a)$$

$$\therefore A = \frac{f(b) - f(a)}{b - a}$$

$$\therefore \psi(x) = f(x) - \left(\frac{f(b) - f(a)}{b - a} \right) x$$

- i) $\psi(x)$ is continuous on $[a, b]$
- ii) $\psi(x)$ is derivable on (a, b)
- iii) $\psi(a) = \psi(b)$

\therefore by Rolle's theorem

$$\psi'(c) = f'(c) - \left(\frac{f(b) - f(a)}{b - a} \right)$$

$$f'(c) = \left(\frac{f(b) - f(a)}{b - a} \right)$$



Theorem 6.2.7 (Cauchy Mean Value theorem). *If f, g defined on $[a, b]$*

- i) *continuous on $[a, b]$*
- ii) *derivable on (a, b)*
- iii) $g'(x) \neq 0, \quad \forall x \in (a, b) \exists \quad c \in (a, b) \quad \ni$

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

Proof. Let us define function $\psi(x)$ on $[a, b] \ni$

$$\psi(x) = f(x) - Ag(x)$$

- i) $\psi(x)$ is continuous on $[a, b]$

ii) $\psi(x)$ is derivable on (a, b)

iii) $\psi(a) = \psi(b)$

$$\Rightarrow f(a) + A.g(a) = f(b) - A.g(b)$$

$$\therefore f(b) - f(a) = A(g(b) - g(a))$$

$$A = \frac{f(b) - f(a)}{g(b) - g(a)}$$

\therefore by rolles theorem,

$$\psi'(c) = 0$$

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$



6.3 Taylor's Theorem

Theorem 6.3.1 (Taylor's Theorem). *If a function f defined on $[a, a + h]$ is such that*

i) $(n - i)^t h$ derivative f^{n-1} is continuous on $[a, a + h]$ and

ii) $n^t h$ derivative f^n exists on $(a, a + h)$ then \exists atleast one real number θ between 0 & 1 ($0 < \theta < 1$) that,

$$f(a+h) = f(a) + hf'(a) + \left(\frac{h^2}{2!}\right)f''(a) + \left(\frac{h^3}{3!}\right)f'''(a) + \dots + \left(\frac{h^{n-1}}{(n-1)!}\right)f^{n-1}(a) + \left(\frac{h^n(1-\theta)^{n-p}}{p[(n-1)!]}\right)f^n(a +$$

θh)

where p is given positive integer \mathbb{R}_n
forms of remainder form-

$$i) R_n = \left(\frac{h^n (1 - \theta)^{n-p}}{p[(n-1)!]} \right) f^n(a + \theta h)$$

$$ii) R_n = \left(\frac{h^n (1 - \theta)^{n-1}}{(n-1)!} \right) f^n(a + \theta h) \Rightarrow \text{Cauchy}$$

$$iii) R_n = \left(\frac{h^n}{n!} \right) f^n(a + \theta h) \Rightarrow \text{Called as Lagranges Forms of remainder}$$

Theorem 6.3.2 (Maclaurins Theorem). $f(x) = f(0) + x f'(0) + \left(\frac{x^2}{2!} \right) f''(0) + \left(\frac{x^3}{3!} \right) f'''(0) + \dots + \left(\frac{x^{n-1}}{(n-1)!} \right) f^{n-1}(0) + \left(\frac{x^n (1 - \theta)^{n-p}}{p[(n-1)!]} \right) f^n(\theta x)$

Example 34:

$$f(x) = e^x$$

\therefore By Maclaurins theorem,

$$f(x) = f(0) + x f'(0) + \left(\frac{x^2}{2!} \right) f''(0) + \dots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \dots$$

Example 35:

$$f(x) = \sin(x)$$

\therefore By Maclaurins theorem,

$$f(x) = \sin 0 + x \cos 0 + \frac{x^2}{2!}(-\sin 0) + \frac{x^3}{3!}(-\cos 0) + \dots$$

$$\therefore \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$$

Example 36:

$$f(x) = \log(1+x)$$

$$f'(x) = \frac{1}{1+x}, \quad f''(x) = \frac{-1}{(1+x)^2}, \quad f'''(x) = \frac{2}{(1+x)^3}$$

\therefore By Maclaurins theorem,

$$f(x) = f(0) + x f'(0) + \left(\frac{x^2}{2!}\right) f''(0) + \left(\frac{x^3}{3!}\right) f'''(0) + \dots$$

$$f(x) = 0 + x(1) + \left(\frac{x^2}{2!}\right)(-1) + \left(\frac{x^3}{3!}\right)(2) + \dots$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

6.4 Maximum or Minimum for function of two variables

$f(a, b)$ is extreme value of $f(x, y)$. if

$$\text{i) } f_x(a, b) = 0 = f_y(a, b)$$

$$\text{ii) } f_{xx}(a, b) = f_{yy}(a, b) - [f_{xy}(a, b)]^2 > 0$$

and this extreme value is maximum or minimum according as $f_{xx}(a, b)$ or $f_{yy}(a, b)$ is negative or positive.

Further investigation needed if,

$$f_{xx}(a, b) \cdot f_{yy}(a, b) - [f_{xy}(a, b)]^2 = 0$$

Example 37:

find maximum and minimum of

$$f(x, y) = x^3 + y^3 - 3x + 12y + 20 = 0$$

$$\text{Proof. } f_x(x, y) = 0$$

$$\text{i.e } 3x^2 - 3 = 0$$

$$x^2 = 1$$

$$x = \pm 1$$

$$f_y(x, y) = 0$$

$$\text{i.e } 3y^2 + 12 = 0$$

$$y^2 = 4$$

$$y = \pm 2$$

$$f_{xx}(x, y) = 6x, \quad f_{yy}(x, y) = 6y, \quad f_{xy}(x, y) = 0$$

for $x = 1, y = 2$

$$f_{xx}(x, y) = 6x = 6, \quad f_{yy}(x, y) = 6y = 12, \quad f_{xy}(x, y) = 0$$

for $x = -1, y = -2$

$$f_{xx}(x, y) = 6x = -6, \quad f_{yy}(x, y) = 6y = -12, \quad f_{xy}(x, y) = 0$$

for $x = -1, y = 2$

$$f_{xx}(x, y) = 6x = -6, \quad f_{yy}(x, y) = 6y = 12, \quad f_{xy}(x, y) = 0$$

for $x = 1, y = -2$

$$f_{xx}(x, y) = 6x = 6, \quad f_{yy}(x, y) = 6y = -12, \quad f_{xy}(x, y) = 0$$

minimum=(1,2)

maximum=(-1,-2)



Sequence and Series of Function

7.1 Sequence of Function

Definition 7.1.1 (Sequence of Function): Let $A \subseteq \mathbb{R}$ be given and suppose that for each $n \in \mathbb{N}$ $\exists f_n : A \rightarrow \mathbb{R}$ we shall say that (f_n) is a sequence of function on A to \mathbb{R}

Definition 7.1.2 (Pointwise Convergent): Let f_n be a sequence of function on $A \subseteq \mathbb{R}$ to \mathbb{R} . let $A_0 \subseteq A$ & let $f_n : A_0 \rightarrow \mathbb{R}$ we say that the sequence f_n converges on A_0 to f if for each $x \in A_0$ the sequence $f_n(x)$ converges to f

The sequence $f_n : A \rightarrow \mathbb{R}$ converges to function $f_n : A_0 \rightarrow \mathbb{R}$ on A_0 iff for each $\varepsilon > 0$ & $x \in A_0$ $\exists k(\varepsilon, x) \in \mathbb{N} \quad \ni \quad |f_n(x) - f(x)| < \varepsilon, \quad \forall n \geq k(\varepsilon, x)$

Example 38:

$$f_n(x) = \frac{x}{n}, \quad f(x) = 0$$

$$|f_n(x) - f(x)| < \varepsilon$$

$$\text{i.e. } \left| \frac{x}{n} - 0 \right| < \varepsilon \Rightarrow \left| \frac{x}{n} \right|$$

$$\therefore \frac{|x|}{n} < \varepsilon$$

$$\therefore n > \frac{|x|}{\varepsilon}$$

Example 39:

$$f_n(x) = x^n$$

$$\left| f_n(x) - f(x) \right| < \varepsilon$$

$$|x^n - 0| < \varepsilon, \quad -1 < x < 1$$

$$|x^n| < \varepsilon$$

$$n \log x < \log \varepsilon$$

$$n < \log \left(\frac{\varepsilon}{x} \right)$$

$$\therefore n > \log \left(\frac{x}{\varepsilon} \right)$$

Example 40:

$$f_n(x) = \frac{x^2 + nx}{n}, \quad x \in \mathbb{R}, \quad f(x) = x$$

$$\left| f_n(x) - f(x) \right| < \varepsilon$$

$$\left| \frac{x^2}{n} + x - x \right| < \varepsilon \quad \left| \frac{x^2}{n} \right| < \varepsilon$$

$$\therefore \frac{x^2}{\varepsilon} < n$$

$$\therefore n > \frac{x^2}{\varepsilon}$$

Definition 7.1.3 (Uniform Convergence): A sequence of function on $A \subseteq \mathbb{R}$ to \mathbb{R} converges uniformly on $A_0 \subseteq A$ to a function $f : A_0 \rightarrow \mathbb{R}$ if for each $\varepsilon > 0$ there is a natural number $k(\varepsilon)$ (depending on ε but not on $x \in A_0$) \exists

$$|f_n(x) - f(x)| < \varepsilon, \quad \forall n \geq k(\varepsilon)$$

denoted by, $f_n(x) \rightrightarrows f(x)$ on A_0

Lemma 7.1.1. A sequence f_n of function on $A \subseteq \mathbb{R}$ does not converges uniformly on $A_0 \subseteq A$ to a function $f : A_0 \rightarrow \mathbb{R}$ iff for some $\varepsilon_0 > 0 \exists$ subsequence f_{n_k} of f_n and a sequence x_k in A_0 such that

$$|f_{n_k}(x_k) - f(x_k)| \geq \varepsilon_0, \quad \forall k \in \mathbb{N}$$

Example 41:

$$f_n(x) = \frac{x_k}{n_k}, \quad f(x) = 0, \quad x_k = k, \quad n_k = k$$

$$|f_{n_k}(x_k) - f(x_k)| \geq \varepsilon_0$$

$$\left| \frac{x_k}{n_k} - 0 \right| \Rightarrow \left| \frac{k}{k} - 0 \right|$$

$$\Rightarrow |1 - 0|$$

$$\Rightarrow |1| \geq \varepsilon$$

Example 42:

$$f_n(x) = \frac{x^2 + nx}{n}, \quad f(x) = x, x_k = k, n_k = -k$$

$$|f_{n_k}(x_k) - f(x_k)| \geq \varepsilon_0$$

$$\left| \frac{x_k^2}{n_k} + x_k - x_k \right| \geq \varepsilon_0 \Rightarrow$$

$$\left| \frac{k^2}{k} \right| \geq \varepsilon_0$$

$$\therefore |k| > \varepsilon$$

\therefore not uniformly convergent

Example 43:

$$f_n(x) = x^n$$

$$f(x) = \begin{cases} 0 & ; 0 \leq x < 1 \\ 1 & ; x = 1 \end{cases}$$

$$x_k = \left(\frac{1}{2}\right)^{\left(\frac{1}{k}\right)}, \quad n_k = k$$

$$|f_{n_k}(x_k) - f(x_k)| \geq \varepsilon_0$$

$$\therefore |x_k^{n_k} - 0| \geq \varepsilon_0$$

$$\left| \left(\frac{1}{2}\right)^{\left(\frac{1}{k}\right)} - 0 \right| \geq \varepsilon_0$$

$$\therefore \left| \frac{1}{2} \right| > \varepsilon$$

\therefore Not uniformly convergent

Definition 7.1.4 (Uniform Norm): If $A \subseteq \mathbb{R}$ & $\psi : A \rightarrow \mathbb{R}$ is a function we say that ψ is bounded on A . If the set $\psi(A)$ is bounded subset of \mathbb{R} if ψ is bounded we define the uniform norm of ψ on A by, $\|\psi\|_A = \sup\{|\psi(x)| : x \in A\}$

Note that, it follows that if $\varepsilon > 0$,

$$\|\psi\|_A \leq \varepsilon \Leftrightarrow |\psi(x)| \leq \varepsilon, \quad \forall x \in A$$

Lemma 7.1.2. A sequence f_n of bounded function on $A \subseteq \mathbb{R}$ uniformly on A to f if and only if $\|f_n - f\|_A \rightarrow 0$

Example 44:

$$f(x) = x \quad [0, 1]$$

$$\sup\{|\psi(x)| : x \in A\} = 1$$

$$\|f\|_A = 1$$

Example 45:

$$f_n(x) = \frac{x}{n}, \quad f(x) = 0, \quad [0, 1]$$

$$|f_n(x) - f(x)| = |x|$$

$$\therefore \|f_n - f\|_A = \frac{1}{n} \rightarrow 0$$

Example 46:

$$f_n(x) = x^n \quad [0, k], \quad f(x) = 0$$

$$|f_n(x) - f(x)| = |x^n|$$

$$\therefore \|f_n - f\|_A = |k^n|$$

Example 47:

$$f_n(x) = x^n(1 - x) \quad x \in [0, 1], \quad f(x) = 0$$

$$|f_n(x) - f(x)| < \varepsilon$$

$$|x^n(1 - x) - 0| = |x^n(1 - x)|$$

$$f_n(x) = x^n - x^{n+1}$$

$$\therefore f'_n(x) = nx^{n-1} - (n+1)x^n = 0$$

$$\Rightarrow nx^{n-1} = (n+1)x^n$$

$$\Rightarrow \frac{n}{n+1} = x$$

$$x = \frac{1}{\left(1 + \frac{1}{n}\right)}$$

$$\therefore \|f_n - f\|_A = \frac{1}{\left(1 + \frac{1}{n}\right)^n} \left(1 - \frac{1}{\left(1 + \frac{1}{n}\right)}\right)$$

$$= \left(1 + \frac{1}{n}\right)^{-n} \left(\frac{\frac{1}{n}}{1 + \frac{1}{n}}\right)$$

$$= \left(1 + \frac{1}{n}\right)^{-n} \left(\frac{1}{1 + \frac{1}{n}}\right) \rightarrow 0$$

7.2 Cauchy Criteria for Uniform Convergence

Theorem 7.2.1. Let f_n be a sequence of bounded function on $A \subseteq \mathbb{R}$ then this sequence converges uniformly on A to a bounded function f iff for each $\varepsilon > 0 \quad \exists \quad H(\varepsilon) \in \mathbb{N} \ni$

$$\|f_m - f_n\|_A \leq \varepsilon, \quad \forall m, n \geq H(\varepsilon)$$

Proof. If $f_n(x) \rightrightarrows f(x)$ then for $\varepsilon > 0 \quad \exists k\left(\frac{\varepsilon}{2}\right) \ni$

$$\|f_n - f\|_A \leq \left(\frac{\varepsilon}{2}\right) \quad \forall n \geq k\left(\frac{\varepsilon}{2}\right)$$

Hence, if both $m, n \geq k\left(\frac{\varepsilon}{2}\right)$

$$\begin{aligned} & |f_m(x)' - f_n(x)| \\ &= |f_m(x) - f(x) + f(x)f_n(x)| \\ &\leq |f_m(x) - f(x)| + |f(x)f_n(x)| \end{aligned}$$

$$\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

$$\leq \varepsilon \quad \forall m, n \geq k\left(\frac{\varepsilon}{2}\right)$$

Conversly,

Suppose, $\varepsilon > 0$, $\exists H(\varepsilon) \in \mathbb{N}$

$\ni \|f_m - f_n\|_A \leq \varepsilon, \quad \forall m, n \geq H(\varepsilon)$

\therefore for each $x \in A$

$$|f_m(x) - f_n(x)| \leq \|f_m(x) - f_n(x)\|_A \leq \varepsilon, \quad \forall m, n \geq H(\varepsilon)$$

$\Rightarrow f_m(x)$ is cauchy sequence and hence convergent.

$$\therefore \exists f(x) = \lim_{n \rightarrow \infty} f_n(x) \quad \forall x \in A$$

We have $|f_m(x) - f_n(x)| \leq \varepsilon, \quad \forall m \geq H(\varepsilon)$

$\therefore f_n(x) \rightrightarrows f(x)$ on A

□

7.3 Series of Function

If f_n is sequence of function defined on subset D of \mathbb{R} with values in \mathbb{R} , the sequence of partial sums S_n of infinite series $\sum f_n$ is defined for x in D by,

$$S_1(x) = f_1(x)$$

$$S_2(x) = f_2(x) + S_1(x)$$

$$\vdots$$

$$\vdots$$

$$S_{n+1}(x) = S_n(x) + f_{n+1}(x)$$

$$\vdots$$

$$\vdots$$

- In the case sequence S_n of functions f_n converges to function f on D we say that $\sum f_n$ converges on D to f
- If the series $\sum |f_n(x)|$ converges for each $x \in D$, we say that $\sum f_n$ converges absolutely on D .
- if (S_n) sequence of partial sums is uniformly convergent on D to f , we say that $\sum f_n$ is uniformly converges on D
- If f_n is continuous on $D \subseteq \mathbb{R}$ to \mathbb{R} for each $n \in \mathbb{N}$ and if $\sum f_n$ converges f on D uniformly, then f is continuous on D

Definition 7.3.1 (Cauchy Criterion): f_n be a sequence of f_n on $D \subseteq \mathbb{R}$ to \mathbb{R} , the series $\sum f_n$ is uniformly convergent on D iff if for every uniformly $\varepsilon > 0$, $\exists M(\varepsilon)$

$$\exists |f_{n+1}(x) + f_{n+2}(x) + \dots + f_m(x)| < \varepsilon, \quad \forall m > n \leq M(\varepsilon)$$

Theorem 7.3.1 (Weistress M-test). Let M_n be a sequence of positive real numbers such that $|f_n(x)| \leq M_n \quad \forall x \in D \quad \forall n \in \mathbb{N}$. If the series M_n is convergent then $\sum f_n$ is uniformly convergent on D

Proof. M_n is convergent,

By cauchy criterion for series,

for any $\varepsilon > 0$, $\exists k(\varepsilon) \in \mathbb{N}$

$$\exists M_{n+1} + M_{n+2} + \dots + M_m < \varepsilon, \quad \forall m > n \leq k(\varepsilon)$$

$$\exists |f_{n+1}(x)| + |f_{n+2}(x)| + \dots + |f_m(x)| < M_{n+1} + M_{n+2} + \dots + M_m < \varepsilon$$

Also,

$$|f_{n+1}(x) + f_{n+2}(x) + \dots + f_m(x)| \leq |f_{n+1}| + |f_{n+2}| + \dots + |f_m| < \varepsilon \quad \forall m > n \geq k(\varepsilon)$$

\therefore By Cauchy criterion,

$\sum f_n$ is uniformly convergent on D . □

Definition 7.3.2 (Power Series): A series of real function $\sum f_n$ is said to be power series around $x = c$ if the function has the form $f_n(x) = a_n(x - c)^n$ where a_n and $c \in \mathbb{R}$ and where $n = 0, 1, 2, \dots$

Example 48:

Power Series

$$\sum_{n=0}^{\infty} a_n x^n = a_0 x^0 + a_1 x + \dots + a_n x^n + \dots$$

$$\sum_{n=0}^{\infty} n! x^n \quad \sum_{n=0}^{\infty} x^n \quad \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Definition 7.3.3 (Radius of Convergence): $\sum a_n x^n$ be a power series if a sequence $|a_n|^{\frac{1}{n}}$ is bounded, we set $\rho = \limsup |a_n|^{\frac{1}{n}}$ if this sequence is not bounded, we set $\rho = +\infty$

We define radius of converge of $\sum a_n x^n$ to be given by,

$$R = \begin{cases} 0 & ; \text{if } \rho = +\infty \\ \frac{1}{\rho} & ; \text{if } 0 < \rho < +\infty \\ \infty & ; \rho = 0 \end{cases}$$

The interval of convergence is the open interval $(-R, R)$

Example 49:

$$\sum \frac{x^n}{2^n} \Rightarrow \left| \frac{1}{2^n} x^n \right|$$

$$\Rightarrow a_n \cdot x^n$$

$$\rho = \limsup |a_n|^{\frac{1}{n}}$$

$$\Rightarrow \lim Sup \left| \frac{1}{2^n} \right|^{\frac{1}{n}}$$

$$\Rightarrow \frac{1}{2}$$

$$\Rightarrow R = \frac{1}{\rho} = 2$$

Example 50:

$$\sum nx^n \Rightarrow a_n x^n$$

$$R = \lim \left| \frac{a_n}{a_{n+1}} \right|$$

$$R = \lim \left| \frac{n}{n+1} \right|$$

$$R = \lim \left| \frac{1}{1 + \frac{1}{n}} \right|$$

$$R = 1$$

Riemann Integral

8.1 Introduction

Riemann Integral

If $I = [a, b]$ be closed bounded interval in \mathbb{R} then partition of I is a finite ordered set $\mathbb{P} = (x_0, x_1, \dots, x_{n-1}, x_n)$ of points in I such that $a = x_0 < x_1 < x_2 < \dots < x_{n-1} < x_n = b$.

The points of P are used to divide $I = [a, b]$ into non-overlapping sub-intervals.

$$I_1 = [x_0, x_1], I_2 = [x_1, x_2], \dots, I_n = [x_{n-1}, x_n]$$

Norm of $P = ||p|| = \max\{|x_i - x_{i-1}|, i = 1, 2, \dots, n\}$

The norm of partition is merely the length of largest sub-interval into which the partition divide if point t_i has been choosen from each sub-interval $I_i = [x_{i-1}, x_i] = \forall i = 1 : n$ then the points are called as tages of sub- intervals $I - i$.

A set of ordered pairs

$\dot{p} = \{[x_{i-1}, x_i], t_i\}_{i=1}^n$ is tagged partition of $[a, b]$

Definition 8.1.1 (Riemann Sum): If \dot{p} is the tagged partition, we define Riemann sum of function. $f : [a, b] \rightarrow \mathbb{R}$ corresponding to \dot{p} to be the number,

$$S(f, \dot{p}) = \sum_{i=1}^n f(t_i)(x_i - x_{i-1})$$

Definition 8.1.2 (Riemann Integral): A function $f : [a, b] \rightarrow \mathbb{R}$ is said to be Riemann integrable on $[a, b]$ if there exists a number $L \in \mathbb{R}$ such that for $\varepsilon > 0 \quad \exists \quad \delta_\varepsilon > 0 \quad \ni$ if \dot{p} is any tagged partition of $[a, b]$ with $||\dot{p}|| < \delta_\varepsilon$ then

$$|S(f, \dot{p}) - L| < \varepsilon$$

The set of all Riemann integrable functions on $[a, b]$ will be denoted by $R[a, b]$
i.e $||\dot{p}|| \rightarrow 0 \Rightarrow S(f, \dot{p}) \rightarrow L$

Definition 8.1.3: If $f \in R[a, b]$ then the number L is uniquely determined and called as Riemann Integral of f over $[a, b]$

$$L = \int_a^b f(x) dx$$

Theorem 8.1.1. If $f \in R[a, b]$ then the value of the integral is uniquely determined.

Proof. Assume that L' & L'' both satisfy the definition and

let $\varepsilon > 0 \quad \exists \quad \delta'_{\frac{\varepsilon}{2}} > 0 \quad \ni$ if \dot{p}_1 is tagged partition with $||\dot{p}_1|| < \delta'_{\frac{\varepsilon}{2}}$ then

$$|S(f, \dot{p}_1) - L'| < \frac{\varepsilon}{2}$$

Similarly, $\exists \quad \delta''_{\frac{\varepsilon}{2}} > 0 \quad \ni$ if \dot{p}_2 is tagged partition with $||\dot{p}_2|| < \delta''_{\frac{\varepsilon}{2}}$ then

$$|S(f, \dot{p}_2) - L''| < \frac{\varepsilon}{2}$$

Now, let $\delta_\varepsilon = \min\left(\delta'_{\frac{\varepsilon}{2}}, \delta''_{\frac{\varepsilon}{2}}\right)$

let \dot{p} be tagged partition with $\|\dot{p}\| < \delta_\varepsilon$

$$\Rightarrow |S(f, \dot{p}) - L'| < \frac{\varepsilon}{2} \text{ and}$$

$$|S(f, \dot{p}_2) - L''| < \frac{\varepsilon}{2}$$

$$\text{So, } |L' - L''| = |L' - S(f, \dot{p}) + S(f, \dot{p}) - L''|$$

$$\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

$$\leq \varepsilon$$

As ε is arbitrary, $L' = L''$

□

Theorem 8.1.2. Every constant function on $[a, b]$ is in $R[a, b]$.

Proof. Let $f(x) = k \quad \forall \quad x \in [a, b]$ be the constant function, if $\dot{p} = \{[x_{i-1}, x_i], t_i\}_{i=1}^n$ is any tagged partition on $[a, b]$

$$S(f, \dot{p}) = \sum_{i=1}^n k(x_i - x_{i-1}) = k(b - a)$$

Hence, for any $\varepsilon > 0$, we can choose $\delta_\varepsilon > 0 \quad \ni \quad \|\dot{p}\| < \delta_\varepsilon \text{ \& } S(f, \dot{p}) - k(b - a) = 0 < \varepsilon$

$$\int_a^b f(x) dx = k(b - a)$$

$\therefore f(x)$ is an Riemann integrable $f \in R[a, b]$

□

8.2 Some Properties of Integral

Theorem 8.2.1. Suppose that f & g are in $R[a, b]$ then

a) If $k \in \mathbb{R}$, the function $k.f$ is in $R[a, b]$ and $\int_a^b kf = k \int_a^b f$

b) the function f & g is in $R[a, b]$ and $\int_a^b f + g = \int_a^b f + \int_a^b g$

c) $f(x) \leq g(x) \quad \forall \quad x \in [a, b]$ then $\int_a^b f \leq \int_a^b g$

Theorem 8.2.2. If $f \in [a, b]$ then f is bounded on $[a, b]$

Proof. Assume that f is unbounded on $[a, b]$

As $f \in [a, b]$, then for any $\varepsilon > 0 \quad \exists \delta_\varepsilon > 0$

such that $\|\dot{p}\| < \delta_\varepsilon$ then $|S(f, \dot{p}) - L| < \varepsilon$

Now, let $Q = \{[x_{i-1}, x_i]\}_{i=1}^n$ be partition on $[a, b]$ with $\|Q\| < \delta$. Since $|f|$ is not bounded on $[a, b]$, \exists atleast one sub-interval $[x_{k-1}, x_k]$ on $[a, b]$ which $|f|$ is not bounded.

Let tag Q by $t_i = x_i$ for $i \neq k$ and $k_k \in [x_{k-1}, x_k]$ such that,

$$|f(t_k).(x_k - x_{k-1})| > |L| + \varepsilon + \left| \sum_{i \neq k}^n f(t_i)(x_i - x_{i-1}) \right|$$

By triangular inequality $|a + b| > |a| - |b|$

$$|S(f, Q)| \geq |f(t_k)(x_k - x_{k-1})| - \left| \sum_{i \neq k}^n f(t_i)(x_i - x_{i-1}) \right| > |L| + \varepsilon$$

\therefore which is contradict to our assumption.

$\therefore f$ is bounded on $[a, b]$ □

Definition 8.2.1 (Cauchy Criterion for Riemann Integrable function): *A function $f : [a, b] \rightarrow \mathbb{R} \in R[a, b]$ if and only if for every $\varepsilon > 0, \exists n_\varepsilon > 0$ if \dot{p} & \dot{Q} are any tagged partitions of $[a, b]$ with $\|\dot{p}\| < n_\varepsilon$ & $\|\dot{Q}\| < n_\varepsilon$ then,*

$$|S(f, \dot{p}) - S(f, \dot{Q})| < \varepsilon$$

Theorem 8.2.3 (Squeeze theorem). *Let $f : [a, b] \rightarrow \mathbb{R}$ then $f \in R[a, b]$ if and only if for every $\varepsilon > 0 \exists$ function α_ε & w_ε in $R[a, b]$ with*

$$\alpha_\varepsilon(x) \leq f(x) \leq w_\varepsilon \quad \forall x \in [a, b] \text{ \& such that } \int_a^b w_\varepsilon - \alpha_\varepsilon < \varepsilon$$

Proof. \iff Take $\alpha_\varepsilon = w_\varepsilon = f \quad \forall \varepsilon > 0$

\iff Let $\varepsilon > 0$, Since $\alpha_\varepsilon, w_\varepsilon \in R[a, b]$

$\exists \delta_\varepsilon > 0 \quad \exists \|\dot{p}\| < \delta_\varepsilon$ then

$$\left| S(\alpha_\varepsilon, \dot{p}) - \int_a^b \alpha_\varepsilon \right| < \varepsilon \text{ \& } \left| S(w_\varepsilon, \dot{p}) - \int_a^b w_\varepsilon \right| < \varepsilon$$

$$\Rightarrow \int_a^b \alpha_\varepsilon - \varepsilon < S(\alpha_\varepsilon, \dot{p}) \quad \& \quad S(w_\varepsilon, \dot{p}) < \int_a^b w_\varepsilon + \varepsilon$$

As $\alpha_\varepsilon \leq f \leq w_\varepsilon$

$$S(\alpha_\varepsilon, \dot{p}) \leq S(f, \dot{p}) \leq S(w_\varepsilon, \dot{p})$$

$$\Rightarrow \int_a^b \alpha_\varepsilon - \varepsilon \leq S(f, \dot{p}) \leq \int_a^b w_\varepsilon + \varepsilon$$

Consider another partition $\|\dot{Q}\| < \delta_\varepsilon$

$$\Rightarrow \int_a^b \alpha_\varepsilon - \varepsilon \leq S(f, \dot{Q}) \leq \int_a^b w_\varepsilon + \varepsilon$$

$$\Rightarrow |S(f, \dot{Q}) - S(f, \dot{p})| < \int_a^b (w_\varepsilon - \alpha_\varepsilon) + 2\varepsilon \leq 3\varepsilon$$

Since, $\varepsilon > 0$, is arbitrary, $f \in R[a, b]$ □

Theorem 8.2.4. If $f : R[a, b] \rightarrow \mathbb{R}$ is continuous on $[a, b]$ then $f \in R[a, b]$

Proof. As f is continuous on closed bounded interval $[a, b]$, f is uniformly continuous on $[a, b]$

\therefore for any $\varepsilon > 0$, $\exists \delta_\varepsilon > 0 \quad \ni$ if $u, v \in [a, b]$

$$|u - v| < \delta_\varepsilon$$

$$\Rightarrow |f(u) - f(v)| < \frac{\varepsilon}{b - a}$$

Let $p = \{I_i\}_{i=1}^n$ be a partition such that $\|p\| < \delta_\varepsilon$, let $u_i \in I_i$ be a point where f attains minimum value on I_i & $v_i \in I_i$ be a point where f attains maximum value on I_i

Let α_ε be the step function

$$\alpha_\varepsilon(x) = f(u_i) \quad \forall \quad x \in [x_{i-1}, x_i] (i = 1 : n-1)$$

Let w_ε be the step function

$$w_\varepsilon(x) = f(v_i) \quad \forall \quad x \in [x_{i-1}, x_i] (i = 1 : n-1)$$

$$\text{so, } \alpha_\varepsilon(x) \leq f(x) \leq w_\varepsilon(x) \quad \forall x \in [a, b]$$

$$0 \leq \int_a^b (w_\varepsilon - \alpha_\varepsilon) = \sum_{i=1}^n (f(v_i) - f(u_i))(x_i - x_{i-1})$$

$$< \sum_{i=1}^n \left(\frac{\varepsilon}{(b-a)} (x_i - x_{i-1}) \right) = \varepsilon$$

\therefore by squeeze theorem,

$$f \in R[a, b]$$

□

Theorem 8.2.5. *If $f : R[a, b] \rightarrow \mathbb{R}$ is monotone on $[a, b]$ then $f \in R[a, b]$*

Proof. Suppose f is I on $[a, b]$

Assume $a < b, \varepsilon > 0$

$$h = \frac{f(b) - f(a)}{q} < \frac{\varepsilon}{(b-a)}$$

$$\text{let } y_k = f(a) + k.h \quad \forall \quad k = 0, 1, \dots, q$$

$$\text{let } A_k = f^{-1}[y_{k-1}, y_k] \quad \forall \quad k = 0, 1, \dots, q-1$$

The sets A_k are pairwise disjoint and have union $[a, b]$

so A_k is either

a) empty

b) single point set

c) non degenerate interval in $[a, b]$

We discard the sets for which a) holds and relabel remaining ones if we adjoin the end points of the remaining intervals A_k , we obtain closed intervals I_k

So we now have step functions α_ε & w_ε

$$\alpha_\varepsilon(x) = y_{k-1}, \quad w_\varepsilon(x) = y_k \quad \forall x \in A_k$$

$$\alpha_\varepsilon(x) \leq f(x) \leq w_\varepsilon(x) \quad \forall x \in [a, b]$$

$$\int_a^b (w_\varepsilon - \alpha_\varepsilon)$$

$$= \sum_{k=1}^q (y_k - y_{k-1})(x_k - x_{k-1})$$

$$= \sum_{k=1}^q h(x_k - x_{k-1})$$

$$= h.(b - a)$$

so, by squeeze theorem,

$$f \in R[a, b]$$



8.3 Fundamental theorem of Integral calculus

Theorem 8.3.1. Suppose, there is finite set E in $[a, b]$ and function $f : [a, b] \rightarrow \mathbb{R}$ such that

1. f is continuous on $[a, b]$

$$2. F'(x) = f(x) \quad \forall \quad x \in [a, b]^E$$

$$3. f \in R[a, b] \text{ then } \int_a^b f = f(b) - f(a)$$

Proof. Let $\varepsilon > 0$, since $f \in R[a, b] \quad \exists \quad \delta_\varepsilon > 0$

\exists if p is any tagged partition $\|\dot{p}\| < \delta_\varepsilon$

$$\left| S(f, \dot{p}) - \int_a^b f \right| < \varepsilon$$

If the sub-intervals in p are $[x_{i-1}, x_i]$ then

by MVT, $\exists \quad u_i \in (x_{i-1}, x_i)$

$$F'(u_i) = \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}} \quad \forall \quad i = 1 : n$$

adding $i = 1 : n$

$$\sum_{i=1}^n f(x_i) - f(x_{i-1}) = \sum_{i=1}^n F'(u_i)(x_i - x_{i-1})$$

$$F(a) - F(b) = \sum_{i=1}^n f'(u_i)(x_i - x_{i-1}) = S(f, \dot{p})$$

Assuming $\dot{p}_u = \{[x_i - x_{i-1}], u_i\}_{i=1}^n$

$$\Rightarrow \left| F(a) - F(b) - \int_a^b f \right| < \varepsilon$$

$$\Rightarrow \int_a^b f = F(a) - F(b)$$

□

8.4 Indefinite Integral

Definition 8.4.1 (Indefinite Integral): If $f \in R[a, b]$ then $f(z) = \int_a^z f \quad \forall \quad z \in [a, b]$

Theorem 8.4.1. The indefinite integral F is continuous on $[a, b]$. In fact, if $|f(x)| \leq M \quad \forall x \in [a, b]$ then $|F(z) - F(w)| \leq M|z - w| \quad \forall z, w \in [a, b]$

Proof. If $z, w \in [a, b], \quad w \leq z$

$$F(z) = \int_a^z f = \int_a^w f + \int_w^z f = f(w) + \int_w^z f$$

$$\Rightarrow \int_w^z f = F(z) - F(w)$$

$$\text{if } -M \leq f(x) \leq M \quad \forall x \in [a, b]$$

$$-M(z - w) \leq \int_w^z f \leq M(z - w)$$

$$\Rightarrow |F(z) - F(w)| \leq \left| \int_w^z f \right| \leq M|z - w|$$

□

8.5 Examples

Example 51:

$$f(x) = x$$

$$g(x) = \frac{1}{x}$$

$$f \circ g = f(g(x)) = f\left(\frac{1}{x}\right) = \frac{1}{x}$$

$$g \circ f = g(f(x)) = g(x) = \frac{1}{x}$$

$$f \circ g = g \circ f$$

Example 52:

$$A_n = \{(n+1)k, \quad k \in \mathbb{N}\}$$

$$A_1 = \{2k, \quad k \in \mathbb{N}\}$$

$$A_2 = \{3k, \quad k \in \mathbb{N}\}$$

$$A_1 \cap A_2 = \{6k, \quad k \in \mathbb{N}\}$$

$$\cap A_i = \{\phi\}$$

$$\cup A_i = \mathbb{N} - \{1\}$$

Example 53:

$$\lim_{n \rightarrow \infty} \frac{n^2}{n!}$$

$$\lim_{n \rightarrow \infty} \frac{n \cdot n}{n \cdot (n-1)!}$$

$$\lim_{n \rightarrow \infty} \frac{n}{(n-2)(n-1)!}$$

$$\lim_{n \rightarrow \infty} \frac{1}{\left(1 - \frac{1}{n}\right)} \lim_{n \rightarrow \infty} \frac{1}{(n-2)!}$$

$$= (1)(0)$$

$$0$$

Example 54:

Result:- $\lim_{x \rightarrow \infty} (1 + a^x)^{\frac{1}{x}} = e^a$

$$x_n = (a^n + b^n)^{\frac{1}{n}}, \quad a < b$$

$$= \lim_{n \rightarrow \infty} (a^n + b^n)^{\frac{1}{n}}$$

$$= \lim_{n \rightarrow \infty} b \left(\frac{a^n}{b^n} + 1 \right)^{\frac{1}{n}}$$

$$= \lim_{n \rightarrow \infty} b \left(\left(\frac{a}{b} \right)^n + 1 \right)^{\frac{1}{n}}$$

$$= b.e^{\frac{a}{b}}$$

$\therefore (a^n + b^n)^{\frac{1}{n}}$ is convergent, bounded and cauchy.

Example 55:

$$\sum x_n = \frac{1}{1} - \frac{1}{5} + \frac{1}{7} - \frac{1}{13} + \frac{1}{17}$$

$$\sum |x_n| = \frac{1}{1} + \frac{1}{5} + \frac{1}{7} + \frac{1}{13} + \frac{1}{17}$$

$$\sum x_n = \frac{(-1)^{n-1}}{4n - (-1)^n}$$

Example 56:

$$f_n(x) = \frac{1}{nx+1}, \quad x \in (0, 1), f(x) = 0$$

$$|f_n(x) - f(x)| < \varepsilon$$

$$\left| \frac{1}{nx+1} - 0 \right| < \varepsilon$$

$$\therefore \left| \frac{1}{nx+1} \right|$$

$$|nx+1| > \frac{1}{\varepsilon}$$

Example 57:

Examine convergent of $\sum \left(\frac{1}{2^n} + \frac{1}{3^n} \right)$

$$\sum \left(\frac{1}{2^n} + \frac{1}{3^n} \right) = \sum \frac{1}{2^n} + \sum \frac{1}{3^n}$$

$$\sum \left(\frac{1}{2^n} + \frac{1}{3^n} \right) = \sum \left(\frac{1}{2} \right)^n + \sum \left(\frac{1}{3} \right)^n$$

$$\sum r_1^n + \sum r_2^n \quad r_1 = \frac{1}{2} < 1, r_2 = \frac{1}{3} < 1$$

which is convergent

Example 58:

$$f_n(x) = \frac{1}{x^n} \quad x \in (0, 1)$$

$$f(x) = \begin{cases} \text{not defined} & x = -1 \\ \frac{1}{2} & x = 1 \\ 0 & x > 1 \end{cases}$$

Example 59:

$$\lim_{n \rightarrow \infty} |x_n - x_{n+1}| = 0$$

x_n does not converges for given example j

Example 60:

$$\sum \frac{1}{\sqrt{n^3 + 4}} \text{ Use comparision test}$$

$$n < n^{\frac{3}{2}}, \quad n > 1$$

$$\frac{1}{n} > \frac{1}{n^{\frac{3}{2}}}$$

As $\frac{1}{n}$ is divergent $\Rightarrow \frac{1}{n^{\frac{3}{2}}}$ is also divergent.

Definition 8.5.1 (Taylors expansion for two variables): $f(x, y) =$
 $f(a, b) + \left[(x - a)\frac{\partial}{\partial x} + (y - b)\frac{\partial}{\partial y} \right] f(a, b) + \frac{1}{2!} \left[(x - a)\frac{\partial}{\partial x} + (y - b)\frac{\partial}{\partial y} \right]^2 f(a, b) + \dots$
 $\dots\dots\dots \frac{1}{(n-1)!} \left[(x - a)\frac{\partial}{\partial x} + (y - b)\frac{\partial}{\partial y} \right]^{n-1} f(a, b) + R_n$

Example 61:

a_n is bounded, decreasing sequence.

b_n is bounded, increasing sequence

$$x_n = a_n + b_n$$

$$\sum |x_n - x_{n+1}|$$

$$= \sum |a_n + b_n - a_{n+1} - b_{n+1}|$$

$$= \sum |a_n - a_{n+1} + b_n - b_{n+1}|$$

$$\leq |a_n - a_{n+1}| + |b_n - b_{n+1}|$$

$$\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

$$\leq \varepsilon$$

$$\sum |x_n - x_{n+1}| \rightarrow 0$$

Example 62:

$$\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}, \quad p > 0$$

$$\log n < n$$

$$\frac{1}{\log n} > \frac{1}{n}$$

$$\left(\frac{1}{\log n} \right)^p > \frac{1}{n^p}$$

$$\frac{1}{n(\log n)^p} > \frac{1}{n^{p+1}}, \quad p+1 > 1$$

\therefore by comparison test,

As $\sum \frac{1}{n^{p+1}}$ convergent $\Rightarrow \sum \frac{1}{n(\log n)^p}$ is convergent.

Example 63:

$$\sum x_n = \frac{1}{2} + \frac{1}{2 \cdot 2^2} + \frac{1}{n \cdot 2^n}$$

$$\begin{aligned} \left| \frac{x_{n+1}}{x_n} \right| &= \left| \frac{\frac{1}{(n+1)2^{n+1}}}{\frac{1}{n2^n}} \right| \\ &= \left| \frac{n \cdot 2^n}{(n+1)2^{n+1}} \right| \end{aligned}$$

$$= \left| \left(\frac{n}{n+1} \right) \frac{1}{2} \right|$$

$$\lim_{n \rightarrow \infty} \left| \frac{x_{n+1}}{x_n} \right| = \lim_{n \rightarrow \infty} \left| \left(\frac{1}{1 + \frac{1}{n}} \right) \frac{1}{2} \right|$$

by ratio test

$$= \frac{1}{2} < 1$$

$\sum x_n = \frac{1}{n2^n}$ is convergent.

Example 64:

$$S = \left\{ 1 + \frac{(-1)^n}{n}, n \in \mathbb{N} \right\}$$

limit point of $S = 1$

Example 65:

$$\sum x_n = \sum \frac{1}{\sqrt{n} + \sqrt{n-1}}$$

$$n > \sqrt{n}$$

$$\frac{1}{n} < \frac{1}{\sqrt{n}}$$

by Ratio test,

$$\left| \frac{x_{n+1}}{x_n} \right| = \left| \frac{\frac{1}{\sqrt{(n+1)} + \sqrt{n}}}{\frac{1}{\sqrt{n} + \sqrt{n-1}}} \right|$$

$$= \left| \frac{\sqrt{n} + \sqrt{n-1}}{\sqrt{n+1} + \sqrt{n}} \right|$$

$$= \left| \frac{n^{\frac{1}{2}}(1 + \sqrt{1 - \frac{1}{n}})}{n^{\frac{1}{2}}(1 + \sqrt{1 + \frac{1}{n}})} \right|$$

$$\therefore \lim \left| \frac{x_{n+1}}{x_n} \right| = \lim \left| \frac{n^{\frac{1}{2}}(1 + \sqrt{1 - \frac{1}{n}})}{n^{\frac{1}{2}}(1 + \sqrt{1 + \frac{1}{n}})} \right| = 1$$

\therefore Ratio test fails here

$$\sum \frac{1}{\sqrt{n} + \sqrt{n-1}} \times \frac{\sqrt{n} - \sqrt{n-1}}{\sqrt{n} - \sqrt{n-1}} = \sum \sqrt{n} - \sqrt{n-1}$$

$\therefore S_n = \sqrt{n}$ which divergent

$\therefore \sum \frac{1}{\sqrt{n} + \sqrt{n-1}}$ is divergent.

Example 66:

$$\sum \frac{(2n-1)}{n(n+1)(n+2)} = \frac{1}{1.2.3} + \frac{3}{2.3.4} + \dots$$

$$\left| \frac{x_{n+1}}{x_n} \right| = \left| \frac{\frac{(2n-1)}{n(n+1)(n+2)(n+3)}}{\left(\frac{2n-1}{n(n+1)(n+2)} \right)} \right|$$

$$= \left| \frac{(2n+1)n}{(2n-)(n+3)} \right|$$

$$= \left| \frac{\left(2 + \frac{1}{n}\right)}{\left(2 - \frac{1}{n}\right)\left(1 + \frac{3}{n}\right)} \right|$$

$$\therefore \lim \left| \frac{x_{n+1}}{x_n} \right| = 1$$

\therefore Ratio test fails here.

$$\sum \left(\frac{2n-1}{n(n+1)(n+2)} \right) = \sum \frac{2n}{n(n+1)(n+2)} - \sum \frac{1}{n(n+1)(n+2)}$$

$$= \sum \frac{2}{(n+1)(n+2)} - \sum \frac{1}{n(n+1)(n+2)}$$

$\therefore x_n$ is convergent.