Real Analysis

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Contents

Chapter 1

Introduction to Real Analysis

1.1 The Algebraic Properties of $\mathbb{R}$

Algebraic Properties of $\mathbb{R}$ On the set $\mathbb{R}$ of real numbers there are two binary operations, denoted by $+$ and $\cdot$ and called addition and multiplication, respectively. These operations satisfy the following properties :

(A1) $a+b=b+a$ for all $a,b\in\mathbb{R}$ (commutative property of addition);

(A2) $(a+b)+c=a+(b+c)$ for all $a,b,c\in\mathbb{R}$ (associative property of addition) ;

(A3) there exists an element $0\in\mathbb{R}$ such that $0+a=a$ and $a+0=a$ for all $a\in\mathbb{R}$ (existence of a zero element) ;

(A4) for each $a\in\mathbb{R}$ there exists an element $a\in\mathbb{R}$ such that $a+(-a)=0$ and $(-a)+a=0$ (existence of negative elements) ;

(M1) $ab=ba$ for all $a,b\in\mathbb{R}$ (commutative property of multiplication) ;

(M2) $(ab)c = a(bc)$ for all $a, b, c \in \mathbb{R}$ (associative property of multiplication) ;

(M3) there exists an element $1 \in \mathbb{R}$ distinct from 0 such that $1a = a$ and $a1 = a$ for all $a \in \mathbb{R}$ (existence of a unit element) ;

(M4) for each $a \in \mathbb{R} - \{0\}$ there exists an element $1/a \in \mathbb{R}$ such that $a(1/a) = (1/a)a = a$ and

(D) $a(b + c) = (ab) + (ac)$ and $(b + c)a = (ba) + (ca)$ for all $a, b, c \in \mathbb{R}$ (distributive property of multiplication over addition).

1.2 The Order Properties of $\mathbb{R}$

There is a nonempty subset $\mathbb{R}^+$ of $\mathbb{R}$, called the set of positive real numbers, that satisfies the following properties :

1. If $a, b \in \mathbb{R}^+$, then $a + b \in \mathbb{R}$.
2. 1f $a, b \in \mathbb{R}^+$, then ab belongs to $\mathbb{R}$.
3. If $a \in \mathbb{R}$, then exactly one of the following holds : $a \in \mathbb{R}^+$ OR $a = 0$ OR $(-a) \in \mathbb{R}^+$.

1. Let $a, b, c \in \mathbb{R}$
   if $a > b$ and $b > c$ then $a > c$
   Given that,
   $a > b$ and $b > c$
   $\therefore a - b > 0$ and $b - c > 0 \cdots$ i.e $(a - b), (b - c) \in \mathbb{R}^+$

$\therefore (a-b)+(b-c)>0\cdots\left(1^{st}\text{ order prop}\right)$
$\therefore a-c>0 \Rightarrow a>c$

2. If $a>b$ then $a+c>b+c$
   Given that,
   $a>bi.e\cdot a-b>0$
   $\therefore a-b\in\mathbb{R}^{+}$
   $\therefore a+c-c-b>0$
   $\therefore (a+c)-(b+c)>0$
   $\therefore a+c>b+c$

3. If $a>b$ and $c>0$ then, $ca>cb$
   Given that, $a>b$ & $c>0 \therefore (a-b)>0\&c>0$
   i.e $(a-b),c\in\mathbb{R}^{+}$
   $\therefore (a-b)\cdot c\in\mathbb{R}^{+}\ldots\left(2^{\text{nd}}\text{ order prop}\right)$
   $\therefore (a-b)\cdot c>0 \Rightarrow a\cdot c-bc>0 \Rightarrow ac>bc$

4. If $a>b$ and $c<0$ then, $ca<cb$
   Given that, $a>b$ & $c<0$
   $\therefore (a-b)\in\mathbb{R}^{+}\&-c\in\mathbb{R}^{+}\ldots\left(3^{\text{rd}}\text{ order prop}\right)$
   $\therefore -c(a-b)>0$
   $\therefore -ca+cb>0$
   $\therefore cb>ca$

$\therefore ca < cb$

1.3 Absolute Value and Real Line

Absolute value and Real line

Absolute value:-

$$|a| = \begin{cases} a & \text{if } a > 0 \\ -a & \text{if } +a < 0 \\ 0 & \text{if } a = 0 \end{cases}$$

Theorem 1.3.1. *For* $a, b \in \mathbb{R}$

a) $|ab| = |a|.|b| \forall a, b \in \mathbb{R}$

b) $|a|^2 = a^2 \forall a \in \mathbb{R}$

c) $\textit{if } c > 0 \textit{ then } |a| \leqslant c \textit{ iff} -c \leqslant a \leqslant c$

d) $-|a| \leqslant a \leqslant |a| \forall a \in \mathbb{R}$

Proof. a) $|ab| = |a|.|b| \forall a, b \in \mathbb{R}$

- if $a = 0$ or $b = 0 \Rightarrow ab = 0 = |ab| = |a| \cdot |b|$
- if $a > 0$ or $b > 0 \Rightarrow ab > 0$
  $|ab| = a \cdot b = |a| \cdot |b|$

- if $a > 0$ or $b < 0 \Rightarrow ab < 0$
  $\therefore |ab| = -a \cdot b = (-a) \cdot b = a \cdot (-b) = |a| \cdot |b|$
- if $a < 0$ or $b > 0 \Rightarrow ab < 0$
  $\therefore |ab| = -ab = |a| \cdot |b|$
- if $a < 0$ or $b < 0 \Rightarrow ab > 0$
  $\therefore |ab| = ab = |a|.|b|$
  - Hence proved -

b) $|a|^2 = a^2 \forall a \in \mathbb{R}$
$\forall a \in \mathbb{R}, a^2 \in \mathbb{R}$ i.e $a^2 \geqslant 0$
let $|a^2|^2 = |a| \cdot |a| = a \cdot a = a^2$, if $a > 0$
$(-a) \cdot (-a) = a^2$, if $a < 0$
Hence, $|a|^2 = a^2$

c) if $c > 0$ then $|a| \leqslant c$ iff $-c \leqslant a \leqslant c$
Given that,
$c > 0$ & $|a| \leqslant c$
i) To show $-c \leqslant a \leqslant c$
Now, $|a| = \max(a, -a) \leqslant c$
$\Rightarrow a \leqslant c \& -a \leqslant c$
$\Rightarrow a \leqslant c \& a \geqslant -c$

$-c \leqslant a \leqslant c$

ii) Given that, $-c \leqslant a \leqslant c$ & To show:- $|a| \leqslant c$

$\Rightarrow a \leqslant c \& -a \leqslant -c$

$\therefore |a| \leqslant c \ldots (|a| = max(a, -a))$

d) For $a \neq 0 \in \mathbb{R}, |a| > 0 \ldots |a| = \max(a, -a)$

Put $c = |a| > 0$ in **c)**

$\therefore -c \leqslant a \leqslant c \Rightarrow -|a| \leqslant a \leqslant |a|$

□

1.4 Triangular Inequality

Triangular Inequality:-

Theorem 1.4.1. *If $a, b \in \mathbb{R}$ then $|a + b| \leqslant |a| + |b|$*

Proof. if $a, b \in \mathbb{R}$ then

$$\begin{array}{ll} & -|a| \leqslant a \leqslant |a| \\ + & -|b| \leqslant b \leqslant |b| \\ \hline & -(|a| + |b|) \leqslant a + b \leqslant (|a| + |b|) \ldots \text{(Themorem:-1.1.1-d)))} \end{array}$$

let $|a| + |b| = c$

$$\therefore -c \leqslant a+b \leqslant c$$
$$\Rightarrow |a+b| \leqslant c \ldots\ldots (Th^m - 1.1.1 - c)$$
$$\therefore |a+b| \leqslant |a| + |b|$$

□

Corollary 1.4.1.1. *If* $a, b \in \mathbb{R}$

a) $||a| - |b|| \leqslant |a-b|$

b) $|a-b| \leqslant |a| + |b|$

Proof. a) We know that, $a, b \in \mathbb{R}$

$a = a - b + b$

$\therefore |a| = |a-b+b| \leqslant |a-b| + |b|$

$$\Rightarrow |a| - |b| \leqslant |a-b| \tag{1.1}$$

Also, $b = b - a + a$

$|b| = |b-a+a| \leqslant |b-a| + |a|$

$$\therefore |b| - |a| \leqslant |a-b| \tag{1.2}$$

from equation (1.1) & (1.2)

$||a| - |b|| \leqslant |a-b|$

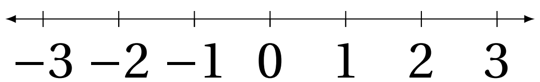
b)

$$\begin{aligned}
&\text{if } a, b, c \in \mathbb{R} \\
&\therefore |a+c| \leqslant |a| + |c| \\
&\text{Put } c = -b, |c| = |-b| = |b| \\
&\therefore |a+(-b)| \leqslant |a| + |-b| \\
&\therefore |a-b| \leqslant |a| + |b|
\end{aligned}$$

□

Corollary 1.4.1.2. *If $a_1, a_2 \dots a_n$ are any real no then $|a_1 + a_2 + \cdots + a_n| \leqslant |a_1| + |a_2| + \cdots \dots |a_n|$*

Definition* 1.4.1** (Real line):*** *A convenient and Familiar interpretation of real no system is the real line.*



Definition* 1.4.2** (ϵ-Neighbourhood:-):*** *let $a \in \mathbb{R}$ & $\in > 0$, then ϵ- neighbourhood of a is the set* $V_\epsilon(a) = \{x | x \in \mathbb{R}, |x-a| < \epsilon\} \dots 0 \leqslant |x-a| < \epsilon$

$$\therefore V_\epsilon(a) = (a-\epsilon, a+\epsilon) \subseteq \mathbb{R}$$

Since $|x-a| < \epsilon \Rightarrow -\epsilon < x-a < \epsilon \Rightarrow a-\epsilon < x < a+\epsilon)$

Definition* 1.4.3** (Deleted-ϵ-Neighbourhood:-):*** $\delta_\epsilon(a) = v_\epsilon(a) - \{a\}$
$= (a-\epsilon, a+\epsilon) - \{a\}$
$i.e 0 < |x-a| < \epsilon$

Example 1:

If $a, b \in \mathbb{R}$. Show that $|a+b| = |a| + |b|$ if and only if $ab \geqslant 0$

Proof. i) Given that $ab \geqslant 0$, To prove- $|a+b| = |a| + |b|$

if $ab \geqslant 0 \Rightarrow a \geqslant 0, b \geqslant 0 or a \leqslant 0, b \leqslant 0$

$$\begin{aligned} a+b &\geqslant 0 \\ \therefore |a| = a, |b| &= b \\ |a+b| &= a+b \\ &= |a| + 1b\,| \end{aligned}$$

$$\begin{aligned} a+b &\leqslant 0 \\ \therefore |a| = -a, |b| &= -b \\ |a+b| &= -(a+b) \\ &= -a-b \\ &= |a| + |b| \end{aligned}$$

ii) Given that $|a+b| = |a| + |b|$, To prove $ab \geqslant 0$

$|a+b|^2 = (|a| + |b|)^2$

$\therefore a^2 + b^2 + 2ab = |a|^2 + |b|^2 + 2 \cdot |a| \cdot |b|$

$\therefore 2ab = 2|a|.|b| \ldots. \left(\because |a|^2 = a^2\right)$

$ab = |a| \cdot |b|$

$ab = |ab| \ldots$ (Theorem:- 1.1.1-a))
$\therefore ab \geqslant 0$

$\square$

Example 2:
Show that if $a, b \in \mathbb{R}$ then

i) $\max\{a, b\} = \frac{1}{2}(a + b + |a - b|)$, $\min\{a, b\} = \frac{1}{2}(a + b - |a - b|)$

ii) $\min\{a, b, c\} = min\{\min\{a, b\}, c\}$

Proof. i) let $a > b \Rightarrow = |a - b| = a - b$

$$max(a, b) = a \tag{1.3}$$

Consider, RHS
$= \frac{1}{2}(a + b - |a - b|)$
$= \frac{1}{2}(a + b + a - b)) \ldots$ from (1.3) $(a - b) \geqslant 0$
$= a$
$=$ LHS

$$\text{let} \min(a, b) = b \tag{1.4}$$

Consider, $RHS = \frac{1}{2}(a+b-|a-b|)$
$= \frac{1}{2}(a+b-(a-b))$
$= b$
$=$ LHS

ii) Suppose, $a > b > c$
LHS=$\min\{a, b, c\} = c$
RHS= $min\{\min\{a, b\}, c\} = \min\{b, c\}$
RHS=C
$=$ LHS
Hence, $\min\{a, b, c\} = min\{\min\{a, b\}, c\}$

□

Example 3:
If $x, y, z \in \mathbb{R}$ & $x \leqslant z$, Show that $x \leqslant y \leqslant z$ if and only if $|x-y|+|y-z| = |x-z|$
$x \leqslant z \Rightarrow x - z \leqslant 0 \therefore |x-z| = z - x$

Proof. i) Given that $x \leqslant y \leq z, x, y, z \in \mathbb{R}$
$\therefore |x-y| = y - x \& |y-z| = z - y$
To show $|x-y|+|y-z| = |x-z|$

$$\begin{aligned}
\text{Consider, LHS} &= |x-y|+|y-z| \\
&= y-x+z-y \\
&= z-x \\
&= |x-z| \\
&= \text{RHS}
\end{aligned}$$

ii) Given that $|x-y|+|y-z|=|x-z|$
To show, $x \leqslant y \leqslant z$
let $a=(x-y), b=(y-z)$
$\therefore |(x-y)+(y-z)|=|x-y|+|y-2|$
$\Rightarrow (x-y)(y-z) \geqslant 0 \ldots (\because \text{ if } |a+b|=|a|+|b| \Leftrightarrow ab \geqslant 0)$

$$\begin{aligned}
&\therefore a, b \geqslant 0 \\
&\text{i.e}(x-y),(y-z) \geqslant 0 \\
&x \geqslant y, y \geqslant z \\
&\therefore x \geqslant y \geqslant z
\end{aligned}$$

which is not possible Since $x \leqslant z$-(given)

$$\begin{aligned}
&a, b \leqslant 0 \\
&(x-y),(y-z) \leqslant 0 \\
&\therefore x \leqslant y, y \leqslant z \\
&\therefore x \leqslant y \leqslant z
\end{aligned}$$

□

Example 4:
If $a < x < b, a < y < b$. Show that $|x - y| < b - a$.

Proof. Given that,

$$a < x < b, a < y < b$$

$$0 < x - a < b - a \tag{1.5}$$

$$0 < y - a < b - a \tag{1.6}$$

multiplying by (-1) to (1.6) and add in (1.5)

$$\begin{array}{lr} & 0 \leqslant -a \leqslant b - a \\ + & -(b-a) \leqslant a - y \leqslant b - a \\ \hline & -(b-a) \leqslant x - y \leqslant (b-a) \Rightarrow |x - y| < (b-a) \end{array}$$

□

Definition* 1.4.4** (Upper bound):*** *Let $S \neq \phi \subseteq \mathbb{R}$, the set s is said to be bounded above if $\exists a \in \mathbb{R} \overset{s.t}{\Rightarrow} x \leqslant a \forall x \in S$ Each such ' a ' is called as upper bound of S.*

Definition* 1.4.5** (Lower bound):*** *Let $S \neq \phi \subseteq \mathbb{R}$. The set S is said to be bounded below if $\exists b \in \mathbb{R} \overset{s.t}{\Rightarrow} x \geqslant b \forall x \in S$ Each such b is called as lower bound of S.*

Definition* 1.4.6** (Bounded Set):*** *If both lower and upper bound exist.*

Definition* 1.4.7** (Unbounded set):*** *If set S is not bounded.*

Definition* 1.4.8** (Supremum & Infimum):*** *Let S be a non-empty subset of $\mathbb{R}$ if S is bounded above/below then a no u is said to be supremum/Infimum (least upper bound or greatest lower bound) of S if it satisfies the conditions:-*

i) *u is an upper(lower) bound of S.*

ii) *if v is any upper(lower) bound of S then $u \leqslant v(u \geqslant v)$.*

1.5 Completeness Property

Statement:-If set is bounded below then its infimum must be exists and if set is bounded above then its supremum must be exists this property is known as completeness property.
let $\mathbb{N} = 1, 2, \ldots 4$ bounded below
Unbounded=$\{\infty\}$ = Supremum
Lower bound= $\{\infty, \ldots, -1, 0, 1\}$ = Infimum = 1

Example 5:
Let $A \subseteq B$ then Prove that,

I) $\inf A \geqslant \inf B$

II) $\sup A \leqslant \sup B$

Proof. I) Given that, $A \subseteq B, x \in A \Rightarrow x \in B$
also, $\inf A = u$ and $\inf B = v \ldots$ (assume)
if u is inf A then, by definition,

i) u is lower bound, $x \geqslant u \forall x \in A$

ii) if u_1 is another lower bound, then $u_1 < u \forall u_1$ Assume that, $\inf B \geqslant inf A$

Assume that, $\inf B \geqslant \inf A$

$$\text{i.e } v \geqslant u$$
$$\text{i.e } x \geqslant v \geqslant u, \forall x \in B$$
$$\therefore x \geqslant v \geqslant u, \forall x \in A$$

$\Rightarrow$ if u is inf, we can not have lower bound greater than u.
So, our assumption is wrong.
Hence, $u \geqslant V$ i.e inf $A \geqslant$ inf B

II) let $\sup A = u$ and $\sup B = v$
if u is supremum of A then, by definition

i) u is upper bound of A i.e $x \leqslant u, \forall x \in A$

ii) if u_1 is any other upper bound then $u \leqslant u_1 \forall u_1$

Assume that, Sup $A \geqslant$ sup B
$u \geqslant v$

i.e $v \leqslant u$

$$x \leqslant v \leqslant u, \forall x \in B$$
$$x \leqslant v \leqslant u, \forall x \in A$$

$\Rightarrow$ if u is sup of A then we can not have upper bound less than u. So assumption is wrong.
Hence, $u \leqslant V$ i.e sup A sup B

$\square$

Example 6:

$$S = 1 - \frac{(-1)^n}{n}, n \in \mathbb{N}. \text{ Find infimum \& suptemum}$$
$$S = \{2, 1/2, 1+1/3, 1-1/4, 1+1/5, 1-1/6, \ldots\}$$
$$\therefore \inf s = 1/2 \text{ of } \sup s = 2$$

Example 7:

$$S = \frac{(-1)^n}{n}, n \in \mathbb{N}$$
$$S = \{-1, 1/2, -1/3, 1/4, -1/5, \ldots.\}$$
$$LB = (-\infty, -1] \Rightarrow \text{Inf} = -1 \in S,$$
$$UB = [1/2, \infty) \Rightarrow \sup = 1/2 \in S$$

Example 8:

$$S = \{\frac{1}{m} - \frac{1}{n}, m, n \in \mathbb{N}\}$$
$$S = \{0, 1/2, -1/2, 1 - 1/3, , -2/3, 1, -1, \ldots.\}$$
$$LB = (-\infty, -1] \Rightarrow \text{Inf} = -1 \notin S,$$
$$UB = (1, \infty) \Rightarrow \sup = 1 \notin S$$

Example 9:

$$S = \{\frac{n-1}{n}, n \in \mathbb{N}\} = \{1 - \frac{1}{n}, n \in \mathbb{N}\}$$
$$S = \{0, 1 - 1/2, 1 - 1/3, 1 - 1/4, \ldots.\}$$
$$LB = (-\infty, 0] \Rightarrow \text{Inf}\, S = 0 \in S,$$
$$UB = [1, \infty) \Rightarrow \sup S = 1 \notin S$$

Chapter 2

Sets Operations

2.1 Set Operations

1. Union $A \bigcup B = \{x / x \in A or x \in B\}$
2. Intersection $A \bigcap B = \{x / x \in A \& x \in B\}$
3. Complement $A^c = \{x / x \in A, x \in \Omega\}$
4. Substraction $A - B = A \backslash B = A \bigcap B^c = \{x / x \in A\ but\ x \notin B\}$

Theorem 2.1.1. *if A, B, C are sets then,*

a) $A \backslash (B \bigcup C) = (A \backslash B) \bigcap (A \backslash C)$

b) $A \backslash (B \bigcap C) = (A \backslash B) \bigcup (A^{\backslash C})$

Proof. To Prove:-

i) $A \backslash (B \bigcup C) \subseteq (A \backslash B \bigcap A \backslash C)$

ii) $(A \setminus B \bigcap A \setminus C) \subseteq A \setminus (B \bigcup C)$

i) $A \setminus (B \bigcup C) \subseteq (A \setminus B \bigcap A \setminus C)$

i) let $x \in A \setminus (B \bigcup C)$i.e$x \in A \bigcap (B \bigcup C)^C$
$\Rightarrow x \in A$ and $x \in (B \bigcup C)^C$
$\Rightarrow x \in A$ and $x \notin (B \bigcup C)$
$\Rightarrow x \in A$ and $(x \notin B$ and$x \notin C)$
$\Rightarrow (x \in A \& x \notin B)$ and $(x \in A \& x \notin C)$
$\Rightarrow x \in A \bigcap B^C$ and $x \in A \bigcap C^C$
$\Rightarrow x \in (A \setminus B) \bigcap (A \setminus C)$

$$\therefore A \setminus (B \bigcup C) \subseteq (A \setminus B) \bigcap (A \setminus C) \tag{2.1}$$

ii) $x \in (A \setminus B) \bigcap (A \setminus C)$
$\Rightarrow x \in A \setminus B$ and $x \in A \setminus C$
$\Rightarrow x \in (A \bigcap B^C)$ and $x \in (A \bigcap C^C)$
$\Rightarrow (x \in A \& x \notin B)$ and $(x \in A \& x \notin C)$
$\Rightarrow x \in A$ and $(x \notin B$ and$x \notin C)$
$\Rightarrow x \in A$ and $(x \in (B \bigcup C)^C)$
$\Rightarrow x \in A \bigcap (B \bigcup C)$

$$A \setminus B \bigcap A \setminus C \subseteq A \setminus (B \bigcup C) \tag{2.2}$$

from (2.1) & (2.2)

$$A \backslash (B \bigcup C) = A \backslash B \bigcap A \backslash C$$

ii) $A \backslash (B \bigcap C) = (A \backslash B) \bigcup (A \backslash C)$

To Prove:-

i) $A \backslash (B \bigcap C) \subseteq (A \backslash B) \bigcap (A \backslash C)$

ii) $A \backslash B \bigcup A \backslash C \subseteq A \backslash (B \bigcup C)$

i) let $x \in A \backslash (B \bigcap C)$ i.e $x \in A \bigcap (B \bigcap C)^C$
$\Rightarrow x \in A$ and $x \in (B \bigcap C)^C$
$\Rightarrow x \in A$ and $x \notin (B \bigcap C)$
$\Rightarrow x \in A$ and $(x \notin B$ or $x \notin C)$
$\Rightarrow (x \in A \& x \notin B)$ or $(x \in A \& x \notin C)$
$\Rightarrow x \in A \bigcap B^C$ or $x \in A \bigcap C^C$
$\Rightarrow x \in (A \backslash B) \bigcup (A \backslash C)$

$$\therefore A \backslash (B \bigcap C) \subseteq A \backslash B \bigcup A \backslash C \tag{2.3}$$

ii) $x \in A \backslash B \bigcup A \backslash C$
$\Rightarrow x \in A \backslash B$ or $x \in A \backslash C$
$\Rightarrow x \in (A \bigcap B^C)$ or $x \in (A \bigcap C^C)$
$\Rightarrow (x \in A \& x \notin B)$ or $(x \in A \& x \notin C)$
$\Rightarrow x \in A$ and $(x \notin B$ or $x \notin C)$

$\Rightarrow x \in A$ and $(x \in (B \bigcap C)^C)$
$\Rightarrow x \in A \bigcap (B \bigcap C)^C$
$\Rightarrow x \in A \backslash (B \bigcap C)$

$$A \backslash B \bigcup A \backslash C \subseteq A \backslash (B \bigcap C) \tag{2.4}$$

from (2.3) & (2.4)

$$A \backslash (B \bigcap C) = A \backslash B \bigcup A \backslash C$$

-Hence Proved- □

2.2 Distributive Law

Distributive Law:-

a) $A \bigcup (B \bigcap C) = (A \bigcup B) \bigcap (A \bigcup C)$

b) $A \bigcap (B \bigcup C) = (A \bigcap B) \bigcup (A \bigcap C)$

Proof. a) To Prove:-

i) $A \bigcup (B \bigcap C) \subseteq (A \bigcup B) \bigcap (A \bigcup C)$

let $x \in A \bigcup (B \bigcap C)$
$\Rightarrow x \in A$ or $x \in B \bigcap C$
$\Rightarrow x \in A$ or $x \in B$ and $x \in C$
$\Rightarrow x \in A$ or $x \in B$ and $x \in A$ or $x \in C$
$\Rightarrow x \in A \bigcup B$ and $x \in A \bigcup C$
$\Rightarrow x \in (A \bigcup B) \bigcap (A \bigcup C)$

$$\therefore A \bigcup (B \bigcap C) \subseteq (A \bigcup B) \bigcap (A \bigcup C) \tag{2.5}$$

ii) $(A \bigcup B) \bigcap (A \bigcup C) \subseteq A \bigcup (B \bigcap C)$
let $x \in (A \bigcup B) \bigcap (A \bigcup C)$
$\Rightarrow x \in (A \bigcup B)$ and $x \in (A \bigcup C)$
$\Rightarrow (x \in A$ or $x \in B)$ and $(x \in A$ or $x \in C)$
$\Rightarrow x \in A$ or $(x \in B$ and $x \in C)$
$\Rightarrow x \in A \bigcup (B \bigcap C)$

$$(A \bigcup B) \bigcap (A \bigcup C) \subseteq A \bigcup (B \bigcap C) \tag{2.6}$$

from (2.5) & (2.6)

$$A \bigcup (B \bigcap C) = (A \bigcup B) \bigcap (A \bigcup C)$$

b) To Prove:-

i) $A\bigcap(B\bigcup C) \subseteq (A\bigcap B)\bigcup(A\bigcap C)$

let $x \in A\bigcap(B\bigcup C)$

$\Rightarrow x \in A \text{and } x \in (B\bigcup C)$

$\Rightarrow x \in A$ and ($x \in B$ or $x \in C$)

$\Rightarrow$ ($x \in A$ and $x \in B$) or ($x \in A$ and $x \in C$)

$\Rightarrow x \in (A\bigcap B)\text{or}(x \in A\bigcap C)$

$\Rightarrow x \in (A\bigcup B)\bigcup(A\bigcap C)$

$$\therefore A\bigcap(B\bigcup C) \subseteq (A\bigcap B)\bigcup(A\bigcap C) \tag{2.7}$$

ii) $(A\bigcap B)\bigcup(A\bigcap C) \subseteq A\bigcap(B\bigcup C)$

let $x \in (A\bigcap B)\bigcup(A\bigcap C)$

$\Rightarrow x \in (A\bigcap B)\text{or} x \in (A\bigcap C)$

$\Rightarrow$ ($x \in A$ and $x \in B$) or ($x \in A \text{and} x \in C$)

$\Rightarrow x \in A\text{and}(x \in B \text{or} x \in C)$

$\Rightarrow x \in A\text{and}(x \in B\bigcup C)$

$\Rightarrow x \in A\bigcap(B\bigcup C)$

$$(A\bigcap B)\bigcup(A\bigcap C) \subseteq A\bigcap(B\bigcup C) \tag{2.8}$$

from (2.7) & (2.8)

$$A\bigcap(B\bigcup C) = (A\bigcap B)\bigcup(A\bigcap C)$$

□

Theorem 2.2.1. *If A & B are sets, Show that $A \subseteq B$ if and only if $A \bigcap B = A$*

Proof. i) Assume that $A \subseteq B$ to Prove that $A \bigcap B = A$

let $x \in A \Rightarrow x \in B$

$\therefore x \in B \ldots (\because A \subseteq B)$

$\Rightarrow x \in A \bigcap B$

$$\therefore A \subseteq A \bigcap B \tag{2.9}$$

Also, by definition,

$$A \bigcap B \subseteq A \tag{2.10}$$

from (2.9) and (2.10)

$$A = A \bigcap B \tag{2.11}$$

ii) Assume that $A \bigcap B = A$, to prove $A \subseteq B$

We know that, $A \bigcap B \subseteq B$

$$\Rightarrow A \subseteq B \tag{2.12}$$

from (2.11) and (2.12)

$$A \subseteq B \text{iff} A = A \bigcap B \tag{2.13}$$

-Hence Proved- □

2.3 Basic Notatioons Theory

$A \times B = \{<x,y> / x \in A, y \in B\}$ ✓ ordered

$A = \{2,3,4\}$ $B = \{1,5,6\}$

Definition 2.3.1 (Cartesian Product)**:** *let $A\&B$ be two sets,* $A =< 2,3,4 > \& < 1,5,6 >$ *then cartesian prodct is given by*
$A \times B = \{< 2,1 >,< 2,5 >,< 2,6 >,< 3,5 >,< 3,6 >,< 4,1 >,< 4,5 >,< 4,6 >\}$

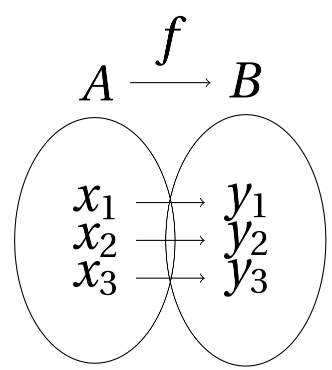
~~set~~

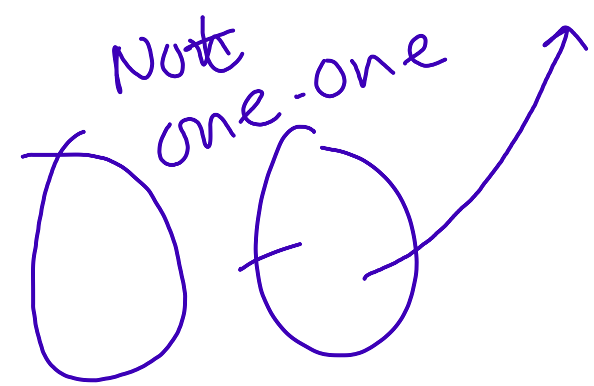
Real func

Definition 2.3.2 (Function)**:** *Let $A\&B$ be sets then a function from A to B is a set f of ordered pairs in $A \times B$ such that for each $a \in A$ then there exists a unique $b \in B$ with $(a,b) \in f$.*
In other words if $< a,b >\in f \& < a,b' >\in f \Rightarrow b = b'$.

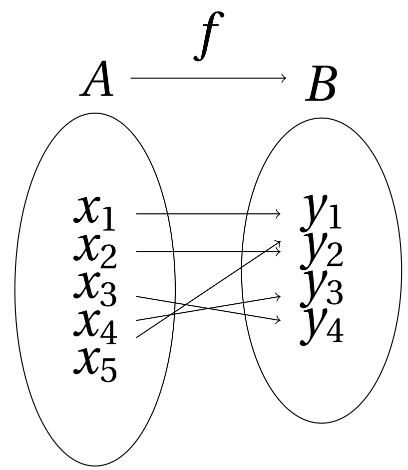
Types of Function

Definition 2.3.3 (One-One (Injective) Function)**:** *The Function f is said to be injective (or One-One) if whenever $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$.*

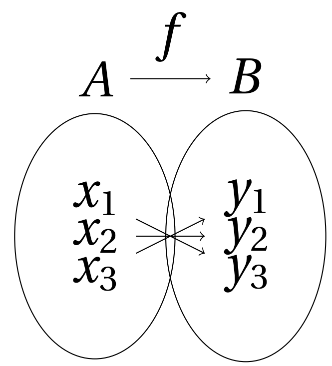




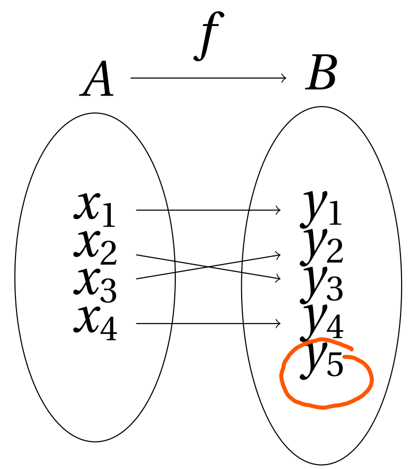
Definition 2.3.4 (Onto (Surjective) Function)**:** *The function f is said to be Surjective if $f(A) = B$ i.e if the range $R(f) = B$.*



Definition* 2.3.5** (One-One & Onto (Bijective) Function):*** *The Function f is both one-one and onto then it is said to be bijective.*



Definition* 2.3.6** (Into Function):*** *If f is not onto then it is called as into function.*



B→C

Definition* 2.3.7** (Composite Function):*** *If* $f : A \to B$ *and* $\cancel{g : A \to B}$ *and if* $R(f) \subseteq D(g) = B$ *then the composite function* $g \circ f$ *is the function from* $A \to C$
$g \circ f : A \to C$ *is composite function if* $g \circ f(x) = g(f(x))x \in A$ Draw diagram

Domain

Example 10:

$f : \mathbb{R} \to \mathbb{R}, g : \mathbb{R} \to \mathbb{R}$

$f(x) = 2x, g(y) = 3y^2 - 1$

Proof. Given that, $f(x)) = 2x, g(y) = 3y^2 - 1$

$$g \circ f(x) = g(f(x))$$
$$= g(2x)$$
$$= 3(2x)^2 - 1$$
$$= 12x^2 - 1$$

$f \circ g(y) = f(g(y))$
$= f(3y^2 - 1)$
$= 2(3y^2 - 1)$
$= 6y^2 - 2$
$\therefore g \circ f \neq f \circ g$ $\square$

Example 11:
Show that if $f : A \to B$ ~~then~~, E, F are subsets of A then,

a) $f(E \bigcup F) = f(E) \bigcup f(F)$ and

b) $f(E \bigcap f) \subseteq f(E) \bigcap f(F)$

Proof. a) $f : A \to B, E, F \subseteq A$

$f(E) = \{y / y = f(x), x \in E \subseteq A\} \subseteq B$
$f(F) = \{y / y = f(x), x \in F \subseteq A\} \subseteq B$
$f(E \bigcup F) = \{y / y = f(x), x \in E \bigcup F\}$
To Prove,

i) $f(E \bigcup F) \subseteq f(E) \bigcup f(F)$

ii) $f(E) \bigcup f(F) \subseteq f(E \bigcup F)$

let $y \in f(E \bigcup F)$
$\Leftrightarrow y = f(x), x \in E \bigcup F$

$$\Leftrightarrow y = f(x), x \in E \text{or} x \in F$$
$$\Leftrightarrow y = f(x), x \in E \subseteq A \text{or} y = f(x), x \in F \subseteq A$$
$$\Leftrightarrow y \in f(E) \text{or} y \in f(F)$$
$$\Leftrightarrow y \in f(E) \bigcup f(F)$$

$$\therefore f(E \bigcup F) \subseteq f(E) \bigcup f(F) \& f(E) \bigcup f(F) \subseteq f(E \bigcup F)$$
$$f(E \bigcup F) = f(E) \bigcup f(F)$$

To Prove,

b) $f(E \bigcap f) \subseteq f(E) \bigcap f(F)$

let $y \in f(E \bigcap F)$

$$\Rightarrow y = f(x), x \in E \bigcap F$$
$$\Rightarrow y = f(x), x \in E \text{and} x \in F$$
$$\Rightarrow y = f(x), x \in E \text{and} y = f(x), x \in F$$
$$\Rightarrow y \in f(E) \text{and} y \in f(F)$$
$$\Rightarrow y \in f(E) \bigcap f(F)$$

$$\therefore f(E \bigcap F) \subseteq f(E) \bigcap f(F)$$

□

Example 12:

Example for $f(E)\bigcap f(F) \subsetneq f(E\bigcap F)$?

let $f(x) = x^2$

$E = \{1,2\}, f(E) = \{1,4\}$

$F = \{-2,4\}, f(E) = \{4,16\}$

$E\bigcap F = \{\phi\}, f(E)\bigcap f(F) = \{4\}$

$f(E\bigcap F) = \{\phi\}$

$f(E)\bigcap f(F) \subsetneq f(E\bigcap F)$

Example 13:

Show that if $f : A \to B$ and G, H are subsets of B then,

a) $f^{-1}(G\bigcup H) = f^{-1}(G)\bigcup f^{-1}(H)$ and

b) $f^{-1}(G\bigcap H) = f^{-1}(G)\bigcap f^{-1}(H)$

Proof. a) $f : A \to B$

$f^{-1}(G) = \{x/ f(x) \in G\} \subseteq A$

$f^{-1}(H) = \{x/ f(x) \in H\} \subseteq A$

let $x \in f^{-1}(G\bigcup H)$

$\Leftrightarrow f(x) \in G\bigcup H$

$\Leftrightarrow f(x) \in G \text{or} f(x) \in H$

$\Leftrightarrow x \in f^{-1}(G) \text{or} x \in f^{-1}(H)$

$\Leftrightarrow x \in f^{-1}(G) \bigcup f^{-1}(H)$

$$\therefore f^{-1}(G \bigcup H) = f^{-1}(G) \bigcup f^{-1}(H)$$

-Hence Proved-

b) let $x \in f^{-1}(G \bigcap H)$
$\Leftrightarrow f(x) \in G \bigcap H$
$\Leftrightarrow f(x) \in G \text{and} f(x) \in H$
$\Leftrightarrow x \in f^{-1}(G) \text{and} x \in f^{-1}(H)$
$\Leftrightarrow x \in f^{-1}(G) \bigcap f^{-1}(H)$

$$\therefore f^{-1}(G \bigcap H) = f^{-1}(G) \bigcap f^{-1}(H)$$

-Hence Proved-

□

Example 14:
one-one
Show that if $f : A \to B$ is injective & $E \subseteq A$ then $f^{-1}(f(E)) = E$. Give an example to show that equality need not hold if f is not injective.

Proof. Given that, $f : A \to B$ is injective
i.e if $x \neq y \Rightarrow f(x) \neq f(y) \forall x, y \in A$

$E = \{x / x \in E, f(x) \in B\} \subseteq A$
$f(E) = \{y / y = f(x) \in f(E), x \in A\} \subseteq B$
To prove $f^{-1}(f(E)) = E$
let$x \in f^{-1}(f(E))$
$\Rightarrow f(x) \in f(E)$
$\Rightarrow x \in E \ldots (\because f$is one-one function)

$$f^{-1}(f(E)) \subseteq E \tag{2.14}$$

Now, let $x \in E$
$\Rightarrow f(x) \in f(E) f^{-1}(H) = \{x / f(x) \in H, x \in A\}$
$x \in f^{-1}(f(E))$

$$E \subseteq f^{-1}(f(E)) \tag{2.15}$$

from (2.14) & (2.15)

$$f^{-1}(f(E)) = E$$

□

Example 15:
let $f(x) = x^2$
$E\{1,2\} \Rightarrow f(E)\{1,4\}$

$E = \{1, 2\}$ $f(E) = \{1, 4\}$

$f^{-1}(f(E) = \{(1, -2, -2)\}$
$f^{-1}(f(E) \neq E$

Example 16:

Show that if $f : A \to B$ is surjective and $E \subseteq A$ then $f(f^{-1}(H)) = H$. Give an example to show that equality need not hold if f is not surjective.

Proof. $f : A \to B, H \subseteq B$ and f is surjective i.e every element in B has inverse image in A
To prove: $f(f^{-1}(H)) = H$
let $y \in f(f^{-1}(H))$
$\Rightarrow f(x) \in f(f^{-1}(H))$
$\Rightarrow x \in f^{-1}(H)$
$\Rightarrow y = f(x) \in H$

$$\therefore f(f^{-1}(H)) \subseteq H \tag{2.16}$$

let $y \in H$ then
$\exists x \in A$such that,
$y = f(x) \in H \ldots (\because$ f is onto)
$\Rightarrow x \in f^{-1}(H)$
$\Rightarrow f(x) \in f(f^{-1}(H)) \ldots (x \in E \Rightarrow f(x) \in f(E))$
$\Rightarrow y \in f^{-1}(H)$

$$\therefore H \subseteq f(f^{-1}(H)) \tag{2.17}$$

from (2.16) & (2.17)

$$f(f^{-1}(H)) = H$$

□

***Definition* 2.3.8** (Finite & Infinite Sets)**:** *1. The empty set ϕ is said to have zero elements*

2. If $n \in \mathbb{N}$, a set S is said to have n elements if there exists a bijection from set $\mathbb{N} = \{1, 2, \ldots n\} on S$

3. A set S is said to be finite if it is either empty or it has n elements for some $n \in \mathbb{N}$

4. A set S is said to be infinite if it is not finite

Theorem 2.3.1 (Uniqueness Theorem)**.** *If S is finite set, then the number of elements in S is unique number in $\mathbb{N}$.*
The set $\mathbb{N}$ of natural numbers is an infinite set.

Theorem 2.3.2. *Suppsose that $S \& T$ are sets and $T \subseteq S$*

a) If S is finite Set, then T is a finite Set.

b) If T is an infinite set then S is an infinite Set.

Proof. a) $T \subseteq S$ and S is finite Set

i) Suppose $S = \phi \Rightarrow T = \phi \Rightarrow T$ is finite

ii) When $S \neq \phi$ then there are two possibilities.

1) $T = \phi \Rightarrow T$ is a finite Set **or**

2) $T \neq \phi$

We will prove this by method of mathematical induction.

- $\#(S) = 1$ and as $T \neq \phi \Rightarrow S = T$
  Hence as S is finite $\Rightarrow T$ is finite
- Now assume that this statement is true for $\#(S) = k$
  i.e $\#(S) = k \& T \subseteq S \Rightarrow T$ is finite set.
- Now, lets prove it for $\#(S) = k+1$
  As S is finite, it has bijection with N_{k+1}

$$S = \{f(1), f(2), \ldots f(k+1)\} \tag{2.18}$$

lets define, $S_1 = S - f(k+1)$
$\therefore \#(S)_1 = k$ and $T_1 = T - f(k+1)$
Now, if $f(k+1) \notin T \Rightarrow T_1 = T \subseteq S_1$
and as $\#(S)_1 = k \& T \subseteq S_1 \subseteq T$ is finite
if $f(k+1) \in T_1 \Rightarrow T_1 = T - f(k+1) \subseteq S_1$
$\therefore T_1 \subseteq S_1, \#(S_1) = k \Rightarrow T_1$ is finite $\Rightarrow T$ is finite.
-Hence Proved-

b) (b) is a contrapositive statement to (a).
Hence, if T is infinite $\Rightarrow$ S is also infinite.

□

Definition* 2.3.9** (Countably Infinite):*** *A set is said to be denumerable or countably infinite if there exists bijection of* $\mathbb{N}$ *onto* S.

Definition* 2.3.10** (Countable Set):*** *A set* S *is said to be countable if it is either finite or denumerable.*

Definition* 2.3.11** (Uncountable Set):*** *A set* S *is said to be uncountable if it is not countable.*

The following statements are equivalent

1. S is a countable set
2. $\exists$ surjection of $\mathbb{N}$onto$S$
3. $\exists$ injection of Sonto$\mathbb{N}$

Example

1. Set of even/odd numbers are denumerable
2. Set of all integers(denumerable)
3. The union of two disjoint denumerable sets is again denumerable
4. The sets $\mathbb{N}, \mathbb{N}^2, \mathbb{N}^n$ are denumerable

Theorem 2.3.3. *Suppsose that $S\&T$ are sets and $T \subseteq S$*

a) If S is countable, then T is a countable set.

b) If T is an uncountable then S is an uncountable Set.

Theorem 2.3.4. *The Set $\mathbb{Q}$ of rational numbers is denumerable.*

Proof. lets prove it for $\mathbb{Q}^+$ first.
$\mathbb{Q} = \left\{\frac{p}{q}, q \neq 0\right\}, \mathbb{Q}^+ = \left\{1, \frac{1}{2}, \ldots \frac{2}{1}, \frac{2}{2}, \frac{2}{3}\right\}$
We can map $\mathbb{Q}^+$ with $\mathbb{N}^2$ however, mapping will not be injection as
$\frac{1}{1} = \frac{2}{2} = \frac{3}{3} \ldots$ or $\frac{1}{2} = \frac{2}{4} = \frac{3}{6} \ldots$
To proceed $\mathbb{N}^2 = \mathbb{N} \times \mathbb{N}$ is countable.

lets define, $g : \mathbb{N} \times \mathbb{N} \to \mathbb{Q}^+$ is mapping of ordered pairs $< m, n >$ into rational no $\frac{m}{n}$

$\frac{1}{1}\frac{1}{2}\frac{1}{3}\frac{1}{4}\ldots$

$\frac{2}{1}\frac{2}{2}\frac{2}{3}\frac{2}{4}\ldots$

$\frac{3}{1}\frac{3}{2}\frac{3}{3}\frac{3}{4}\ldots$

$\vdots$

$\vdots$

$\Rightarrow \mathbb{Q}^+$ is countable
Similarly, $\mathbb{Q}^-$ is also countable
So, $\mathbb{Q} = \mathbb{Q}^+ \bigcup \mathbb{Q}^-$ is countable.... ($\because$ Union of two disjoint denumerable sets is again denumerabel)

□

- Countable union of countable sets again countable.

2.4 Archemedian Property

If $x \in \mathbb{R}$, then there exists $n_x \in \mathbb{N}$subject to$x < n_x$.

Proof. By method of contradication,
$x \in \mathbb{R}, n_x < x \forall n_x \in \mathbb{N}$
$\therefore x$ is upper bound for set $\mathbb{N}$
By completeness property, the set which has upper bound must have supremum (says)

$$
\begin{aligned}
n_x &< u n_x \in \mathbb{N} \\
n_{x+1} &\leqslant u \forall n_x \\
n_x &\leqslant u - 1 \forall n_x \\
\therefore u - 1 \quad &\text{is also upper bound} < \text{u (by definition)}
\end{aligned}
$$

But we know that, Supremum is the least upper bound i.e there exits no other upper bound which is less than u.

So our assumption is wrong
Hence, $x < n_x, x \in \mathbb{R}$ □

Corollary 2.4.0.1. *If* $S = \left\{\frac{1}{n}, n \in \mathbb{N}\right\}$ *then inf*$S = 0$

Proof. $S \neq \phi$ and 0 is lower bound of S.
$\therefore$ By completeness Property, set S has infimum (v)
Let, $\varepsilon \in \mathbb{R}, \frac{1}{\varepsilon} > 0 \Rightarrow \frac{1}{\varepsilon} \in \mathbb{R}$
$\therefore$ By archemedian property
$\exists n \in \mathbb{N}, 0 < \frac{1}{\varepsilon} < n \Rightarrow 0 < \frac{1}{n} < \varepsilon \Rightarrow 0 is \inf(S)$ □

Corollary 2.4.0.2. *If* $t > 0, \quad \exists \quad n_t \in \mathbb{N} \Rightarrow 0 < \frac{1}{n_t} < t$

Proof. $t > 0, \frac{1}{t} > 0 \Rightarrow \frac{1}{t} \in \mathbb{R}$
$\therefore$ By archemedian property, $\exists n \in \mathbb{N}$ subject to $\frac{1}{t} < n_t, \exists n_t \in \mathbb{N}$
$\Rightarrow 0 < \frac{1}{n_t} < t$ □

Corollary 2.4.0.3. *If* $y > 0, \quad \exists \quad n_y \in \mathbb{N} \quad \exists \quad n_{y-1} \leqslant y < n_y$

Proof. Given that $y > 0$ i.e $y \in \mathbb{R}$
$y < n_y, \exists n_y \in \mathbb{N} \ldots$By archemedian property

$E_y = \{n / y < n, n \in \mathbb{N}\}$
$\Rightarrow y$ is lower Bound of E_y
$\Rightarrow$ least element of E_y is $\inf(n_y)$
$\Rightarrow n_{y-1} \leqslant y < n_y$ □

Theorem 2.4.1 (Density Theorem)**.** *If $x \& y$ are any real numbers with $x < y, then \exists$ a ratioanl numbers $r \in \mathbb{Q}$ such that $x < r < y$*

Proof. assume $x > 0, x \in \mathbb{R}$
Given, $x > y \Rightarrow y - x > 0, y - x \in \mathbb{R}$
$\exists n \in \mathbb{N}, \dfrac{1}{n} < y - n \ldots (corollarly 2.4.0.2)$

$$1 < ny - nx$$

$$nx + 1 < ny \tag{2.19}$$

Also, $x > 0 \Rightarrow n_x > 0$ then $\exists m \in \mathbb{N}$ such that $m - 1 \leqslant n_x < m, \ldots (corolllarly 2.4.0.3)$
from (2.19)
$n_x < m \leqslant n_{x+1} < n_y$
$\Rightarrow n_x < m < n_y$
$\Rightarrow x < \dfrac{m}{n} < y$
$\Rightarrow x < r < y, \text{where} r = \dfrac{m}{n} =$ ratiional number
-Hence Proved- □

Corollary 2.4.1.1. *If x and y are any real numbers with $x < y$ then $\exists$ an irrational number $r \in \mathbb{Q}^c \ni x < r < y$*

Proof. By density theorem,
If $x < y$ then $\exists \quad r_1 \in \mathbb{Q} \ni x < r_1 < y$. Here $x < y$

$$\therefore \sqrt{2}x < \sqrt{2}y$$
$$\sqrt{2}x < r_1 < \sqrt{2}y$$
$$x < \frac{r_1}{\sqrt{2}} < y$$
$$x < r < y \quad \text{where} \quad r = \frac{r_1}{\sqrt{2}} = \text{irrational number}$$

-Hence Proved- □

Intervals:-

- $[a, b] = \{x / a \leqslant x \leqslant b\}$= Closed
- $(a, b) = \{x / a \leqslant x \leqslant b\}$= Open
- $[a, b) = \{x / a \leqslant x \leqslant b\}$= Half Closed- Half Open
- $(a, b] = \{x / a \leqslant x \leqslant b\}$= Half Closed- Half Open

Intersection:-

Finite :- $\bigcap_{i=1}^{n}\left[0,\frac{1}{n}\right]$

Arbitrary:-

$$\bigcap_{i=1}^{\infty} = \{0\}$$

$$\bigcap_{n=1}^{\infty}\left(0,\frac{1}{n}\right) = \phi$$

$$\bigcap_{n=1}^{\infty}(n,\infty) = \phi$$

$$\bigcup_{n=1}^{\infty}(-n,n) = \{-\infty,\infty\}$$

$$\bigcap_{n=1}^{\infty}(-n,n) = \{-1,1\}$$

$$\bigcap_{n=1}^{\infty}\left[-1,1+\frac{1}{n}\right) = [-1,1]$$

$$\bigcup_{n=1}^{\infty}\left[-1,1-\frac{1}{n}\right) = [-1,1)$$

$$\bigcap_{n=1}^{\infty}[-n,n] = [-1,1]$$

$$\bigcap_{n=1}^{\infty}[-n,n]=(-\infty,\infty)$$

Theorem 2.4.2. $\mathbb{R}$ *is uncountable.*

Proof. Assume that $\mathbb{R}$ is countable so does $(0,1)$ is countable.
We can write one-one correspondence with $\mathbb{N}$ as,
$b_1 = 0.a_{11}a_{12}a_{13}\ldots \neq C$
$b_2 = 0.a_{21}a_{22}a_{23}\ldots \neq C$
$b_3 = 0.a_{31}a_{32}a_{33}\ldots \neq C$
:
:
:
$b_i = 0.a_{i1}a_{i2}a_{i3}\ldots a_{ii} \neq C$
$b_i = 0.C_1C_2C_3\ldots \in (0,1)$
$C_1 \neq a_{11}$
$C_2 \neq a_{22}$
$C_3 \neq a_{33}$
:
:
$C_i \neq a_{ii}$
As $C_i \neq a_i$ there does not exists any $C_i \neq C$

$\Rightarrow$ Our counting Scheme is wrong.
$\Rightarrow$ Our assumption is wrong.
$\Rightarrow$ $(0,1)$ must be uncountable .
$\Rightarrow$ $\mathbb{R}$ is uncountable. $\square$

2.5 Cauchy Schwartz Inequality

Let $a_i, b_i \in \mathbb{R} \forall i$ then

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leqslant \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right)$$

Proof. let $x \in \mathbb{R}$ then,
$a_i x + b_i \in \mathbb{R} \ldots \because a_i, b_i \in \mathbb{R}$
$\therefore (a_i x + b_i)^2 \geqslant 0$
$a_i^2 x^2 + 2a_i x b_i + b_i^2 \geqslant 0$

$$\Rightarrow (\sum_{i=1}^{n} a_i^2)x^2 + 2(\sum_{i=1}^{n} a_i b_i)x + \sum_{i=1}^{n} b_i^2 \geqslant 0$$

$$Ax^2 + 2Bx + C \geqslant 0 \tag{2.20}$$

where,

$$A=\sum_{i=1}^{n}a_i^2, B=\sum_{i=1}^{n}a_ib_i,, C=\sum_{i=1}^{n}b_i^2$$

let $x=\frac{-B}{A}$
$\therefore$ from (2.20)
$A\left(\dfrac{B}{A}\right)^2+2B\left(\dfrac{-B}{A}\right)+C\geqslant 0\Rightarrow \dfrac{B^2}{A}-\dfrac{2B^2}{A}+C\geqslant 0$
$\Rightarrow \dfrac{-B^2}{A}+C\geqslant 0$
$\Rightarrow C\geqslant \frac{B^2}{A}$
$\Rightarrow A\cdot C\geqslant B^2$
$\Rightarrow B^2\geqslant A\cdot C$

$$\therefore \left(\sum_{i=1}^{n}a_ib_i\right)^2\leqslant\left(\sum_{i=1}^{n}a_i^2\right)\left(\sum_{i=1}^{n}b_i^2\right)$$

-Hence Proved-

□

Note:-
Equality hold if a_i and b_i is equal to zero.
If $x=\dfrac{-B\pm\sqrt{B^2-4AC}}{2A}$ then $Ax^2+2Bx+C=0$

Chapter 3

Elements of Point Set Topology

3.1 Terminology and Notations

Definition* 3.1.1** (Member of a set):*** *If an element x is in a set A, we write $x \in A$ and say that x is a member of A, or that x belongs to A. If x is not in A, we write $x \notin A$*

Definition* 3.1.2** (Subset):*** *If every element of a set A also belongs to a set B, we say that A is a subset of B and write $A \subseteq B$ or $B \supseteq A$*

Definition* 3.1.3** (Proper Subset):*** *We say that a set A is a proper subset of a set B if $A \subset B$, but there is at least one element of B that is not in A.*

Definition* 3.1.4** (Equal Sets):*** *Two sets A and B are said to be equal, and we write $A = B$, if they contain the same elements. i.e. $A \subseteq B$ and $B \supseteq A$.*

A set is normally defined by either listing its elements explicitly, or by specifying a property that determines the elements of the set.

- The set of natural numbers $\mathbb{N} := \{1, 2, 3, \cdots,$
- The set of integers $\mathbb{Z} := \{0, 1, -1, 2, -2, 3, -3, \cdots,$
- The set of rational numbers $\mathbb{Q} = \{m/n : m, n \in \mathbb{Z} and n \neq 0\}$,
- The set of real numbers $\mathbb{R}$

***Definition* 3.1.5** (Open Set)**:** *A subset G of $\mathbb{R}$ is open in $\mathbb{R}$ if for each $x \in G$ there exists a neighbourhood $\forall$ of x such that $v \subseteq G$.*

***Definition* 3.1.6** (Closed set)**:** *A subset f of $\mathbb{R}$ is closed in $\mathbb{R}$ if the complement f^C is open in* $\mathbb{R}$
G is open iff for $x \in G \exists \in x > 0$
$x \in (x - \varepsilon_x, x + \varepsilon_x) \subseteq G$

e.g.$(-\infty, \infty) = \mathbb{R}$ - open as well as closed
$(0, 1)$ - open
(a, b) - open
$[a, \infty)$ - not open but closed
$[a, b]$ - not open but closed
ϕ - open and closed
$[a, b)$ - neither open nor closed
$(a, b]$ - neither open nor closed
$\mathbb{Q}$ - not closed not open

$\mathbb{N}$ - closed but not open
$\mathbb{I}$ - closed but not open

***Definition* 3.1.7** (Interior point)**:** *For some $x \in s$ if $\exists$ open interval $I_x \ni x \in I_x \subseteq$ then x is called interior point of set S.*

***Definition* 3.1.8** (Interior of Set)**:** *Collection of all interior point is called interior of set (S_i). example $S = \{[0,1],[0,1),(0,1]\}, S_i(0,1)$*

Theorem 3.1.1. *Finite union of open sets is open.*

Proof. let A and B be two finite open sets.
Claim- $A \cup B$ is open set.
$\therefore A$ & B be two open set.
$\Rightarrow \forall x \in A, \exists I_x \subseteq A$ and $\forall x \in B\ \exists\ I_x \subseteq B$
let $x \in A \cup B$
$x \in A \text{or} x \in B$
$\therefore x \in I_x \subseteq A \text{or} x \in I_x \subseteq B$
$\Rightarrow x \in I_x \subseteq A \cup B$
$\Rightarrow A \cup B$ is open set. □

Theorem 3.1.2. *Finite intersection of open set is open.*

Proof. let A & B be two open sets.
claim- $A \cap B$ is open.

let $x \in A \cap B$

$\therefore x \in A \text{or} x \in B$

$\Rightarrow \exists I_x \ni x \in I_x \subseteq A \text{and} x \in I_x \subseteq B$

$\Rightarrow x \in I_x \subseteq A \cap B$

$\Rightarrow A \cap B$ is open set. □

Theorem 3.1.3. *Arbitary union of open sets is open.*

Proof. $let \{A_i\}_{i=1}^{\infty}$ be collection of open sets.

claim- $\bigcup_{i=1}^{\infty} A_i$ is open set

$let x \in \bigcup_{i=1}^{\infty} A_i$

$\Rightarrow x \in A_j$, for some $j \in I$

$\Rightarrow x \in I_x \subseteq A_j$, for some $j \in I$

$\Rightarrow x \in I_x \subseteq A_j \subseteq \cup_{i=1}^{\infty} A_i$

$\therefore \bigcup_{i=1}^{\infty} A_i$ is open set. □

Theorem 3.1.4. *Arbitary intersection of open sets may or may not be open set.*

Proof. Set $S_n = \left(1 - \frac{1}{n}, 1 + \frac{1}{n}\right)$

$\bigcap_{n=1}^{\infty} S_n = \{1\}$ which is not open set. □

Theorem 3.1.5. *Finite union of two closed set is closed.*

Proof. let A & B closed set.
Claim- $A \cup B$ is closed set.
Since, A^C & B^C are open sets.
$\Rightarrow A^C \cap B^C$ is open set.
$\Rightarrow (A \cup B)^C$ is open set
$\Rightarrow A \cup B$ is closed set □

Theorem 3.1.6. *Finte intersection of two closed set is closed.*

Proof. let A & B two closed set.
$\Rightarrow A^C$ & B^C are two open sets.
$\Rightarrow A^C \cup B^C$ is again open set.
$\Rightarrow (A \cap B)^C$ is open set
$\Rightarrow A \cap B$ is closed set □

Theorem 3.1.7. *Arbitary union of closed sets may not be closed.*

Example 17:
[Counter example] $A_n = [0, n], \cup A_n [0, \infty)$-closed
$A_n = \left[0, 1 - \frac{1}{n}\right]$
$A_1 = \{0\}$

$A_2 = \left[0, \frac{1}{2}\right] \cup A_n[0,1)$ -not closed
$A_3 = \left[0, 1 - \frac{1}{3}\right] \ldots (\because (-\infty, 0) \cup [1, \infty)$ -not open)

Theorem 3.1.8. *Every open set is union of open intervals.*

Proof. Suppose $S = \{x_1, x_2, x_3\}$
let S be an open set, $S = \{x_1, x_2, x_3 \ldots\} = \{x_i\}$
for each $x_i \in I_{x_i} \subseteq S$
$\{x_i\} \subseteq I_{x_i} \subseteq S$
$S = \cup\{x_i\} \subseteq \cup_{i \in I} \subseteq I_{x_i} \subseteq S$
Hence, Every open set is union of open intervals. □

Theorem 3.1.9. *Interior of set is open set.*

Proof. Given that, Let S^i is interior.
S is open set.
Claim- $x \in S^i, \exists I_x \in S^i \ni x \in I_x \subseteq S^i$
let $x \in S^i$
$\Rightarrow x$ is interior point of S
$\Rightarrow x \in I_x \subseteq S^i$
let $y \in I_x \Rightarrow y \in S \Rightarrow y \in I_x \subseteq S$
$\Rightarrow y \in S^i, y \in I_x$

$\therefore y$ is also interior point of S
this is true for all $y \in I_x$
$\therefore I_x \subseteq S^i \Rightarrow x \in I_x \subseteq S^i$
$\Rightarrow S^i$ is open set. □

Theorem 3.1.10. *Interior of set is largest open subset of set.*

Proof. let $S \subseteq \mathbb{R}$, S^i is interior set of S.
Claim:- $S^i \subseteq S$ is largest open set.
We prove this by method of contradiction
Assume that, T is largest open subset of set S.
(S^i is not largest) i.e $S^i \subseteq T \subseteq S$
S^i is proper subset of T
Since, $S^i \in T$
$\exists$ some $x \in T, x \notin S^i$
Now, $x \in T \subseteq S \Rightarrow x$ is interior point of S
This contradicts to our assumption that $x \notin S^i$
$\therefore$ Our assumption is wrong.
Hence, Interior of set is largest open subset. □

***Definition* 3.1.9** (Limit point of set)**:** *Let c be the limit point of set S if for any* $\varepsilon > 0$, $\exists\, x \in S \ni$
$0 < |x - c| < \varepsilon$
$i.e\ -\varepsilon < x - c < \varepsilon$

$i.e\ c-\varepsilon < x < c+\varepsilon$
$i.e\ x \in \delta_\varepsilon(c)$
$\Rightarrow \#(\delta_\varepsilon \cap A) \neq 0$
example- $S = \left\{\frac{1}{n}, n \in \mathbb{R}\right\}, 0 is limit point of S.$

Definition* 3.1.10** (Derived Set):*** *The set of all limit points of Set S is called the derived set of S and denoted by* S'
$S' = \{c / c \text{ is limit point of } S\}$

Definition* 3.1.11** (Closed Set):*** *The set S is said to be closed set if it contains all of its limit points (i.e* $S' \subseteq S$*)*

Definition* 3.1.12** (Closure Set):*** $S = S \cup S'$

Example 18:

1. $S = \left\{\frac{1}{n}, n \in \mathbb{N}\right\}, S = \{0\} \notin S$ [Neither open nor closed]
   $\overline{S} = \left\{\frac{1}{n}, n \in \mathbb{N}\right\} \cup \{0\}$
2. $S = \mathbb{Q}, S' = \mathbb{R}\overline{S} = \mathbb{R}$

3. $S = \mathbb{I}, S' = \phi \overline{S} = \mathbb{I}$
4. $S = \mathbb{N}, S' = \phi \overline{S} = \mathbb{N}$

Note:- If S is closed then $S = \overline{S}$

Theorem 3.1.11. *Let $S \subseteq T$ then $S' \subseteq T'$*

Proof. Let $c \in S'$
$\Rightarrow c$ is limit point of S
for any $\varepsilon > 0, \delta_\varepsilon(c) \cap S \neq \phi$
$\Rightarrow \delta_\varepsilon(c) \cap T \neq \phi$ as $S \subseteq T$
$\Rightarrow c$ is limit point of T
$\Rightarrow c \in T'$
$\therefore S' \subseteq T'$ □

Theorem 3.1.12. *Show that* $(S \cup T)' = S' \cup T'$

Proof. To prove, $(S \cup T)' = S' \cup T'$
i.e

a) $(S \cup T)' \subseteq S' \cup T'$

b) $S' \cup T' \subseteq (S \cup T)'$

- first we prove part b)
  $S \subseteq S \cup T \Rightarrow S' \subseteq (S \cup T)'$

$T \subseteq S \cup T \Rightarrow T' \subseteq (S \cup T)'$

$$\Rightarrow S' \cup T' \subseteq (S \cup T)' \tag{3.1}$$

- a) let $c \in (S \cup T)'$
  $\Rightarrow c$ is limit points of $S \cup T$
  $\Rightarrow \exists\ S \cup T \ni x \in \delta_\varepsilon(c)$
  $\Rightarrow x \in S \ni x \in \delta_\varepsilon(c)$ or$x \in T \ni x \in \delta_\varepsilon(c)$
  $\Rightarrow c$ is limit point of S or c is limit point of T
  $\Rightarrow c \in S'$or$c \in T'$
  $\Rightarrow c \in S' \cup T'$

$$(S \cup T)' \subseteq S' \cup T' \tag{3.2}$$

from (3.1) and (3.2)

$$(S \cup T)' = S' \cup T'$$

□

Theorem 3.1.13. *Finite intersection of two closed set is closed.*

Proof. let S & T be two closed sets.
$\therefore S' \subseteq S$ and $T' \subseteq T$
Claim: $S \cap T$ is closed
i.e $(S \cap T)' \subseteq (S \cap T)$
We know,
$S \cap T \subseteq S \Rightarrow (S \cap T)' \subseteq S' \subseteq S$
$S \cap T \subseteq T \Rightarrow (S \cap T)' \subseteq T' \subseteq T$
$(S \cap T)' \subseteq (S \cap T)$
$\therefore S \cap T$ is closed set. □

Theorem 3.1.14. *let S & T be subsets of $\mathbb{R}$, $S' \cap T'$ may or may not be subset of $S \cap T'$*

Proof. $\therefore S' = [1,2], T'[2,3]$
$(S \cap T) = \phi$ & $S' \cap T' = \{2\}$
$\Rightarrow (S' \cap T')' = \phi$

$$\therefore S' \cap T' \nsubseteq (S' \cap T')'$$

□

Definition* 3.1.13** (Dense Set):*** *A Subset $A \subseteq \mathbb{R}$ is said to be dense set in $\mathbb{R}$ if every point of $\mathbb{R}$ is point of A or limit point of $\mathbb{R}$ or equivalently if closure of A is $\mathbb{R}$*
$\overline{A} = A' \cup A = \mathbb{R}$

- A set A is said to be dense in itself if $\overline{A} = A$
- A set A is said to be nowhere dense relative to $\mathbb{R}$ if no neighborhood of $\mathbb{R}$ is contained in the closure of A
- A set is said to be perfect if it is identical with its derived set or equivalently a set which is closed and dense in itself.

Theorem 3.1.15. *Set is closed if and only if its complement is open.*

Proof. a) let S be closed set
To prove- S^c is open.
let $x \in S^c$
$\Rightarrow x$ is not limit point of $S(\overline{S} = S)$
for some $\varepsilon > 0, V_\varepsilon(x) \cap S = \phi$
$(x-\varepsilon, x+\varepsilon) \subseteq S^c$
$\therefore S^c$ is open.

b) let S^c is open set
To prove- S is closed set
By method of contradiction,
Assume that S is not closed.
$\therefore \exists$ some limit point of x of $S \ni x \notin S$
$\Rightarrow x \in s^c$

for some $\varepsilon > 0 \ \exists \ (x-\varepsilon, x+\varepsilon) \subseteq S^c \ldots (\because S^c$ is open set)
$\therefore v_\varepsilon(x) \cap S = \phi$
which is not possible as x is limit point
$\Rightarrow$ Our Assumption is that $x \ni S$ is wrong
$\Rightarrow$ All limit point of S are in S
$\Rightarrow$ is closed set.

□

Theorem 3.1.16. *Derived set of set is closed.*

Proof. let $S \subseteq \mathbb{R}$, S' is derived set of S.
To prove- S' is closed i.e $(S')' \subseteq S' = S'\prime$
let $c \in S'' \Rightarrow c$ is limit point of S'
i.e every $\varepsilon - neighborhood\ v$ of c contains atleast one point x of $S' \ni x \neq c$
i.e $x \in S' \Rightarrow x$ is limit point of set S.
$\therefore$ Every ε neighborhood v of x contains atleast one point of S.
As $x \in v$, v is also a ε neighborhood of x
$\therefore v$ contains atleast one point of S.
In this way, we can prove that, every ε neighborhood v of c contains infintly many points of S.
$\therefore C$ is limit point of set S .
Also $c \in S'$
As $c \ \in S'' \Rightarrow c \in S', S'' \subseteq S' \Rightarrow S'$ is closed set when $S'' = \phi$

then $S'' \subseteq S' \Rightarrow S'$ is closed set. □

3.2 Compact Set

Definition* 3.2.1** (Open Cover):*** *Let A be a subset of $\mathbb{R}$. An open cover of A is an collection $G = \{G_\alpha\}$ of open sets in $\mathbb{R}$ whose union contains A i.e*

$$A \subseteq \cup_\alpha G_\alpha$$

Definition* 3.2.2** (Subcover):*** *if G' is subcollection of sets from G such that the union of sets in G' also contains A then G' is called a subcover of G*

Definition* 3.2.3** (Finite Subcover):*** *A subset k of $\mathbb{R}$ is said to be compact if every open cover of $\mathbb{R}$ has finite subcover.*

Example 19:

1. $S = (0,1), G_i = \left(0, 1 - \frac{1}{1}\right)$
   $\cap G_i = (0,1) \supseteq (0,1)$
   $\cap G_i = \left(0, 1 - \frac{1}{n}\right) \not\subseteq (0,1)$
   $\therefore (0,1)$ is not compact
2. $\mathbb{N}$ is not compact

3.3 Heine Borel theorem

Theorem 3.3.1 (Heine Borel theorem)**.** *The set k is compact set if and only if it is closed & bounded.*

Proof. Given that, k is compact set.
i.e Every open cover exists finite subcover.
claim- k is bounded & closed.

1. k is bounded
   $G_i = (-i, i), G = \mathbb{R}$
   $\cup_{i=1}^{n} G_i = (-n, n), k \subseteq (-n, n)$
   $\therefore k$ is bounded

2. k is closed i.e k^c is open
   let $x \in k^c$
   $$G_n = \left(-\infty, x - \frac{1}{n}\right) \cup \left(x + \frac{1}{n}, \infty\right)$$
   $$G_1 = (-\infty, x - 1) \cup (x + 1, \infty)$$
   $$G_2 = \left(-\infty, x - \frac{1}{2}\right) \cup \left(x + \frac{1}{2}, \infty\right)$$
   $\dots$
   $\dots$

$\ldots$

$$G_n = \left(-\infty, x - \frac{1}{n}\right) \cup \left(x + \frac{1}{n}, \infty\right)$$

$\therefore k$ is closed.

Hence, from a) and b),

k is compact if and only if it is closed and bounded.

$\square$

Chapter 4

Sequence and Series

Definition* 4.0.1** (Sequence and Series):*** *A sequence of real numbers is function defined on the set* $\mathbb{N}$ *whose range is contained in the set* $\mathbb{R}(x : \mathbb{N} \to \mathbb{R})$
Denoted by $x, (x_n), (x_n, n \in \mathbb{N})$
example $\frac{1}{n}, \frac{1}{n^2}, 2n, n^2 + 1, n^2 - n$

- *Constant Sequence-* $x_n = x, \forall n \in \mathbb{N}$
- *Increasing Sequence-*$x_n \leqslant x_{n+1}, \forall n \in \mathbb{N}$
- *Strictly increasing sequence-* $x_n < x_{n+1} \forall n \in \mathbb{N}$
- *Decreasing Sequence-* $x_n \geqslant x_{n+1}, \forall n \in \mathbb{N}$
- *Strictly Decreasing Sequence-* $x_n > x_{n+1}, \forall n \in \mathbb{N}$

Definition* 4.0.2** (Fibonacci Sequence):*** x_1, x_2, $x_{n+2} = x_{n+1} + x_n$

- *Limit of Sequence- A Sequence* $(x_n) \in \mathbb{R}$ *is said to be converage to* $x \in \mathbb{R}$ *or* x *is said to be limit of* (x_n) *if for every* $\varepsilon > 0 \; \exists > 0 \; k(\varepsilon) \in \mathbb{N}$ *such that*
  $|x_n - x| < \varepsilon, \forall n \geqslant k(\varepsilon)$
  If sequence has limit, we say that sequence is convergent. IF it has no limit has no limit, we say that is divergent.
  $lim(x_n) = x$ *or* $x_n \to x$

- *Ocillating Sequence:* $(x_n) = (-1)^n, n \in \mathbb{N}$ - *(non convergent)*
  $(x_n) = \dfrac{(-1)^n}{n}, n \in \mathbb{N}$

Definition* 4.0.3** (Uniqueness of limit point):*** *A sequence in* $\mathbb{R}$ *have atmost limit point one.*

let x_1 & x_2 be two limit points of x_n
$\therefore$ for any $\varepsilon > 0 \; \forall, \; n \geqslant k_1(\varepsilon)$ & $|x_n - x_1| < \varepsilon$
$\exists \; k_1(\varepsilon) \in \mathbb{N} \ni |x_n - x_1| < \varepsilon, \; \forall n \geqslant k_1(\varepsilon)$
$\exists \; k_2(\varepsilon) \in \mathbb{N} \ni |x_n - x_2| < \varepsilon, \; \forall n \geqslant k_2(\varepsilon)$
$k(\varepsilon) = max(k_1(\varepsilon), k_2(\varepsilon))$
$\forall n \in \mathbb{N} \ni n \geqslant k(\varepsilon)$
$|x_1 - x_2|$
$= |x_1 - x_n + x_n - x_2|$
$\leqslant |x_n - x_1| + |x_n - x_2|$

$\leqslant \varepsilon + \varepsilon$
$\leqslant 2\varepsilon$
As this statement is true for any $\varepsilon > 0, x_1 = x_2$
Hence, Sequence have atmost one limit point.

***Definition* 4.0.4** (Tail Sequence)**:** *If $\{x_1, x_2, \ldots\}$ is sequnce of real numbers and if m is given natural number then $m - tail$ of x_n is sequence*
$x_m = \{x_{m+n} / x_{m+1}, x_{m+2} \ldots\}$

Theorem 4.0.1. *Let x_n be sequence of real numbers and let $m \in \mathbb{N}$ then $m - tail$ x_m of x_n converges if & only if x_n converges.*

Proof. Let $x_n \to x$ i.e $\lim\limits_{n\to\infty} x_n = x$
$\Rightarrow$ for any $\varepsilon > 0$, $\exists\ k(\varepsilon) \in \mathbb{N}$
such that $|x_n - x| < \varepsilon$, $\forall n \geqslant k(\varepsilon)$
$\Rightarrow x - \varepsilon < x_n < x + \varepsilon, \ \forall n \geqslant k(\varepsilon)$
$\Rightarrow x - \varepsilon < x_k, x_{k+1}, \ldots < x + \varepsilon$
let $y_n = x_{m+n}, n$
$\Rightarrow x - \varepsilon < y_{k-m}, y_{k+1-m}, \ldots < x + \varepsilon$
$\Rightarrow x - \varepsilon < y_n < x + \varepsilon \forall n \geqslant k(\varepsilon) - m = k_1(\varepsilon)$
$\Rightarrow |y_n - x| < \varepsilon \forall n \geqslant k_1(\varepsilon)$
$y_n \to x$
-Hence proved- □

Theorem 4.0.2. *Let x_n be a sequence of real numbers and $x \in \mathbb{R}$ if a_n is sequence of positive real numbers with* $\lim a_n = 0$ *and if for some constant* $c > 0$ *and some* $m \in \mathbb{N}$, *we have* $|x_n - x| \leqslant c a_n, \forall n \geqslant m$ *then it follows that* $\lim x_n = x$

Proof. Given that $lim a_n = 0$
i.e $a_n \to 0$
$\therefore$ by definition, for any $\varepsilon > 0, \frac{\varepsilon}{c} (\because c > 0)$
$\exists\, k(\varepsilon) \in \mathbb{N}$ such that
$|a_n - 0| < \frac{\varepsilon}{c}$
$a_n < \frac{\varepsilon}{c} \ldots (\because a_n > 0)$
let $k_1(\varepsilon) = max(m_1 k_1(\varepsilon))$
$\forall n \geqslant k_1(\varepsilon)$
$|x_n - x|$
$\leqslant c a_n$
$\leqslant c(\varepsilon/c)$
$\leqslant \varepsilon, \forall n \geqslant k_1(\varepsilon)$
$\therefore x_n \to x$ □

Definition* 4.0.5** (Bounded Sequence):*** *A Sequence of real numbers x_n is said to be bounded if* $\exists\, m > 0$ *such that* $|x_n| \leqslant m, \forall n \in \mathbb{N}$

Theorem 4.0.3. *The Convergent sequence of real numbers is bounded.*

Proof. let $x_n \to x$
$\therefore$ by definition, for any $\varepsilon > 0$, $\exists\ k(\varepsilon) \in \mathbb{N}$
such that $|x_n - x| < \varepsilon, \forall n \geqslant k(\varepsilon)$
$\therefore x - \varepsilon < x_n < x + \varepsilon, \forall n \geqslant k(\varepsilon)$
let
$M = max\{|x_1|, |x_2|, \dots |x_k|, x + \varepsilon\}$
$\therefore |x_n| \leqslant M, \forall n$
$\Rightarrow x_n$ is bounded.
-Hence Proved- □

Theorem 4.0.4. *a) Let x_n and y_n be sequence of real numbers that converges to x and y respectively and let $c \in \mathbb{R}$ then, the sequence $X + Y, X - Y, XY$ and CX converges to $x + y, x - y, xy$ and cx*

b) If $x_n \to x$ and z_n is sequence of non-zero real numbers that converges to z and if $z \neq 0$ then $\dfrac{X}{Z} \to \dfrac{x}{z}$

Proof. a) given that $x_n \to x$
$\therefore$ by definition, for any $\varepsilon > 0, \dfrac{\varepsilon}{2} > 0$
$\exists\ k_1(\varepsilon) \in \mathbb{N}$ such that
$|x_n - x| < \varepsilon/2, \forall n \geqslant k_1(\varepsilon)$
also, $y_n \to y$

$\therefore$ by definition, for any $\varepsilon > 0, \dfrac{\varepsilon}{2} > 0$
$\exists\, k_2(\varepsilon) \in \mathbb{N}$ such that
$|y_n - y| < \varepsilon/2, \forall n \geqslant k_2(\varepsilon)$
let $k(\varepsilon) = max(k_1(\varepsilon), k_2(\varepsilon))$
$\therefore \forall n \geqslant k(\varepsilon)$

i) $|(x_n + y_n) - (x + y)| = |x_n - x + y_n - y|$
$\leqslant |x_n - x| + |y_n - y| \ldots$ (triangular inequality)
$\leqslant \dfrac{\varepsilon}{2} + \dfrac{\varepsilon}{2}$

$\leqslant \varepsilon$
$\therefore (x_n + y_n) \to (x + y)$

ii) $|(x_n - y_n) - (x + y)| = |x_n - x - y_n - y|$
$\leqslant |x_n - x| + |y_n - y| \ldots$ (triangular inequality)
$\leqslant \dfrac{\varepsilon}{2} + \dfrac{\varepsilon}{2}$

$\leqslant \varepsilon$
$\therefore (x_n - y_n) \to (x - y)$

iii) $x_n \to x$
$\therefore$ by definition, for any $\varepsilon > 0, \dfrac{\varepsilon}{2M} > 0, \ldots (\because M > 0)$

$\exists\ k_1(\varepsilon) \in \mathbb{N}$ such that
$|x_n - x| < \varepsilon/2M, \forall n \geqslant k_1(\varepsilon)$
also, $y_n \to y$
$\therefore$ by definition, for any $\varepsilon > 0, \dfrac{\varepsilon}{2|x|} > 0, \ldots (\because |x| > 0)$
$\exists\ k_2(\varepsilon) \in \mathbb{N}$ such that
$|y_n - y| < \varepsilon/2|x|, \forall n \geqslant k_2(\varepsilon)$
let $k(\varepsilon) = max(k_1(\varepsilon), k_2(\varepsilon))$
$\therefore \forall n \geqslant k(\varepsilon)$
$|(x_n y_n) - (xy)| = |x_n y_n - x y_n + x y_n - xy|$
$\leqslant |y_n||x_n - x| + |x_n||y_n - y| \ldots$ (triangular inequality)
$\leqslant M\frac{\varepsilon}{2M} + |x|\frac{\varepsilon}{2}$

$\leqslant \varepsilon$
$\therefore x_n y_n \to xy$

iv) $x_n \to x$
$\therefore$ by definition, for any $\varepsilon > 0, \dfrac{\varepsilon}{|c|} > 0, \ldots (\because |c| > 0)$
$\exists\ k(\varepsilon) \in \mathbb{N}$ such that
$|x_n - x| < \frac{\varepsilon}{|c|}, \forall n \geqslant k(\varepsilon)$
$|(cx_n - cx)| = |c|.|x_n - x|$

$\leqslant |c|.\frac{\varepsilon}{|c|}$

$\leqslant \varepsilon$

$\therefore cx_n \to cx$

b) $x_n \to x$ and $z_n \to z$

$\therefore$ by definition, for any $\varepsilon > 0, \varepsilon.|z|.m > 0$

$\exists\ k(\varepsilon) \in \mathbb{N}$ such that

$|z_n - z| < \varepsilon.|z|.m, \forall n \geqslant k(\varepsilon)$

let $y_n = \dfrac{1}{z_n}$

consider, $|(y_n - x)| = \left|\dfrac{1}{z_n} - \dfrac{1}{z}\right|$

$= \dfrac{|z - z_n|}{|z_n.z|}$

$\leqslant \dfrac{\varepsilon.|z|.m}{|z_n|.|z|}$

$\leqslant \dfrac{\varepsilon.m}{|z_n|}$

$\leqslant \varepsilon \ldots (\because z_n$ is bounded $m < z_n < m)$

$\therefore \dfrac{1}{x_n} \to \dfrac{1}{z}$
$\therefore y_n \to y$
we know that, $x_n y_n \to xy \ldots (\because$ if $x_n \to x$ & $y_n \to y$ then $x_n y_n \to xy)$
$\therefore \dfrac{x_n}{z_n} \to \dfrac{x}{y}$
-Hence Proved- □

Theorem 4.0.5. *If $x_n \to x$ and if $x_n \geqslant 0, \forall n \in \mathbb{N}$ then $x = lim x_n \geqslant 0$*

Proof. Given that, $x_n \to x$
$\therefore$ by definition, for any $\varepsilon > 0$
$\exists\ k(\varepsilon) \in \mathbb{N}$
such that $|x_n - x| < \varepsilon, \forall n \geqslant k(\varepsilon)$
we will prove this by method of contradiction.
let if possible $x < 0$
$\therefore -x > 0$
Assume, $0 < \varepsilon < -x$
$\therefore x - \varepsilon < 0$ and $x + \varepsilon < 0$ &
$\therefore x - \varepsilon < x_n < x + \varepsilon, \forall n \geqslant k(\varepsilon)$
$\therefore x_n < 0$
which contradicts to given statement that $x_n \geqslant 0$
$\therefore$ Our assumption is wrong.

$\therefore x = \lim x_n \geqslant 0$
-Hence Proved- □

Theorem 4.0.6. *If $x_n \to x$, $y_n \to y$ are convergent sequence of real numbers and if $x_n \leqslant y_n, \forall n \in \mathbb{N}$ then $lim x_n \leqslant \lim y_n$*

Proof. Given that, $x_n \to x$, and $y_n \to y$ also, $x_n \leqslant y_n, \forall n$
$\Rightarrow y_n - x_n \geqslant 0$
$\Rightarrow z_n \geqslant 0$
Now, $y_n - x_n \to y - x(say z)$
As, $z_n \geqslant 0, z_n \to z$
$\therefore z \geqslant 0 \ldots$ (by above theorem)
$\therefore y - x \geqslant 0$
$\therefore y \geqslant x$
$\therefore x \leqslant y$
-Hence Proved □

Theorem 4.0.7. *If x_n is convergent to some $x \in \mathbb{R}$ and $a \leqslant x_n \leqslant b, \forall n$ then $a \leqslant x \leqslant b$*

Proof. Given that, $x_n \to x$ and $a \leqslant x_n \leqslant b$
let $a_n = a \& b_n = b$
i.e $a_n \to a i.e b_n \to b$
$\therefore a_n \leqslant x_n \leqslant b_n$
i.e $a_n \leqslant x_n \& x_n \leqslant b_n$

$lim a_n \leqslant \lim x_n \& \lim x_n \leqslant b_n \ldots$ (by above theorem)
$a \leqslant x and x \leqslant b \therefore a \leqslant x \leqslant b$
-Hence Proved- □

4.1 Squeeze Theorem

Theorem 4.1.1. *Suppose x_n, y_n and z_n are sequence of real numbers $\ni x_n \leqslant y_n \leqslant z_n, \forall n \in \mathbb{N}$ and* $\lim x_n \leqslant \lim y_n$ *then y_n is convergent and* $\lim x_n = \lim y_n = \lim z_n$.

Proof. Given that, $x_n \leqslant y_n \leqslant z_n, \forall n$
let, $lim x_n = \lim z_n = w$
i.e $x_n \to w$ and $z_n \to w$
$\therefore$ by definition, for any $\varepsilon > 0 \ \exists$
$k_1(\varepsilon) \in \mathbb{N}$ and $k_2(\varepsilon) \in \mathbb{N}$ such that
$|x_n - w| < \varepsilon, \forall n \geqslant k_1(\varepsilon)$ and $|z_n - w| < \varepsilon, \forall n \geqslant k_2(\varepsilon)$
$\therefore w - \varepsilon \leqslant x_n \leqslant w + \varepsilon$ and $w - \varepsilon \leqslant z_n \leqslant w + \varepsilon$
$\therefore w - \varepsilon \leqslant x_n \leqslant y_n$ and $y_n \leqslant z_n \leqslant w + \varepsilon$
$\therefore w - \varepsilon \leqslant x_n \leqslant y_n \leqslant z_n \leqslant w + \varepsilon$
i.e $w - \varepsilon \leqslant y_n \leqslant w + \varepsilon$
i.e $|y_n - w| < \varepsilon, \forall \in k(\varepsilon) = max(k_1(\varepsilon), k_2(\varepsilon))$
$\therefore y_n \to w$
$\therefore \lim x_n = \lim y_n = \lim z_n = w$ □

Theorem 4.1.2. *Given that, $x_n \to x$ then Show that,* *a)* $|x_n| \to |x|$

b) $\sqrt{x_n} \to \sqrt{x}$

Proof. Given that, $x_n \to x$
$\therefore$ by definition, for any $\varepsilon > 0$, $\exists\ k(\varepsilon) \in \mathbb{N}$
such that $|x_n - x| < \varepsilon, \forall n \geqslant k(\varepsilon)$
consider,
$||x_n| - |x||$
$\leqslant |x_n - x| \ldots$ (by corollary of triangular inequality)
$\leqslant \varepsilon$
$|x_n| \to |x|$

Given that, $x_n \to x$
$\therefore$ by definition, for any $\varepsilon > 0$, $\sqrt{x} > 0$, $\dfrac{\varepsilon}{\sqrt{x}} > 0$, $\varepsilon\sqrt{x} > 0$
$\exists\ k(\varepsilon) \in \mathbb{N}$ such that
$|x_n - x| \leqslant \varepsilon\sqrt{x},\ \forall\ k(\varepsilon) \in \mathbb{N}$
As, $\sqrt{x} > 0$
$\therefore 0 < \sqrt{x} < \sqrt{x_n} + \sqrt{x}$

$$\frac{1}{\sqrt{x}} > \frac{1}{\sqrt{x_n} + \sqrt{x}} \tag{4.1}$$

$|\sqrt{x_n} - \sqrt{x}|$

$= \dfrac{|\sqrt{x_n} - \sqrt{x}|.|\sqrt{x_n} + \sqrt{x}|}{|\sqrt{x_n} - \sqrt{x}|}$

$= \dfrac{|x_n - x|}{\sqrt{x_n} + \sqrt{x}}$

$\leqslant \dfrac{|x_n - x|}{\sqrt{x}} \ldots$ (from 4.1)

$\leqslant \dfrac{\varepsilon.\sqrt{x}}{\sqrt{x}}$

$\leqslant \varepsilon$

$\therefore \sqrt{x_n} \to \sqrt{x}$

□

4.2 Monotone Sequence

- *Monotone decreasing:* $x_n \geqslant x_{n+1}, \forall n$
- *Monotone increasing:* $x_n \leqslant x_{n+1}, \forall n$
  x_n is called as monotone if it is increasing or decreasing.

Theorem 4.2.1 (Monotone Convergence theorem)**.** *A monotone sequence of real numbers is convergent if and only if*

a) If x_n is bounded increasing sequence
$lim(x_n) = Sup\{x_n, n \in \mathbb{N}\}$

b) If x_n is bounded decreasing sequence
$lim(x_n) = Inf\{x_n, n \in \mathbb{N}\}$

Proof. We know that, Convergent sequence must be bounded.
Conversly, let x_n be monotone bounded sequence.

a) Assume x_n is increasing and bounded.
As x_n is bounded $M \in \mathbb{R}, |x_n| \leqslant M, \forall n$
let, $S = \{x_n, \forall n \in \mathbb{N}\}$
M upper bound of S
$\therefore$ By completeness property, $\exists\ x^* \in \mathbb{R}$
$\ni x^* = Sup\{x_n, n \in \mathbb{N}\}$
$\therefore x_n \leqslant x^* \forall \mathbb{N}$
for any $\varepsilon > 0$ $x^* - \varepsilon$ is not supremum of S
$\therefore x^* - \varepsilon < x_k \leqslant x^*$, for some k
$\Rightarrow x^* - \varepsilon < x_k \leqslant x_{k+1} \leqslant x_{k+2} \leqslant \ldots \leqslant x^*$
$\therefore x^* - \varepsilon < x_n < x^*, \forall n \geqslant k(\varepsilon)$
$\therefore x^* - \varepsilon < x_n < x^* + \varepsilon$
$\therefore x^* = \lim x_n$
i.e x_n is convergent sequence.

b) Assume x_n is decreasing and bounded.
As x_n is bounded $M \in \mathbb{R}, |x_n| \leqslant M, \forall n$
let, $S = \{x_n, \forall n \in \mathbb{N}\}$
$-M$ lower bound of S
$\therefore$ By completeness property, $\exists\ x^* \in \mathbb{R}$
$\ni\ x^* = Inf\{x_n, n \in \mathbb{N}\}$
$\therefore x_n \geqslant x^* \forall \mathbb{N}$
for any $\varepsilon > 0$ $x^* + \varepsilon$ is not lower bound of S
$\therefore x^* < x_k < x^* + \varepsilon$, for some k
$\Rightarrow\ x^* < \ldots \leqslant x_{k+2} \leqslant x_{k+1} \leqslant x_k < x^* + \varepsilon$
$\therefore x^* < x_n < x^* + \varepsilon$
$\therefore x^* - \varepsilon < x_n < x^* + \varepsilon$
$\therefore x^* = \lim x_n$
i.e x_n is convergent sequence.

□

Theorem 4.2.2. *If x_n converges to x then any subsequences x_{n_k} of x_n also converges to x.*

Proof. for any $\varepsilon > 0$ $\exists$ $k(\varepsilon) \in \mathbb{N}$ such that,
$|x_n - x| < \varepsilon, \forall n \geqslant k(\varepsilon)$
let subsequence $x_{n_k} = \{x_{n_1}, x_{n_2}, x_{n_3}, \ldots\}$
As $x_n \to x \Rightarrow x - \varepsilon < x_n < x + \varepsilon$

Also, $n_k \geqslant n \geqslant k(\varepsilon)$
$\rightarrow x-\varepsilon < x_{n_k} < x+\varepsilon, \forall n_k \geqslant k(\varepsilon)$
$\therefore x_{n_k} \rightarrow x$ □

Theorem 4.2.3 (Monotone Subsequence theorem)**.** *If x_n is sequence of real numbers then there is subsequence of x_n that is monotone.*

Proof. We will say that m^{th} term x_m is a peak if $x_m \geqslant x_n \forall n \geqslant m$.
Note that, In a decreasing sequence, every term is peak while in increasing sequence, no term is peak.
Case-1:-
x_n has infinitely many peaks. In this case, we list the peaks by,
$x_{m_1} \geqslant x_{m_2} \geqslant x_{m_3} \ldots \geqslant x_{m_k}, \ldots$
$\therefore$ subsequence x_{m_k} is decreasing subsequnce of x_n.
Case-2:-
x_n has finitely number of peaks.
let these peaks be denoted by,
$x_{m_1}, x_{m_2}, x_{m_3} \ldots x_{m_r}$
let $S_1 = m_r + 1$ be the first index beyond the last peak since x_{S_1} is not peak $\exists\, S_2 > S_1$
$\ni x_{S_1} < x_{S_2}$ since x_{S_2} is not peak $\exists\, S_3 > S_2$
$\ni x_{S_2} < x_{S_3}$ continuing this way, we obtain an increasing sequence. □

Theorem 4.2.4 (Bozano- Weistress theorem)**.** *A bounded sequence of real numbers has convergent subsequence.*

Proof. Let x_n be bounded sequence.
$\therefore$ by monotone subsequence theorem,
$\exists\ x_{n_k}$ subsequence of x_n that is monotone.
As x_n is bounded x_{x_k} is also bounded
$\therefore$ by monotone convergence theorem,
x_{x_k} is monotone and bounded so convergent. □

4.3 Cauchy Sequence

***Definition* 4.3.1** (Cauchy Sequence)**:** *A sequence of real numbers is said to be cauchy if for every* $\varepsilon > 0, \exists\ H(\varepsilon) \in \mathbb{N} \ni |X_n - x_m| < \varepsilon, \forall n, m \geqslant H(\varepsilon)$

Theorem 4.3.1. *Every convergent sequence is cauchy.*

Proof. let $x_n \to x$
for any $\frac{\varepsilon}{2} > 0, \exists\ k(\varepsilon) \in \mathbb{N}$
$\ni |X_n - x| < \frac{\varepsilon}{2}, \forall n \geqslant k(\varepsilon)$ let, $k_1, k_2 \in \mathbb{N}$ such that $\forall k_1, k_2 \geqslant k(\varepsilon)$
$|X_{k_1} - x_{k_2}|$
$\leqslant |X_{k_1} - x| + |X_{k_2} - x|$
$\leqslant \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$
$\leqslant \varepsilon$
Hence, every convergent sequence is cauchy. □

Theorem 4.3.2. *A cauchy sequence of real numbers is bounded.*

Proof. let x_n be cauchy sequence and
let $\varepsilon = 1$ if $H = H(1)$ and $n \geqslant H$ then $n \geqslant H$.
$M = sup\{|x_1|, |x_2|, |x_3|, \dots |x_{H-1}|, |x_H| + 1\}$
then it follows that $|x_n| \leqslant M \forall n$
$\therefore$ cauchy sequence of real numbers is bounded. □

Definition* 4.3.2** (Cauchy convergence criterion):*** *A Sequence of real numbers is convergent if and only if it is cauchy sequence.*

Definition* 4.3.3** (Contractive Sequence):*** *We say that the sequence x_n of real numbers is contractive sequence if there exists a constant c, $0 < c < 1$ such that,*
$|x_{n+2} - x_{n+1}| \leqslant c.|x_{n+1} - x_n|, \forall n$

Theorem 4.3.3. *Contractive sequence is cauchy sequence.*

Proof. let x_n is contractive sequence
$\therefore \exists\ c, 0 < c < 1$ such that
$|x_{n+2} - x_{n+1}| \leqslant c.|x_{n+1} - x_n|, \forall n$
for $\varepsilon > 0$ choose $k(\varepsilon) \in \mathbb{N} \ni$ for $m > n$
$|x_m - x_n|$
$= |x_m - x_{m-1} + x_{m-1} + \dots + x_{n+1} - x_n|$
$\leqslant |x_m - x_{m-1}| + |x_{m-1} - x_{m-2}| + \dots + |x_{n+1} - x_n|$
$\leqslant c|x_{m-1} - x_{m-2}| + c|x_{m-2} - x_{m-3}| + \dots + c|x_n - x_{n-1}|$
$\leqslant c^2|x_{m-2} - x_{m-3}| + c^2|x_{m-3} - x_{m-4}| + \dots + c|x_n - x_{n-1}|$

$\leqslant (c^{m-n} + c^{m-n-1} + \ldots + c)|x_n - x_{n-1}|$
$\leqslant \dfrac{c(1-c^{m-n}}{1-c}|x_n - x_{n-1}|$
$\leqslant \varepsilon \quad \because \dfrac{c(1-c^{m-n})}{(1-c)} < 1$
$\therefore x_n$ is cauchy sequence. □

Divergent Sequence Let x_n be sequence of real numbers

a) $x_n \to +\infty$ and $\lim x_n = +\infty$
if every $\alpha \in \mathbb{R}$ there exists a natural number $k(\alpha)$ such that if $n \geqslant k(\alpha)$, then $x_n > \alpha$.

b) $x_n \to -\infty$ and $\lim x_n = -\infty$
if every $\beta \in \mathbb{R}$ there exists a natural number $k(\beta)$ such that if $n \geqslant k(\beta)$, then $x_n < \beta$.
We say that x_n is properly divergent if $\lim x_n = +\infty$ or $-\infty$

4.4 Infinite Series

***Definition* 4.4.1** (Infinite Series)**:** *If x_n is sequence in $\mathbb{R}$, then the infinite series generated by x_n is sequence S_n*

$S_1 = x_1$
$S_2 = x_1 + x_2$
.

.

.

$S_n = x_1 + x_2 + \ldots x_n$

Denoted by $\sum x_n$ or $\sum_{n=1}^{\infty} x_n$

Example 20:

1. $\sum_{n=0}^{\infty} r_n = 1 + r + r^2 + \ldots$
2. $\sum_{n=1}^{\infty} (-1^n) = (-1) + 1 + (-1) + \ldots$
3. $\sum \frac{1}{n(n+1)} = \frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4}$

Theorem 4.4.1 (The n^{th} term test)**.** *if* $\sum x_n$ *converges then* $lim x_n = 0$

Proof. By definition $\sum x_n$ converges if S_n converges,
Since $= \sum_{i=1}^{n} x_i$
$\therefore x_n = S_n - S_{n-1}$
$\therefore \lim x_n = \lim S_n - \lim S_{n-1} = 0$ □

***Definition* 4.4.2** (Cauchy Criterion for Series)**:** *The series* $\sum x_n$ *converges if and only if* $\forall\ \varepsilon > 0$, $\exists\ M(\varepsilon) \in \mathbb{N} \ni$ *if* $m > n \geqslant M(\varepsilon)$ *then*
$|S_m - S_n| = |x_{n+1} + x_{n+2} + \ldots + x_m| < \varepsilon$

Theorem 4.4.2. *let x_n be a sequence of non-negative real numbers then the series $\sum x_n$ converges if and only if the sequence S_k of partial sum is bounded.*

$$\sum x_n = \lim S_k = \sup\{S_k : k \in \mathbb{N}\}$$

Theorem 4.4.3. *Show that,* $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$

Proof. Suppose,
$S_{n+1} = 1 + r + \ldots + r^n$
$S_n = 1 + r + \ldots + r^{n-1}$
$rS_n = (r + r^2 + \ldots + r^n)$
$\therefore S_{n+1} - rS_n = 1$
$\therefore \lim_{n\to\infty}(S_{n+1} - rS_n) = 1 \lim_{n\to\infty} 1 = 1$
$\therefore (S - rS) = 1(\ldots \text{where } S \sum_{n=0}^{\infty})$
$S(1-r) = 1$
$S = \frac{1}{(1-r)}$ $\square$

Theorem 4.4.4. *The p Series* $\sum \frac{1}{n^p}$ *converges when* $p > 1$

Proof. if $k_1 = 2 - 1 = 1, S_{k_1} = 1$
$k_1 = 2^2 - 1 = 3, 2^p < 3^p$

$$S_{k_2} = \frac{1}{1^p} + \left(\frac{1}{2^p} + \frac{1}{3^p}\right) < \frac{1}{1^p} + \frac{2}{2^p} = 1 + \frac{1}{2^{p-1}}$$

further, if $k_3 = 2^3 - 1$ then

$S_{k_3} < S_{k_2} + \frac{4}{4^p} < 1 + \frac{1}{2^{p-1}}\frac{1}{4^{p-1}}$

finally, let $r = \frac{1}{2^{p-1}}$ Since $p > 1$

Using mathematical induction

we can show that if $k_j = 2^j - 1$

$$0 < S_{k_j} < 1 + r + r^2 + \ldots + r^{j-1} < \frac{1}{1-r}$$

$\Rightarrow$ The p-series converges if $p > 1$ □

The alternating harmonic series

$$\sum \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{(-1)^{n+1}}{n} + \ldots$$

is convergent

let $S_{2n} = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \ldots + \left(\frac{1}{2n-1} - \frac{1}{2n}\right)$

$$S_{2n+1} = 1 - \left(\frac{1}{2} - \frac{1}{3}\right) - \left(\frac{1}{4} - \frac{1}{5}\right) - \ldots - \left(\frac{1}{2n} - \frac{1}{2n+1}\right)$$

Since $0 < S_{2n} < S_{2n} + \dfrac{1}{2n+1} = S_{2n+1} \leqslant 1$
S_{2n} and S_{2n+1} both bounded and monotone, so by monotone convergence theorem, must be convergent and to same point.
$\sum \dfrac{(-1)^{n+1}}{n}$ must be convergent.

Theorem 4.4.5 (The Comparision Test)**.** *Let x_n and y_n be real sequence and for some $k \in \mathbb{N}$ $0 \leqslant x_n \leqslant y_n, \forall n \geqslant k$*

a) Convergent of $\sum y_n \Rightarrow$ Convergence of $\sum x_n$

b) Divergence of $\sum x_n \Rightarrow$ divergence of $\sum y_n$

Proof. a) Suppose $\sum y_n$ is convergent,
i.e for any $\varepsilon > 0, \exists M(\varepsilon) \in \mathbb{N} \ni m > n \geqslant M(\varepsilon)$
$|y_{n+1} + \ldots + y_m| < \varepsilon$
if $m > Sup(k, M(\varepsilon))$
$0 \leqslant x_{n+1} + \ldots + x_m \leqslant y_{n+1} + \ldots + y_m < \varepsilon$
$\Rightarrow \sum x_n$ converges.

b) This statement i contrapositive to a)

□

Theorem 4.4.6 (Limit Comparison Test)**.** *Suppose x_n and y_n are strictly positive sequence and Suppose following limit exists*

$r = \lim \left(\dfrac{x_n}{y_n}\right)$

a) If $r \neq 0$ then $\sum x_n$ convergent iff $\sum y_n$ convergent.

b) If $r = 0$ then if, $\sum y_n$ convergent then $\sum x_n$ convergent.

Proof. a) Given $r = \lim \dfrac{x_n}{y_n}$

$\therefore$ by defination, For any $\varepsilon > 0$, $\exists$, $k(\varepsilon) \in \mathbb{N}$

such that $\left|\dfrac{x_n}{y_n} - r\right| < \varepsilon, \forall n \geqslant k(\varepsilon)$

As $r \neq 0, \Rightarrow r > 0 \Rightarrow \varepsilon\dfrac{r}{2}$

$r - \varepsilon < \dfrac{x_n}{y_n} < r + \varepsilon$

$\left(\dfrac{r}{2}\right) y_n < x_n < \left(\dfrac{3r}{2}\right) y_n$

$\therefore \left(\dfrac{r}{2}\right) y_n < x_n$

$\Rightarrow$ if x_n converges then $\sum y_n$ also converges. ...(by comparison test)

$\therefore x_n < \left(\dfrac{3r}{2}\right) Y_n$

$\Rightarrow$ If $\sum y_n$ converges then x_n also converges. ... (by comparison test)

$r = 0$ i.e $\lim\left(\frac{x_n}{y_n}\right) = 0$
$\therefore$ by defination, For any $\varepsilon > 0$, $\exists$, $k(\varepsilon) \in \mathbb{N}$
such that

$$\left|\frac{x_n}{y_n} - 0\right| < \varepsilon$$
$$\left|\frac{x_n}{y_n}\right| < \varepsilon$$
$$\frac{x_n}{y_n} < \varepsilon$$

$0 < x_n < \varepsilon y_n$
$\therefore$ By comparison test,
$\sum x_n$ converges if $\sum y_n$ converges. □

***Definition* 4.4.3** (Absolute Convergence)**:** *let x_n be sequence in $\mathbb{R}$. We say that $\sum x_n$ is absolutely convergent if the series $\sum |x_n|$ is convergent . A series is said to be conditionally convergent if it is convergent but not absolutely convergent.*

Example 21:

$\sum \frac{(-1)^n}{n}$ is convergent but $\sum \left|\frac{(-1)^n}{n}\right| = \sum \frac{1}{n}$ is not convergent
$\therefore \sum \frac{(-1)^n}{n}$ is conditionally convergent.

Theorem 4.4.7. *If a series is absolutely convergent then it is convergent.*

Proof. $\sum |x_n|$ is convergent
$\therefore$ for any $\varepsilon > 0$ $M(\varepsilon) \in \mathbb{N}$
$||x_{n+1}| + |x_{n+2}| + \ldots + |x_m|| < \varepsilon \forall m > n > M(\varepsilon)$

$$|x_{n+1} + x_{n+2} + \ldots + x_m| \leqslant \varepsilon$$

$$|x_{n+1}| + |x_{n+2}| + \ldots + |x_m| \leqslant \varepsilon \forall m > n > M(\varepsilon)$$

$\Rightarrow \sum x_n$ is convergent. □

Theorem 4.4.8 (Limit Comparison Test- II-)**.** *Suppose x_n and y_n are non-zero real sequence and Suppose that following limit exists in $\mathbb{R}$*
$r = \lim \left(\frac{x_n}{y_n}\right)$
a) If $r \neq 0$ then $\sum x_n$ absolutely convergent iff $\sum y_n$ is absolutely convergent.

b) If $r = 0$ and $\sum y_n$ is absolutely convergent then $\sum x_n$ absolutely convergent.

Theorem 4.4.9 (Root test)**.** *Let x_n be sequence in $\mathbb{R}$. Suppose that the limit $r = \lim |x_n|^{\frac{1}{n}}$ exists in $\mathbb{R}$ then $\sum x_n$ is absolultely convergent when $r < 1$ and is divergent when $r > 1$.*

Proof. $r < 1$, $r = \lim |x_n|^{\frac{1}{n}}, \exists r_1, r_1 \in (r, 1)$
$|x_n|^{\frac{1}{n}} \leqslant r_1$
$\therefore |x_n| \leqslant \sum r_1^n$
by comparison test,
$|x_n| < (r_1)^n$ it is convergent
$|x_n| < (r_1)^n$ it is absoultely convergent. □

Theorem 4.4.10 (Ratio Test)**.** *Let x_n be non-zero sequnence in $\mathbb{R}$. Suppose $r = \lim \left|\frac{x_{n+1}}{x_n}\right|$ exists then $\sum x_n$ is absolutely convergent when $r < 1$ and divergent when $r > 1$*

Proof. $r < 1$, $r_1 \in (r, 1)$
$\left|\frac{x_{n+1}}{x_n}\right| \leqslant r_1, \forall n > k(\varepsilon)$

$|x_{n+1}| \leqslant r_1 |x_n|$

$|x_{n+1}| \leqslant r_1|x_n| < r_1.r_1|x_{n-1}| < \ldots < r_1^n|x_1|$
$\therefore |x_{n+1}| < r_1^n.c$
$\therefore \sum |x_{n+1}| < \sum r_1^n.c$
$\therefore$ by comparison test,
$\sum x_n$ is absolutely convergent □

4.5 Establish the converges/divergence of series

Example 22:

$\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)}$

$$\sum_{n=1}^{\infty} = \frac{1}{2.3} + \frac{1}{3.4} + \frac{1}{4.5} + \dots$$

The series is converges to zero

or $(n+1)(n+2) > n.n$

$$\therefore \frac{1}{(n+1)(n+2)} < \frac{1}{n^2}$$

$$\therefore 0 < \frac{1}{(n+1)(n+2)} < \frac{1}{n^2}$$

by comparison test,

$\sum \frac{1}{(n+1)(n+2)}$ is convergent.

Example 23:

$2^{(\frac{-1}{n})}$

$$\lim_{n\to\infty} 2^{\left(\frac{-1}{n}\right)} = 1 \neq 0$$

$\therefore$ by n^{th} term test

$2^{\left(\frac{-1}{n}\right)}$ is divergent

Example 24:

$$\frac{n}{2^n}$$

Applying ratio test

$$\left|\frac{x_{n+1}}{x_n}\right| = \left|\frac{(n+1)/2^{(n+1)}}{n/2^n}\right| = \left|\frac{n+1}{n}\right|.\frac{1}{2}$$

$$\lim_{n\to\infty}\left|\frac{x_{n+1}}{x_n}\right| = \frac{1}{2} < 1$$

$\therefore \dfrac{\sum n}{2^n}$ is convergent.

Definition* 4.5.1** (Integral test):*** *Let f be a positive decreasing function on $\{t, t > 1\}$ then the* *series* $\sum_{k=1}^{\infty} f(k)$ *converges if and only if the improper integral*

$$\int_1^{\infty} f(t)dt = \lim_{b\to\infty}\int_1^b f(t)dt$$

exists. In the case of convergence, the partial sum

$S_n = \sum_{k=1}^{n} = f(k)$ *and sum* $S = \sum_{k=1}^{\infty} = f(k)$ *satisfy the estimates*

$$\int_{n+1}^{\infty} f(t)dt \leqslant S - S_n \leqslant \int_1^{\infty} f(t)dt$$

Example 25:

$\sum_{n=1}^{\infty} \frac{1}{n^p}$

$= \int_1^{\infty} \frac{1}{t^p}dt, \quad x_n\frac{1}{n^p}$

$= \left[\frac{t^{-p+1}}{-p+1}\right]_1^{\infty}$

$= \frac{1}{1-p}\left[\frac{1}{t^{p-1}}\right]_1^{\infty}$

$\frac{1}{p-1}, p > 1$

$\therefore \sum \frac{1}{n^p}$ is convergent

Definition* 4.5.2** (Raabies Test):*** *Let x_n be non-zero sequence in $\mathbb{R}$ and let* $a = \lim n\left(1 - \left|\frac{x_{n+1}}{x_n}\right|\right)$ *whenever this limit exists then $\sum x_n$ absoultely convergent when $a > 1$ and is not absoultely convergent when $a < 1$*

Example 26:

$$x_n = \frac{1}{n^p}$$

$$\left|\frac{x_{n+1}}{x_n}\right| = \left|\frac{\frac{1}{n+1}^p}{\frac{1}{n^p}}\right| = \left|\frac{n^p}{(n+1)^p}\right| = \left|\frac{1}{1+\frac{1}{n}}\right|$$

$$\therefore \lim n\left(1 - \left|\frac{x_{n+1}}{x_n}\right|\right) = \lim n\left(1 - \left|\left(\frac{1}{1+\frac{1}{n}}\right)^p\right|\right)$$

$$= \lim \left(\frac{\left(1+\frac{1}{n}\right)^p - 1}{\frac{1}{n}\left(1+\frac{1}{n}\right)^p} \right)$$

$$= \lim \left(\frac{p\left(1+\frac{1}{n}\right)^{p+1}\left(-\frac{1}{n^2}\right)}{\frac{1}{n}\left(1+\frac{1}{n}\right)^p} \right)$$

$$= \lim \left(\frac{-\frac{p}{n}\left(1+\frac{1}{n}\right)}{\left(1+\frac{1}{n}\right)^p} \right)$$

$$= \lim_{n\to\infty} \left(\frac{-p\left(1+\frac{1}{n}\right)}{n} \right)$$

$$= \lim_{n\to\infty} p\left(-\frac{1}{n}-\frac{1}{n^2}\right)$$

$$= p$$

Example 27:

$$x_n = \frac{1}{n(n+1)}$$

$$\left|\frac{x_{n+1}}{x_n}\right| = \left|\frac{\frac{1}{(n+1)(n+2)}}{\frac{1}{n(n+1)}}\right| = \left|\frac{n}{n+2}\right| = \left|\frac{1}{1+\frac{2}{n}}\right|$$

$$\therefore \lim\left|\frac{x_{n+1}}{x_n}\right| = \left|\frac{1}{1+\frac{2}{n}}\right| = 1$$

$\therefore$ Ratio test fails $(\because r = 1)$

we know $n(n+1) > n.n$

$$\frac{1}{n(n+1)} < \frac{1}{n^2}$$

$$\therefore \frac{1}{n(n+1)} < \frac{1}{n^2} \quad (0 < x_n < y_n)$$

by comparison test

As $\sum \frac{1}{n^2}$ is convergent , $\sum \frac{1}{n(n+1)}$ is also convergent.

Example 28:

$\frac{n!}{n^n}$

Using raabies test , we have,

consider, $\left|\frac{x_{n+1}}{x_n}\right| = \frac{1}{\left(1+\frac{1}{n}\right)^n}$

$$\Rightarrow \left(1 - \frac{1}{\left(1+\frac{1}{n}\right)^n}\right)$$

$$= \frac{\left(1+\frac{1}{n}\right)^n - 1}{\left(1+\frac{1}{n}\right)^n}$$

$$= n\left(1 - \frac{1}{\left(1+\frac{1}{n}\right)^n}\right)$$

as $n \to \infty, \quad r = \lim n\left(1 - \left|\frac{x_{n+1}}{x_n}\right|\right) = 0 < 1$

$\therefore \sum x_n = \sum \frac{n!}{n^n}$ is not absoultely convergent. i.e divergent.

Example 29:

$$\frac{n^2}{\sqrt{n+1}} = \left|\frac{x_{n+1}}{x_n}\right| = \frac{(n+1)^2}{\sqrt{n+2}}.\frac{\sqrt{n+1}}{n^2}$$

$$= \left(1+\frac{1}{n}\right)^2 \frac{\sqrt{1+\frac{1}{n}}}{1-\frac{2}{n}}$$

$$\Rightarrow n\left(1-\left|\frac{x_{n+1}}{x_n}\right|\right)$$

$$= \frac{\sqrt{1+\frac{2}{n}}-\left(1+\frac{1}{n}\right)^2\sqrt{1+\frac{1}{n}}}{\sqrt{\frac{1}{n^2}+\frac{2}{n^3}}}$$

$\therefore \sum \frac{n^2}{\sqrt{n+1}}$ is not absloultely convergent. i.e divergent.

4.6 Test for Non-Absolute Convergence

***Definition* 4.6.1** (Alternative Series)**:** *A sequence of non-zero real numbers is said to be alternating if the terms* $(-1)^{(n+1)}x_n, \quad n \in \mathbb{N}$ *are all positive (or all negative) real numbers. If the sequence* x_n *is alternating, we say that the series* $\sum x_n$ *is alternating series.*

Theorem 4.6.1 (Alternating Series test)**.** *Let* z_n *be decreasing sequence with strictly positive numbers with* $\lim z_n = 0$ *then the alternating series* $\sum(-1)^{n+1}z_n$ *is convergent.*

Proof. Given that z_n decreasing sequence and let $S_n = \sum(-1)^{n+1}z_n$
We have
$S_{2n} = (z_1 - z_2) + (z_3 - z_4) + \ldots + (z_{2n-1} - z_{2n})$
and Since $(z_k - z_{k+1}) \geqslant 0$, it follows that S_{2n} is increasing sequence
$S_{2n} = z_1 - (z_2 - z_3) + \ldots - (z_{n-2} - z_{n-1}) - z_{2n}$
$\therefore S_{2n} \leqslant z_1$
$\therefore$ bounded by MCT, S_{2n} must be convergent to some number $c \in \mathbb{R}$.
We have to show that entire $S_n \to c$ if $\varepsilon > 0$, let $k \in \mathbb{N}$. if $n \geqslant k$
$|S_{2n} - c| \leqslant \dfrac{\varepsilon}{2}$ and $z_{2n+1} \leqslant \dfrac{\varepsilon}{2}$
$|S_{2n+1} - c| = |S_{2n} + z_{n+1} - c|$
$\leqslant |S_{2n} - c| + |z_{n+1}$

$\leqslant \dfrac{\varepsilon}{2} + \dfrac{\varepsilon}{2} = \varepsilon$
$S_n \to c$

$S_n = \sum(-1)^{n+1} z_n$ is convergent. □

Lemma 4.6.2 (Abels Lemma)**.** *$x_n, y_n \in \mathbb{R}$ $S_n = \sum_{i=1}^{n}$ with $S_0 = 0$ if $m > n$ then,*

$$\sum_{k=n+1}^{m} x_k y_k = (x_m S_m - x_{n+1} S_n) + \sum_{k=n+1}^{m-1} (x_k - x_{k+1}) S_k$$

Proof. $y_k = S_k - S_{k-1}$ $\left(\because S_k = \sum_{i=1}^{k} y_i \& S_{k-1} = \sum_{i=1}^{k-1} y_i\right)$

$x_k y_k = x_k S_k - x_k S_{k-1}$

$$\sum_{k=n+1}^{m} x_k y_k$$

$$= \sum_{k=n+1}^{m-1} (x_k S_k - x_k S_{k-1})$$

$$= x_{n+1}.S_{n+1} - x_{n+1} S_n + x_{n+2} S_{n+2} - x_{n+2} S_{n+1} + \ldots + x_m S_m - x_m S_{m-1}$$

$$= (x_m S_m - x_{n+1} S_n) + \sum_{k=n+1}^{m-1} (x_k - x_{k+1}) S_k$$ □

Theorem 4.6.3 (Diricblet's Test)**.** *If x_n is decreasing 0 , if $S_n = \sum y_i$ is bounded then $x_n y_n$ is convergent.*

Proof. Let $S_n \leqslant B, \quad \forall n \in \mathbb{N}$. if $m > n$, by abels lemma and $x_k - x_{k+1} > 0$ (as x_n is decreasing)
Consider,

$$\left|\sum_{k=n+1}^{m} x_k y_k\right|$$

$$= \left|(x_m S_m - x_{n+1} S_n) + \sum_{k=n+1}^{m-1} (x_k - x_{k+1}) S_k\right|$$

$$\leqslant |(x_m S_m - x_{n+1} S_n)| + \sum_{k=n+1}^{m-1} (x_k - x_{k+1}) |S_k|$$

Suppose,

$S_m, S_n, S_k = B$

$$\leqslant |x_m - x_{n+1}| B + B \sum_{k=n+1}^{m-1} (x_k - x_{k+1}) |S_k|$$

$$\leqslant \frac{\varepsilon}{2B} B + B \frac{\varepsilon}{2B}$$

$$\leqslant \varepsilon$$

$\therefore \sum x_n y_n$ is convergent. □

Theorem 4.6.4 (Abel's Test)**.** *If x_n convergent monotone sequence and y_n is convergent then the series is $x_n y_n$ also convergent.*

Proof. Let x_n is decreasing x
$u_n = x_n - x$ decreasing 0
$\sum u_n y_n$ is convergent by diricblets test
$$\sum_n x_n y_n$$
$$= \sum_n (x + u_n) y_n$$
$$= x \sum_n y_n + \sum_n u_n y_n$$
$\sum_n x_n y_n$ is convergent sequence. □

Example 30:
$\sum a_n$ convergent then

1. $\sum b_n = \dfrac{a_n}{n}$ is convergent sequence.
2. $\sum n^{1/n} a_n$ is divergent sequence.
3. $\sum a_n \sin n$ is divergent sequence.
4. $\sum \dfrac{\sqrt{a_n}}{n}$ is convergent sequence.
5. $\sum \sqrt{a_n}$ is divergent sequence.

Chapter 5

Function and Continuity

Definition* 5.0.1** (Cluster Point):*** *Let $A \subseteq \mathbb{R}$. A point $c \in \mathbb{R}$ is cluster point of A if every $\delta > 0$ $\exists$ atleast one point $x \in A$, $x \neq c \ni |x - c| < \delta$*

Theorem 5.0.1. *The number $c \in \mathbb{R}$, is cluster point of $A \subseteq \mathbb{R}$ if and only if $\exists$ sequence a_n in A such that* $\lim(a_n) = c$ *and $a_n \neq c, \quad \forall n$*

Proof. If c is cluster point of A then for any $n \in \mathbb{N}$ the $\dfrac{1}{n}$ neighbourhood $v_{1/n}(c)$ contains atleast one point a_n in A distinct from c, then $a_n \in A, \quad a_n \neq c$ & $|a_n - c| < \dfrac{1}{n} \Rightarrow \lim a_n = c$ conversly, if $\exists$ a sequence a_n in $A^{\setminus\{c\}}$ with $\lim(a_n) = c$, then for any $\delta > 0, \quad \exists$ k such that if $n \geqslant k$, then $a_n \in v_\delta(c)$. Therefor, δ neighbourhood $v_\delta(c)$ contains the point $a_n, \quad \forall n \geqslant k$ which belong to A and are distinct from c. □

Definition* 5.0.2** (Limit of Function):*** *Let $A \subseteq \mathbb{R}$ and let $c \in \mathbb{R}$ be the cluster point of A. for a function $f : A \to \mathbb{R}$ a real number L is said to be limit of f at c if, given any $\varepsilon > 0, \quad \exists \quad \delta > 0, \ni$*

$x \in A$ and $0 < |x - c| < \delta$ then $|f(x) - L| < \varepsilon$ then we say f converges to L at c.

Theorem 5.0.2. *If $f : A \to \mathbb{R}$ and if c is a cluster point of A, then f can have only one limit at c.*

Proof. We will prove this by method of contradiction.
Let L and L' be limits of f at c
For any $\varepsilon > 0$, $\exists \quad \delta\left(\frac{\varepsilon}{2}\right) > 0 \ni x \in A$ and $0 < |x - c| < \delta\left(\frac{\varepsilon}{2}\right)$

$\Rightarrow |f(x) - L| < \frac{\varepsilon}{2}$

Also, $\exists\, \delta'\left(\frac{\varepsilon}{2}\right) > 0 \quad \ni \quad x \in A$ and $|x - c| < \delta'\left(\frac{\varepsilon}{2}\right)$

$\Rightarrow |f(x) - L'| < \frac{\varepsilon}{2}$

$|L - L'|$
$= |L - f(x) + f(x) - L'|$
$\leqslant |L - f(x)| + |f(x) - L'|$

$\leqslant \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$

$\leqslant \varepsilon$
Since, $\varepsilon > 0$ is arbitary, $L = L'$ □

Theorem 5.0.3 (Sequential Criterion)**.** *Let $f : A \to \mathbb{R}$ and let c be a cluster point of A then the following are equivalent.*

1. $\lim\limits_{x \to c} f(x) = L$
2. *for every x_n in A, $x_n \to c$, $x_n \neq c$, $\quad \forall n \in \mathbb{N} \Rightarrow f(x_n) \to L$.*

***Definition* 5.0.3** (Divergence Criterion)**:** *Let $A \subseteq \mathbb{R}$ let $f : A \to \mathbb{R}$ and let $c \in \mathbb{R}$ be cluster point of A.*

a) *If $L \in \mathbb{R}$ then f does not have limit L at c iff $\quad \exists$ sequence x_n in A with $x_n \neq c$, $\quad \forall n \in \mathbb{N}$ such that sequence x_n converges to c. but the sequence $f(x_n)$ does not converges to L*

b) *the function does not have a limit L at c iff $\quad \exists\, x_n$ in A with $x_n \neq c$, $\quad \forall n \in \mathbb{N}$ such that the sequence x_n converges to c but the sequence $f(x_n)$ does not converges in $\mathbb{R}$*

f(x)=

$$f(x) = \begin{cases} +1 & if\, x > 0 \\ -0 & if\, x = 0 \\ -1 & if\, x < 0 \end{cases}$$

Theorem 5.0.4 (Limit Theorem)**.** *Let $A \subseteq \mathbb{R}$. and $c \in \mathbb{R}$, be cluster point of A we say that f is bounded on neighbourhood of c if $\exists$ a δ neighbourhood of $v_\delta(c)$ of c and constant $M > 0 \ni$ $|f(x)| \leqslant M \quad \forall x \in A \cap V_\delta(c)$*

Theorem 5.0.5. *If $A \subseteq \mathbb{R}$ and $f : A \to \mathbb{R}$ has limit at $c \in \mathbb{R}$ then f is bounded on some neighbourhood of c*

Proof. If $L = \lim_{x \to c} f$ then for $\varepsilon = 1, \exists\, \delta_c < 0$
Such that $0 < |x - c| < \delta \Rightarrow |f(x) - L| < 1$
$|f(x)| - |L| \leqslant |f(x) - L| < 1$
if $x \in A \cap V_\delta(c), \quad x \neq c$ then,
$|f(x)| < |L| + 1$
if $c \notin A, \quad$ Take $M = |L| + 1$
while if $c \in A$, Take $M = Sup\{|f(x)|, |L| + 1\}$
$\therefore |f(x)| \leqslant M$
$\therefore$ by limit theorem
$\therefore f$ is bounded on neighbourhood of c. □

***Definition* 5.0.4:** *Let $A \subseteq \mathbb{R}$ and let f & g be function defined on A to $\mathbb{R}$. We define the sum $f + g$, the differnce $f - g$ and the product $f.g$ on $A \to \mathbb{R}$ to be function from A to $\mathbb{R}$ given by ,*
$(f + g)(x) = f(x) + g(x)$
$(f - g)(x) = f(x) - g(x)$
$(f.g)(x) = f(x).g(x)$
Further if $b \in \mathbb{R}$
$(bf)(x) = b.f(x)$
finally, if $h(x) =\neq 0$,

$$\left(\frac{f}{h}\right)(x) = \frac{f(x)}{h(x)}$$

Theorem 5.0.6. *Let $A \subseteq \mathbb{R}$ let f & g be function on $A \to \mathbb{R}$ and le $c \in \mathbb{R}$ be a cluster point of $A \to \mathbb{R}$ & let*

1. *If* $\lim\limits_{x\to c} f = L$ & $\lim\limits_{x\to c} g = M$ *then*

   $$\lim_{x\to c}(f \pm g) = L \pm M$$

   $$\lim_{x\to c}(f \cdot g) = L \cdot M$$

   $$\lim_{x\to c}(b \cdot f) = b \cdot L$$

2. $\lim\limits_{x\to c}\left(\frac{f}{c}\right) = \frac{L}{H}$
   where, $h(x) =\neq 0$ *and* $\lim\limits_{x\to c} h(x) = H \neq 0$

Theorem 5.0.7. *Let $A \subseteq \mathbb{R}$ let $f : A \to \mathbb{R}$ and let $c \in \mathbb{R}$ be the cluster point of A.*
if $a \leqslant f(x) \leqslant b\ \forall x \in A, x \neq c$ and if $\lim\limits_{x\to c} f$ *exists then* $a \leqslant \lim\limits_{x\to c} f \leqslant b$

Proof. Given, $f : A \to \mathbb{R}$ and c is cluster point of A.
let $x_n \in A$ such that $x_n \to c$

$\therefore f(x_n) \to L = \lim_{x\to c} f(x_n) = \lim_{x\to c} f(x)$

Also,

$a \leqslant f(x) \leqslant b$

$a \leqslant f(x_n) \leqslant b$

$a \leqslant \lim_{x\to c} f(x_n) \leqslant b$

$a \leqslant \lim_{x\to c} f(x) \leqslant b$

$a \leqslant L \leqslant b$ □

Theorem 5.0.8 (Squeez Theorem)**.** *Let* $A \subseteq \mathbb{R}$ *let* $f, g, h :\to \mathbb{R}$ *&* $c \in \mathbb{R}$ *be a cluster point of* A*. If* $f(x) \leqslant g(x) \leqslant h(x), \quad \forall x \in A, x \neq c$ *&* $\lim_{x\to c} f = \lim_{x\to c} h$ *then,* $\lim_{x\to c} g = L$*.*

Proof. Given, $f, g, h :\to \mathbb{R}$ & c is cluster point of A $x_n \in A \Rightarrow x_n \to c$

$f(x_n) \to L = \lim_{x\to c} f(x_n) = \lim_{x\to c} f(x) = \lim_{x\to c} h(x) = \lim_{x\to c} h(x_n)$

i.e $f(x_n) \to L$ & $h(x_n) \to L$

Also,

$f(x) \leqslant g(x) \leqslant h(x)$

$f(x_n) \leqslant g(x_n) \leqslant h(x_n)$

$$\lim_{x_n\to c} f(x_n) \leqslant \lim_{x_n\to c} g(x_n) \leqslant \lim_{x_n\to c} h(x_n)$$

$$\therefore L \leqslant \lim_{x\to c} g(x_n) \leqslant L$$

$\lim_{x\to c} g(x_n) = L$

i.e $g(x_n) \to L$

i.e $\lim_{x\to c} g = L$
-Hence Proved- $\square$

***Definition* 5.0.5:** *Let $A \in \mathbb{R}$ & let $f L A \to \mathbb{R}$*

1. *If $c \in \mathbb{R}$ is a cluster point of the set $A \cap (c,\infty) = \{x \in A, \quad x > c\}$ then we say that $L \in \mathbb{R}$ is right hand limit of f at c*
   $\lim_{x\to c^+} f(x) = L$ *If given any* $\varepsilon > 0 \quad \exists \quad \delta(\varepsilon) > 0 \quad \ni \quad \forall x \in A$ *with* $0 < x - c < \delta$ *then* $|f(x) - L| < \varepsilon$
2. *If $c \in \mathbb{R}$ is a cluster point of the set $A \cap (-\infty, 0) = \{x \in A, \quad x < c\}$ then we say that $L \in \mathbb{R}$ is left hand limit of f at c*
   $\lim_{x\to c^-} f(x) = L$ *If given any* $\varepsilon > 0 \quad \exists \quad \delta(\varepsilon) > 0 \quad \ni \quad \forall x \in A$ *with* $0 < -x + c < \delta$ *then* $|f(x) - L| < \varepsilon$

5.1 Continuous Function

***Definition* 5.1.1** (Continuous Function)**:** *Let $A \subseteq \mathbb{R}$ let $f : A \to \mathbb{R}$ & let $c \in A$ we say that f is continuous at c if given any $\varepsilon > 0, \quad \exists \quad \delta(\varepsilon) > 0 \quad \ni$ if x is any point of A satisfying $|x - c| < \delta$, then $|f(x) - f(c)| < \varepsilon$ if f fails to be continuous at c then we say that f is discountinous at c*

Theorem 5.1.1. *A function $f : A \to \mathbb{R}$ is continuous at point $c \in A$ if and only if given any $\varepsilon > 0$, $v_\varepsilon(f(c))$ of $f(c)$ $\exists$ of c such tat if x is any point of $A \cap v_\delta(c)$ then $f(x) \in v_\varepsilon(f(c))$ i.e $A \cap v_\delta(c) \subseteq v_\varepsilon(f(c))$*

Proof. $\therefore \lim\limits_{x \to c} = L$
i.e any $\varepsilon > 0, \quad \exists \quad \delta(\varepsilon) > 0 \quad \ni$
$|x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon$
and $\lim f(x) = f(c)$
for any $\varepsilon > 0, \quad \exists \quad \delta(\varepsilon) > 0 \quad \ni$
$|x - c| < \delta, \Rightarrow |f(x) - f(c)| < \varepsilon$
$\therefore x \in A \cap v_\delta(c) \Rightarrow f(x) \in v_\varepsilon(f(c)), \quad \forall x$
$\therefore f(A \cap v_\delta(c)) \subseteq v_\varepsilon(f(c)) \ldots (\because$ if A $x \in A \Rightarrow x \in B$ then $A \subseteq B)$ □

***Definition* 5.1.2** (Cobmbination of Continuous function)**:** *Let $A \subseteq \mathbb{R}$. Let f & g be function on A to $\mathbb{R}$, let $b \in \mathbb{R}$, Suppose that $c \in A$ & that f & g are continuous at c*

a) then $f + g$, $f - g$, $f \cdot g$ and $b \cdot f$ are contiunous at c

b) if $h : A \to \mathbb{R}$ is continuous at $c \in A$ & if $h(x) \neq 0, \quad x \in A$, then $\left(\frac{f}{h}\right)$ is also continuous at c

***Definition* 5.1.3** (Continuous Point)**:** *Let $A \subseteq \mathbb{R}$ & $f : A \to \mathbb{R}$. if $B \subseteq A$ we say that f is continuous on set B if f is continuous at every point of B*

Example 31:
Continuous

- $f(x) = x, \quad x \in \mathbb{R}$
- $f(x) = x^2, \quad x \in \mathbb{R}$
- $f(x) = \frac{1}{x}, \quad x \in \mathbb{R}^+, \{0\}$
- $f(x) =$ Polynomial function $\quad x \in \mathbb{R}$
- $f(x) =$ Rational function
- $f(x) =$ Trignometric function
- $f(x) = \sqrt{f}, \quad x \in \mathbb{R}$

Example 32:
Discontinuous

- $\psi(x) = \frac{1}{x}, \quad x = 0$
- $\psi(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \in \mathbb{Q}^c \\ \text{discount everywhere} \end{cases}$

- $\sin(x) = \begin{cases} 1, & x > 0 \\ 0, & x < 0 \\ 0, & x = 0 \\ \text{discount at } x = 0 \end{cases}$
- $\psi(x) = [x] =$ greatest integer function discount at integer

Theorem 5.1.2. *Let $A \subseteq \mathbb{R}$. Let $f : A \to \mathbb{R}$ & let $|f|$ be defined by $|f|x = |f(x)| \quad \forall x \in A$*

1. *If f is continuous at point $c \in A$ then $|f|$ is countinuous at c*
2. *If f is continuous on A then $|f|$ is continous on A.*

Theorem 5.1.3. *Let $A, B \in \mathbb{R}$ & let $f : A \to \mathbb{R}$ & $g : B \to \mathbb{R}$ be function such that $f(A) \subseteq B$ if f is countinuous at point $c \in A$ and g is continuous at $b = f(c) \in B$ then the composition $g \circ f : A \to \mathbb{R}$ is continuous at c.*

Proof. Let W be $\varepsilon-$ neighbourhood of $g(b)$. since g is continuous at b there is a $\delta-$ neighbourhood of v of $b = f(c)$ such that if $y \in B \cap v$ then $g(y) \in W$. Since f is also continuous at c, ther is a v-neighbourhood v of $c \ni \quad \in A \cap U$ then $f(x) \in v$
Since $f(A) \subseteq B$, it follows that if $x \in A \cap U$ then $f(x) \in B \cap v$ so that $g \circ f(x) = g(f(x)) \in W$ But, Since W is an arbitary $\varepsilon - neighbourhood$ of $g(b)$ this implies $g \circ f$ is continuous at c. □

5.2 Continuous function on Interval

***Definition* 5.2.1** (Bounded Function)**:** *A function $f : A \to \mathbb{R}$ is said to be bounded on A if $\exists$ a constant $M > 0$ such that $|f(x)| \leqslant M$ for all $x \in A$*

Theorem 5.2.1 (Boundedness Theorem-)**.** *Let $I = [a, b]$ be a closed bounded interval and let $f : I \to \mathbb{R}$ be continuous on I then f is bounded on I.*

Proof. Suppose f is bounded on I.
then, for any $n \in \mathbb{N}, \quad \exists \quad x_n \in I \quad \ni |f(x_n)| > k$.
Since, I is bounded, sequence x_n is bounded.
$\therefore$ By Bolzano weistress theorem,
$\exists$ subsequence x_{nk} that converges to some x
Since, I is closed, elements of sequence $x_{nk} \in I \Rightarrow x \in I$.
then, f is continuous at x so that $f(x_{nk})$ converges to $f(x)$.
$\Rightarrow |f(x_{nk})| > n_k > k \quad \forall k \in \mathbb{N}$
$\therefore$ Our assumption is wrong.
Hence, f must be bounded. □

***Definition* 5.2.2** (Absolute Extremum)**:** *Let $A \subseteq \mathbb{R}$ and let $f : A \to \mathbb{R}$.We say that f has an absolute maximum on A if there is $x^* \in A$ such that*
$f(x^*) \geqslant f(x), \quad \forall x \in A$
We say that f has absolute minimum on A if there is $x^* \in A$ such that
$f(x^*) \geqslant f(x), \quad \forall x \in A$

Theorem 5.2.2 (Maximum-Minimum Theorem). *Let $I = [a, b]$ be closed bounded interval and let $f : I \to \mathbb{R}$ be continuous on I then f has an absolute maximum and absolute minimum on I.*

Proof. $f(I) = \{f(x); \quad x \in I\}$
I is a closed bounded and f is continuous on I then $f(x)$ is also bounded is $\subseteq \mathbb{R}$
$\therefore$ By completness property, it has suremum and infimum
$\therefore S^* = Sup\{f(I)\}, \quad S_* = Inf\{f(I)\}$
claim- To show , $\exists x^*, x_* \in I$
$\ni S^* = f(x^*) =$ absolute maximum
$S_* = f(x_*) =$ absolute minimum
$S_* = Inf\{f(I)\}$
if $n \in \mathbb{N}$ then $S^* - \dfrac{1}{n}$ is not upper bound
$\therefore S^* - \dfrac{1}{n} < f(x_n) < S^*, \quad \forall n \in \mathbb{N}$
Since, I is bounded x_n is bounded By Bolzano weistress theorem,
$\exists \quad x_{n_k}$ subsequence of x_n and $x_{n_k} \to somex^*$
Also, As I is closed and $x_{n_k} \in I \Rightarrow x^*$ must be in I
$\Rightarrow f$ is continuous at $x^*, \lim f(x_{n_k}) = f(x^*)$
$S^* - \dfrac{1}{n} < f(x_{n_r}) \leqslant S^*, \quad \forall r \in \mathbb{N}$
$\therefore$ by squeeze theorem

$\lim f(x_{n_r}) = S^*$
$\therefore S^* = f(x^*)$ i.e $f(x^*) \geqslant f(x), \quad \forall x$
$\therefore x^*$ is absolute maximum
Similarly, we show x_* is absolute minimum □

Theorem 5.2.3 (Location of Root)**.** *Let $I = [a,b]$ & let $f : I \to \mathbb{R}$ be continuous on I. If $f(a) < 0 f(b)$ or $f(b) < 0 < f(a)$, then* $\quad \exists \quad c \in (a,b) \ni f(c) = 0.$

Proof. Assume that $f(a) < 0 f(b)$
Let $I_1 = [a_1, b_1]$ where, $\quad a_1 = a, b_1 = b$
let $P_= \dfrac{a+b}{2}$ if $f(P_1) = 0$ then $c = P_1$
if $P_1 \neq 0$, then wither $f(P_1) > 0$ or $f(P_1) < 0$
if $f(P_1) > 0$ then $a_2 = a_1, b_2 = P_1$ and if $f(P_1) < 0$
$a_2 = P_1, b_2 = b_1$ thus , we get $I = [a_2, b_2] \in I_1$
continuing this bisectins, we obtain intervals $I_1, I_2, \dots I_k$
In this process, we terminate by locating a point $P_n \quad \ni \quad f(P_n) = 0$
if process does not terminate, we obtain nested sequence of bounded interval
$I_n = [a_n, b_n]$
$\ni f(a_n) < 0$ & $f(b_n) > 0$
& length of interval $b_n - a_n = \dfrac{(b-a)}{2^{n-1}}$
$\Rightarrow \exists \quad a$ point $c \in I_n \quad \forall n \in \mathbb{N}$
$a_n \leqslant c \leqslant b_n, \quad \forall n \in \mathbb{N}$

$$\Rightarrow 0 \leqslant c - a_n \leqslant b_n - a_n$$

$$\Rightarrow 0 \leqslant c - a_n \leqslant \frac{(b-a)}{2^{n-1}}$$
$$\Rightarrow \lim f(a_n) = \lim f(b_n) = f(c)$$
$$\Rightarrow 0 \leqslant b_n - c \leqslant b_n - a_n$$

$$\Rightarrow 0 \leqslant b_n - c \leqslant \frac{(b-a)}{2^{n-1}}$$

□

Theorem 5.2.4 (Bolzano's Intermediate Theorem)**.** *Let I be an interval and let $f : I \to \mathbb{R}$ be continuous on I if $a, b \in I$ and if $k \in \mathbb{R}$ satisfies $f(a) < k < f(b)$ then a point $c \in I$ between a & $b \ni f(c) = k$.*

Proof. 1. Assume that, $a < b, a, b, \in I, f$ continuous on I

Define $g(x) = f(x) - k$

As $f(x)$ is continuous, $g(x)$ is also continuous on I

Also, $f(a) < k < f(b)$

$f(a) - k < 0 < f(b) - k$

$g(a) < 0 < g(b)$

$\therefore$ by location of root theorem

$\exists c \ni g(c) = 0$

i.e $f(c) - k = 0$

$\therefore f(c) = k$

2. Assume that, $a > b, a, b, \in I, f$ continuous on I
   Define $h(x) = k - f(x)$
   As $f(x)$ is continuous, $h(x)$ is also continuous on I
   Also, $f(a) < k < f(b)$
   $k - f(a) < 0 < k - f(b)$
   $h(a) < 0 < h(b)$
   $\therefore$ by location of root theorem
   $\exists c \ni h(c) = 0$
   i.e $k - f(c) = 0$
   $\therefore f(c) = k$

□

Corollary 5.2.4.1. *Let $I - [a, b]$ be a closed bounded interval. Let $f : I \to \mathbb{R}$ be continuous on I if $k \in \mathbb{R}$ is any number satisfying $Inf\, f(I) \leqslant k \leqslant Sup\, f(I)$ then $\exists$ a number $c \in I \ni f(c) = k$*

Proof. Given that, I is a closed bounded interval and $f : I \to \mathbb{R}$ is continuous on I
$\therefore$ By maximum- minimum theorem,
$\exists \quad x^*, x_* \in I$ such that $f(x^*) = Sup\{f(I)\}$
$f(x_*) = Inf\{f(I)\}$

Also, Given that, $Inf f(I) \leqslant k \leqslant Sup f(I)$
i.e $f(x^*) \leqslant k \leqslant f(x_*)$
$\therefore$ by Bolzano intermediate theorem,
$\exists \quad c \in I \quad \ni \quad f(c) = k$
-Hence Proved- $\square$

Theorem 5.2.5. *Let I be closed bounded interval and let $f : I \to \mathbb{R}$ be continuous on I then, the set $f(I) = \{f(x) : x \in I\}$ be closed bounded interval.*

Proof. let ,
$m = Inf\{f(I)\}$
$M = Sup\{f(I)\}$
by maximum - minimum theorem, $m, M \in f(I)$
$f(I) \subseteq [m, M]$
if $k \in [m, M]$
$\therefore$ by bolzano-itermediate theorem
$\exists \quad c \in I, \quad f(c) = I$
Hence, $k \in f(I)$
$\Rightarrow [m, M] \subseteq f(I)$
$\therefore$, $f(I)$ is the interval $m, M]$ $\square$

5.3 Continuity

***Definition* 5.3.1** (Uniform Continuous)**:** *Let $A \subseteq \mathbb{R}$ & let $f : A \to \mathbb{R}$. We say that f is uniformly continuous on A if for each $\varepsilon > 0$, $\exists \quad \delta(\varepsilon) > 0 \quad \ni \quad$ if $\quad x, y, \in A$ are any numbers satisfying $|x - y| < \delta(\varepsilon) \Rightarrow |f(x) - f(y)| < \varepsilon$*

***Definition* 5.3.2** (Non- Uniform Continuity)**:** *Let $A \subseteq \mathbb{R}$ & let $f : A \to \mathbb{R}$ then follwoing statements are equivalent.*

i) *f is not uniformly continuous on A.*

ii) $\exists \quad a_n \quad \varepsilon_0 > 0 \quad \ni$ *for every $\delta > 0$ there are points x_δ, y_δ in A such that,* $|x_\delta - y_\delta| < \delta$ *and* $|f(x_\delta) - f(y_\delta)| \geqslant \varepsilon_0$

iii) $\exists \quad a_n \quad \varepsilon_0 > 0$ *and two sequence x_n & y_n in A such that* $\lim x_n - y_n = 0$ *and* $|f(x_n) - f(y_n)| \geqslant \varepsilon_0, \quad \forall n \in \mathbb{N}$

Theorem 5.3.1 (Uniform Continuity Theorem)**.** *Let I be closed bounded interval and let $f : I \to \mathbb{R}$ be continuous on I then f is uniform continuous on I.*

Proof. If f is not uniform continuous on I then,
$\exists \quad \varepsilon_0 > 0$ and two seqeunce $x_n, y_n \in I$
$|x_n - y_n| < \dfrac{1}{n}$ & $|f(x_n) - f(y_n)| \geqslant \varepsilon_0$
Since I is bounded x_n, y_n are bounded.
$\exists \quad$ subsequnece x_{n_k} of x_n that converges to some elements $z \in I$ (as I closed) as

$|x_n - y_n| < \dfrac{1}{n} \quad \forall n$
Subsequence y_{n_k} of y_n also converges to z
$|y_{n_k} - z|$
$= |y_{n_k} - x_{n_k} + x_{n_k} - z|$
$\leqslant |y_{n_k} - x_{n_k}| + |x_{n_k} - z|$
$\therefore y_{n_k}$ is also converges to z
Now if f is continuous at z both $f(x_{n_k})$ and $f(y_{n_k})$ must converges $f(z)$
But this not possible as $|f(x_n) - f(y_n)| \geqslant \varepsilon_0$
$\therefore$ Our assumption is wrong. □

***Definition* 5.3.3** (Lipschitz Function)**:** *Let $A \subseteq \mathbb{R}$ and $f : A \to \mathbb{R}$ if there exists a constant $k > 0$ such that*
$|f(x) - f(u)| < k|x - u| \quad \forall x, u \in A$ *then f is said to be a Lipschitz function on A*

Theorem 5.3.2. *Lipschitz function is an uniformly continuous function always.*

Proof. for Lipschitz function
$|f(x) - f(u)| < k|x - u|$
Now, $|x - u| < \dfrac{\varepsilon}{k} = \delta, \quad | < \dfrac{\varepsilon}{k} > 0 ask > 0$
$|f(x) - f(u)| < k.\dfrac{\varepsilon}{k}$
$< \varepsilon$
$|x - u| < \delta \Rightarrow |f(x) - f(u)| < \varepsilon$

Lipschitz function is always uniformly continuous function. □

Theorem 5.3.3. *If $f : A \to \mathbb{R}$ is uniformly continuous on subset A of $\mathbb{R}$ and if x_n is a cauchy sequence in A, then $f(x_n)$ is cauchy sequence in $\mathbb{R}$.*

Proof. let x_n is a cauchy sequence in A and let $\varepsilon > 0$ choose $x, y \in A, \quad \delta > 0$
$|x - u| < \delta \Rightarrow |f(x) - f(u)| < \varepsilon$
Since, x_n is a cauchy sequence $\quad \exists H(\delta)$
$|x_n - x_m| < \delta, \quad \forall n, m \geqslant H(\delta)$
(as f is uniformly continuous)
$|f(x_n) - f(x_m)| < \varepsilon$
Therefore, the sequence $f(x_n)$ is cauchy sequence. □

Theorem 5.3.4 (Continuous Extension Theorem)**.** *A function f is uniformly continuous on (a, b) iff it can be defined at the end points a & b such that the extended function is continuous on $[a, b]$.*

Proof. Assume that function f is continuous on $[a, b]$
$\therefore$ by defination,
for any $\varepsilon > 0, \quad \dfrac{\varepsilon}{2} > 0, \quad \exists \quad \delta(\varepsilon) > 0$ such that,
$|x - c| < \delta \Rightarrow |f(x) - f(c)| < \varepsilon$
let x_1 & $x_2 \quad \in [a, b]$
by defination,

for any $\varepsilon > 0$, $\quad \dfrac{\varepsilon}{2} > 0, \quad \exists \quad \delta(\varepsilon) > 0$ such that,

$|x_1 - c| < \dfrac{\delta}{2} \Rightarrow |f(x_1) - f(c)| < \dfrac{\varepsilon}{2}$

and,

for any $\varepsilon > 0$, $\quad \dfrac{\varepsilon}{2} > 0, \quad \exists \quad \delta(\varepsilon) > 0$ such that,

$|x_2 - c| < \dfrac{\delta}{2} \Rightarrow |f(x_2) - f(c)| < \dfrac{\varepsilon}{2}$

consider,

$|x_1 - x_2|$

$= |x_1 - c + c - x_2|$

$\leqslant |x_1 - c| + |x_2 - c|$

$\leqslant \dfrac{\delta}{2} + \dfrac{\delta}{2}$

$\leqslant \delta$

and,

$|f(x_1) - f(x_2)|$

$= |f(x_1) - f(c) + f(c) - f(x_2)|$

$\leqslant |f(x_1) - f(c)| + |f(x_2) - f(c)|$

$\leqslant \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$

$\leqslant \varepsilon$
$\therefore |x_1 - x_2| \leqslant \delta \Rightarrow |f(x_1) - f(x_2)| \leqslant \varepsilon$
$\therefore f$ is uniformly continuous on (a, b)
Conversly, Suppose f is uniformly continuous on (a, b). Lets define $f(a)$ & $f(b)$
Lets x_n be sequence in $(a, b) \quad \ni \quad \lim x_n = a$
$\Rightarrow x_n$ is cauchy sequence and as f is uniformly continuous on (a, b) and $x_n \in (a, b)$
$\therefore$ by sequential criteria , $\lim f(x_n) = L$ exists if y_n is any other sequence in (a, b) that converges to a then
$\lim x_n - y_n = a - a = 0$
$\lim f(y_n) = \lim(f(y_n) - f(x_n) + f(x_n)) = L$
So we define, $L = f(a)$
then f is continuous at a
Similarly, we can find some $M = f(b)$ and we can say that f is continuous on extended $[a, b]$ □

***Definition* 5.3.4** (Step Function)**:** *$I \subseteq \mathbb{R}$ be an interval and let $S : I \to \mathbb{R}$ then S is called a step function if it has only a finite number of distinct values.*

5.4 Continuity And Gauges

Definition* 5.4.1** (Partition):*** *A partition of an interval* $I = [a,b]$ *is collection* $P = \{I_1, I_2, \dots I_n\}$ *of non-over-lapping closed intervals whose union is* $[a,b]$. *We generally denote* $I_i = [x_{i-1}, x_i]$ *where* $a = x_0 < \dots < x_{i-1} < x_i < \dots < x_n = b$
The points $x_i \quad (i = 0,1,2,\dots n)$ *are called the partition points of* p. *If a point* t_i *has been choosen from each interval* I_i, *for* $(i = 0,1,2,\dots n)$ *then the points* t_i *are called tages and set of ordered pairs* $\dot{p} = \{(I_1, t_1), (I_2, t_2), \dots (I_n, t_n)\}$ *is called as tagged partition o I*

Definition* 5.4.2*: *A gauge on* I *is a strictly positive function defined on* I. *if* δ *is a gauge on* I, *then a tagged partition* $\dot{p}$ *is said to be* $\delta - fine$ *if*
$t_i \in I_i \subseteq [t_i - \delta(t_i), t_i + \delta(t_i)]$
If a partition p of $I = [a,b]$ *is a* $\delta - fine$ & $x \in I$, *then* $\exists$ *a tag* t_i *in* p *such that* $|x - t_i| \leqslant \delta(t_i)$

Alternative proof of Boundedness Theorem

Proof. Since f is continuous on I, then for each $t \in I \quad \exists \quad \delta(t) > 0 \quad \ni \quad$ if $x \in I$ and $|x - t| < \delta(t)$ then $|f(x) - f(t)| < 1$
Thus, $\delta - gauge on I$ let $\{(I_i, t_i)\}_{i=1}^{n}$ be $\delta - fine$ partition on I and let
$k = max\{|f(t_i)| \quad i = 1,2,\dots n\}$
Given any $x \in I \quad \exists \quad i$with $\quad |x - t_i| \leqslant \delta(t_i)$
$|f(x)|$
$= |f(x) - f(t_i) + f(t_i)|$

$\leqslant 1+k$

Since $x \in I$ is arbitary, f is bounded. □

***Definition* 5.4.3** (Monotone and Inverse Function)**:** *If $A \subseteq \mathbb{R}$, then a function $f : A \to \mathbb{R}$ is said to be increasing on A if whenver $x_1, x_2 \in A$ and $x_1 < x_2$ then $f(x_1) \leqslant f(x_2)$*
if $x_1, x_2 \in A$ and $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$ then f is called strictly increasing function.
Similarly, for decreasing fucntion,
$x_1 < x_2$ then $f(x_1) \geqslant f(x_2)$ and strictly decreasing function
$x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$

Theorem 5.4.1. *Let $I \subseteq \mathbb{R}$ be an interval and let $f : I \to \mathbb{R}$ be increasing on I. Suppose $c \in I$ is not endpoint of I then ,*

1. $\lim_{x \to c^-} = Sup\{f(x)/x \in I, x < c\}$
2. $\lim_{x \to c^+} = Sup\{f(x)/x \in I, x > c\}$

Proof. 1. Let $x \in I$ & $x < c \Rightarrow f(x) < f(c)$
So, for set $\{f(x)/x \in I, x > c\}$, $f(c)$ is uppeer bound, So by completeness property,
$\exists$ Supremum, say L.
if $\varepsilon > 0$, then $L - \varepsilon$ is not upper bound
Hence, $\exists \quad y_\varepsilon \in I, \quad y_\varepsilon < c$
$\ni \quad L - \varepsilon < f(y_\varepsilon) \leqslant L$
Since, f is increasing, if $\delta_\varepsilon = c - y_\varepsilon$ and if

$0 < c - y < \delta_c$ then
$y_\varepsilon < y < c$
So that, $t - \varepsilon < f(y_\varepsilon) \leqslant f(y) \leqslant L$
$\Rightarrow |f(y) - L| < \varepsilon$ when $0 < c - y < \delta_c$
Similarly we can prove (ii)

□

Theorem 5.4.2 (Continuous Inverse Function)**.** *Let $I \subseteq \mathbb{R}$ be an interval and let $f : I \to \mathbb{R}$ be strictly monotone and continuous on I then the function $g - inverse$ to f is strictly monotone and continuous on $I = f(I)$*

Proof. Let f is strictly increasing
Since f is continuous on I
By preservation of interval theorem,
$J = f(I)$ is also an interval. Also,
$f : I \to \mathbb{R}$ is strictly monotone and injective on I, threfore, inverse function $g : J \to \mathbb{R}$ exists if
$y_1, y_2 \in J$ with $y_1 < y_2$ then
$y_1 = f(x_1), \quad y_2 = f(x_2)$ for some $x_1, x_2 \in I$
$\Rightarrow x_1 < x_2$ as function is increasing
$\Rightarrow x_1 = g(y_1) < g(y_2) = x_2$
Since, y_1, y_2 arbitary elements of J with
$y_1 < y_2$, we conclude that g is strictly increasing on J.
Now, we have to show that g is continuous on J.

As $g(J) = I$ is an interval.
Indeed, if g is discontinuous at a point $c \in J$, then the jump at c is non-zero so that $\lim_{y\to c^-} g < \lim_{y\to c^+} g$
if we choose any number $x \neq g(c)$ satisfying $\lim_{y\to c^-} g < x \lim_{y\to c^+} g$
then, $x \neq g(y), \quad$ for any $y \in J$
Hence, $x \notin I$ which contradicts to our given condition that I is interval.
$\therefore$ The inverse function g is continuous on J. $\square$

Chapter 6

Differentiation

6.1 Derivative

***Definition* 6.1.1** (Derivative)**:** *Let $I \subseteq \mathbb{R}$ be an interval. let $f : I \to \mathbb{R}$ and let $c \in I$. We say that a real number L is derivative of f at c if given any $\varepsilon > 0 \quad \exists \quad \delta(\varepsilon) > 0 \quad \ni$ if $x \in I$ satisfies*
$0 < |x - c| < \delta \Rightarrow \left| \dfrac{f(x) - f(c)}{x - c} - L \right| < \varepsilon$

We say, f is differntiable at c.
$f'(c) = \lim\limits_{x \to c} \dfrac{f(x) - f(c)}{x - c}$

Theorem 6.1.1. *If $f : I \to \mathbb{R}$ has a derivative at $c \in I$, then f is continuous at c.*

Proof. $\forall \quad x \in I, x \neq c$
$f(x) - f(c) = \dfrac{f(x) - f(c)}{x - c}(x - c)$

$$\lim_{x\to c} f(x) - f(c)$$

$$= \lim_{x\to c} \frac{f(x) - f(c)}{x - c}(x - c)$$

$$= f'(c).0$$
$$= 0$$
$$\lim_{x\to c} f(x) = f(c)$$

$\therefore f$ is continuous at point c

if $f : I \to \mathbb{R}$ is continuous at point c then f may or may not be derivable at c. $\square$

Example 33:

$f(x) = |x|$ is continuous at 0 but not differntiable at 0.

Theorem 6.1.2. *Let $I \subseteq \mathbb{R}$ be an interval, let $c \in I$ & let $f : I \to \mathbb{R}$ & $g : I \to \mathbb{R}$ be function that are diffrentiable at c then*

a) $(\alpha f)'(c) = \alpha f'(c), \quad \alpha \in \mathbb{R}$

b) $(f + g)'(c) = f'(c) + g'(c)$

c) $(fg)'(c) = f'(c)g(c) + f(c)g'(c)$

d) $\left(\frac{f}{c}\right)'(c) = \frac{f'(c)g(c) - f(c)g'(c)}{(g(c))^2} \quad (g(c) \neq 0)$

Proof. a) $(\alpha f)'(c)$

$$= \lim_{x\to c} \frac{(\alpha f)(x) - (\alpha f)(c)}{x-c}$$

$$= \lim_{x\to c} \frac{\alpha f(x) - \alpha f(c)}{x-c}$$

$$= \alpha \lim_{x\to c} \frac{f(x) - f(c)}{x-c}$$

$$\underline{(\alpha f)'(c) = \alpha f'(c)}$$

b) $\lim_{x\to c} \frac{(f+g)(x) - (f+g)(c)}{x-c} = (f+g)'(c)$

$\therefore (f+g)'(c)$

$$= \lim_{x\to c} \frac{(f(x)+g(x)) - (f(c)+g(c))}{x-c}$$

$$= \lim_{x\to c} \frac{(f(x)-f(c)) - (g(x)+g(c))}{x-c}$$

$$= \lim_{x\to c}\left\{\frac{f(x)-f(c)}{x-c}\right\} + \lim_{x\to c}\left\{\frac{g(x)-g(c)}{x-c}\right\}$$

$$\underline{(f+g)'(c) = f'(c) + g'(c)}$$

c) Let $h(x) = fg(x)$

$$h'(c) = \lim_{x\to c}\frac{h(x)-h(c)}{x-c}$$

$$(fg)'(c)$$

$$= \lim_{x\to c}\frac{fg(x)-fg(c)}{x-c}$$

$$= \lim_{x\to c}\frac{f(x).g(x)-f(c).g(c)}{x-c}$$

$$= \lim_{x\to c}\frac{f(x).g(x)-f(c).g(x)+f(c).g(x)-f(c).g(c)}{x-c}$$

$$= \lim_{x\to c}\frac{(f(x)-f(c))g(x)+f(c).(g(x)-g(c))}{x-c}$$

$$= \lim_{x\to c}\frac{f(x)-f(c)}{x-c}.\lim_{x\to c}g(x) + \lim_{x\to c}\frac{g(x)-g(c)}{x-c}.f(c)$$

$$\underline{(fg)'(c) = f'(c)g(c) + f(c)g'(c)}$$

d) let $h = \dfrac{f}{g}$

$$\therefore h'(c) = \lim_{x \to c} \frac{h(x) - h(c)}{x - c}$$

$$\therefore \left(\frac{f}{g}\right)'(c)$$

$$= \lim_{x \to c} \frac{\left(\frac{f}{g}\right)(x) - \left(\frac{f}{g}\right)(c)}{x - c}$$

$$= \lim_{x \to c} \frac{\frac{f(x)}{g(x)} - \frac{f(c)}{g(c)}}{x - c}$$

$$= \lim_{x \to c} \frac{f(x).g(c) - f(c).g(x)}{g(x).g(c)(x - c)}$$

$$= \lim_{x \to c} \frac{f(x).g(c) - f(c).g(c) + f(c).g(c) - f(c).g(x)}{g(x).g(c)(x - c)}$$

$$= \lim_{x\to c} \frac{(f(x)-f(c)).g(c)+f(c).(g(x)-g(c))}{g(x).g(c)(x-c)}$$

$$= \lim_{x\to c}\left(\frac{1}{g(x).g(c)}\right)\left[\lim_{x\to c}\left(\frac{f(x)-f(c)}{x-c}\right).g(c)-\lim_{x\to c}\left(\frac{g(x)-g(c)}{x-c}\right)f(c)\right]$$

$$= \frac{1}{(g(c))^2}[f'(c).g(c)-g'(c)f(c)]$$

$$\underline{\left(\frac{f}{c}\right)'(c) = \frac{f'(c)g(c)-f(c)g'(c)}{(g(c))^2} \quad (g(c)\neq 0)}$$

□

Theorem 6.1.3. *Let f be defined on an interval I containing point c. Then f is differntial at c iff $\exists$ a function ψ on I that is continuous at c and satisfies $f(x)-f(c)=\psi(x)(x-c) \quad x\in I$ In this case, $\psi(c)=f'(c)$*

Proof. If $f'(c)$ exists we can define,

$$\psi(x) = \begin{cases} \dfrac{f(x)-f(c)}{x-c} & for\, x\neq c, x\in I \\ f'(c) & for\, x=c \end{cases}$$

$$\lim_{x\to c}\psi(x) = f'(c)$$

Now, assume that ψ function is continuous at c and satisfies
$f(x) - f(c) = \psi(x).(x - c)$

$\lim\limits_{x\to c} \dfrac{f(x) - f(c)}{x - c} = \lim\limits_{x\to c} \psi(c) = \psi(c)$ exists
$\therefore f$ is differntiable at c and $\psi(c) = f'(c)$ □

6.2 Chain Rule

Theorem 6.2.1 (Chain Rule)**.** *Let I, J be intervals in $\mathbb{R}$. Let $g : I \to \mathbb{R}$ & $f : J \to \mathbb{R}$ be function $\ni$ $f(J) \subseteq I$ and let $c \in J$*
If f is differntiable at c & g is differntiable at $f(c)$ then the composite function $g \circ f$ is diffrentiable at c and $(g \circ f)'(c) = g'(f(c)).f'(c)$

Proof. Given that f is diffrentiable at c
$\therefore$ $\exists$ function ψon $J \ni$
$f(x) - f(c) = \psi(x).(x - c)$ & $f'(c) = \psi(c)$
Also, g is diffrentiable at $f(c)$
$\exists$ function ψ on $I \ni$
$g(f(x)) - g(f(c)) = \psi(f(x)).(f(x) - f(c))$ & $g'(f(c)) = \psi(f(c))$
Consider,
$g \circ f(x) - g \circ f(c)$
$= g(f(x)) - g(f(c))$
$= \psi(f(x)).(f(x) - f(c))$

$= \psi(f(x)).(\psi(x).(x-c))$
$= [\psi(f(x)).\psi(x)].(x-c)$
$\therefore g \circ f$ is diffrentiable at c
Also, $\lim_{x \to c} \frac{g \circ f(x) - g \circ f(c)}{(x-c)}$
$= \lim_{x \to c} [\psi(f(x)).\psi(x)]$
$= \psi(f(c)).\psi(c)$
$(g \circ f)'(c) = g'(f(c)).f'(c)$ □

***Definition* 6.2.1** (Inverse Function)**:** *Let I be an interval in $\mathbb{R}$ and $f : I \to \mathbb{R}$ strictly monotone and continuous on I. let $J = f(I)$ and let $g : J \to \mathbb{R}$ be strictly monotone and continuous function inverse to f.*

Theorem 6.2.2. *If f is differentiable at c, $c \in I$ & $f'(c) \neq 0$ then g is diffrentiable at $d = f(c)$ &* $g'(d) = \frac{1}{f(c)} = \frac{1}{f'(g(d))}$

Proof. Given that, f is differentiable at $c \in I$
$\therefore \exists\, \psi$ on I continuous at $c \ni$
$f(x) - f(c) = \psi(x).(x-c)$ & $\psi(c) = f'(c)$
Since $f'(c) \neq 0 \Rightarrow \psi(c) \neq 0$
$\exists$ neighbourhood of c, $v = (c-\delta, c+\delta)$
$\ni \quad \psi(x) \neq 0 \quad \forall x \in v \cap I$
If $U = f(v \cap I)$ then inverse function g satisfies

$f(g(y)) = y, \quad \forall y \in U$
$y - d = f(g(y)) - f(c) = \psi(g(y)).(g(y) - g(d))$
since, $\psi(g(y)) \neq 0, \quad \forall y \in U$
$g'(y) - g(d) = \dfrac{1}{\psi(g(y))}(y - d)$
Since, $\psi(g(y))$ is continuous at d
$\therefore g'(d)$ exists and
$g'(d) = \dfrac{1}{\psi(g(d))} = \dfrac{1}{\psi(c)} = \dfrac{1}{f'(c)}$ □

Theorem 6.2.3. *Prove that if $f : \mathbb{R} \to \mathbb{R}$ is an even function [i.e $f(-x) = f(x) \forall x$] and has derivative at every point, then the derivative f' is an odd function. Also, prove that if $g : \mathbb{R} \to \mathbb{R}$ is a diffrentiable odd function, then g' is even function.*

Proof. a) Given that f is even function
$f(x) = f(-x) \forall x$
Also, f is differntiable at c
$\therefore f'(c) = \lim\limits_{x \to c} \dfrac{f(x) - f(c)}{x - c}$ exists
To prove, f' is odd function
i.e $f'(-c) = -f'(c)$
consider,
$f'(-c)$

$$= \lim_{x\to -c} \frac{f(x)-f(-c)}{x+c}$$

$$= \lim_{-x\to c} \frac{f(x)-f(c)}{-x+c}$$

$$= \lim_{-x\to c} \frac{f(x)-f(c)}{-(x-c)}$$

$= -f'(c)$

$\therefore f'$ is odd function.

b) Given that g is odd function

$g(x) = g(-x) \forall x$

Also, g is differntiable at c

$\therefore g'(c) = \lim_{x\to c} \frac{g(x)-g(c)}{x-c}$ exists

To prove, g' is even function

i.e $g'(-c) = -g'(c)$

consider,

$g'(-c)$

$$= \lim_{x\to -c} \frac{g(x)-g(-c)}{x+c}$$

$$= \lim_{-x\to c} \frac{-g(x)+g(c)}{-x+c}$$

$$= \lim_{-x\to c} \frac{g(x) - g(c)}{(x-c)}$$

$= g'(c)$

$\therefore g'$ is even function.

□

Theorem 6.2.4 (Interior Extremum)**.** *Let c be an interior point of the interval I at which $f : I \to \mathbb{R}$ has a relative extremum. If derivative of f at c exists, then $f'(c) = 0$*

Proof. Let f has relative maximum at c

if $f'(c) > 0, \quad \exists \quad V_\varepsilon(c) \subseteq I$

$$\frac{f(x) - f'(c)}{x - c} > 0, \quad \forall x \in v_\varepsilon(c), x \neq c$$

if $x \in v, x > c$

$$f(x) - f(c) = (x-c).\frac{f(x) - f'(c)}{x - c} > 0$$

$\therefore f(x) > f(c) \quad \forall x > 0, \quad x \in v_\varepsilon(c)$

but, f has relative maximum at c.

So, our assumption is wrong that $f'(c) > 0$

Similarly, we can show that $f'(c) < 0$

$\therefore f'(c) = 0$ □

Corollary 6.2.4.1. *Let $f : I \to \mathbb{R}$ be continuous on an interval I and suppose that f has relative* *extremum at an interior at c of I then either the derivative of f at c does not exists or it is equal to* 0

Theorem 6.2.5 (Rolle's theorem)**.** *If a function f defined on $[a,b]$ is*

1. *Continuous on $[a,b]$*
2. *derivable on (a,b)*
3. *$f(a) = f(b)$*

then $\exists \quad c \in \mathbb{R}, c \in (a,b) \ni f'(c) = 0$

Proof. Since, f is continuous $[a,b] \Rightarrow f$ is bounded
$\therefore$ by maximum- minimum theorem,
If $m = inf\{f(I)\}$ and $M = Sup\{f(I)\}$ then $\exists \quad c,d \in (a,b)$
$f(c) = m$ & $f(d) = M$
there are two possibilities $m = M$ or $m \neq M$
If $m = M$
$\Rightarrow inf\{f(I)\} = Sup\{f(I)\} \Rightarrow f$ is continuous
$\Rightarrow f'(c) = 0, \quad \forall c \in (a,b)$
If $m \neq M$
$\Rightarrow f(c) = m \neq f(a) \Rightarrow c \neq a$
$\Rightarrow f(c) = m \neq f(b) \Rightarrow c \neq b$
$\Rightarrow c$ lies in (a,b)

Now, we have to show $f'(c) = 0$
IF $f'(c) < 0, \quad \exists \quad (c, c+\delta), \delta_1 > 0$ for every x of which $f(x) < f(c) = m$ which contradicts to our assumption that infimum attains at c.
Simillarly, $f'(c) > 0$ is not possible
$\therefore f'(c) = 0$ □

Theorem 6.2.6 (Langrange's Mean Value theorem)**.** *If a function f defined on $[a, b]$*

i) Continuous on $[a, b]$

ii) differentiable on (a, b)

then $\exists\ c \in (a, b) \ni f'(c) = \dfrac{f(b) - f(a)}{b - a}$

Proof. Let us define function ψ on $[a, b]$ such that
$\psi(x) = f(x) - Ax$,where A is constant.
As $f(x)$ is continuous on $[a, b]$ and differentiable on (a, b),
$\psi(x)$ is also continuous on $[a, b]$ and differentiable on (a, b)
Assume, $\psi(a) = \psi(b)$
$f(a) - A.a = f(b) - A.b$
$f(b) - f(b) = A(b - a)$
$\therefore A = \dfrac{f(b) - f(a)}{b - a}$
$\therefore \psi(x) = f(x) - \left(\dfrac{f(b) - f(a)}{b - a}\right) x$

i) $\psi(x)$ is continuous on $[a,b]$

ii) $\psi(x)$ is derivable on (a,b)

iii) $\psi(a)=\psi(b)$

$\therefore$ by rolle's theorem

$$\psi'(c)=f'(c)-\left(\frac{f(b)-f(a)}{b-a}\right)$$

$$f'(c)=\left(\frac{f(b)-f(a)}{b-a}\right)$$ □

Theorem 6.2.7 (Cauchy Mean Value theorem)**.** *If $f.g$ defined on $[a,b]$*

i) continuous on $[a,b]$

ii) derivable on (a,b)

iii) $g'(x)\neq 0, \quad \forall x\in(a,b)\ \exists \quad c\in(a,b) \quad \ni$

$$\frac{f'(c)}{g'(c)}=\frac{f(b)-f(a)}{g(b)-g(a)}$$

Proof. Let us deine function $\psi(x)$ on $[a,b]\ni$

$\psi(x)=f(x)-Ag(x)$

i) $\psi(x)$ is continuous on $[a,b]$

ii) $\psi(x)$ is derivable on (a,b)

iii) $\psi(a)=\psi(b)$

$\Rightarrow f(a)+A.g(a)=f(b)-A.g(b)$

$\therefore f(b)-f(a)=A(g(b)-g(a))$

$$A=\frac{f(b)-f(a)}{g(b)-g(a)}$$

$\therefore$ by rolles theorem,

$\psi'(c)=0$

$$\frac{f'(c)}{g'(c)}=\frac{f(b)-f(a)}{g(b)-g(a)}$$

□

6.3 Taylor's Theorem

Theorem 6.3.1 (Taylor's Theorem)**.** *If a function f defined on $[a,a+h]$ is such that*

i) $(n-i)^th$ derivative f^{n-1} is continuous on $[a,a+h]$ and

ii) n^th derivative f^n exists on $(a,a+h)$ then $\exists$ atleast one real number θ between 0 & 1 $(0<\theta<1)$ that,

$$f(a+h)=f(a)+hf'(a)+\left(\frac{h^2}{2!}\right)f''(a)+\left(\frac{h^3}{3!}\right)f'''(a)+\ldots\ldots\ldots+\left(\frac{h^{n-1}}{(n-1)!}\right)f^{n-1}(a)+\left(\frac{h^n(1-\theta)^{n-p}}{p[(n-1)!]}\right)f^n(a+$$

$\theta h)$

where p is given positve integer $\mathbb{R}_n$
forms of remainder form-

$$i)\ R_n = \left(\frac{h^n(1-\theta)^{n-p}}{p[(n-1)!]}\right) f^n(a+\theta h)$$

$$ii)\ R_n = \left(\frac{h^n(1-\theta)^{n-1}}{(n-1)!}\right) f^n(a+\theta h) \Rightarrow Cauchy$$

$$iii)\ R_n = \left(\frac{h^n}{n!}\right) f^n(a+\theta h) \Rightarrow \textit{Called as Langrages Forms of remainder}$$

Theorem 6.3.2 (Maclaurins Theorem)**.** $f(x) = f(0)+xf'(0)+\left(\frac{x^2}{2!}\right)f''(0)+\left(\frac{x^3}{3!}\right)f'''(0)+\ldots\ldots\ldots+$
$\left(\frac{x^{n-1}}{(n-1)!}\right)f^{n-1}(0)+\left(\frac{x^n(1-\theta)^{n-p}}{p[(n-1)!]}\right)f^n(\theta x)$

Example 34:
$f(x) = e^x$
$\therefore$ By Maclaurins theorem,
$f(x) = f(0)+xf'(0)+\left(\frac{x^2}{2!}\right)f''(0)+\ldots\ldots$

$$e^x = 1 + x + \frac{x^2}{2!} + \ldots\ldots$$

Example 35:

$f(x) = \sin(x)$

$\therefore$ By Maclaurins theorem,

$$f(x) = \sin 0 + x\cos 0 + \frac{x^2}{2!}(-\sin 0) + \frac{x^3}{3!}(-\cos 0) + \ldots\ldots$$

$$\therefore \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots\ldots$$

Example 36:

$f(x) = \log(1 + x)$

$$f'(x) = \frac{1}{1+x}, \quad f''(x) = \frac{-1}{(1+x)^2}, \quad f'''(x) = \frac{2}{(1+x)^3}$$

$\therefore$ By Maclaurins theorem,

$$f(x) = f(0) + xf'(0) + \left(\frac{x^2}{2!}\right)f''(0) + \left(\frac{x^3}{3!}\right)f'''(0) + \ldots\ldots\ldots$$

$$f(x) = 0 + x(1) + \left(\frac{x^2}{2!}\right)(-1) + \left(\frac{x^3}{3!}\right)(2) + \ldots\ldots\ldots$$

$$\log(1+x) = x - \frac{x^2}{2}\frac{x^3}{3} + \frac{x^4}{4} + \ldots\ldots$$

6.4 Maximum or Minimum for function of two variables

$f(a,b)$ is extreme value of $f(x,y)$. if

i) $f_x(a,b) = 0 = f_y(a,b)$

ii) $f_{xx}(a,b) = f_{yy}(a,b) - [f_{xy}(a,b)]^2 > 0$

and this extreme value is maximum or minimum according as $f_{xx}(a,b)$ or $f_{yy}(a,b)$ is negative or positive.
Further investigation needed if,
$f_{xx}(a,b).f_{yy}(a,b) - [f_{xy}(a,b)] = 0$
Example 37:
find maximum and minimum of
$f(x,y) = x^3 + y^3 - 3x + 12y + 20 = 0$

Proof. $f_x(x,y) = 0$
i.e $3x^2 - 3 = 0$
$x^2 = 1$
$x = \pm 1$
$f_y(x,y) = 0$
i.e $3y^2 + 12 = 0$
$y^2 = 4$
$x = \pm 2$

$f_{xx}(x,y) = 6x\ ,\quad f_{yy}(x,y) = 6y,\quad f_{xy}(x,y) = 0$
for $x = 1, y = 2$
$f_{xx}(x,y) = 6x = 6\ ,\quad f_{yy}(x,y) = 6y = 12,\quad f_{xy}(x,y) = 0$
for $x = -1, y = -2$
$f_{xx}(x,y) = 6x = -6\ ,\quad f_{yy}(x,y) = 6y = -12,\quad f_{xy}(x,y) = 0$
for $x = -1, y = 2$
$f_{xx}(x,y) = 6x = -6\ ,\quad f_{yy}(x,y) = 6y = 12,\quad f_{xy}(x,y) = 0$
for $x = 1, y = -2$
$f_{xx}(x,y) = 6x = 6\ ,\quad f_{yy}(x,y) = 6y = -12,\quad f_{xy}(x,y) = 0$
minimum=$(1,2)$
maximum=$(-1,-2)$ $\square$

Chapter 7

Sequence and Series of Function

7.1 Sequence of Function

Definition* 7.1.1** (Sequence of Function):*** *Let $A \subseteq \mathbb{R}$ be given and suppose that for each $n \in \mathbb{N}$ $\exists \quad f_n : A \to \mathbb{R}$ we shall say that (f_n) is a sequence of function on A to $\mathbb{R}$*

Definition* 7.1.2** (Pointwise Convergent):*** *Let f_n be a sequence of function on $A \subseteq \mathbb{R}$ to $\mathbb{R}$. let $A_0 \subseteq A$ & let $f_n : A_0 \to \mathbb{R}$ we say that the sequence f_n converges on A_0 to f if for each $x \in A_0$ the sequence $f_n(x)$ converges to f*
The sequence $f_n : A \to \mathbb{R}$ converges to function $f_n : A_0 \to \mathbb{R}$ on A_0 iff for each $\varepsilon > 0$ & $x \in A_0$ $\exists$ $k(\varepsilon_1 x) \in \mathbb{N} \quad \ni \quad |f_n(x) - f(x)| < \varepsilon, \quad \forall n \geqslant k(\varepsilon_1 x)$

Example 38:

$f_n(x) = \dfrac{x}{n}, \quad f(x) = 0$

$\left| f_n(x) - f(x) \right| < \varepsilon$

i.e $|\dfrac{x}{n} - 0| < \varepsilon \Rightarrow \left|\dfrac{x}{n}\right|$

$\therefore \dfrac{|x|}{n} < \varepsilon$

$\therefore n > \dfrac{|x|}{\varepsilon}$

Example 39:

$f_n(x) = x^n$

$\left|f_n(x) - f(x)\right| < \varepsilon$

$|x^n - 0| < \varepsilon, \quad -1 < x < 1$

$|x^n| < \varepsilon$

$n \log x < \log \varepsilon$

$n < \log\left(\dfrac{\varepsilon}{x}\right)$

$\therefore n > \log\left(\dfrac{x}{\varepsilon}\right)$

Example 40:

$f_n(x) = \dfrac{x^2 + nx}{n}, \quad x \in \mathbb{R}, \quad f(x) = x$

$\left|f_n(x) - f(x)\right| < \varepsilon$

$\left|\frac{x^2}{n}+x-x\right|<\varepsilon\left|\frac{x^2}{n}\right|<\varepsilon$

$\therefore \frac{x^2}{\varepsilon}<n$

$\therefore n>\frac{x^2}{\varepsilon}$

***Definition* 7.1.3** (Uniform Convergence)**:** *A sequence of function on $A \subseteq \mathbb{R}$ to $\mathbb{R}$ converges uniformly on $A_0 \subseteq A$ to a function $f : A_0 \to \mathbb{R}$ if for each $\varepsilon > 0$ there is a natural number $k(\varepsilon)$ (depending on ε but not on $x \in A_0$) $\ni$*

$\left|f_n(x)-f(x)\right|<\varepsilon, \quad \forall n \geqslant k(\varepsilon)$

denoted by, $f_n(x) \rightrightarrows f(x)$ on A_0

Lemma 7.1.1. *A sequence f_n of function on $A \subseteq \mathbb{R}$ does not converges uniformly on $A_0 \subseteq A$ to a function $f : A_0 \to \mathbb{R}$ iff for some $\varepsilon_0 > 0$ $\exists$ subsequence f_{n_k} of f_n and a sequence x_k in A_0 such that*

$\left|f_{n_k}(x_k)-f(x_k)\right| \geqslant \varepsilon_0, \quad \forall k \in \mathbb{N}$

Example 41:

$f_n(x)=\frac{x_k}{n_k}, \quad f(x)=0, x_k=k, n_k=k$

$\left|f_{n_k}(x_k)-f(x_k)\right| \geqslant \varepsilon_0$

$$\left|\frac{x_k}{n_k}-0\right| \Rightarrow \left|\frac{k}{k}-0\right|$$

$\Rightarrow |1-0|$

$\Rightarrow |1| \geqslant \varepsilon$

Example 42:

$$f_n(x)=\frac{x^2+nx}{n}, \quad f(x)=x, x_k=k, n_k=-k$$

$$\left|f_{n_k}(x_k)-f(x_k)\right| \geqslant \varepsilon_0$$

$$\left|\frac{x_k^2}{n_k}+x_k-x_k\right| \geqslant \varepsilon_0 \Rightarrow$$

$$\left|\frac{k^2}{k}\right| \geqslant \varepsilon_0$$

$\therefore |k| > \varepsilon$

$\therefore$ not uniformly convergent

Example 43:

$f_n(x)=x^n$

$$f(x)=\begin{cases}0 & ;0\leqslant x<1\\ 1 & ;x=1\end{cases}$$

$$x_k = \left(\frac{1}{2}\right)^{\left(\frac{1}{k}\right)}, \quad n_k = k$$

$$\left|f_{n_k}(x_k) - f(x_k)\right| \geqslant \varepsilon_0$$

$$\therefore |x_k^{n_k} - 0| \geqslant \varepsilon_0$$

$$\left|\left(\frac{1}{2}\right)^{\left(\frac{1}{k}\right)} - 0\right| \geqslant \varepsilon_0$$

$$\therefore \left|\frac{1}{2}\right| > \varepsilon$$

$\therefore$ Not uniformly convergent

***Definition* 7.1.4** (Uniform Norm)**:** *If $A \subseteq \mathbb{R}$ & $\psi : A \to \mathbb{R}$ is a function we say that ψ is bounded on A. If the set $\psi(A)$ is bounded subset of $\mathbb{R}$ if ψ is bounded we define the uniform norm of ψ on A by, $||\psi||_A = Sup\{|\psi(x)| : x \in A\}$*
Note that, it follows that if $\varepsilon > 0$,
$||\psi||_A \leqslant \varepsilon \Leftrightarrow |\psi(x)| \Longleftrightarrow \varepsilon, \quad \forall x \in A$

Lemma 7.1.2. *A sequence f_n of bounded function on $A \subseteq \mathbb{R}$ uniformly on A to f if and only if $||f_n - f||_A \to 0$*

Example 44:

$f(x) = x \quad [0,1]$
$Sup\{|\psi(x)| : x \in A\} = 1$
$||f||_A = 1$

Example 45:

$f_n(x) = \frac{x}{n}, \quad f(x) = 0, \quad [0,1]$

$|f_n(x) - f(x)| = |x|$

$\therefore ||f_n - f||_A = \frac{1}{n}^n \to 0$

Example 46:

$f_n(x) = x^n \quad [0,k], \quad f(x) = 0$
$|f_n(x) - f(x)| = |x^n|$
$\therefore ||f_n - f||_A = |k^n|$

Example 47:

$f_n(x) = x^n(1-x) \quad x \in [0,1], f(x) = 0$
$|f_n(x) - f(x)| < \varepsilon$
$|x^n(1-x) - 0| = |x^n(1-x)|$
$f_n(x) = x^n - x^{n+1}$
$\therefore f_n'(x) = nx^{n-1} - (n+1)x^n = 0$
$\Rightarrow nx^{n-1} = (n+1)x^n$

$$\Rightarrow \frac{n}{n+1} = x$$

$$x = \frac{1}{\left(1+\frac{1}{n}\right)}$$

$$\therefore ||f_n - f||_A = \frac{1}{\left(1+\frac{1}{n}\right)^n}\left(1 - \frac{1}{\left(1+\frac{1}{n}\right)}\right)$$

$$= \left(1+\frac{1}{n}\right)^{-n}\left(\frac{\frac{1}{n}}{1+\frac{1}{n}}\right)$$

$$= \left(1+\frac{1}{n}\right)^{-n}\left(\frac{1}{1+\frac{1}{n}}\right) \to 0$$

7.2 Cauchy Criteria for Uniform Convergence

Theorem 7.2.1. *Let f_n be a sequence of bounded function on $A \subseteq \mathbb{R}$ then this seqence converges uniformly on A to a bounded function f iff for each $\varepsilon > 0 \quad \exists \quad H(\varepsilon) \in \mathbb{N} \ni$ $||f_m - f_n||_A \leqslant \varepsilon, \quad \forall m, n \geqslant H(\varepsilon)$*

Proof. If $f_n(x) \rightrightarrows f(x)$ then for $\varepsilon > 0 \quad \exists k\left(\frac{\varepsilon}{2}\right) \ni$

$||f_n - f||_A \leqslant \left(\frac{\varepsilon}{2}\right) \quad \forall n \geqslant k\left(\frac{\varepsilon}{2}\right)$

Hence, if both $m, n \geqslant k\left(\frac{\varepsilon}{2}\right)$

$|f_m(x)' - f_n(x)|$
$= |f_m(x) - f(x) + f(x) f_n(x)|$
$\leqslant= |f_m(x) - f(x)| + |f(x) f_n(x)|$

$\leqslant \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$

$\leqslant \varepsilon \quad \forall m, n \geqslant k\left(\frac{\varepsilon}{2}\right)$
Conversly,
Suppose, $\varepsilon > 0, \quad \exists H(\varepsilon) \in \mathbb{N}$
$\ni ||f_m - f_n||_A \leqslant \varepsilon, \quad \forall m, n \geqslant H(\varepsilon)$

$\therefore$ for each $x \in A$
$|f_m(x) - f_n(x)| \leqslant ||f_m(x) - f_n(x)||_A \leqslant \varepsilon, \quad \forall m, n \geqslant H(\varepsilon)$
$\Rightarrow f_m(x)$ is cauchy sequence and hence convergent.
$\therefore \exists \quad f(x) = \lim_{n\to\infty} f_n(x) \quad \forall x \in A$
We have $|f_m(x) - f_n(x)| \leqslant \varepsilon, \quad \forall m \geqslant H(\varepsilon)$
$\therefore f_n(x) \rightrightarrows f(x)$ on A $\square$

7.3 Series of Function

If f_n is sequence of function defined on subset D of $\mathbb{R}$ with values in $\mathbb{R}$, the sequence of partial sums S_n of infinite series $\sum f_n$ is defined for x in D by,
$S_1(x) = f_1(x)$
$S_2(x) = f_2(x) + S_2(x)$
:
:
$S_{n+1}(x) = S_n(x) + f_{n+1}(x)$
:
:

- In the case sequence S_n of functions f_n converges to function f on D we say that $\sum f_n$ converges on D to f
- If the series $\sum |f_n(x)|$ converges for each $x \in D$, we say that $\sum f_n$ converges absolutely on D.
- if (S_n) sequence of partial sums is uniformly convergent on D to f, we say that $\sum f_n$ is uniformly converges on D
- If f_n is continuous on $D \subseteq \mathbb{R}$ to $\mathbb{R}$ for each $n \in \mathbb{N}$ and if $\sum f_n$ converges f on D uniformly, then f is continuous on D

Definition* 7.3.1** (Cauchy Criterion):*** *f_n be a sequence of f_n on $D \subseteq \mathbb{R}$ to $\mathbb{R}$, the series $\sum f_n$ is uniformly convergent on D iff if for every uniformly $\varepsilon > 0, \quad \exists M(c)$*
$\ni |f_{n+1}(x) + f_{n+2}(x) + \ldots\ldots + f_m(x)| < \varepsilon, \quad \forall m > n \leqslant M(\varepsilon)$

Theorem 7.3.1 (Weistress M-test)**.** *Let M_n be a sequence of positive real numbers such that $|f_n(x)| \leqslant M_n \quad \forall \quad x \in D \quad \forall n \in \mathbb{N}$. If the series M_n is convergent then $\sum f_n$ is uniformly convergent on D*

Proof. M_n is convergent,
By cauchy criterion for series,
for any $\varepsilon > 0, \quad \exists\ k(\varepsilon) \in \mathbb{N}$
$\ni M_{n+1} + M_{n+2} + \ldots\ldots + M_m < \varepsilon, \quad \forall m > n \leqslant k(\varepsilon)$
$\ni |f_{n+1}(x)| + |f_{n+2}(x)| + \ldots\ldots + |f_m(x)| < M_{n+1} + M_{n+2} + \ldots\ldots + M_m < \varepsilon$
Also,
$|f_{n+1}(x) + f_{n+2}(x) + \ldots\ldots + f_m(x)| \leqslant |f_{n+1}| + |f_{n+2}| + \ldots\ldots + |f_m| < \quad \forall m > n \geqslant k(\varepsilon)$
$\therefore$ By Cauchy criterion,
$\sum f_n$ is uniformly convergent on D. □

Definition* 7.3.2** (Power Series):*** *A series of real function $\sum f_n$ is said to be power series around $x = c$ if the function has the form $f_n(x) = a_n(x-c)^n$ where a_n and $c \in \mathbb{R}$ and where $n = 0, 1, 2, \ldots$*

Example 48:
Power Series

$\sum a_n x^n = a_0 x^0 + a_1 x + \ldots + a_n x^n + \ldots$

$\sum_{n=0}^{\infty} n! x^n \quad \sum_{n=0}^{\infty} x^n \quad \sum_{n=0}^{\infty} \frac{x^n}{n!}$

Definition* 7.3.3** (Radius of Convergence):*** *$\sum a_n x^n$ be a power series if a sequence $|a_n|^{\frac{1}{n}}$ is bounded, we set $\rho = \lim Sup |a_n|^{\frac{1}{n}}$ if this sequence is not bounded, we set $\rho = +\infty$*
We define radius of convergece of $\sum a_n x^n$ to be given by,

$$R = \begin{cases} 0 & ; if \quad \rho = +\infty \\ \frac{1}{\rho} & ; if \quad 0 < \rho < +\infty \\ \infty & ; \quad \rho = 0 \end{cases}$$

The interval of convergence is the open interval $(-R, R)$

Example 49:

$\sum \frac{x^n}{2^n} \Rightarrow \left| \frac{1}{2^n} x^n \right|$

$\Rightarrow a_n . x^n$

$\rho = \lim Sup |a_n|^{\frac{1}{n}}$

$$\Rightarrow \lim Sup \left|\frac{1}{2^n}\right|^{\frac{1}{n}}$$

$$\Rightarrow \frac{1}{2}$$

$$\Rightarrow R = \frac{1}{\rho} = 2$$

Example 50:

$$\sum nx^n \Rightarrow a_n x^n$$

$$R = \lim \left|\frac{a_n}{a_{n+1}}\right|$$

$$R = \lim \left|\frac{n}{n+1}\right|$$

$$R = \lim \left|\frac{1}{1+\frac{1}{n}}\right|$$

$$R = 1$$

Chapter 8

Riemann Integral

8.1 Introduction

Riemann Integral

If $I = [a, b]$ be closed bounded interval in $\mathbb{R}$ then partition of I is a finite ordered set $\mathbb{P} = (x_0, x_1, \ldots, x_{n-1}, x_n)$ of points in I such that $a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b$.
The points of P are used to divide $I = [a, b]$ into non-overlapping sub-intervals.
$I_1 = [x_0, x_1]$,$I_2 = [x_1, x_2], \ldots, I_n = [x_{n-1}, x_n]$

Norm of $P = ||p|| = max\{|x_i - x_{i-1}|, i = 1, 2, \ldots, n\}$
The norm of partition is merely the length of largest sub-interval into which the partition divide if point t_i has been choosen from each sub-interval $I_i = [x_{i-1}, x_i] = \quad \forall i = 1 : n$ then the points are called as tages of sub- intervals $I - i$.
A set of ordered pairs
$\dot{p} = \{[x_{i-1}, x_i], t_i\}_{i=1}^{n}$ is tagged partition of $[a, b]$

Definition* 8.1.1** (Riemann Sum):*** *If $\dot{p}$ is the tagged partition, we define Riemann sum of function. $f:[a,b]\to\mathbb{R}$ corresponding to $\dot{p}$ to be the number,*

$$S(f,\dot{p})=\sum_{i=1}^{n} f(t_i)(x_i-x_{i-1})$$

Definition* 8.1.2** (Riemann Integral):*** *A function $f:[a,b]\to\mathbb{R}$ is said to be Riemann integrable on $[a,b]$ if there exists a number $L\in\mathbb{R}$ such that for $\varepsilon>0 \quad \exists \quad \delta_\varepsilon>0 \quad \ni$ if $\dot{p}$ is any tagged partition of $[a,b]$ with $||\dot{p}||<\delta_\varepsilon$ then*

$$|S(f,\dot{p})-L|<\varepsilon$$

The set of all Riemann integrable functions on $[a,b]$ will be denoted by $R[a,b]$

i.e $||\dot{p}||\to 0 \Rightarrow S(f,\dot{p})\to L$

Definition* 8.1.3*: *If $f\in R[a,b]$ then the number L is uniquely determined and called as Riemann Integral of f over $[a,b]$*

$$L=\int_a^b f(x)\,dx$$

Theorem 8.1.1. *If $f\in R[a,b]$ then the value of the integral is uniquely determined.*

Proof. Assume that L' & L'' both satisfy the definition and

let $\varepsilon>0 \quad \exists \quad \delta'_{\frac{\varepsilon}{2}}>0 \quad \ni$ if $\dot{p}_1$ is tagged partition with $||\dot{p}_1||<\delta'_{\frac{\varepsilon}{2}}$ then

$$|S(f,\dot{p}_1)-L'|<\frac{\varepsilon}{2}$$

Similarly, $\exists \quad \delta''_{\frac{\varepsilon}{2}}>0 \quad \ni$ if $\dot{p}_2$ is tagged partition with $||\dot{p}_2||<\delta''_{\frac{\varepsilon}{2}}$ then

$|S(f,\dot{p}_2)-L''|<\frac{\varepsilon}{2}$
Now, let $\delta_\varepsilon = min\left(\delta'_{\frac{\varepsilon}{2}},\delta''_{\frac{\varepsilon}{2}}\right)$
let $\dot{p}$ be tagged partition with $||\dot{p}||<\delta_\varepsilon$
$\Rightarrow |S(f,\dot{p})-L'|<\frac{\varepsilon}{2}$ and
$|S(f,\dot{p}_2)-L''|<\frac{\varepsilon}{2}$
So, $|L'-L''|=|L'-S(f,\dot{p})+s(f,\dot{p})-L''|$

$$\leqslant \frac{\varepsilon}{2}+\frac{\varepsilon}{2}$$

$$\leqslant \varepsilon$$

As ε is arbitary, $L'=L''$ □

Theorem 8.1.2. *Every constant function on $[a,b]$ is in $R[a,b]$.*

Proof. Let $f(x)=k \quad \forall \quad x\in[a,b]$ be the constant function, if $\dot{p}=\{[x_{i-1},x_i],t_i\}_{i=1}^n$ is any tagged partition on $[a,b]$

$$S(f,\dot{p})=\sum_{i=1}^{n}k(x_i-x_{i-1})=k(b-a)$$

Hence, for any $\varepsilon>0$, we can choose $\delta_\varepsilon>0 \quad \ni \quad ||\dot{p}||<\delta_\varepsilon$ & $S(f,\dot{p}-k(b-a)|=0<\varepsilon$

$$\int_a^b f(x)dx=k(b-a)$$

$\therefore f(x)$ is an Riemmann integrable $f\in R[a,b]$ □

8.2 Some Properties of Integral

Theorem 8.2.1. *Suppose that f & g are in $R[a,b]$ then*

a) If $k \in \mathbb{R}$, the function $k.f$ is in $R[a,b]$ and $\displaystyle\int_a^b kf = k\int_a^b f$

b) the function f & g is in $R[a,b]$ and $\displaystyle\int_a^b f+g = \int_a^b f + \int_a^b g$

c) $f(x) \leqslant g(x) \quad \forall \quad x \in [a,b]$ *then* $\displaystyle\int_a^b f \leqslant \int_a^b g$

Theorem 8.2.2. *If $f \in [a,b]$ then f is bounded on $[a,b]$*

Proof. Assume that f is unbounded on $[a,b]$
As $f \in [a,b]$, then for any $\varepsilon > 0 \quad \exists \delta_\varepsilon > 0$
such that $||\dot{p}|| < \delta_\varepsilon$ then $|S(f,\dot{p}) - L| < \varepsilon$
Now, let $Q = \{[x_{i-1}, x_i\}_{i=1}^n$ be partition on $[a,b]$ with $||Q|| < \delta$. Since $|f|$ is not bounded on $[a,b]$, $\exists$ atlest one sub-interval $[x_{k-1}, x_k]$ on $[a,b]$ which $|f|$ is not bounded.
Let tag Q by $t_i = x_i$ for $i \neq k$ and $k_k \in [x_{k-1}, x_k]$ such that,

$$|f(t_k).(x_k - x_{k-1})| > |L| + \varepsilon + \left|\sum_{i\neq k}^{n} f(t_i)(x_i - x_{i-1})\right|$$

By triangular inequality $|a+b| > |a| - |b|$

$$S(f,Q)| \geqslant |f(t_k)(x_k - x_{k-1})| - + \left|\sum_{i\neq k}^{n} f(t_i)(x_i - x_{i-1})\right| > |L| + \varepsilon$$

$\therefore$ which is contradict to our assumpsion.

$\therefore f$ is bounded on $[a,b]$ □

***Definition* 8.2.1** (Cauchy Criterion for Riemann Integrable function)**:** *A function $f : [a,b] \to \mathbb{R} \in R[a,b]$ if and only if for every $\epsilon > 0, \exists \quad n_\varepsilon > 0$ if $\dot{p}$ & Q are any tagged partitions of $[a,b]$ with $||\dot{p}|| < n_\varepsilon$ & $||\dot{Q}|| < n_\varepsilon$ then,*

$|S(f,\dot{p}) - S(f,\dot{Q})| < \varepsilon$

Theorem 8.2.3 (Squeez theorem)**.** *Let $f : [a,b] \to \mathbb{R}$ then $f \in R[a,b]$ if and only if for every $\varepsilon > 0 \,\exists$ function α_ε & w_ε in $R[a,b]$ with*

$$\alpha_\varepsilon(x) \leqslant f(x) \leqslant w_\varepsilon \quad \forall \in R[a,b] \text{ \& such that } \int_a^b w_\varepsilon - \alpha_\varepsilon < \varepsilon$$

Proof. $\Longleftrightarrow$ Take $\alpha_\varepsilon = w_\varepsilon = f \quad \forall \varepsilon > 0$

$\Longleftrightarrow$ Let $\varepsilon > 0$, Since $\alpha_\varepsilon, w_\varepsilon \in R[a,b]$

$\exists \quad \delta_\varepsilon > 0 \quad \ni ||\dot{p}|| < \delta_\varepsilon$ then

$$\left|S(\alpha_\varepsilon,\dot{p}) - \int_a^b \alpha_\varepsilon\right| < \varepsilon \;\&\; \left|S(w_\varepsilon,\dot{p}) - \int_a^b w_\varepsilon\right| < \varepsilon$$

$$\Rightarrow \int_a^b \alpha_\varepsilon - \varepsilon < S(\alpha_\varepsilon,\dot{p}) \quad \& \quad S(w_\varepsilon,\dot{p}) < \int_a^b w_\varepsilon + \varepsilon$$

As $\alpha_\varepsilon \leqslant f \leqslant w_\varepsilon$

$S(\alpha_\varepsilon, \dot{p}) \leqslant S(f, \dot{p}) \leqslant S(w_\varepsilon, \dot{p})$

$$\Rightarrow \int_a^b \alpha_\varepsilon - \varepsilon \leqslant S(f, \dot{p}) \leqslant \int_a^b w_\varepsilon + \varepsilon$$

Consider anothr partition $||\dot{Q}|| < \delta_\varepsilon$

$$\Rightarrow \int_a^b \alpha_\varepsilon - \varepsilon \leqslant S(f, \dot{Q}) \leqslant \int_a^b w_\varepsilon + \varepsilon$$

$$\Rightarrow |S(f, \dot{Q}) - S(f, \dot{p})| < \int_a^b (w_\varepsilon - \alpha_\varepsilon) + 2\varepsilon \leqslant 3\varepsilon$$

Since, $\varepsilon > 0$, is arbitary, $f \in R[a, b]$ □

Theorem 8.2.4. *If $f : R[a, b] \to \mathbb{R}$ is continuous on $[a, b]$ then $f \in R[a, b]$*

Proof. As f is continuous on closed bounded interval $[a, b]$, f is uniformly continuous on $[a, b]$

$\therefore$ for any $\varepsilon > 0$, $\quad \exists \quad \delta_\varepsilon > 0 \quad \ni$ if $u, v \in [a, b]$

$|u - v| < \delta_\varepsilon$

$$\Rightarrow |f(u) - f(v)| < \frac{\varepsilon}{b - a}$$

Let $p = \{I_i\}_{i=1}^n$ be a partition such that $||p|| < \delta_\varepsilon$, let $u_i \in I_i$ be a point where f attains minimum value on I_i & $v_i \in I_i$ be a point where f attains maximum value on I_i

Let α_ε be the step function

$\alpha_\varepsilon(x) = f(u_i) \quad \forall \quad x \in [x_{i-1}, x_i](i = 1 : n-1)$
Let w_ε be the step function
$w_\varepsilon(x) = f(v_i) \quad \forall \quad x \in [x_{i-1}, x_i](i = 1 : n-1)$
so, $\alpha_\varepsilon(x) \leqslant f(x) \leqslant w_\varepsilon(x) \quad \forall x \in [a, b]$

$$0 \leqslant \int_a^b (w_\varepsilon - \alpha_\varepsilon) = \sum_{i=1}^{n} (f(v_i) - f(u_i))(x_i - x_{i-1})$$

$$< \sum_{i=1}^{n} (\frac{\varepsilon}{(b-a)}(x_i - x_{i-1}) = \varepsilon$$

$\therefore$ by squeez theorem,
$f \in R[a, b]$ □

Theorem 8.2.5. *If $f : R[a, b] \to \mathbb{R}$ is monotone on $[a, b]$ then $f \in R[a, b]$*

Proof. Suppose f is I on $[a, b]$
Assume $a < b, \varepsilon > 0$

$$h = \frac{f(b) - f(a)}{q} < \frac{\varepsilon}{(b-a)}$$

let $y_k = f(a) + k.h \quad \forall \quad k = 0, 1, \ldots q$
let $A_k = f^{-1}[y_{k-1}, y_k] \quad \forall \quad k = 0, 1, \ldots q-1$
The sets A_k are pairwise disjpoint and have union $[a, b]$
so A_k is either

a) empty

b) single point set

c) non degenerate interval in $[a, b]$

We discard the sets for which a) holds and relabel remaining ones if we adjoin the end points of the remaining intervals A_k, we obtain closed intervals I_k
So we how have step functions α_ε & w_ε
$\alpha_\varepsilon(x) = y_{k-1}, \quad w_\varepsilon(x) y_k \quad \forall x \in A_k$
$\alpha_\varepsilon(x) \leqslant f(x) \leqslant w_\varepsilon(x) \quad \forall x \in [a, b]$

$$\int_a^b (w_\varepsilon - \alpha_\varepsilon)$$

$$= \sum_{k=1}^{q} (y_k - y_{k-1})(x_k - x_{k-1})$$

$$= \sum_{k=1}^{q} h(x_k - x_{k-1})$$

$$= h.(b-a)$$

so, by squeez theorem,
$f \in R[a, b]$ □

8.3 Fundamental theorem of Integral calculus

Theorem 8.3.1. *Suppose, there is finite set E in $[a, b]$ and function f $F : [a, b] \to \mathbb{R}$ such that*

1. *F is continuous on $[a, b]$*

2. $F'(x) = f(x) \quad \forall \quad x \in [a, b]^{\backslash E}$

3. $f \in R[a, b]$ *then* $\int_a^b f = f(b) = f(a)$

Proof. Let $\varepsilon > 0$, since $f \in R[a, b] \quad \ni \quad \delta_\varepsilon > 0$
$\ni$ if p is any tagged partition $||\dot{p}|| < \delta_\varepsilon$

$$\left| S(f, \dot{p}) - \int_a^b f \right| < \varepsilon$$

If the sub-intervals in p are $[x_{i-1}, x_i]$ then
by MVT, $\exists \quad u_i \in (x_{i-1}, x_i)$

$$F'(u_i) = \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}} \quad \forall \quad i = 1 : n$$

adding $i = 1 : n$

$$\sum_{i=1}^{n} f(x_i) - f(x_{i-1}) = \sum_{i=1}^{n} F'(u_i)(x_i - x_{i-1})$$

$$F(a) - F(b) = \sum_{i=1}^{n} f'(u_i)(x_i - x_{i-1}) = S(f, \dot{p})$$

Asuuming $\dot{p}_u = \{[x_i - x_{i-1}], u_i\}_{i=1}^{n}$

$$\Rightarrow \left| F(a) - F(b) - \int_a^b f \right| < \varepsilon$$

$$\Rightarrow \int_a^b f = F(a) - F(b)$$

□

8.4 Indefinite Integral

***Definition* 8.4.1** (Indefinite Integral)**:** *If* $f \in R[a,b]$ *then* $f(z) = \int_a^z f \quad \forall \quad z \in [a,b]$

Theorem 8.4.1. *The indefinite integral* F *is continuous on* $[a,b]$*. In fact, if* $|f(x)| \leqslant M \quad \forall x \in [a,b]$ *then* $|F(z) - F(w)| \leqslant M|z-w| \quad \forall z, w \in [a,b]$

Proof. If $z, w \in [a,b], \quad w \leqslant z$

$F(z) = \int_a^z f = \int_a^w f + \int_w^z f = f(w) + \int_w^z f$

$\Rightarrow \int_w^z f = F(z) - F(w)$

if $-M \leqslant f(x) \leqslant M \quad \forall x \in [a,b]$

$-M(z-w) \leqslant \int_w^z f \leqslant M(z-w)$

$\Rightarrow |F(z) - F(w)| \leqslant \left| \int_w^z f \right| \leqslant M|z-w|$ □

8.5 Examples

Example 51:

$f(x) = x$

$g(x) = \frac{1}{x}$

$$f \circ g = f(g(x)) = f\left(\frac{1}{x}\right) = \frac{1}{x}$$
$$g \circ f = g(f(x)) = g(x) = \frac{1}{x}$$
$$f \circ g = g \circ f$$

Example 52:

$$A_n = \{(n+1)k, \quad k \in \mathbb{N}\}$$
$$A_1 = \{2k, \quad k \in \mathbb{N}\}$$
$$A_2 = \{3k, \quad k \in \mathbb{N}\}$$
$$A_1 \cap A_2 = \{6k, \quad k \in \mathbb{N}\}$$
$$\cap A_i = \{\phi\}$$
$$\cup A_i = \mathbb{N} - \{1\}$$

Example 53:

$$\lim \frac{n^2}{n!}$$
$$\lim \frac{n.n}{n.(n-1)!}$$
$$\lim \frac{n}{(n-2)(n-1)!}$$

$$\lim_{n\to\infty} \frac{1}{\left(1-\frac{1}{n}\right)} \lim_{n\to\infty} \frac{1}{(n-2)!}$$
$= (1)(0)$

0

Example 54:

Result:- $\lim_{x\to\infty} (1 + a^x)^{\frac{1}{x}} = e^a$

$$x_n = (a^n + b^n)^{\frac{1}{n}}, \quad a < b$$

$$= \lim_{n\to\infty} (a^n + b^n)^{\frac{1}{n}}$$

$$= \lim_{n\to\infty} b\left(\frac{a^n}{b^n} + 1\right)^{\frac{1}{n}}$$

$$= \lim_{n\to\infty} b\left(\left(\frac{a}{b}\right)^n + 1\right)^{\frac{1}{n}}$$

$= b.e^{\frac{a}{b}}$

$\therefore (a^n + b^n)^{\frac{1}{n}}$ is convergent, bounded and cauchy.

Example 55:

$\sum x_n = \frac{1}{1} - \frac{1}{5} + \frac{1}{7} - \frac{1}{13} + \frac{1}{17}$

$\sum |x_n| = \frac{1}{1} + \frac{1}{5} + \frac{1}{7} + \frac{1}{13} + \frac{1}{17}$

$$\sum x_n = \frac{(-1)^{n-1}}{4n-(-1)^n}$$

Example 56:

$$f_n(x) = \frac{1}{nx+1}, \quad x \in (0,1), f(x) = 0$$

$$|f_n(x) - f(x)| < \varepsilon$$

$$\left|\frac{1}{nx+1} - 0\right| < \varepsilon$$

$$\therefore \left|\frac{1}{nx+1}\right|$$

$$|nx+1| > \tfrac{1}{\varepsilon}$$

Example 57:

Examine convergent of $\sum \left(\frac{1}{2^n} + \frac{1}{3^n}\right)$

$$\sum \left(\frac{1}{2^n} + \frac{1}{3^n}\right) = \sum \frac{1}{2^n} + \sum \frac{1}{3^n}$$

$$\sum\left(\frac{1}{2^n}+\frac{1}{3^n}\right)=\sum\left(\frac{1}{2}\right)^n+\sum\left(\frac{1}{3}\right)^n$$

$\sum r_1^n + \sum r_2^n \quad r_1 = \frac{1}{2} < 1, r_2 = \frac{1}{3} < 1$
which is convergent

Example 58:

$$f_n(x) = \frac{1}{x^n} \quad x \in (0,1)$$

$$f(x) = \begin{cases} \text{not defined} & x = -1 \\ \frac{1}{2} & x = 1 \\ 0 & x > 1 \end{cases}$$

Example 59:

$$\lim_{n\to\infty} |x_n - x_{n+1}| = 0$$

x_n does not converges for given example j

Example 60:

$\sum \frac{1}{\sqrt{n^3+4}}$ Use comparision test

$n < n^{\frac{3}{2}}, \quad n > 1$

$$\frac{1}{n} > \frac{1}{n^{\frac{3}{2}}}$$

As $\frac{1}{n}$ is divergent $\Rightarrow \frac{1}{n^{\frac{3}{2}}}$ is also divergent.

***Definition* 8.5.1** (Taylors expansion for two variables)**:** $f(x,y) =$
$f(a,b) + \left[(x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right] f(a,b) + \frac{1}{2!}\left[(x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right]^2 f(a,b) + \ldots$
$\ldots\ldots \frac{1}{(n-1)!}\left[(x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right]^{n-1} f(a,b) + R_n$

Example 61:

a_n is bounded, decreasing sequence.
b_n is bounded, increasing sequence
$x_n = a_n + b_n$
$\sum |x_n - x_{n+1}|$
$= \sum |a_n + b_n - a_{n+1} - b_{n+1}|$
$= \sum |a_n - a_{n+1} + b_n - b_{n+1}|$
$\leqslant |a_n - a_{n+1}| + |b_n - b_{n+1}|$
$\leqslant \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$

$\leqslant \varepsilon$
$\sum |x_n - x_{n+1}| \to 0$

Example 62:

$$\sum_{n=2}^{\infty} \frac{1}{n(log)^p}, \quad p > 0$$

$$\log n < n$$

$$\frac{1}{\log n} > \frac{1}{n}$$

$$\left(\frac{1}{\log n}\right)^p > \frac{1}{n^p}$$

$$\frac{1}{n(\log n)^p} > \frac{1}{n^{p+1}}, \quad p+1 > 1$$

$\therefore$ by comparision test,

As $\sum \frac{1}{n^{p+1}}$ convergent $\Rightarrow \sum \frac{1}{n(\log n)}^p$ is convergent.

Example 63:

$$\sum x_n = \frac{1}{2} + \frac{1}{2.2^2} + \frac{1}{n.2^n}$$

$$\left|\frac{x_{n+1}}{x_n}\right| = \left|\frac{\frac{1}{n+1^{2^{n+1}}}}{n2^n}\right|$$

$$= \left|\frac{n.2^n}{(n+1)2^{n+1}}\right|$$

$$= \left|\left(\frac{n}{n+1}\right)\frac{1}{2}\right|$$

$$\lim_{n\to\infty}\left|\frac{x_{n+1}}{x_n}\right| = \lim_{n\to\infty}\left|\left(\frac{1}{1+\frac{1}{n}}\right)\frac{1}{2}\right|$$

by ratio test

$$= \frac{1}{2} < 1$$

$\sum x_n = \dfrac{1}{n2^n}$ is convergent.

Example 64:

$$S = \left\{1 + \frac{(-1)^n}{n}, n \in \mathbb{N}\right\}$$

limit point of $S = 1$

Example 65:

$$\sum x_n = \sum \frac{1}{\sqrt{n}+\sqrt{n-1}}$$

$$n > \sqrt{n}$$

$$\frac{1}{n} < \frac{1}{\sqrt{n}}$$

by Ratio test,

$$\left|\frac{x_{n+1}}{x_n}\right| = \left|\frac{\frac{1}{\sqrt{(n+1)}+\sqrt{n}}}{\frac{1}{\sqrt{n}+\sqrt{n-1}}}\right|$$

$$= \left| \frac{\sqrt{n} + \sqrt{n-1}}{\sqrt{n+1} + \sqrt{n}} \right|$$

$$= \left| \frac{n^{\frac{1}{2}}(1 + \sqrt{1 - \frac{1}{n}})}{n^{\frac{1}{2}}(1 + \sqrt{1 + \frac{1}{n}})} \right|$$

$$\therefore \lim \left| \frac{x_{n+1}}{x_n} \right| = \lim \left| \frac{n^{\frac{1}{2}}(1 + \sqrt{1 - \frac{1}{n}})}{n^{\frac{1}{2}}(1 + \sqrt{1 + \frac{1}{n}})} \right| = 1$$

$\therefore$ Ratio test fails here

$$\sum \frac{1}{\sqrt{n} + \sqrt{n-1}} \times \frac{\sqrt{n} - \sqrt{n-1}}{\sqrt{n} - \sqrt{n-1}} = \sum \sqrt{n} - \sqrt{n-1}$$

$\therefore S_n = \sqrt{n}$ which divergent

$\therefore \sum \frac{1}{\sqrt{n} + \sqrt{n-1}}$ is divergent.

Example 66:

$$\sum \frac{(2n-1)}{n(n+1)(n+2)} = \frac{1}{1.2.3} + \frac{3}{2.3.4} + \ldots$$

$$\left|\frac{x_{n+1}}{x_n}\right| = \left|\frac{\frac{(2n-1)}{n(n+1)(n+2)(n+3)}}{\left(\frac{2n-1}{n(n+1)(n+2)}\right)}\right|$$

$$= \left|\frac{(2n+1)n}{(2n-)(n+3)}\right|$$

$$= \left|\frac{\left(2+\frac{1}{n}\right)}{\left(2-\frac{1}{n}\right)\left(1+\frac{3}{n}\right)}\right|$$

$$\therefore \lim \left|\frac{x_{n+1}}{x_n}\right| = 1$$

$\therefore$ Ratio test fails here.

$$\sum \left(\frac{2n-1}{n(n+1)(n+2)}\right) = \sum \frac{2n}{n(n+1)(n+2)} - \sum \frac{1}{n(n+1)(n+2)}$$

$$= \sum \frac{2}{(n+1)(n+2)} - \sum \frac{1}{n(n+1)(n+2)}$$

$\therefore x_n$ is convergent.