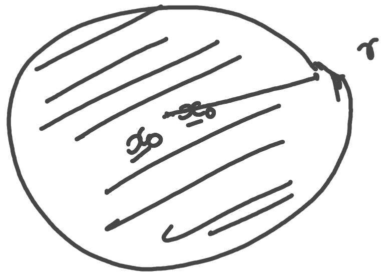
open ball $= \{\underline{x} / \|x - x_0\| < r\}$



Solution. Let $\underline{x}, \underline{y} \in B(\underline{x}_0, r)$ where $\underline{x}_0 \in \mathbb{R}^n$ and $r \geq 0$. Then $\|\underline{x} - \underline{x}_0\| \leq r$ and $\|\underline{y} - \underline{x}_0\| \leq r$. Let $0 \leq \lambda \leq 1$. To show that $\lambda \underline{x} + (1-\lambda)\underline{y} \in B(\underline{x}_0, r)$.

$$
\begin{aligned}
\|\lambda \underline{x} + (1-\lambda)\underline{y} - \underline{x}_0\| &= \|\lambda \underline{x} - \lambda \underline{x}_0 + (1-\lambda) y - \underline{x}_0 + \lambda \underline{x}_0\| \\
&= \|\lambda(\underline{x} - \underline{x}_0) + (1-\lambda)(\underline{y} - \underline{x}_0)\| \\
&\leq |\lambda| \|\underline{x} - \underline{x}_0\| + |1-\lambda| \|\underline{y} - \underline{x}_0\| \\
&= \lambda r + r - \lambda r \\
&= r.
\end{aligned}
$$

This shows that $\lambda \underline{x} + (1-\lambda)\underline{y} \in B(\underline{x}_0, r)$.

Example 1.7 A closed ball in $\mathbb{R}^n$ is a set of type $\{\underline{x} \in \mathbb{R}^n : \|\underline{x} - \underline{x}_0\| \leq r\}$, where $r > 0$ is a convex set.

Solution. Let $S = \{\underline{x} : \|\underline{x} - \underline{x}_0\| \leq r\}$. Let $\underline{x}, \underline{y} \in S$ and $0 \leq \lambda \leq 1$. To show that $\lambda \underline{x} + (1-\lambda)\underline{y} \in S$. Consider

$$
\begin{aligned}
\left\|\lambda \underline{x} + (1-\lambda)\underline{y} - \underline{x}_0\right\| &= \left\|\lambda \underline{x} - \lambda \underline{x}_0 + \lambda \underline{x}_0 + \underline{y} - \lambda \underline{y} - \underline{x}_0\right\| \\
&= \left\|\lambda(\underline{x} - \underline{x}_0) + (1-\lambda)(\underline{y} - \underline{x}_0)\right\| \\
&= \left\|\lambda(\underline{x} - \underline{x}_0) + (1-\lambda)(\underline{y} - \underline{x}_0)\right\| \\
&\leq \lambda r + (1-\lambda) r = r.
\end{aligned}
$$

It shows that

$$\Rightarrow \left\|\lambda \underline{x} + (1-\lambda)\underline{y} - \underline{x}_0\right\| \leq r \Rightarrow \lambda \underline{x} + (1-\lambda)\underline{y} \in S.$$

Theorem 1.1 If C is a convex set then λC is convex set.

Proof. Let C is a convex set. To show λC is convex set. Let $\lambda C = \{\lambda c : c \in C\}$. Let $\underline{x}, \underline{y} \in \lambda C$. Then $\underline{x} = \lambda \underline{c}_1$ and $\underline{y} = \lambda \underline{c}_2$ for some $\underline{c}_1, \underline{c}_2 \in C$. Let $0 \leq \mu \leq 1$. To show that $\mu \underline{x} + (1-\mu)\underline{y} \in \lambda C$. Now

$$
\begin{aligned}
\mu \underline{x} + (1-\mu)\underline{y} &= \mu \lambda \underline{c}_1 + \lambda \underline{c}_2 - \mu \lambda\, \underline{c}_2 \\
&= \lambda(\mu \underline{c}_1 + (1-\mu)\underline{c}_2) \\
&= \lambda \underline{c}_3
\end{aligned}
$$

where $\underline{c}_3 = \mu\underline{c}_1 + (1-\mu)\underline{c}_2 \in C$. This show that $\mu\underline{x} + (1-\mu)\underline{y} \in \lambda C$. Thus λC is a convex set.

Theorem 1.2 If C and D are convex sets, then $C + D$ is also convex set.

Proof. Let C and D be convex sets.Then $C + D = \{\underline{x} + \underline{y} : \underline{x} \in C, \underline{y} \in D\}$. Let $\underline{x}, \underline{y} \in C + D$ and $0 \le \mu \le 1$. Then $\underline{x} = \underline{x}_1 + \underline{y}_1$ and $\underline{y} = \underline{x}_2 + \underline{y}_2$. To show $\mu\underline{x} + (1-\mu) \in C + D$. Consider

$$\begin{aligned}\mu\underline{x} + (1-\mu)\underline{y} &= \mu\left(\underline{x}_1 + \underline{y}_1\right) + (1-\mu)\left(\underline{x}_2 + \underline{y}_2\right)\\ &= \mu\underline{x}_1 + \mu\underline{y}_1 + (1-\mu)\underline{x}_2 + (1-\mu)\underline{y}_2\\ &= \mu\,\underline{x}_1 + (1-\mu)\underline{x}_2 + \mu\underline{y}_1 + (1-\mu)\underline{y}_2\\ &= \underline{x}_3 + \underline{y}_3.\end{aligned}$$

where $\underline{x}_3 = \mu\underline{x}_1 + (1-\mu)\underline{x}_2 \in C$ and $\underline{y}_3 = \mu\underline{y}_1 + (1-\mu)\underline{y}_2 \in D$, so $\underline{x}_3 + \underline{y}_3 \in C + D$ and hence $\mu\underline{x} + (1-\mu)\underline{y} \in C + D$. Thus C + D are convex set.

Theorem 1.3 The intersection of any convex sets is a convex set.

Proof. Let $A = \bigcap_\alpha S_\alpha$where each S_α is convex set. To show that A is a convex set. Let $0 \le \lambda \le 1$ and $\underline{x}, \underline{y} \in A$. Then $\underline{x}, \underline{y} \in S_\alpha$ for each α. Since each S_α is convex set implies $\lambda\underline{x} + (1-\lambda)\underline{y} \in S_\alpha$ and hence $\lambda\underline{x} + (1-\lambda)\underline{y} \in \bigcap_\alpha S_\alpha = A$. Thus intersection of any convex set is convex set.

Theorem 1.4 If $C \subseteq \mathbb{R}^n$ is convex, then $Cl(C)$, the closure of C, is also convex.

Proof. Suppose $\underline{x}, \underline{y} \in Cl(C)$. Then there exist sequences $\{\underline{x}_n\}$ and $\{\underline{y}_n\}$ in C such that $\underline{x}_n \to x$ and $\underline{y}_n \to \underline{y}$ as $n \to \infty$. Let $0 \le \lambda \le 1$. Consider $\underline{z}_n = \lambda\underline{x}_n + (1-\lambda)\underline{y}_n$. Then, by convexity of C, $\underline{z}_n \in C$. Moreover

$$\lim_{n\to\infty}\underline{z}_n = \lim_{n\to\infty}\left(\lambda\underline{x}_n + (1-\lambda)\underline{y}_n\right) = \lambda\underline{x} + (1-\lambda)\underline{y}.$$

Hence $\lambda\underline{x} + (1-\lambda)\underline{y} \in Cl(C)$.

Theorem 1.5 A set $S \subseteq \mathbb{R}^n$ is convex if and only if every convex combination of any finite number of points of S is contained in S.

Proof. Assume that every convex combination of any finite number of points of S is contained in S. To show that the set $S \subseteq \mathbb{R}^n$ is convex set. Let $\underline{x}_1, \underline{x}_2 \in S$ and $\lambda_1 + \lambda_2 = 1$. Then by hypothesis $\lambda_1\underline{x}_1 + \lambda_2\underline{x}_2 \in S$. Thus S is a convex set.

Conversely, suppose that S is a convex set. To prove that every convex combination of any finite number points of S is a point of S. We prove this result by mathematical induction. For $n = 2$, clearly the result is true, since the convex combination of two points of S is contained in S.

Suppose, the result is true for $n = k$, i.e. the convex combination of k points of S is contained in S. Now, we show that the result is true for $n = k+1$. Let $\underline{x}_1, \underline{x}_2, \cdots, \underline{x}_{k+1} \in S$ and $0 \le \lambda_1, \lambda_2, \cdots, \lambda_{k+1} \le 1$ such that $\sum_{i=1}^{k+1} \lambda_i = 1$. Suppose $0 < \lambda_{k+1} < 1$. Now consider

$$\begin{aligned}\lambda_1 \underline{x}_1 + \lambda_2 \underline{x}_2 + \cdots + \lambda_k \underline{x}_k + \lambda_{k+1} \underline{x}_{k+1} &= \sum_{i=1}^{k} \lambda_i \underline{x}_i + \lambda_{k+1} \underline{x}_{k+1} \\ &= (1-\lambda_{k+1}) \left(\sum_{i=1}^{k} \frac{\lambda_i \underline{x}_i}{(1-\lambda_{k+1})} \right) + \lambda_{k+1} \underline{x}_{k+1}.\end{aligned}$$

Clearly $\sum_{i=1}^{k} \frac{\lambda_i}{(1-\lambda_{k+1})} = 1$ and by assumption $\sum_{i=1}^{k} \frac{\lambda_i}{(1-\lambda_{k+1})} \underline{x}_i \in S$. Since S is convex, then $(1-\lambda_{k+1}) \underline{x}_i + \lambda_{k+1} \underline{x}_{k+1} \in S$ implies $\lambda_1 \underline{x}_1 + \lambda_2 \underline{x}_2 + \cdots + \lambda_{k+1} \underline{x}_{k+1} \in S$. Thus every convex combination of any finite number of points of S is contained in S.

Definition 1.11 Let $S \subseteq \mathbb{R}^n$ be a convex set. A point $\underline{x} \in S$ is called an extreme point or vertex of S if there exists no points $\underline{x}_1$ and $\underline{x}_2$ in S such that $x = \lambda \underline{x}_1 + (1-\lambda) \underline{x}_2, 0 < \lambda < 1$.

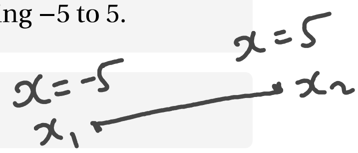
Definition 1.12 Let $S \subseteq \mathbb{R}^n$. The intersection of all the convex sets containing the set S is called the convex hull of S and it is denoted by $Co(S)$.

Definition 1.13 Let $S = \{\underline{x}, \underline{y}\} \subseteq R^2$. Then $Co(S)$ is the line segment joining $\underline{x}$ and y.

Example 1.8 Let $S = \{\underline{x}, \underline{y}, \underline{z}\} \subseteq R^2$. Convex hull of these three points is the solid part of triangle.

Example 1.9 If S is convex set then convex hull of this set is itself i.e. $Co(S) = S$.

Example 1.10 If $S = \{\underline{x} \in \mathbb{R} : |x| = 5\}$ then $Co(s)$ is the line segment joining -5 to 5.



Example 1.11 If $S = \{\underline{x} : ||\underline{x}|| > 5\} \subseteq \mathbb{R}^2$ then $Co(S) = \mathbb{R}^2$.