Unit IV: Integral Calculus

Multiple Integrals: Evaluation of double and triple integrals, evaluation of double integrals by change of order of integration, changing into polar coordinates. Applications to find Area and Volume by double integral. Problems.

Beta and Gamma functions: Definitions, properties, relation between Beta and Gamma functions. Problems

I. Evaluate the following:

1. $\int_1^2 \int_1^3 xy^2\, dx\, dy$
2. $\int_0^1 \int_0^6 xy\, dx\, dy$
3. $\int_0^5 \int_0^{x^2} x(x^2+y^2)\, dx\, dy$
4. $\int_0^1 \int_x^{\sqrt{x}} (x^2+y^2)\, dx\, dy$
5. $\int_0^1 \int_x^{\sqrt{x}} xy\, dx\, dy$
6. $\int_0^1 \int_0^{\sqrt{1-y^2}} x^3 y\, dy\, dx$
7. $\int_0^2 \int_1^3 \int_1^2 xy^2 z\, dx\, dy\, dz$
8. $\int_0^a \int_0^b \int_0^c (x^2+y^2+z^2)\, dx\, dy\, dz$
9. $\int_{-c}^{c} \int_{-b}^{b} \int_{-a}^{a} (x^2+y^2+z^2)\, dx\, dy\, dz$
10. $\int_{-1}^{1} \int_0^z \int_{x-z}^{x+z} (x+y+z)\, dx\, dy\, dz$
11. $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} xyz\, dx\, dy\, dz$
12. $\int_0^{\log 2} \int_0^x \int_0^{x+\log y} e^{x+y+z}\, dx\, dy\, dz$
13. $\int_0^a \int_0^x \int_0^{x+y} e^{x+y+z}\, dx\, dy\, dz$
14. $\int_0^1 \int_0^2 \int_0^{2-x-y} dx\, dy\, dz$
15. $\int_1^e \int_1^{\log y} \int_1^{e^x} \log z\, dx\, dy\, dz$

II. Evaluate the following integrals by changing the order of integration:

1. $\int_0^1 \int_x^{\sqrt{x}} xy\, dx\, dy$
2. $\int_0^1 \int_{\sqrt{y}}^1 xy\, dx\, dy$
3. $\int_0^1 \int_x^1 \frac{x}{x^2+y^2}\, dx\, dy$
4. $\int_0^{4a} \int_{x^2/4a}^{2\sqrt{ax}} xy\, dx\, dy$
5. $\int_0^1 \int_0^{\sqrt{1-x^2}} y^2\, dx\, dy$
6. $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\, dx\, dy$
7. $\int_0^a \int_y^a \frac{x}{x^2+y^2}\, dx\, dy$

III. Evaluate the following integrals by changing into polar coordinates:

1. $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\, dx\, dy$
2. $\int_{-a}^{a} \int_0^{\sqrt{a^2-x^2}} \sqrt{x^2+y^2}\, dx\, dy$
3. $\int_0^a \int_0^{\sqrt{a^2-y^2}} y\sqrt{x^2+y^2}\, dx\, dy$

IV. Application to Area and Volume:

1. Find the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$
2. Show that the area between the parabolas $y^2 = 4ax$ and $x^2 = 4ay$ is $16a^2/3$
3. Find the volume of the solid bounded by the planes x = 0, y = 0, z = 0 and x + y +z = 1
4. Find the volume of the tetrahedron bounded by the planes x = 0, y = 0, z = 0 and x/a + y/b +z/c = 1

Beta and Gamma Function

1. Define Gamma function and show that $\Gamma(n+1) = n\Gamma(n)$ and $\Gamma(n+1) = n!$.
2. Define Beta function and prove that $\beta(m,n) = \beta(n,m)$, $\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$(No proof)
3. Evaluate $\beta\left(\frac{9}{2},\frac{7}{2}\right)$
4. Prove that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ (No proof)
5. Evaluate $\int_0^{\frac{\pi}{2}} \sin^5\theta \cos^7\theta \, d\theta$
6. Evaluate $\int_0^{\frac{\pi}{2}} \sqrt{tan\theta} \, d\theta$.
7. Evaluate $\int_0^{\frac{\pi}{2}} \sqrt{cot\theta} \, d\theta$.
8. Prove that $\int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{sin\theta}} d\theta \int_0^{\frac{\pi}{2}} \sqrt{sin\theta} \, d\theta = \pi$.
9. Evaluate $\int_0^1 \frac{dx}{\sqrt{1-x^4}}$.
10. Evaluate $\int_0^4 x^{\frac{3}{2}}(4-x)^{\frac{5}{2}} dx$.
11. Show that $\int_{-1}^{1}(1+x)^{p-1}(1-x)^{q-1}dx = 2^{p+q-1}\beta(p,q)$.
12. Express $\int_0^2 (4-x^2)^{\frac{3}{2}} dx$ in terms of beta function.
13. Prove that $\int_0^1 \frac{x^2}{\sqrt{1-x^4}} dx \int_0^1 \frac{x^2}{\sqrt{1+x^4}} dx = \frac{\pi}{4\sqrt{2}}$.
14. Express $\int_0^\infty 4x^4 e^{-x^4} dx$ in terms of Gamma function.
15. Evaluate $\int_0^\infty \sqrt{x}\, e^{-x^3} dx$.
16. Prove that $\int_0^\infty x\, e^{-x^8} dx \int_0^\infty x^2 e^{-x^4} dx = \frac{\pi}{16\sqrt{2}}$.
17. Prove that $\int_0^\infty \frac{e^{-x^2}}{\sqrt{x}} dx \int_0^\infty x^2 e^{-x^4} dx = \frac{\pi}{4\sqrt{2}}$.