UNIT – II QUESTION BANK

SERIES EXPANSION AND MULTI VARIABLE CALCULUS

Taylor's and Macluarin's expansions of function of one variable (without proof). Indeterminate forms L' Hospital Rule (without proof). Partial differentiation: Partial derivatives, Total derivative and composite function. Jacobians, Maxima & Minima for a function of two variables.

TAYLOR'S AND MACLAURIN'S EXPANSIONS FOR FUNCTION OF VARIABLE (WITHOUT PROOF):

1. State Taylor's Theorem
2. Maclaurin's series expansion of f(x) up to fifth degree term is...........
3. Expand $f(x)=e^x$ in a series of increasing powers of $(x-1)$ up to Forth degree terms
4. Expand the following function using Maclaurin's series up to the term containing x^4.

   i) Sinx   ii) Cosx
5. Expand $\sqrt{1+\sin 2x}$ by the Maclaurin's series up to and including fifth degree terms
6. Using Maclaurin's series, expand $\tan x$ up to the term containing x^5.
7. Expand log(secx) by the Maclaurin's series up to terms containing x^6.
8. Using Maclaurin's series, expand $tan^{-1}x$ up to the term containing x^5.
9. Expand $\log(1+\sin x)$ by the Maclaurin's series up to and including third degree terms.
10. Expand log(1+cosx) by the Maclaurin's series up to and including third degree terms.
11. Expand $\log(1+e^x)$ by Maclaurin's series, up to the term containing x^4.
12. Expand $e^{\sin x}$ by Maclaurin's series, up to the term containing x^4.

INDETERMINATE FORMS L' HOSPITAL RULE (WITHOUT PROOF):

Evaluate the following limits:

1. $\lim_{x\to 0} x^x$
2. $\lim_{x\to \frac{\pi}{4}} (\tan x)^{\tan 2x}$
3. $\lim_{x\to 0} (a^x + x)^{\frac{1}{x}}$
4. $\lim_{x\to 0} (1+\sin x)^{\cot x}$

5. $\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{\frac{1}{x^2}}$

6,, $\lim_{x\to 0}\left(\frac{\tan x}{x}\right)^{\frac{1}{x^2}}$

7. $\underset{x \to 0}{Lim}\left(\frac{a^x+b^x+c^x+d^x}{4}\right)^{\frac{1}{x}}$

8 $\underset{x \to 0}{Lim}\left(\frac{a^x+b^x+c^x}{3}\right)^{\frac{1}{x}}$

9. $\lim_{x\to \infty}\left(\frac{ax+1}{ax-1}\right)^{x}$

Partial Derivatives:

1. If $u=\log\left(x^3+y^3+z^3-3xyz\right)$, show that i) $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=\frac{3}{x+y+z}$

   ii) $\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u=-9(x+y+z)^{-2}$

1. If $v=\log\left(x^2+y^2+z^2\right)$, prove that $\left(x^2+y^2+z^2\right)\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)=2$

2. If $v=\left(x^2+y^2+z^2\right)^{-\frac{1}{2}}$, prove that $\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}=0$

3. If $z(x+y)=x^2+y^2$, show that $\left(\frac{\partial z}{\partial x}-\frac{\partial z}{\partial y}\right)^2=4\left(1-\frac{\partial z}{\partial x}-\frac{\partial z}{\partial y}\right)$

4. If $z=f(x+ct)+\phi(x-ct)$, prove that $\frac{\partial^2 z}{\partial t^2}=c^2\frac{\partial^2 z}{\partial x^2}$

5. If $u=x^2\tan^{-1}\left(\frac{y}{x}\right)-y^2\tan^{-1}\left(\frac{x}{y}\right)$, show that $\frac{\partial^2 u}{\partial x\partial y}=\frac{x^2-y^2}{x^2+y^2}$

6. If $u=x^y$, show that $\frac{\partial^2 u}{\partial x\partial y}=\frac{\partial^2 u}{\partial y\partial x}$

7. If $z=e^{ax+by}f(ax-by)$, prove that $b\frac{\partial z}{\partial x}+a\frac{\partial z}{\partial y}=2abz$

8. If $u=f(r)$, where $r=\sqrt{x^2+y^2+z^2}$, show that $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=f''(r)+\frac{2}{r}f'(r)$

9. Let $r^2=x^2+y^2+z^2 and \quad v=r^m$, prove that $v_{xx}+v_{yy}+v_{zz}=m(m+1)r^{m-2}$

10. If $v=\log(\tan x+\tan y+\tan z)$, show that $\sin 2x\, u_x+\sin 2y\, u_y+\sin 2z\, u_z=2$

Total Derivative

1. If $u = x^2 + y^2 + z^2$ and $x = e^t, y = e^t \cos t, z = e^t \sin t$ Find $\frac{du}{dt}$ as a total derivative and verify the result by direct substitution.
2. If x increases at the rate of 2 cm/sec at the instant when x = 3cm and y=1 cm at what rate must y be changing in order that the function $2xy - 3x^2 y$ shall be neither increasing nor decreasing?
3. At a given instant the sides of a rectangle are 4 ft. and 3 ft. respectively and they are increasing at the rate of 1.5 ft./sec. and 0.5 ft./sec respectively, find the rate at which the area is increasing at that instant.
4. If $z = 2xy^2 - 3x^2 y$ and if x increases at the rate of 2 cm. per second and it passes through the value x = 3 cm., show that if y is passing through the value y = 1 cm., y must be decreasing at the rate of $2\frac{2}{15}$ cm. per second, on order that z shall remain constant.

Composite function

1. If $z = f(x, y)$ and $x = e^u + e^{-v}$, $y = e^{-u} - e^v$ Prove that $\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$.
2. If $u = f(x, y)$ and $x = r\cos\theta, y = r\sin\theta$,Prove that $\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = \left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2$.
3. If $u = f(2x-3y, 3y-4z, 4z-2x)$,Prove that $\frac{1}{2}\frac{\partial u}{\partial x} + \frac{1}{3}\frac{\partial u}{\partial y} + \frac{1}{4}\frac{\partial u}{\partial z} = 0$.
4. If $u = F(x-y, y-z, z-x)$ prove that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$
5. If $u = u\left(\frac{y-x}{xy}, \frac{z-x}{xz}\right)$ Show that $x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z} = 0$.
6. If $u = f(r, s, t)$ and $r = \frac{x}{y}, s = \frac{y}{z}, t = \frac{z}{x}$ Prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$
7. If $u = f(e^{y-z}, e^{z-x}, e^{x-y})$, Prove that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$.

Jacobians :

1. If $x = u^2 - v^2$ and $y = 2uv$, Show that $J\left(\frac{x, y}{u, v}\right) = 4(u^2 + v^2)$
2. If $x = r\cos\theta, y = r\sin\theta, z = z$, then find $\frac{\partial(x, y, z)}{\partial(r, \theta, z)}$
3. If $x = r\sin\theta\cos\phi, y = r\sin\theta\sin\phi, z = r\cos\theta$, then show that $\frac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = r^2\sin\theta$
4. If $u = x + 3y^2 - z^3, v = 4x^2yz, w = 2z^2 - xy$ Evaluate $\frac{\partial(u, v, w)}{\partial(x, y, z)}$ at(1,-1,0)

5. If $u = \frac{yz}{x}$, $v = \frac{zx}{y}$, $w = \frac{xy}{z}$ Show that $\frac{\partial(u\,v\,w)}{\partial(x\,y\,z)} = 4$

6. If $y_1 = \frac{x_2 x_3}{x_1}$, $y_2 = \frac{x_3 x_1}{x_2}$, $y_3 = \frac{x_1 x_2}{x_3}$, Show that the Jacobian of y_1, y_2, y_3 with respect to x_1, x_2, x_3 is 4.

7. If $u = xyz$, $v = x^2 + y^2 + z^2$, $w = x + y + z$, find $\frac{\partial(x,y,z)}{\partial(u,v,w)}$

8. If $u = x + y + z, uv = y + z, uvw = z$, show that $\frac{\partial(x,y,z)}{\partial(u,v,w)} = u^2 v$

9. If $u = x^2 + y^2 + z^2, v = xy + yz + zx, w = x + y + z$, find $\frac{\partial(u,v,w)}{\partial(x,y,z)}$.

Maxima and Minima for function of Two Variables

Find the extreme values of the following functions:

(1) $f(x,y) = x^3 + y^3 - 3x - 12y + 20$

(2) $f(x,y) = 2xy - 5x^2 - 2y^2 + 4x + 4y - 6$

(3) $f(x,y) = 2(x-y)^2 - x^4 - y^4$

(4) $f(x,y) = x^2 y^2 - 5x^2 - 8xy - 5y^2$

(5) $f(x,y) = x^3 y^2 (1 - x - y)$

(6) $f(x,y) = x^3 + y^3 - 3axy, a > 0$