Study on eigen values and eigen vectors of matrices: An iterative approach, GANIT J. Bangladesh Math. Soc. 37(2017) 39-50. (ISSN Print. 1606-3694 ; online 2224-5111)

1 author:

Abdur Jamali
Khulna University of Engineering and Technology

GANIT *J. Bangladesh Math. Soc.* (ISSN 1606-3694) **37** (2017) 51-61

STUDY ON EIGEN VALUES AND EIGEN VECTORS OF MATRICES : AN ITERATIVE APPROACH

A. R. M. Jalal Uddin Jamali*, Md. Hasibul Haque and Md. Sah Alam
Department of Mathematics, Khulna University of Engineering and Technology (KUET),
*Corresponding author: armjamali@yahoo.com

Received 24.07.2016 Accepted 11.11.2017

ABSTRACT

Power method is frequently used for finding largest Eigen-pair. On the other hand, Inverse Power method is utilized to find smallest Eigen-pair. Using shifting property, Power method and/or Inverse Power method can be used to find out other desired Eigen pairs too. Several lemmas based on Power method with shifting property are presented here. Moreover, a Modified Hybrid Iterative Algorithm based upon both Power method and Inverse Power method is proposed to find both largest and smallest Eigen-pairs simultaneously with ease. Several experiments have been performed to investigate the robustness and effectiveness of the algorithm. The proposed algorithm is able to find both (largest and smallest) Eigen-pairs successfully and efficiently. Moreover, the proposed algorithm is able to find out the nature (positive and negative sign) of the Eigen values and in some cases the algorithm is also able to find out the second largest Eigen pair in consequence.

Keywords: Eigen value, Eigen vector, Power method, Inverse Power method, Iterative method, Modified Hybrid Iterative Algorithm.

1. Introduction

Eigen value problems arise naturally from a wide variety of scientific and engineering applications: such common applications include structural dynamics [17], quantum mechanics [18], electrical networks, control theory and design [14,20], bioinformatics [10], acoustic field simulations [13], electromagnetic modeling of particle accelerators [11], Markov chains [4], pattern recognition [8], stability analysis [14], mathematical physics [3], image processing [7], geophysics, molecular spectroscopy, particle physics [6] and many other areas. Some efficient methods are available in the literatures which are dedicated to find out only Eigen values. But for finding corresponding Eigen vectors much more efforts are necessary. The best known direct method is the QR-algorithm which is able to find all Eigen values. This method is based on the QR decomposition of a matrix and this method is also implemented in Software package like Mat Lab function **eig( ).** The QR iteration algorithm for computing the Eigen values of a general matrix came from an elegantly simple idea that was proposed by Heinz Rutishauser in 1958 and refined by Francis in 1961-1962. But this method is failed to find corresponding Eigen vectors.

On the other hand, Power method for Eigen value problems is implemented in various fields of application where only largest Eigen pairs are important [2, 5, 12 and 16]. Again Inverse Power method is used to find out smallest Eigen pair. Moreover, using shifting property, Power method and/or Inverse Power method is also applicable to find out desired Eigen-pair. It is important to note that various areas of science and engineering seek both largest as well as smallest Eigen-pairs for different reasons other than algorithmic gains. There exist some important fields of application in which second largest (smallest) Eigen-pair are also required.

Beside Power method and Inverse Power method, there are several methods available which are able to find both Eigen values and Eigen vectors. Chang et al. [1] proposed several refinements of the Power method that enable the computation of multiple extreme Eigen-pairs of very large matrices by using Monte Carlo simulation method. Panju [15] examined some numerical iterative methods for computing the Eigen values and Eigen vectors of real matrices. He examined five methods – from the simple Power iteration method to the more complicated QR iteration method. The derivations, procedure, and advantages of each method are briefly discussed there. Besides, Zhang P. et al. [21] have recently developed a general solution strategy for Power method. Recently, Li et al. [19] propose a bound for ratio of the largest Eigen value and second largest Eigen value in module for a higher-order tensor. From this bound, one may deduce the bound of the second largest Eigen value in module for a positive tensor, and the bound can reduce to the matrix cases. In [21], a general solution strategy of the modified power iteration method for calculating higher Eigen-modes has been developed and applied in continuous energy Monte Carlo simulation. A variant of power method is applied to continuous energy Monte Carlo simulation. After the brief literature review, which is presented in section 1 the article is organized as follows:

In Section 2, we present some preliminaries related to the proposed algorithm. The proposed Hybrid Iterative algorithm is presented in Section 3. The experimental study as well as discussion is delivered in Section 4. Finally conclusion is drawn in Section 5.

2. Preliminaries

Before presenting the proposed algorithm it is worthwhile to discuss some related issues. If A be a square matrix, then the matrix Eigen value problem is defined as follows:

$$A\mathbf{x} = \lambda\mathbf{x} \tag{1}$$

where λ is called Eigen value and $\mathbf{x}$ ($\neq 0$) is called corresponding Eigen vector in brief (λ, $\mathbf{x}$) is called Eigen-pair.

It is noted that the Power method is able to find out the largest Eigen value which is denoted here as *First Eigen value*. On the other hand by using shifting property the Power method is able to find out further Eigen value. We call this shifted Eigen value as *Second Eigen* value provided the shifted parameter should be *first Eigen* value. Incorporating these two terminologies we have the properties and results (Lemma 1 – 4) below.

Property (*First* Eigen value): The Eigen value obtained by Power method is the largest in magnitude among all Eigen values but it is not necessarily the largest in actual value.

Property (Shifting Property): If (λ, $\mathbf{x}$) be any Eigen-pair of equation (1) and α be any scalar quantity, then (λ- α) be an Eigen value of $(\mathbf{A}-\boldsymbol{\alpha}\mathbf{I})$. So we have

$$(\mathbf{A}-\boldsymbol{\alpha}\mathbf{I})\mathbf{x}=(\boldsymbol{\lambda}-\boldsymbol{\alpha})\mathbf{x} \tag{2}$$

Here (λ- α, $\mathbf{x}$) be the Eigen-pair of the shifting Eigen value problem (2).

Property (*Second* Eigen value): Using shifting property, shifting with *first* (largest) Eigen value, the Modified Power method is able to find out another Eigen value and corresponding Eigen vector.

Using these properties in power method Jamali and Alam [9] have proposed some Lemmas which are stated below.

Lemma 1: If *first* (largest) Eigen value is positive and *second* Eigen value is also positive (produced by shifting largest one) then *first* Eigen value be the largest both in magnitude as well as in actual value. On the other hand the *second* Eigen value is the smallest both in magnitude as well as in actual value. In consequence, all Eigen values are positive in sign.

Lemma 2: If *first* (largest) Eigen value is positive and *second* Eigen value is negative (produced by shifting largest one) then the *first* Eigen value be the largest both in magnitude as well as in actual value. On the other hand the *second* Eigen value (obtained by the algorithm) is the smallest in actual value but it is the largest in magnitude among all negative Eigen values (if any). In consequence, some Eigen values along with largest Eigen value are positive and some Eigen values are negative in sign (if exist).

Lemma 3: If *first* (largest) Eigen value is negative and *second* Eigen value is also negative (produced by shifting largest one) then *first* Eigen value be the largest in magnitude but smallest in actual value. On the other hand the *second* Eigen value is the smallest in magnitude but largest in actual value. In consequence, all Eigen values are negative in sign.

Lemma 4: If *first* (largest in magnitude) Eigen value is negative and *second* Eigen value is positive (produced by shifting largest one) then *first* Eigen value be the largest in magnitude but smallest in actual value. On the other hand the *second* Eigen value is the largest in actual value and it is also largest among all positive Eigen values (if any). In consequence, some Eigen values along with largest Eigen values are negative in sign and some Eigen values are positive in sign (if exist).

For proofs and detailed discussion see our paper [9]. In the following section, we have developed a Modified Hybrid Iterative Algorithm based on these Lemmas. Actually the proposed approach is the algorithm of sequential uses of Power method and Inverse Power method when necessary. These Lemmas control the flow of execution of the proposed algorithm implicitly. Moreover they help to identify the nature of Eigen values explicitly.

3. Proposed Modified Hybrid Iterative Algorithm

Since our proposed algorithm is based on Power method as well as Inverse Power method it is of importance to have a brief look on the algorithm of these two methods [see Table 1 and Table 2]. It is worthwhile to mention that by using shifting property, the Power method, sometimes, fails to find out absolute smallest Eigen-pair because of the nature of the Eigen spectrum. In such condition, Inverse Power method is necessary to find out the smallest Eigen pair.

Table 1: Algorithm of Power Method

```
Power Method ( ){
    Step (1): read A
                  set   y = x_0
                  set   ξ= ξ_0
                  set   I_max
                  for   k = 1,2, …, I_max do
                        {
    Step (2):       ν = y / ||y||_2
    Step (3):       y = Aν
    Step (4):       θ = ν * y
    Step (5):       if ||y – θ ν||_2 ≤ξ |θ|,
                    set (λ , x )= (θ , ν)
                  else continue
                  }
              end for
    Step (6): accept (λ , x )= (θ , ν)
    }
```

Table 2: Algorithm of Inverse Power Method

```
Inverse Power Method ( ){
    Step (1):  read A
                  set  x = z_0
                   set   ζ= ζ_0
                  set   I_max
                  for k = 1,2,, I_max  do
                  {
     Step (2):    Ay = x
     Step (3):    x = y / ||y||_2
     Step (4):    v = Ax
     Step (5):    θ = x * v
     Step (6):    if ||y –  v||_2 ≤ζ
                  set (λ , x )= (θ , x)
                  else continue
                }
                  end for
     Step (7): accept (λ , x ) =(1/θ, x)
     }
```

By exploiting the lemma and hybridizing both the Power method and Inverse Power method, we have developed the Hybrid Modified Iterative (HMI) algorithm which can find out the largest as well as smallest Eigen-pairs and also the nature of the Eigen spectra. The algorithm of the proposed HMI is given below:

Modified Hybrid Iterative Algorithm ( )

{

Step (1): read **A**

Set **B=A**

Set $\{\lambda, \mathbf{x}\} = \{\lambda_0, \mathbf{x_0}\}$

for r =1, 2 **do**

{

if $\boldsymbol{r}$ =1

{

Step (2): apply **Power method ( )**

output {λ_1, $\mathbf{x_1}$}

Step (3) : find s_1, such that $\lambda_1 = s_1|\lambda_1|$

Step (4): output {λ_1, $\mathbf{x}_1$, s_1}

$r = r+1$

}

else if r =2

{

Step (5): set $\mathbf{B}=\mathbf{A}- \lambda_1\mathbf{I}$

Step (6): apply **Power method ( )**

output {σ_2, $\mathbf{y_2}$}

Step (7) : $\lambda_2 = \sigma_2 + \lambda_1$

find s_2, such that $\lambda_2 = s_2|\lambda_2|$

Step (8): output {λ_2, $\mathbf{x}_2$, s_2}

}

}

end for

Step (9) : **if** ($s_1 = s_2$ and > 0)

{

Output : {(λ_1, $\mathbf{x}_1$), (λ_2, $\mathbf{x}_2$),(all $\lambda_i \geq 0$)}

Step (10) : **Stop**

}

else if ($s_1 = s_2$ and <0)

{

Output : {(λ_1, $\mathbf{x}_1$), (λ_2, $\mathbf{x}_2$),(all $\lambda_i \leq 0$)}

Step (11) : **Stop**

}

else if ($s_1 \neq s_2$ and $s_1 > 0$)

{

Output : {(λ_1, $\mathbf{x}_1$), (λ_2, $\mathbf{x}_2$),(sign of all λ_i)}

Step (12) : **continue**

}

else if ($s_1 \neq s_2$ and $s_1 < 0$)

{

Output : {(λ_1, $\mathbf{x}_1$), (λ_2, $\mathbf{x}_2$),(sign of all λ_i)}

Step (13) : **continue**

}

Step (14) ***:*** Set **B =A**

Set{λ, **x**}={λ_0, $\mathbf{x_0}$}

Step (15) : **Apply Inverse Power method ( )**

output $\{\lambda_3^*, \mathbf{x_3}\}$

Step (16) : find s_3, such that $\lambda_3^* = s_3|\lambda_3^*|$

Step (17): Output : {(λ_1, $\mathbf{x}_1$), (λ_3^*, $\mathbf{x}_3$),(sign of all λ_i)}

Stop (18): **Stop** and **end**

}

The proposed MHI algorithm is able to find out both the largest and the smallest Eigen-pairs, and the nature of Eigen spectrums. Moreover, in some cases the algorithm is also able to find out second smallest Eigen-pairs successfully. In the pseudo-code of the proposed algorithm, we have observed that there is a *for loop*, with index r = 1 and 2, which is analogous to Power method. The first larger loop will start with r =1. When r =1, for the call of function **Power method ( )**, the algorithm will be able to produce largest Eigen pair. In Step (4), the algorithm will determine the sign of largest Eigen value, which is helpful for the identification of second Eigen value. After execution of Step (4), the value of r will be increased to 2. So, in the second iteration within this loop, the algorithm skips step (2) to step (4) and as a result, the algorithm will start execution from Step (5). In Step (5) the original matrix **A** is transformed to **B** by the shifting element λ_1 such that Eigen values of **B**are Eigen values of **A** but shifted by λ_1(the largest Eigen value of **A**). Again the algorithm calls the function **Power method( )**. Therefore, again the function **Power method( )** produces the largest Eigen pair of **B** rather than **A.** Consequently, in step (8), the algorithm finds out second Eigen pairs of the given matrix **A** successfully. As the value of r = 2, the algorithm escape from the first major loop and enter into next step namely Step (9). The Step (9) consists of some conditional arguments. If the sign of both Eigen values are same then without executing the function **Inverse Power method ( )** the algorithm is able to find out both absolutely largest and smallest (ignore the sign) Eigen values, corresponding Eigen vectors and nature of Eigen spectra. But if the sign of both Eigen values are not same, then second Eigen value produced by **Power method ( )** is not absolutely (ignoring sign) smallest Eigen value though smallest in value. Therefore the algorithm proceeds to next steps i.e. Step (14), (15), (16), (17) and finally (18). When the algorithm executes Step (15) the **Inverse Power method ( )** function is run. As a result the absolute smallest Eigen value and corresponding Eigen vector along with nature of the Eigen values are found.

4. Experimental Results and Discussion

Test Prob. 1

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 4 & 5 & 6 & 7 & 8 & 3 & 5 & 8 & 1 & 2 \\ 2 & 2 & 9 & 8 & 5 & 6 & 5 & 2 & 1 & 0 & 2 & 0 \\ 4 & 9 & 1 & 2 & 9 & 5 & 6 & 5 & 0 & 2 & 1 & 2 \\ 5 & 8 & 2 & 9 & 5 & 2 & 1 & 0 & 2 & 1 & 3 & 4 \\ 6 & 5 & 9 & 5 & 5 & 2 & 1 & 2 & 0 & 5 & 8 & 9 \\ 7 & 6 & 5 & 2 & 2 & 1 & 9 & 5 & 6 & 2 & 1 & 4 \\ 8 & 5 & 6 & 1 & 1 & 9 & 1 & 2 & 3 & 4 & 2 & 5 \\ 3 & 2 & 5 & 0 & 2 & 5 & 2 & 2 & 0 & 1 & 0 & 2 \\ 5 & 1 & 0 & 2 & 0 & 6 & 3 & 0 & 1 & 2 & 5 & 9 \\ 8 & 0 & 2 & 1 & 5 & 2 & 4 & 1 & 2 & 6 & 2 & 2 \\ 1 & 2 & 1 & 3 & 8 & 1 & 2 & 0 & 5 & 2 & 1 & 0 \\ 2 & 0 & 2 & 4 & 9 & 4 & 5 & 2 & 9 & 2 & 0 & 5 \end{bmatrix}$$

In order to test the effectiveness of the proposed algorithm and validity of the lemmas simultaneously (as the algorithm incorporates these lemmas implicitly) we have implemented the algorithm and performed several intensive numerical experiments. At first we have performed experiments on a12×12 **Test Prob. 1**and output is reported in the Table 3. To test the validity of the proposed algorithm and the lemmas, the problem is also solved by Mat Lab solver and output is also incorporated in the Table 3 namely last column of the table. It is observed that the proposed algorithm finds the first Eigen value 43.6996and sign of the Eigen value is positive. On the other hand, we have also observed that the largest Eigen value obtained by the Mat Lab solver is also 43.6996. That is the algorithm is successfully able to find out the largest Eigen value. Now we observe that the second Eigen value obtained by the algorithm is ***-15*.8665** and sign of the second Eigen value is negative. So according to the lemma 2, the second Eigen value is not smallest (in magnitude) Eigen value but largest Eigen value among all the negative Eigen values. Moreover, the spectrums of Eigen value consist of both positive and negative values in which the largest value is of positive sign. We observe that the experimental results agree with the lemmas. That is the second Eigen value is not smallest Eigen value regarding magnitude but largest in magnitude among all negative value though smallest Eigen value if we consider sign of each Eigen value. Now since the second Eigen value is not absolutely smallest Eigen value, so for finding out the smallest (in magnitude)

Table 3: Finding Eigen pairs and comparison of values for the **Test Prob. 1**

<table>
<tr><th colspan="4">Eigen pairs and sign of Eigen value obtained by
Modified Hybrid Iterative Algorithm</th><th>Eigen values
(Mat Lab).</th></tr>
<tr><th>Eigen value</th><th>Sign</th><th>Eigen values (λ)</th><th>Eigen vector
(X)</th><th></th></tr>
<tr><td>First Eigen Value
(Largest Eigen value)
(in magnitude)</td><td>+ *ve*</td><td>*43.6996*</td><td>[0.92073, 0.782303, 0.851719, 0.767980, 1.000000, 0.876090 , 0.849523, 0.453403, 0.599790, 0.641932, 0.480074, 0.799947]</td><td rowspan="4">*43.6996*
12.1565
10.8018
7.8891
5.9091
1.8160
-0.7210
-4.9082
-6.5833
-8.4682
- 0.7250
- 15.8666</td></tr>
<tr><td>Second Eigen value
(second Largest negative Eigen value (among the negative values)</td><td>- *ve*</td><td>*-15.8665*</td><td>[-0.708442,-0.233479, -0.394794,0.193943, 1.00000, -0.124613,0.721741, 0.195695, 0.801014,0.0019818, -0.731598, -0.876232]</td></tr>
<tr><td>Smallest Eigen value
(in magnitude)</td><td>- *ve*</td><td>*-0.72098*</td><td>[0.10929, -0.261378, -0.274267, 0.241883, -0.106012,0.0853739, -0.394766, 1.000000, 0.174666, -0.0421989, 0.421505, -0.26894]</td></tr>
<tr><td>Second smallest positive Eigen value</td><td>+ *ve*</td><td>*1.815937*</td><td>[-0.115199 ,0.224341, -0.00357368, -0.416398 , -0.0414743 ,0.189909, 0.21043 , -0.408524,0.382283, -0.228485 ,1 , -0.535562]</td></tr>
</table>

Eigen values and corresponding Eigen vectors, the algorithm needs to execute further steps and the next subsequent step has executed **Inverse Power method ( )** function. Thus the algorithm finds out the Eigen value which value is *-0.72098* and sign is negative**.** It is observed that the smallest Eigen value obtained by the Mat Lab is *-0.7210* which is almost identical with the Eigen value obtained by the proposed algorithm. We also observe that the algorithm is able to find not only Eigen values but also able to find out corresponding Eigen vectors which is displayed in the column 4 of the Table 3.

Now for further experimental study, we have considered the **Test Prob. 2** given below. We observe that in the **Test Prob. 2**, the transformation matrix **F** is of order 8. The experimental results are displayed in the Table 4. To compare the results obtained by our proposed algorithm and Mat Lab solver we have juxtaposed the results in the Table 4 also.

Test Prob. 2

$$\begin{bmatrix} 57 & 37 & 50 & 21 & 25 & 22 & -3 & 20 \\ 37 & 44 & 40 & 28 & 12 & 29 & 27 & 07 \\ 50 & 40 & 85 & 31 & 40 & 36 & 23 & 22 \\ 21 & 28 & 431 & 80 & 7 & 18 & 16 & 15 \\ 25 & 12 & 40 & 07 & 59 & 20 & 15 & 08 \\ 22 & 29 & 36 & 18 & 20 & 34 & 24 & 10 \\ -3 & 27 & 23 & 16 & 15 & 24 & 11 & -3 \\ 20 & 07 & 02 & 15 & 8 & 10 & -3 & 61 \end{bmatrix}$$

It is observed that the sign of both first and second Eigen values obtained by the proposed algorithm are positive. So according to the lemmas the first Eigen value should be absolutely largest Eigen values whereas the second one should be absolutely smallest Eigen value. Moreover according to the lemma all the Eigen values should be positive in sign. By comparing the experimental results with Mat Lab value, we may conclude that the proposed algorithm able to find out largest and smallest Eigen-pairs efficiently and legitimated the lemmas.

Table 4: Finding Eigen pairs and comparison of Eigen values for the **Test Prob. 2**

Eigen pairs and sign of Eigen value obtained by Modified Hybrid Iterative Algorithm				Mat Lab
Eigen value	Eigen values (λ)	Sign	Eigen vector (X)	All Eigen values
1st Eigen value (Largest Eigen value)	*229.048*	+ve	[0.905122,0.789705,0.877491,0.733968,1, 646421,0.474731,0.91332,0.668277]	*229.0477* 108.0800 68.7215 50.3789 38.0092 18.0295 15.2773***2.4559***
2nd Eigen value (Smallest Eigen value)	*2.456*	+ve	[-0.714502, 1, 0.109971, -0.100363, 0.257215, -0.540873 ,-0.191723, 0.156506]	

5. Conclusion

By exploiting the Lemmas we have developed a Hybrid Modified Iterative (HMI) algorithm. In this algorithm we have incorporated both Power method and Inverse Power method to find both the largest and the smallest Eigen pairs simultaneously. Several experiments have been carried out for the validation of the algorithm as well as lemmas. According to the numerical experiments considered here it may be concluded that the proposed algorithm is able to find out both absolutely largest as well as absolutely smallest Eigen values and corresponding Eigen vectors successfully. Moreover the algorithm is able to find out the nature of the Eigen spectrums. But it is worthwhile to mention here that the proposed algorithm carries some shortcomings of Power method namely finding Eigen value when the largest (smallest) Eigen value is duplicated and finding complex Eigen values.

REFERENCES

[1] Chang Q., Wong Y. and Lin C., "Journal of Computational Physics" Elsiver, (19), **227**: 8489-8768, 2008.

[2] Demmel J. and James W., Applied Numerical Linear Algebra, Philadelphia, PA: Society for Industrial and Applied Mathematics, ISBN 0-89871-389-7, MR 1463942,1997.

[3] Dyson F. J., Statistical theory of the energy levels of complex systems i, ii, and iii., Journal of Mathematical Physics, **3**(1): 166-175,1962.

[4] Gade K. K., Numerical Estimation of the Second Largest Eigen value of a Reversible Markov Transition Matrix, Doctor of Philosophy, Department of Computer Science, New York University January. An Inexact Cayley Transform Method for Inverse Eigen value Problems, 2009.

[5] Golub and Van Loan C., *Matrix Computations*, The Johns Hopkins University Press, Baltimore, third edition, 1996.

[6] Gubernatis J. E. and Booth T. E., Multiple Extremal Eigen pairs by the Power Method, Theoretical Division and Applied Physics Division, Los Alamos National Laboratory, Los Alamos, NM 87545 U.S.A. T. E. Booth, LA-UR-07-4046, 2008.

[7] Guillaumin, J. V. and Schmid C., Is that you? Metric learning approaches for face identification. In IEEE 12th International Conference on Computer Vision, 498-505, 2009.

[8] Hoi S. C. H., Liu W., Lyu M. R., and Ma W.Y., Learning distances metrics with contextual constraints for image retrieval. In IEEE Computer Society Conference on Computer Vision and Pattern Recognition, 2072–2078, 2006.

[9] Jamali A.R.M. J.U. and Alam S., Some lemma on spectrum of Eigen value regarding power method. The 4th international conference of mathematical sciences (ICMS4), 15-17 Nov. 2016, AIP conference proceedings 1830, 020005, 2017.

[10] Kato and Nagano N., Metric learning for enzyme active-site search, Bioinformatics, **26**: 2698- 2704, 2010.

[11] Liao B.-S., Bai Z., Lee L.-Q. and Ko K., Nonlinear Rayleigh-Ritz iterative method for solving large scale nonlinear Eigen value problems, *Taiwanese J. Math.,* **14**(3A): 869-883, 2010.

[12] Marcus M. and Minc H., Introduction to Linear Algebra, New York: Dover, 1988.

[13] Mehrmann V. and Schroder C., Nonlinear Eigen value and frequency response problems in industrial practice. *J. Math. Ind.,* **1**(7): 1-18, 2011.

[14] Michiels W. and Niculescu S. I., Stability and Stabilization of Time-Delay Systems, Advances in Design and Control Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA. An Eigen value based approach, (12), 2007.

[15] Panju, M., Iterative Methods for Computing Eigen values and Eigenvectors, University of Waterloo, arXiv preprint tarXiv:1105.1185, http://mathreview.uwaterloo.ca/archive/ voli/ 1/panju.pdf, 2011.

[16] Parlett N. B., *The Symmetric Eigen value Problem,* Prentice-Hall, Englewood Cliffs, NJ. Reprinted as Classics in Applied Mathematics 20, SIAM, Philadelphia, 1997.

[17] Singh K.V. and Ram Y.M., (Transcendental Eigen value problem and its applications. *AIAA Journal,* **40**(7): 1402-1407, 2002.

[18] Thompson J., Mathematical Statistical Mechanics, Princeton University Press, Princeton, 1972.

[19] Li W., Liu W.H., Vong S.W., On the Bound of the Eigen value in Module for a Positive Tensor, J. Oper. Res. Soc. China, **5**: 123-129, 2017.

[20] Yang K., Numerical algorithms for Inverse Eigen value problems arising in control and nonnegative matrices, PhD. Thesis, The Australian National University Canberra, Australia, 2006.

[21] Zhang P., Lee H., Lee D., A general solution strategy of power method for higher mode solutions. Journal of Computational Physics, **305**: 387-402, 2016.