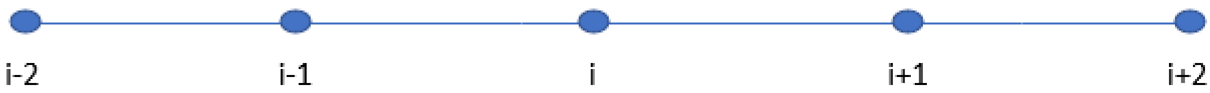
<u>DERIVATION OF FOURTH ORDER APPROXIMATION OF THE SECOND ORDER DERIVATIVE:</u>

CENTRAL DIFFERENCE METHOD:

Numerical Stencil for Central Difference,



For the above stencil, Hypothesis equation is,

$$\frac{\partial^2 u}{\partial x^2}=af(i-2)+bf(i-1)+cf(i)+df(i+1)+ef(i+2) \quad \ldots\ldots\ldots (1)$$

The values of the co-efficient (a, b, c, d, & e), are found using Taylor Table Method.

Taylor Series expansion for *a f (i- 2)* term,

$$af(i-2)=af(i)-\frac{af'(i)(2dx)}{1!}+\frac{af''(i)(2dx)^2}{2!}-\frac{af'''(i)(2dx)^3}{3!}+\frac{af''''(i)(2dx)^4}{4!}-\frac{af'''''(i)(2dx)^5}{5!}+\frac{af''''''(i)(2dx)^6}{6!}+\cdots$$

Simplifying above equation,

$$af(i-2)=af(i)-\frac{2af'(i)(dx)}{1}+\frac{4af''(i)(dx)^2}{2}-\frac{8af'''(i)(dx)^3}{6}+\frac{16af''''(i)(dx)^4}{24}-\frac{32af'''''(i)(dx)^5}{120}+\frac{64af''''''(i)(dx)^6}{720}+\cdots$$

Similarly, Taylor Series expansion for *b f (i- 1)* term,

$$bf(i-1)=bf(i)-\frac{bf'(i)(dx)}{1}+\frac{bf''(i)(dx)^2}{2}-\frac{bf'''(i)(dx)^3}{6}+\frac{bf''''(i)(dx)^4}{24}-\frac{bf'''''(i)(dx)^5}{120}+\frac{bf''''''(i)(dx)^6}{720}+\cdots$$

Similarly, Taylor Series expansion for *c f (i)* term,

$$cf(i)=cf(i)$$

Similarly, Taylor Series expansion for *d f (i+ 1)* term,

$$df(i+1)=df(i)+\frac{df'(i)(dx)}{1}+\frac{df''(i)(dx)^2}{2}+\frac{df'''(i)(dx)^3}{6}+\frac{df''''(i)(dx)^4}{24}+\frac{df'''''(i)(dx)^5}{120}+\frac{df''''''(i)(dx)^6}{720}+\cdots$$

Similarly, Taylor Series expansion for *e f (i+ 2)* term,

$$ef(i-2)=ef(i)+\frac{2ef'(i)(dx)}{1}+\frac{4ef''(i)(dx)^2}{2}+\frac{8ef'''(i)(dx)^3}{6}+\frac{16ef''''(i)(dx)^4}{24}+\frac{32ef'''''(i)(dx)^5}{120}+\frac{64ef''''''(i)(dx)^6}{720}+\cdots$$

The above Taylor Series expansion is in represented in Taylor's Table,

		$f(i)$	$\Delta x f'(i)$	$\Delta x^2 f''(i)$	$\Delta x^3 f'''(i)$	$\Delta x^4 f''''(i)$	$\Delta x^5 f'''''(i)$
Numerical Stencil	$af(i)$	a	0	0	0	0	0
	$bf(i+1)$	b	b	$\frac{b}{2}$	$\frac{b}{6}$	$\frac{b}{24}$	$\frac{b}{120}$
	$cf(i+2)$	c	2c	$\frac{4c}{2}$	$\frac{8c}{6}$	$\frac{16c}{24}$	$\frac{32c}{120}$
	$df(i+3)$	d	3d	$\frac{9d}{2}$	$\frac{27d}{6}$	$\frac{81d}{24}$	$\frac{243d}{120}$
	$ef(i-4)$	e	4e	$\frac{16e}{2}$	$\frac{64e}{6}$	$\frac{256e}{24}$	$\frac{1024e}{120}$
	$gf(i-4)$	g	5g	$\frac{25g}{2}$	$\frac{125g}{6}$	$\frac{625g}{24}$	$\frac{3125g}{120}$
	Sum of co-efficient	0	0	1	0	0	0

Linear equations obtained from the above table is solved using a matrix for a, b, c, d, & e.

A*B =X

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 1 & 2 \\ 2 & 1/2 & 0 & 1/2 & 2 \\ -8/6 & -1/6 & 0 & 1/6 & 8/6 \\ 16/24 & 1/24 & 0 & 1/24 & 16/24 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

X = inv(A)*B

After solving the above matrix, we get the values of all the coefficients as:

```
X =

   -0.0833
    1.3333
   -2.5000
    1.3333
   -0.0833
```

a = -0.0833
b = 1.3333
c = -2.5000
d = 1.3333
e = -0.0833

Substituting the above values in equation 1 and dividing by $(\Delta x)^2$.

$$\frac{\partial^2 u}{\partial x^2}=\frac{(-0.0833)f(i-2)+(1.3333)f(i-1)+(-2.5000)f(i)+(1.3333)f(i+1)+(-0.0833)f(i+2)}{(\Delta x)^2}$$

SKEWED RIGHT-SIDED DIFFERENCE:

The order of accuracy **P** *= Number of nodes in the stencil* **N** *- order of derivative* **Q**

Number of nodes in the stencil **N** *= 4 + 2 = 6*

Numerical Stencil for Skewed Right-Sided Difference,

For the above stencil, Hypothesis equation is,

$$\frac{\partial^2 u}{\partial x^2}=af(i)+bf(i+1)+cf(i+2)+df(i+3)+ef(i+4)+gf(i+5) \quad \ldots\ldots\ldots (2)$$

Expanding all the terms in the above equation (2) using Taylor series expansion and are represented in a tabular form using Taylor's Table Method.

		$f(i)$	$\Delta x f'(i)$	$\Delta x^2 f''(i)$	$\Delta x^3 f'''(i)$	$\Delta x^4 f''''(i)$	$\Delta x^5 f'''''(i)$
Numerical Stencil	$af(i)$	a	0	0	0	0	0
	$bf(i+1)$	b	b	$\frac{b}{2}$	$\frac{b}{6}$	$\frac{b}{24}$	$\frac{b}{120}$
	$cf(i+2)$	c	2c	$\frac{4c}{2}$	$\frac{8c}{6}$	$\frac{16c}{24}$	$\frac{32c}{120}$
	$df(i+3)$	d	3d	$\frac{9d}{2}$	$\frac{27d}{6}$	$\frac{81d}{24}$	$\frac{243d}{120}$
	$ef(i-4)$	e	4e	$\frac{16e}{2}$	$\frac{64e}{6}$	$\frac{256e}{24}$	$\frac{1024e}{120}$
	$gf(i-4)$	g	5g	$\frac{25g}{2}$	$\frac{125g}{6}$	$\frac{625g}{24}$	$\frac{3125g}{120}$
	Sum of co-efficient	0	0	1	0	0	0

Linear equations obtained from the above table is solved using a matrix for a, b, c, d, e, & g.

A*B =X

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 1/2 & 4/2 & 9/2 & 16/2 & 25/2 \\ 0 & 1/6 & 8/6 & 27/6 & 64/6 & 125/6 \\ 0 & 1/24 & 16/24 & 81/24 & 256/24 & 625/24 \\ 0 & 1/120 & 32/120 & 243/120 & 1024/120 & 3125/120 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \\ e \\ g \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

X = inv(A)*B

After solving the above matrix using Matlab, we get the values of all the coefficients as:

```
X =

    3.7500
  -12.8333
   17.8333
  -13.0000
    5.0833
   -0.8333
```

a = 3.7500
b = -12.8333
c = 17.8333
d = -13.0000

$e = 5.0833$
$g = -0.8333$

Substituting the above values in equation 2 and dividing by $(\Delta x)^2$.

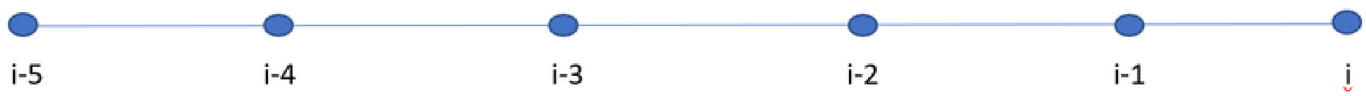
$$\frac{\partial^2 u}{\partial x^2} = \frac{(3.75)f(i)+(-12.8333)f(i+1)+(17.8333)f(i+2)+(-13)f(i+3)+(5.0833)f(i+4)+(-0.8333)f(i+5)}{(\Delta x)^2}$$

SKEWED LEFT-SIDED DIFFERENCE:

The order of accuracy **P** = Number of nodes in the stencil **N** - order of derivative **Q**

Number of nodes in the stencil **N** = 4 + 2 = 6

Numerical Stencil for Skewed Left-Sided Difference,



For the above stencil, Hypothesis equation is,

$$\frac{\partial^2 u}{\partial x^2} = af(i-5)+bf(i-4)+cf(i-3)+df(i-2)+ef(i-1)+gf(i) \quad \text{……… (3)}$$

Expanding all the terms in the above equation (3) using Taylor series expansion and are represented in a tabular form using Taylor's Table Method.

		$f(i)$	$\Delta x f'(i)$	$\Delta x^2 f''(i)$	$\Delta x^3 f'''(i)$	$\Delta x^4 f''''(i)$	$\Delta x^5 f'''''(i)$
Numerical Stencil	$af(i-5)$	a	-5a	$\frac{25a}{2}$	$\frac{-125a}{6}$	$\frac{625a}{24}$	$\frac{-3125a}{120}$
	$bf(i-4)$	b	-4b	$\frac{16b}{2}$	$\frac{-64b}{6}$	$\frac{256b}{24}$	$\frac{-1024b}{120}$
	$cf(i-3)$	c	-3c	$\frac{9c}{2}$	$\frac{-27c}{6}$	$\frac{81c}{24}$	$\frac{-243c}{120}$
	$df(i-2)$	d	-2d	$\frac{4d}{2}$	$\frac{-8d}{6}$	$\frac{16d}{24}$	$\frac{-32d}{120}$
	$ef(i-1)$	e	-e	$\frac{e}{2}$	$\frac{-e}{6}$	$\frac{e}{24}$	$\frac{-e}{120}$
	$gf(i)$	g	0	0	0	0	0
	Sum of co-efficient	0	0	1	0	0	0

Linear equations obtained from the above table is solved using a matrix for a, b, c, d, e, & g.

A*B =X

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ -5 & -4 & -3 & -2 & -1 & 0 \\ 25/2 & 16/2 & 9/2 & 4/2 & 1/2 & 0 \\ -125/6 & -64/6 & -27/6 & -8/6 & -1/6 & 0 \\ 625/24 & 253/24 & 81/24 & 16/24 & 1/24 & 0 \\ -3125/120 & -1024/120 & -243/120 & -32/120 & -1/120 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \\ e \\ g \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

X = inv(A)*B

After solving the above matrix using Matlab, we get the values of all the coefficients as:

```
X =

   -0.8333
    5.0833
  -13.0000
   17.8333
  -12.8333
    3.7500
```

a = -0.8333
b = 5.0833
c = -13.0000
d = 17.8333
e = -12.8333
g = 3.7500

Substituting the above values in equation 3 and dividing by `(Δx)^2`.

`(∂^2 u)/(∂x^2 )= ((-0.8333)f(i-5)+(5.0833)f(i-4)+(-13)f(i-3)+(17.8333)f(i-2)+(-12.8333)f(i-1)+(3.75)f(i))/(Δx)^2`

<u>MATLAB PROGRAM:</u>

MAIN CODE:

```
 % A program to calculate the second-order derivative of the analytical function
exp(x)*cos(x).

clear all
close all
clc

% value of x
x = pi/3;
```

```
% range of dx
dx = linspace(pi/4, pi/4000,20);

% Loop for solving a particular approximation for all values of dx
for i=1:length(dx)

central_difference_error(i) = central_differencing_approximation(x,dx(i));
right_sided_skewed_error(i) = right_skewed_approximation(x,dx(i)) ;
left_sided_skewed_error(i) = left_skewed_approximation(x,dx(i));

end

% Plotting

figure(1)
loglog(dx,central_difference_error,'b')
hold on
loglog(dx,right_sided_skewed_error,'r')
hold on
loglog(dx,left_sided_skewed_error,'g')
xlabel('dx')
ylabel('Error')
legend('Central Difference Error','Skewed Right-Sided Error','Skewed Left-Sided
Error','Location','northwest')

figure(2)
plot(dx,central_difference_error,'b')
hold on
plot(dx,right_sided_skewed_error,'r')
hold on
plot(dx,left_sided_skewed_error,'g')
xlabel('dx')
ylabel('Absolute Error')
legend('Central Difference Error','Skewed Right-Sided Error','Skewed Left-Sided
Error','Location','northwest')
```

FUNCTION CODE FOR CENTRAL DIFFERENCE:

```
function Error_central_differencing = central_differencing_approximation(x,dx)


% analytic function exp(x) *cos(x)
% analytic derivative
% f'(x) = -2*exp(x)*sin(x)
analytical_derivative = -2*exp(x)*sin(x) ;

% Values of all the coefficients
a = -0.0833;
b = 1.3333;
c = -2.5000;
d = 1.3333;
e = -0.0833;

% Central differencing
central_difference =((a*exp(x-2*dx)*cos(x-2*dx))...
    +(b*exp(x-dx)*cos(x-dx))...
    +(c*exp(x)*cos(x))...
    +(d*exp(x+dx)*cos(x+dx))...
    +(e*exp(x+2*dx)*cos(x+2*dx)))/(dx^2);
% Error
Error_central_differencing= abs(central_difference- analytical_derivative);
end
```

FUNCTION CODE FOR SKEWED RIGHT-SIDED DIFFERENCE:

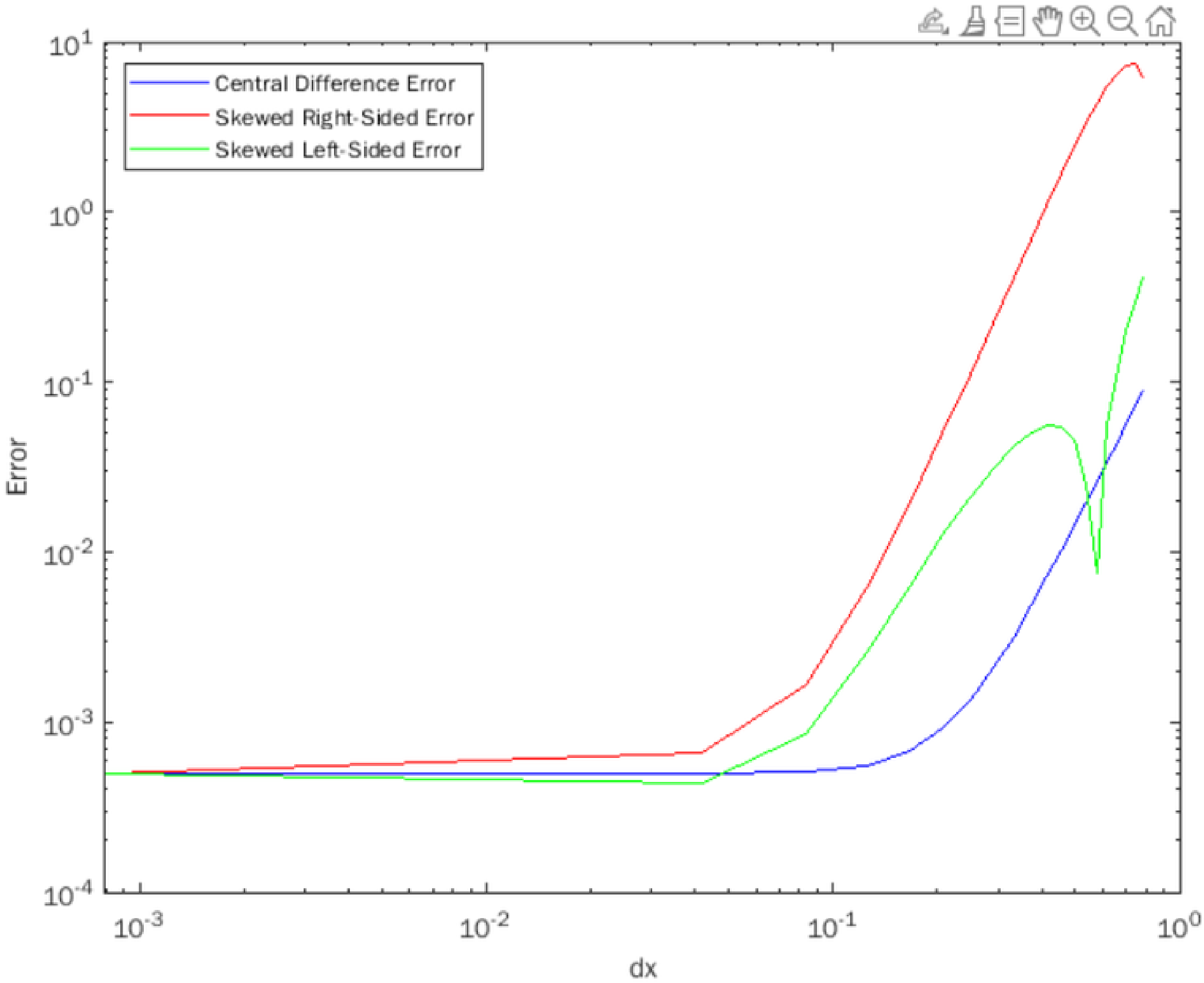
```
function Error_right_skewed = right_skewed_approximation(x,dx)
```

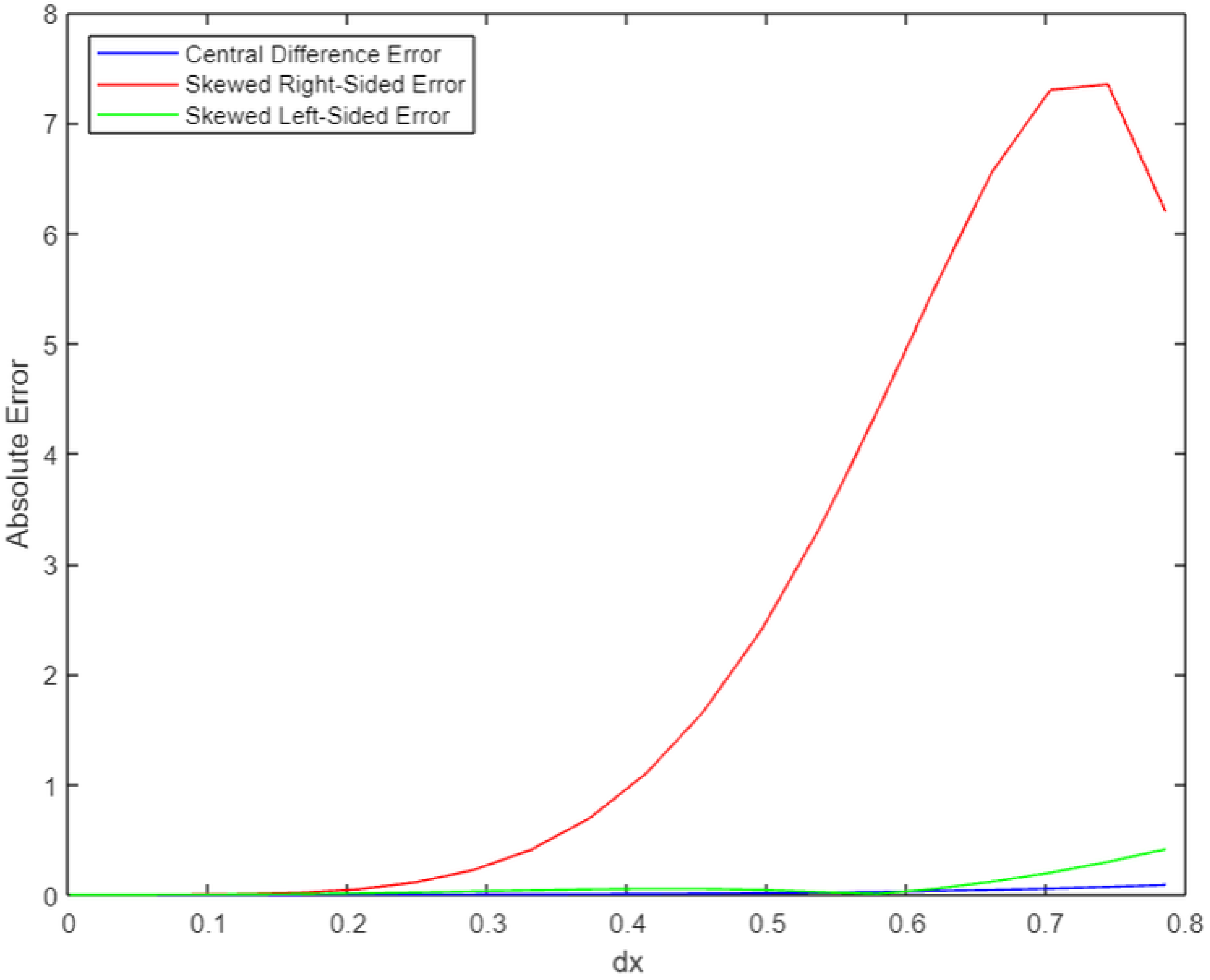
```
% analytic function exp(x) *cos(x)
% analytic derivative
% f'(x) = -2*exp(x)*sin(x)
analytical_derivative = -2*exp(x)*sin(x);

% Values of all the coefficients
 a = 3.7500;
 b = -12.8333;
 c = 17.8333;
 d = -13.0000;
 e = 5.0833;
 g = -0.8333;

% Right Side Skewed
right_skewed =((a*exp(x)*cos(x))...
    +(b*exp(x+dx)*cos(x+dx))...
    +(c*exp(x+2*dx)*cos(x+2*dx))...
    +(d*exp(x+3*dx)*cos(x+3*dx))...
    +(e*exp(x+4*dx)*cos(x+4*dx))...
    +(g*exp(x+5*dx)*cos(x+5*dx)))/(dx^2);
%Error
Error_right_skewed = abs(right_skewed - analytical_derivative);
end
```

FUNCTION CODE FOR SKEWED LEFT-SIDED DIFFERENCE:

```
function Error_left_skewed = left_skewed_approximation(x,dx)

% analytic function exp(x) *cos(x)
% analytic derivative
% f'(x) = -2*exp(x)*sin(x)
analytical_derivative = -2*exp(x)*sin(x) ;

% Values of all the coefficients
  a = -0.8333; b = 5.0833; c = -13.0000;
  d = 17.8333; e = -12.8333; g = 3.7500;

% Left Side Skewed
left_skewed = ((a*exp(x-5*dx)*cos(x-5*dx))...
    +(b*exp(x-4*dx)*cos(x-4*dx))...
    +(c*exp(x-3*dx)*cos(x-3*dx))...
    +(d*exp(x-2*dx)*cos(x-2*dx))...
    +(e*exp(x-dx)*cos(x-dx))...
    +(g*exp(x)*cos(x)))/(dx^2);
% Error
Error_left_skewed= abs(left_skewed- analytical_derivative);
end
```

OUTPUT:


Central Difference Error
Skewed Right-Sided Error
Skewed Left-Sided Error
Error
dx
10^{1}
10^{0}
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-3}
10^{-2}
10^{-1}
10^{0}



CONCLUSION:

From the above plot we can conclude that the error from the Central Difference scheme is less as compared to Skewed scheme.

Why a skewed scheme is useful? What can a skewed scheme do that a CD scheme cannot?

The system which doesn't contain sufficient information on both side of the considered point of interest. In such cases we cannot use central differencing scheme so skewed scheme is useful in such cases. Central differencing scheme cannot not be employed at the boundary nodes due to unavailability of right and left nodes at the same time. In this situation skewed difference method can be used, as in the skewed scheme data from one side can be used to compute numerical approximation at a point.