LIMITING DISTRIBUTION

MODEL

Consider the following social planner's problem:

$$\max_{c_t, k_{t+1}} \mathbb{E} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

s.t $c_t + k_{t+1} = z_t f(k_t)$, where productivity z_t is a stochastic process. $k_0 > 0$ and $z_0 > 0$ are given.

- The Bellman equation can be written as

$$\begin{aligned} v(k_t, z_t) &= \max_{c_t, k_{t+1}} \left[u(c_t) + \beta \, \mathbb{E} \, v(k_{t+1}, z_{t+1})\right] \\ &= \max_{c_t, k_{t+1}} \left[u(c_t) + \beta \sum_{z_{t+1} \mid z^t} v(k_{t+1}, z_{t+1}) \pi(z_{t+1} \mid z^t)\right] \end{aligned}$$

s.t $c_t + k_{t+1} = z_t f(k_t)$.

QUESTION

- The sequence of $\{k_t\}$ is generated by $k_{t+1} = g(k_t, z_t)$.
- Due to the random shocks z_t, k_t will not converge to any single value (steady state). Then what about the limiting distribution of k_t when $t \to \infty$?
- A stochastic analogue to a steady state of a deterministic system is a stationary (invariant) distribution.
    - Under what conditions does the sequence of k_t converge to a stationary limiting distribution?

MATHEMATICS

- To adhere to the principles of recursive theory, it is preferable for the process $k_{t+1} = g(k_t, z_t)$ to be a Markov process.
- For simpilicty, we assume k_t is a finite state Markov chain with a transition function

$$Q(a, b) := \mathrm{Prob}\{k_{t+1} \leq a \mid k_t = b\}.$$

- Then the sequence of distribution functions $\{\psi_t\}$ for the k_t's is given inductively by

$$\psi_{t+1}(a) := \mathrm{Prob}\{k_{t+1} \leq a\} = \int Q(a, b) d\psi_t(b), t = 0, 1, ...,$$

where ψ_0 (the initial distribution of k_0) is given.

- Under what conditions, is H such that the sequnece $\{\psi_t\}$ converges to a invariant limiting distribution function ψ satisfying

$$\psi(k') = \int Q(k', k) d\psi(k).$$

AN I.I.D EXAMPLE

- Suppose policy function has the multiplicative form $k_{t+1} = z_t g(k_t)$ and that z_t is **i.i.d.** over time with cumulative distribution

$$H(z) = \text{Prob}\{z_t \leq z\}\,.$$

- Now consider the cumulative distribution of k at time $t+1$:

$$\begin{aligned}
\psi_{t+1}(k') &= \text{Prob}\{k_{t+1} \leq k'\} = \text{Prob}\{z_t g(k_t) \leq k'\} \\
&= \text{Prob}\{z_t \leq \frac{k'}{g(k_t)}\} = H\left(\frac{k'}{g(k_t)}\right) \\
&= \text{Prob}\{k_{t+1} \leq k' \mid k_t\} = Q(k', k_t)
\end{aligned}$$

- Then the cumulative distribution of k satisfies the law of motion

$$\psi_{t+1}(k') = \int Q(k', k) d\psi_t(k) = \int H\left(\frac{k'}{g(k)}\right) d\psi_t(k)$$

STATIONARY DISTRIBUTION

- Generally, we would have a joint distribution (density) $\lambda(k, z)$ over the state variables (k, z) induced by
  - the policy function $k' = g(k, z)$, and
  - the exogenous condtional density $\pi(z' \mid z)$.
- If z_t is a Markov chain, the pair (k, z) is also a Markov chain.
- Then we are in fact facing a Markov chain defined by $(k \otimes z, P, \lambda_0)$.
- The law of motion for the distribution λ is $\lambda_{t+1}(k', z') = P^{\mathrm{T}} \cdot \lambda_t(k, z)$.
- The above transition can be denoted as a deterministic difference equation:

$$\lambda_{t+1} = P^{\mathrm{T}} \cdot \lambda_t,$$

where $P := [p_{ij}]_{(n_k \times n_z) \times (n_k \times n_z)}$ is the transition matrix (density function) of λ with $p_{ij} = \mathrm{Prob}\{k', z' \mid k, z\}$.

- The stationary distribution λ^* of the Markov chain satisfies $\lambda^* = P^{\mathrm{T}} \cdot \lambda^*$ (or $(I - P^{\mathrm{T}})\lambda^* = 0$), i.e., a fixed point of the difference equation $\lambda_{t+1} = P^{\mathrm{T}} \cdot \lambda_t$.

THEORETICAL ISSUES

- The fixed point λ^* may not exist.
- λ^* is a fixed point, but it may not be unique.
- λ^* is a unique fixed point, but it may not be stable (iterating $\lambda_{t+1} = P^{\mathrm{T}} \cdot \lambda_t$ may not converge to it)

THEOREM 12.12 (SLP)

If the transition function P satisfy 3 conditions:

- Feller property (for existence)
- mixing property (for uniqueness)
- monotonicity (for convergence)

Then by Theorem 12.12 in SLP (1989), there exists a unique invariant probability measure λ^* and P ensures convergence, independently of the starting measure λ_0.

FELLER PROPERTY FOR EXISTENCE

- A transition function P has the Feller property if $f \in C(X)$ implies $Tf \in C(X)$, where $(Tf)(s) = \int f(s')P(s, s')$.
- Intuitively, the Feller property ensures that P is indeed a transition function.

MIXING PROPERTY FOR UNIQUENESS

- Requires that probability of moving away from the worst state in a finite number of periods is positive and vice versa from the best state (American dream, American nightmare).
- counterexample of mixing property:

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

where any initial situation is invariant.

MONOTONICITY FOR CONVERGENCE

- A transition function P is monotone if $f \in B(X)$ is increasing implies Tf is also increasing, where $(Tf)(s) = \int f(s')P(s, s')$. This requires positive autocorrelation of P.
- Intuitively, if $\lambda(k, z)$ is high, then $\lambda'(k, z)$ will be high too.
- Counterexample of monotonicity:

$$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

where two random states keep switching back and forth, therefore no convergence is possible.

MARKOV CHAIN

- Every (finite state) stochastic matrix P has at least one stationary distribution.
- The stochastic matrix P is called irreducible if all states communicate. That is, for all states (x, y), x can eventually be reached from y and vice versa.
- A Markov chain is called periodic if it cycles in a predictable way, and aperiodic otherwise.
- If P is both aperiodic and irreducible, then
  - P has exactly one stationary distribution λ^*.
  - For any initial marginal distribution λ_0, we have $\left\|P^{\mathrm{T}}\lambda_0 - \lambda^*\right\| \to 0$ as $t \to 0$.
- A sufficient condition for a unique stable stationary distribution is that $0 < p_{ij} < 1$ for all i, j.

2×2 EXAMPLE

- Consider a two state Markov chain with transion matrix

$$P = \begin{bmatrix} 1-p & p \\ q & 1-q \end{bmatrix}$$

- Stationary distribution solves (note the transpose)

$$\left(\begin{bmatrix} 1 & \\ & 1 \end{bmatrix} - \begin{bmatrix} 1-p & q \\ p & 1-q \end{bmatrix} \right) \begin{bmatrix} \psi_1^* \\ \psi_2^* \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- Gives

$$\begin{bmatrix} \psi_1^* \\ \psi_2^* \end{bmatrix} = \begin{bmatrix} \frac{q}{p+q} \\ \frac{p}{p+q} \end{bmatrix}$$

ALGORITHM

- How to solve this fix point for the joint distribution?
  - iteration
  - eigenvector
  - ergodicity

EXAMPLE

- Suppose there is a 3-point Markov chain with transition matrix:

$$\Pi = \begin{bmatrix} 0.971 & 0.029 & 0.000 \\ 0.145 & 0.778 & 0.077 \\ 0.000 & 0.508 & 0.492 \end{bmatrix}.$$

- We know that P has at least one stationary distribution. We want to check uniqueness and stability.

```python
import numpy as np
import quantecon as qe
P = np.array(((0.971, 0.029, 0.000),
              (0.145, 0.778, 0.077),
              (0.000, 0.508, 0.492)))
mc = qe.MarkovChain(P)
print("irreducibility:", mc.is_irreducible)
print("aperiodicity:", mc.is_aperiodic)
```

```
irreducibility: True
aperiodicity: True
```

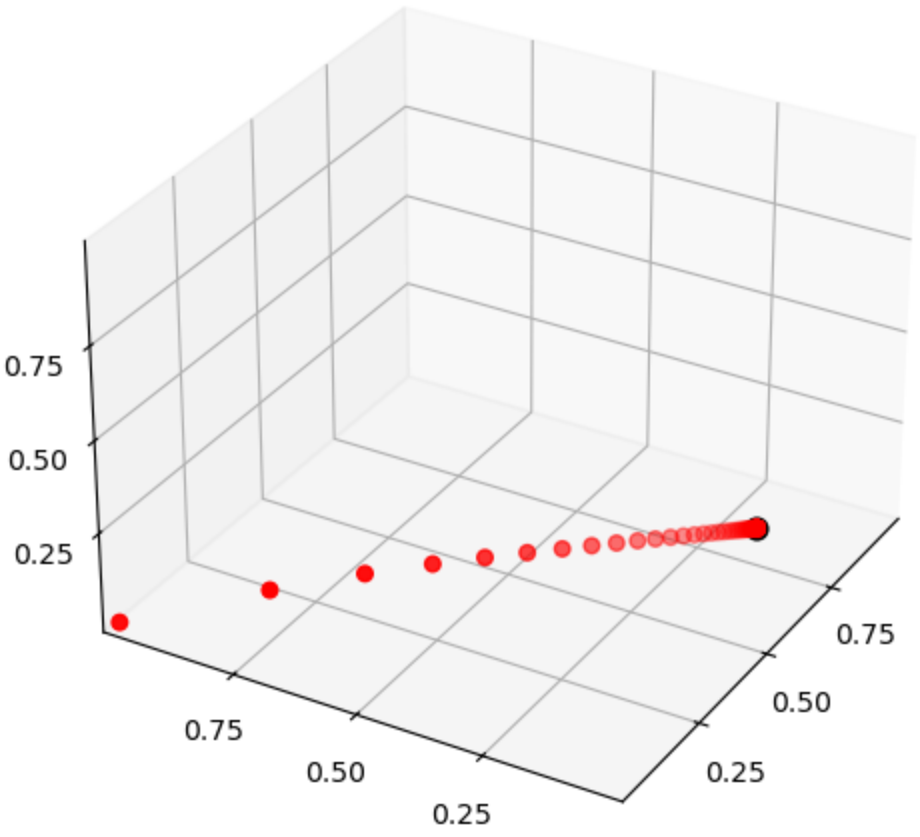
ITERATION

- The trainsion of λ follows: $\lambda_t = (P^{\mathrm{T}})^t \lambda_0$.
- If the fixed point λ^* exists,

$$\lambda^* = \lim_{t \to \infty} \lambda_t = \lim_{t \to \infty} (P^{\mathrm{T}})^t \lambda_0 \, .$$

- Then given any initial λ_0, we can iterate until convergence to get λ^*.

ILLUSTRATION



CODE EXAMPLE

- can be optimized using multigrid tricks (remember to use interpolation between two convergences)

```python
def stationary_distribution_iteration(P, psi0, tol):
    err = np.inf
    max_iter = 5000
    iter = 0
    print_skip = 50
    while iter < max_iter and err > tol:
        psi = P.T @ psi0
        err = np.max(np.abs(psi - psi0))
        iter = iter + 1
        if iter % print_skip == 0:
            print(f"Error at iteration {iter}:", err)
        psi0 = psi
    if err > tol:
        print("Failed to converge!")
    else:
        print(f"\nConverged in {iter} iterations.")
    return psi0
```

RESULT

```
psi0 = np.array((0.0, 1.0, 0.0))
stationary_distribution_iteration(P, psi0, tol=1e-16)
```

```
Error at iteration 50: 4.298089431309382e-05
Error at iteration 100: 1.3944348564720599e-08
Error at iteration 150: 4.523936780742588e-12
Error at iteration 200: 1.4432899320127035e-15

Converged in 222 iterations.

array([0.8128 , 0.16256, 0.02464])
```

EIGENVECTOR

- $\lambda^* = P^{\mathrm{T}} \cdot \lambda^*$ can be written as

$$(I - P^{\mathrm{T}})\lambda^* = 0$$

- We see λ^* is a (right) eigenvector of P^{T} associated with a unit eigenvalue (require a normalization of $\sum_i \lambda_i^* = 1$).

FUNCTIONS

Several functions to calculate eigenvectors (usages differ):

- `numpy.linalg.eig`
- `scipy.linalg.eig`
- `scipy.sparse.linalgeigs`

CODE EXAMPLE

```python
from numpy.linalg import eig


def stationary_distribution(transition_matrix):
    eigenvalues, eigenvectors = eig(transition_matrix.T)
    index = np.argmin(np.abs(eigenvalues - 1.0))
    stationary_vector = np.real(eigenvectors[:, index])
    stationary_distribution = stationary_vector / np.sum(stationary_vector)
    return stationary_distribution.flatten()


stationary_distribution(P)
```

```
array([0.8128 , 0.16256, 0.02464])
```

QUANTECON

```python
import quantecon as qe


mc = qe.MarkovChain(P)
mc.stationary_distributions[0]
```

```
array([0.8128 , 0.16256, 0.02464])
```

ERGODICITY

- A stochastic matrix P that is both aperiodic and irreducible is also called uniformly ergodic. That is, for all states $x \in S$, $\frac{1}{T}\sum_{t=1}^{T} \mathbf{1}\{X_t = x\}$ converges to the stationary distribution ψ^* uniformally as $T \to \infty$.

```
mc = qe.MarkovChain(P)
chain = mc.simulate(ts_length=10000000)
prob = []
for i in range(len(P)):
    res = np.round(np.count_nonzero(chain == i)/len(chain), 4)
    prob.append(res)
print(prob)
```

```
[0.8129, 0.1625, 0.0246]
```

REMAINING QUESTIONS

- How to calculate the transion function P?
    - Once we get P, we can calculate the limiting joint distribution $\lambda^*(k, z)$.
- How to calculate ψ^* (the marginal distribution of k) from $\lambda^*(k, z)$?

CALCULATE P

$$
\begin{aligned}
\lambda_{t+1}(k', z') &= \underbrace{\sum_{z} \sum_{k' = g(k,z)} \pi(z', z)}_{P} \lambda_t(k, z) \\
&:= P^{\mathrm{T}} \cdot \lambda_t(k, z)
\end{aligned}
$$

CODE EXAMPLE

```
def transQ(g):
    n_k, n_z = g.shape
    n = n_k * n_z
    Q = np.zeros((n, n))
    for j in prange(n):
        i_k = j // n_z
        i_z = j % n_z
        diff = np.abs(g[i_k, i_z] - kgrid)
        mark = np.argmin(diff)
        j_prime_start = mark * n_z
        j_prime_end = j_prime_start + n_z
        Q[j, j_prime_start:j_prime_end] = P[i_z, :]
    return Q
```

CALCULATE THE DENSITY OF ψ^*

$$\frac{d\psi^*}{dx} = \sum_z \lambda^*(k, z)$$

```
def capital_marginal(joint_dist):
    tmp = joint_dist.reshape((n_k, n_z))
    k_marginal = tmp.sum(axis=1)
    return k_marginal
```

THE STOCHASTIC GROWTH MODEL

- Suppose the productivity z_t takes values in a 5-point Markov chain

$$z_t \in \{0.9792, 0.9896, 1, 0000, 1.0106, 1.0212\}$$

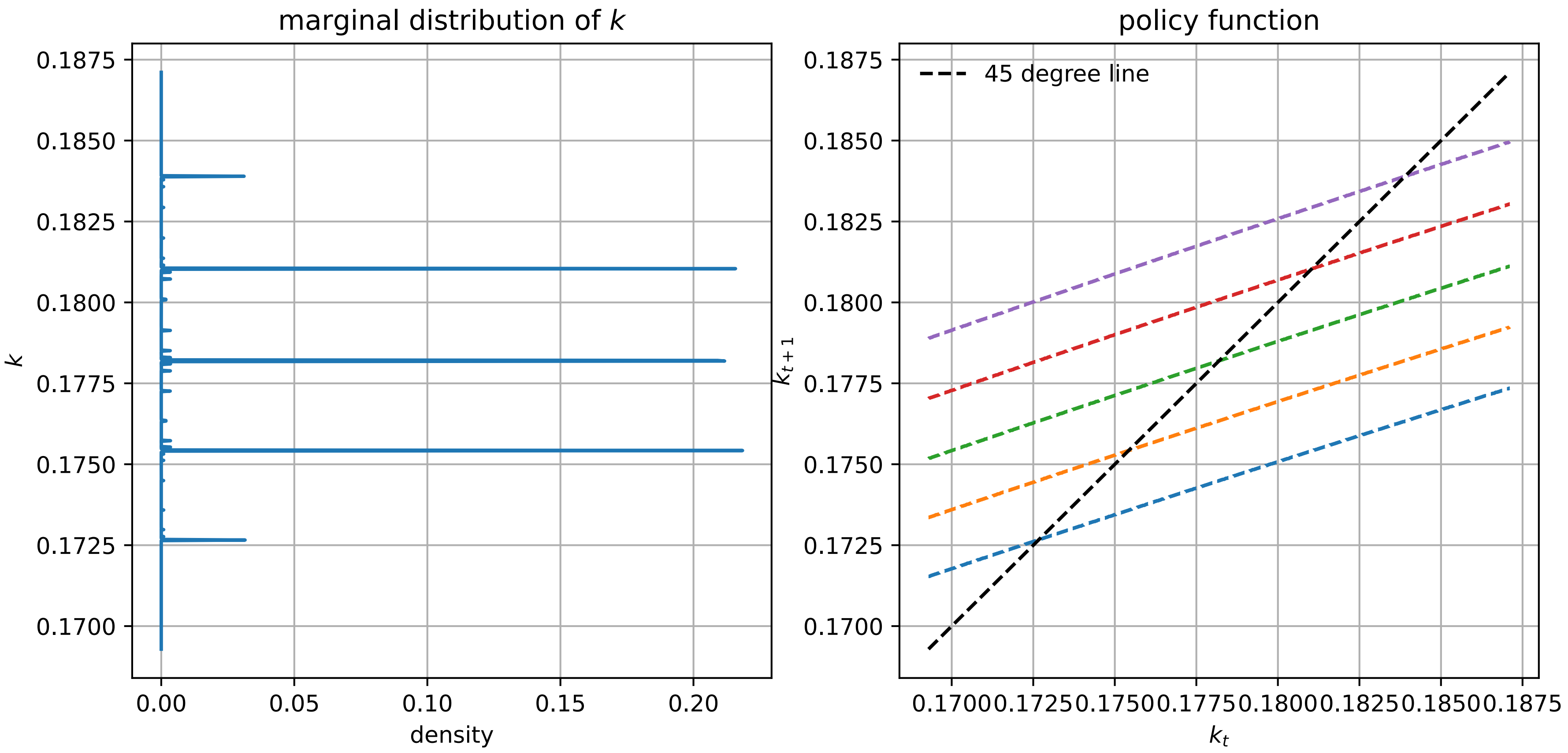
with transition matrix :

$$\Pi = \begin{pmatrix} 0.9727 & 0.0273 & & & \\ 0.0041 & 0.9806 & 0.0153 & & \\ & 0.0082 & 0.9837 & 0.0082 & \\ & & 0.0153 & 0.9806 & 0.0041 \\ & & & 0.0273 & 0.9727 \end{pmatrix}.$$

PROCEDURE

1. solve the policy function g
2. solve the transition function P for the joint distribution $\lambda(k, z)$
3. calculate the limiting joint distribution $\lambda^*(k, z)$
4. calculate ψ^* (the marginal distribution of k) from $\lambda^*(k, z)$

RESULT



marginal distribution of k
0.1875
0.1850
0.1825
0.1800
0.1775
0.1750
0.1725
0.1700
k
0.00
0.05
0.10
0.15
0.20
density
policy function
45 degree line
k_{t+1}
0.17000.17250.17500.17750.18000.18250.18500.1875
k_t