An Intertemporal Capital Asset Pricing Model

Author(s): Robert C. Merton

Source: *Econometrica*, Sep., 1973, Vol. 41, No. 5 (Sep., 1973), pp. 867–887

Published by: The Econometric Society

Stable URL: https://www.jstor.org/stable/1913811

AN INTERTEMPORAL CAPITAL ASSET PRICING MODEL[1]

By Robert C. Merton

An intertemporal model for the capital market is deduced from the portfolio selection behavior by an arbitrary number of investors who act so as to maximize the expected utility of lifetime consumption and who can trade continuously in time. Explicit demand functions for assets are derived, and it is shown that, unlike the one-period model, current demands are affected by the possibility of uncertain changes in future investment opportunities. After aggregating demands and requiring market clearing, the equilibrium relationships among expected returns are derived, and contrary to the classical capital asset pricing model, expected returns on risky assets may differ from the riskless rate even when they have no systematic or market risk.

1. INTRODUCTION

ONE OF THE MORE important developments in modern capital market theory is the Sharpe-Lintner-Mossin mean-variance equilibrium model of exchange, commonly called the capital asset pricing model.[2] Although the model has been the basis for more than one hundred academic papers and has had significant impact on the non-academic financial community,[3] it is still subject to theoretical and empirical criticism. Because the model assumes that investors choose their portfolios according to the Markowitz [21] mean-variance criterion, it is subject to all the theoretical objections to this criterion, of which there are many.[4] It has also been criticized for the additional assumptions required,[5] especially homogeneous expectations and the single-period nature of the model. The proponents of the model who agree with the theoretical objections, but who argue that the capital market operates "as if" these assumptions were satisfied, are themselves not beyond criticism. While the model predicts that the expected excess return from holding an asset is proportional to the covariance of its return with the market

---

[2] See Sharpe [38 and 39], Lintner [19 and 20], and Mossin [29]. While more general and elegant than the capital asset pricing model in many ways, the general equilibrium model of Arrow [1] and Debreu [8, Ch. 7] has not had the same impact, principally because of its empirical intractability and the rather restrictive assumption that there exist as many securities as states of nature (see Stiglitz [41]). The "growth optimum" model of Hakansson [15] can be formulated as an equilibrium model although it is consistent with expected utility maximization only if all investors have logarithmic utility functions (see Samuelson [36] and Merton and Samuelson [27]). However, Roll [32] has shown that the model fits the data about as well as the capital asset pricing model.

[3] For academic references, see Sharpe [39] and the Jensen [17] survey article. For a summary of the model's impact on the financial community, see [42].

[4] See Borch [4], Feldstein [12], and Hakansson [15]. For a list of the conditions necessary for the validity of mean-variance, see Samuelson [34 and 35].

[5] See Sharpe [39, pp. 77–78] for a list of the assumptions required.

portfolio (its "beta"), the careful empirical work of Black, Jensen, and Scholes [3] has demonstrated that this is not the case. In particular, they found that "low beta" assets earn a higher return on average and "high beta" assets earn a lower return on average than is forecast by the model.[6] Nonetheless, the model is still used because it is an equilibrium model which provides a strong specification of the relationship among asset yields that is easily interpreted, and the empirical evidence suggests that it does explain a significant fraction of the variation in asset returns.

This paper develops an equilibrium model of the capital market which (i) has the simplicity and empirical tractability of the capital asset pricing model; (ii) is consistent with expected utility maximization and the limited liability of assets; and (iii) provides a specification of the relationship among yields that is more consistent with empirical evidence. Such a model cannot be constructed without costs. The assumptions, principally homogeneous expectations, which it holds in common with the classical model, make the new model subject to some of the same criticisms.

The capital asset pricing model is a static (single-period) model although it is generally treated as if it holds intertemporally. Fama [9] has provided some justification for this assumption by showing that, if preferences and future invest-ment opportunity sets are not state-dependent, then intertemporal portfolio maximization can be treated as if the investor had a single-period utility function. However, these assumptions are rather restrictive as will be seen in later analysis.[7] Merton [25] has shown in a number of examples that portfolio behavior for an intertemporal maximizer will be significantly different when he faces a changing investment opportunity set instead of a constant one.

The model presented here is based on consumer-investor behavior as described in [25], and for the assumptions to be reasonable ones, it must be intertemporal. Far from a liability, the intertemporal nature of the model allows it to capture effects which would never appear in a static model, and it is precisely these effects which cause the significant differences in specification of the equilibrium relation-ship among asset yields that obtain in the new model and the classical model.

2. CAPITAL MARKET STRUCTURE

It is assumed that the capital market is structured as follows.

ASSUMPTION 1: *All assets have limited liability.*

ASSUMPTION 2: *There are no transactions costs, taxes, or problems with in-divisibilities of assets.*

---

[6] Friend and Blume [14] also found that the empirical capital market line was "too flat." Their explanation was that the borrowing-lending assumption of the model is violated. Black [2] provides an alternative explanation based on the assumption of no riskless asset. Other less important, stylized facts in conflict with the model are that investors do not hold the same relative proportions of risky assets, and short sales occur in spite of unfavorable institutional requirements.

[7] Fama recognizes the restrictive nature of the assumptions as evidenced by discussion in Fama and Miller [11].

ASSUMPTION 3: *There are a sufficient number of investors with comparable wealth levels so that each investor believes that he can buy and sell as much of an asset as he wants at the market price.*

ASSUMPTION 4: *The capital market is always in equilibrium (i.e., there is no trading at non-equilibrium prices).*

ASSUMPTION 5: *There exists an exchange market for borrowing and lending at the same rate of interest.*

ASSUMPTION 6: *Short-sales of all assets, with full use of the proceeds, is allowed.*

ASSUMPTION 7: *Trading in assets takes place continually in time.*

ASSUMPTIONS 1–6 are the standard assumptions of a perfect market, and their merits have been discussed extensively in the literature. Although Assumption 7 is not standard, it almost follows directly from Assumption 2. If there are no costs to transacting and assets can be exchanged on any scale, then investors would prefer to be able to revise their portfolios at any time (whether they actually do so or not). In reality, transactions costs and indivisibilities do exist, and one reason given for finite trading-interval (discrete-time) models is to give implicit, if not explicit, recognition to these costs. However, this method of avoiding the problem of transactions costs is not satisfactory since a proper solution would almost certainly show that the trading intervals are stochastic and of non-constant length. Further, the portfolio demands and the resulting equilibrium relationships will be a function of the specific trading interval that is chosen.[8] An investor making a portfolio decision which is irrevocable ("frozen") for ten years, will choose quite differently than the one who has the option (even at a cost) to revise his portfolio daily. The essential issue is the market structure and not investors' tastes, and for well-developed capital markets, the time interval between successive market openings is sufficiently small to make the continuous-time assumption a good approximation.[9]

3. ASSET VALUE AND RATE OF RETURN DYNAMICS

Having described the structure of the capital market, we now develop the dynamics of the returns on assets traded in the market. It is sufficient for his decision

[8] A simple example from the expectations theory of the term structure will illustrate the point. It is well known (see, e.g., Stiglitz [40]) that bonds cannot be priced to equate expected returns over *all* holding periods. Hence, one must select a "fundamental" period (usually one "trading" period, our h) to equate expected returns. Clearly, the prices which satisfy this relationship will be a function of h. Similarly, the demand functions of investors will depend on h. We have chosen for our interval the smallest h possible. For processes which are well defined for every h, it can be shown that the limit of every discrete-time solution as h tends to zero, will be the continuous solutions derived here (see Samuelson [35]).

[9] What is "small" depends on the particular process being modeled. For the orders of magnitude typically found for the moments (mean, variance, skewness, etc.) of annual returns on common stocks, daily intervals ($h = 1/270$) are small. The essential test is: for what h does the distribution of returns become sufficiently "compact" in the Samuelson [35] sense?

making that the consumer-investor know at each point in time : (i) the transition probabilities for returns on each asset over the next trading interval (the *investment opportunity set*); and (ii) the transition probabilities for returns on assets in future periods (i.e., knowledge of the stochastic processes of the changes in the investment opportunity set). Unlike a single-period maximizer who, by definition, does not consider events beyond the present period, the intertemporal maximizer in selecting his portfolio takes into account the relationship between current period returns and returns that will be available in the future. For example, suppose that the current return on a particular asset is negatively correlated with changes in yields ("capitalization" rates). Then, by holding this asset, the investor expects a higher return on the asset if, ex post, yield opportunities next period are lower than were expected.

A brief description of the supply side of the asset market will be helpful in understanding the relationship between current returns on assets and changes in the investment opportunity set.

An asset is defined as a production technology which is a probability distribution for cash flow (valued in consumption units) and physical depreciation, as a function of the amount of capital, $K(t)$ (measured in physical units, e.g., number of machines), employed at time t. The price per unit capital in terms of the consumption good is $P_k(t)$, and the value of an asset at time t, $V(t)$, equals $P_k(t)K(t)$. The return on the asset over a period of length h will be the cash flow, X, plus the value of undepreciated capital, $(1 - \lambda)P_k(t + h)K(t)$ (where λ is the rate of physical depreciation of capital), minus the initial value of the asset, $V(t)$. The total change in the value of the asset outstanding, $V(t + h) - V(t)$, is equal to the sum of the return on the asset plus the value of gross new investment in excess of cash flow, $P_k(t + h)[K(t + h) - (1 - \lambda)K(t)] - X$.

Each firm in the model is assumed to invest in a single asset and to issue one class of securities, called equity.[10] Hence, the terms "firm" and "asset" can be used interchangeably. Let $N(t)$ be the number of shares of the firm outstanding and let $P(t)$ be the price per share, where $N(t)$ and $P(t)$ are defined by the difference equations,

(1) $P(t + h) \equiv [X + (1 - \lambda)P_k(t + h)K(t)]/N(t)$

and

(2) $N(t + h) \equiv N(t) + [P_k(t + h)[K(t + h) - (1 - \lambda)K(t)] - X]/P(t + h),$

subject to the initial conditions $P(0) = P$, $N(0) = N$, and $V(0) = N(0)P(0)$. If we assume that all dividend payments to shareholders are accomplished by share

---

[10] It is assumed that there are no economies or diseconomies to the "packaging" of assets (i.e., no "synergism"). Hence, any "real" firm holding more than one type of asset will be priced as if it held a portfolio of the "firms" in the text. Similarly, it is assumed that all financial leveraging and other capital structure differences are carried out by investors (possibly through financial intermediaries).

repurchase, then from (1) and (2), $[P(t + h) - P(t)]/P(t)$ is the rate of return on the asset over the period, in units of the consumption good.[11]

Since movements from equilibrium to equilibrium through time involve both price and quantity adjustment, a complete analysis would require a description of both the rate of return and change in asset value dynamics. To do so would require a specification of firm behavior in determining the supply of shares, which in turn would require knowledge of the real asset structure (i.e., technology; whether capital is "putty" or "clay"; etc.). In particular, the current returns on firms with large amounts (relative to current cash flow) of non-shiftable capital with low rates of depreciation will tend to be strongly affected by shifts in capitalization rates because, in the short run, most of the adjustment to the new equilibrium will be done by prices.

Since the present paper examines only investor behavior to derive the demands for assets and the relative yield requirements in equilibrium,[12] only the rate of return dynamics will be examined explicitly. Hence, certain variables, taken as exogenous in the model, would be endogenous to a full-equilibrium system.

From the assumption of continuous trading (Assumption 7), it is assumed that the returns and the changes in the opportunity set can be described by continuous-time stochastic processes. However, it will clarify the analysis to describe the processes for discrete trading intervals of length h, and then, to consider the limit as h tends to zero.

We assume the following:

ASSUMPTION 8: *The vector set of stochastic processes describing the opportunity set and its changes, is a time-homogeneous[13] Markov process.*

ASSUMPTION 9: *Only local changes in the state variables of the process are allowed.*

ASSUMPTION 10: *For each asset in the opportunity set at each point in time t, the expected rate of return per unit time, defined by*

$$\alpha \equiv E_t[(P(t + h) - P(t))/P(t)]/h,$$

---

[11] In an intertemporal model, it is necessary to define two quantities, such as number of shares and price per share, to distinguish between the two ways in which a firm's value can change. The return part, (1), reflects new additions to wealth, while (2) reflects a reallocation of capital among alternative assets. The former is important to the investor in selecting his portfolio while the latter is important in (determining) maintaining equilibrium through time. The definition of price per share used here (except for cash dividends) corresponds to the way open-ended, mutual funds determine asset value per share, and seems to reflect accurately the way the term is normally used in a portfolio context.

[12] While the analysis is not an equilibrium one in the strict sense because we do not develop the supply side, the derived model is as much an equilibrium model as the "exchange" model of Mossin [29]. Because his is a one-period model, he could take supplies as fixed. To assume this over time is nonsense.

[13] While it is not necessary to assume that the processes are independent of calendar time, nothing of content is lost by it. However, when a state variable is declared as constant in the text, we really mean non-stochastic. Thus, the term "constant" is used to describe variables which are deterministic functions of time.

and the variance of the return per unit time, defined by

$$\sigma^2 \equiv E_t[([P(t + h) - P(t)]/P(t) - \alpha h)^2]/h,$$

exist, are finite with $\sigma^2 > 0$, and are (right) continuous functions of h, where "E_t" is the conditional expectation operator, conditional on the levels of the state variables at time t. In the limit as h tends to zero, α is called the instantaneous expected return and σ^2 the instantaneous variance of the return.

Assumption 8 is not very restrictive since it is not required that the stochastic processes describing returns be Markov by themselves, but only that by the "expansion of the state" (supplementary variables) technique [7, p. 262] to include (a finite number of) other variables describing the changes in the transition probabilities, the entire (expanded) set be Markov. This generalized use of the Markov assumption for the returns is important because one would expect that the required returns will depend on other variables besides the price per share (e.g., the relative supplies of assets).

Assumption 9 is the discrete-time analog to the continuous-time assumption of continuity in the state variables (i.e., if $X(t + h)$ is the random state variable, then, with probability one, $\lim_{h \to 0} [X(t + h) - X(t)] = 0$). In words, it says that over small time intervals, price changes (returns) and changes in the opportunity set are small. This restriction is non-trivial since the implied "smoothness" rules out Pareto-Levy or Poisson-type jump processes.[14]

Assumption 10 ensures that, for small time intervals, the uncertainty neither "washes out" (i.e., $\sigma^2 = 0$) nor dominates the analysis (i.e., $\sigma^2 = \infty$). Actually, Assumption 10 follows from Assumptions 8 and 9 (see [13, p. 321]).

If we let $\{X(t)\}$ stand for the vector stochastic process, then Assumptions 8–10 imply that, in the limit as h tends to zero, $X(t)$ is a diffusion process with continuous state-space changes and that the transition probabilities will satisfy a (multi-dimensional) Fokker-Planck or Kolmogorov partial differential equation.

Although these partial differential equations are sufficient for study of the transition probabilities, it is useful to write down the explicit return dynamics in stochastic difference equation form and then, by taking limits, in stochastic differential equation form. From the previous analysis, we can write the returns dynamics as

$$(3) \qquad \frac{P(t + h) - P(t)}{P(t)} = \alpha h + \sigma y(t)\sqrt{h},$$

where, by construction, $E_t(y) = 0$ and $E_t(y^2) = 1$, and $y(t)$ is a purely random process; that is, $y(t)$ and $y(t + s)$, for $s > 0$, are identically distributed and mutually

---

[14] While a similar analysis can be performed for Poisson-type processes (see Kushner [18] and Merton [25]) and for the subordinated processes of Press [30] and Clark [6], most of the results derived under the continuity assumption will not obtain in these cases.

independent.[15] If we define the stochastic process, $z(t)$, by

$$(4) \qquad z(t + h) = z(t) + y(t)\sqrt{h},$$

then $z(t)$ is a stochastic process with independent increments. If it is further assumed that $y(t)$ is Gaussian distributed,[16] then the limit as h tends to zero of $z(t + h) - z(t)$ describes a Wiener process or Brownian motion. In the formalism of stochastic differential equations,

$$(5) \qquad dz \equiv y(t)\sqrt{dt}.$$

In a similar fashion, we can take the limit of (3) to derive the stochastic differential equation for the instantaneous return on the ith asset as

$$(6) \qquad \frac{dP_i}{P_i} = \alpha_i \, dt + \sigma_i \, dz_i.$$

Processes such as (6) are called Itô processes and while they are continuous, they are not differentiable.[17]

From (6), a sufficient set of statistics for the opportunity set at a given point in time is $\{\alpha_i, \sigma_i, \rho_{ij}\}$ where ρ_{ij} is the instantaneous correlation coefficient between the Wiener processes dz_i and dz_j. The vector of return dynamics as described in (6) will be Markov only if α_i, σ_i, and ρ_{ij} were, at most, functions of the P's. In general, one would not expect this to be the case since, at each point in time, equilibrium clearing conditions will define a set of implicit functions between equilibrium market values, $V_i(t) = N_i(t)P_i(t)$, and the $\alpha_i, \sigma_i,$ and ρ_{ij}. Hence, one would expect the changes in required expected returns to be stochastically related to changes in market values, and dependence on P solely would obtain only if changes in N (changes in supplies) were non-stochastic. Therefore, to close the system, we append the dynamics for the changes in the opportunity set over time: namely,

$$(7) \qquad d\alpha_i = a_i \, dt + b_i \, dq_i,$$
$$d\sigma_i = f_i \, dt + g_i \, dx_i,$$

where we do assume that (6) and (7), together, form a Markov system,[18] with dq_i and dx_i standard Wiener processes.

[15] It is sufficient to assume that the $y(t)$ are uncorrelated and that the higher order moments are $o(1/\sqrt{h})$. This assumption is consistent with a weak form of the efficient markets hypothesis of Samuelson [33] and Fama [10]. See Merton and Samuelson [27] for further discussion.

[16] While the Gaussian assumption is not necessary for the analysis, the generality gained by not making the assumption is more apparent than real, since it can be shown that all continuous diffusion processes can be described as functions of Brownian motion (see Feller [13, p. 326] and Itô and McKean [16]).

[17] See Merton [25] for a discussion of Itô processes in a portfolio context. For a general discussion of stochastic differential equations of the Itô type, see Itô and McKean [16], McKean [22], and Kushner [18].

[18] It is assumed that the dynamics of α and σ reflect the changes in the supply of shares as well as other factors such as new technical developments. The particular derivation of the dz_i in the text implies that the ρ_{ij} are constants. However, the analysis could be generalized by appending an additional set of dynamics to include changes in the ρ_{ij}.

Under the assumptions of continuous trading and the continuous Markov structure of the stochastic processes, it has been shown that the instantaneous, first two moments of the distributions are sufficient statistics.[19] Further, by the existence and boundedness of α and σ, P equal to zero is a natural absorbing barrier ensuring limited liability of all assets.

For the rest of the paper, it is assumed that there are n distinct[20] risky assets and one "instantaneously risk-less" asset. "Instantaneously risk-less" means that, at each instant of time, each investor knows with certainty that he can earn rate of return $r(t)$ over the next instant by holding the asset (i.e., $\sigma_{n+1} = 0$ and $\alpha_{n+1} \equiv r(t)$). However, the future values of $r(t)$ are not known with certainty (i.e., $b_{n+1} \neq 0$ in (7)). We interpret this asset as the exchange asset and $r(t)$ as the instantaneous private sector borrowing (and lending) rate. Alternatively, the asset could represent (very) short government bonds.

4. PREFERENCE STRUCTURE AND BUDGET EQUATION DYNAMICS

We assume that there are K consumer-investors with preference structures as described in [25]: namely, the kth consumer acts so as to

$$(8) \qquad \max E_0\left[\int_0^{T^k} U^k[c^k(s), s]\, ds + B^k[W^k(T^k), T^k]\right],$$

where "E_0" is the conditional expectation operator, conditional on the current value of his wealth, $W^k(0) = W^k$ are the state variables of the investment opportunity set, and T^k is the distribution for his age of death (which is assumed to be independent of investment outcomes). His instantaneous consumption flow at age t is $c^k(t)$.[21] U^k is a strictly concave von Neumann-Morgenstern utility function for consumption and B^k is a strictly concave "bequest" or utility-of-terminal wealth function.

Dropping the superscripts (except where required for clarity), we can write the accumulation equation for the kth investor as[22]

$$(9) \qquad dW = \sum_1^{n+1} w_i W\, dP_i/P_i + (y - c)\, dt,$$

where $w_i \equiv N_i P_i/W$ is the fraction of his wealth invested in the ith asset, N_i is the number of shares of the ith asset he owns, and y is his wage income. Substituting

---

[19] Since these are sufficient statistics, if there are $n + 1$ assets and n is finite, then our assumption of a finite vector for X is satisfied.

[20] "Distinct" means that none of the assets' returns can be written as an (instantaneous) linear combination of the other assets' returns. Hence, the instantaneous variance-covariance matrix of returns, $\Omega = [\sigma_{ij}]$, is non-singular.

[21] Because the paper is primarily interested in finding equilibrium conditions for the asset markets, the model assumes a single consumption good. The model could be generalized by making c^k a vector and introducing as state variables the relative prices. While the analysis would be similar to the one-good case, there would be systematic effects on the portfolio demands reflecting hedging behavior against unfavorable shifts in relative consumption goods prices (i.e., in the consumption opportunity set).

[22] See Merton [25] for a derivation of (9).

for dP_i/P_i from (6), we can re-write (9) as

$$(10) \qquad dW = \left[\sum_1^n w_i(\alpha_i - r) + r \right] W \, dt + \sum_1^n w_i W \sigma_i \, dz_i + (y - c) \, dt,$$

where his choice for $w_1, w_2, \ldots, w_n$ is unconstrained because w_{n+1} can always be chosen to satisfy the budget constraint $\Sigma_1^{n+1} w_i = 1$.

From the budget constraint, $W = \Sigma_1^{n+1} N_i P_i$, and the accumulation equation (9), we have that

$$(11) \qquad (y - c) \, dt = \sum_1^{n+1} dN_i(P_i + dP_i),$$

i.e., the net value of new shares purchased must equal the value of savings from wage income.

5. THE EQUATIONS OF OPTIMALITY: THE DEMAND FUNCTIONS FOR ASSETS

For computational simplicity, we will assume that investors derive all their income from capital gains sources (i.e., $y \equiv 0$),[23] and for notational simplicity, we introduce the state-variable vector, X, whose m elements, x_i, denote the current levels of P, α, and σ. The dynamics for X are written as the vector Itô process,

$$(12) \qquad dX = F(X) \, dt + G(X) \, dQ,$$

where F is the vector $[f_1, f_2, \ldots, f_m]$, G is a diagonal matrix with diagonal elements $[g_1, g_2, \ldots, g_m]$, dQ is the vector Wiener process $[dq_1, dq_2, \ldots, dq_m]$, η_{ij} is the instantaneous correlation coefficient between dq_i and dz_j, and v_{ij} is the instantaneous correlation coefficient between dq_i and dq_j.

I have shown elsewhere[24] that the necessary optimality conditions for an investor who acts according to (8) in choosing his consumption-investment program are that, at each point in time,

$$(13) \qquad 0 = \max_{\{c,w\}} \left[U(c,t) + J_t + J_W \left[\left(\sum_1^n w_i(\alpha_i - r) + r \right) W - c \right] \right.$$
$$+ \sum_1^m J_i f_i + \tfrac{1}{2} J_{WW} \sum_1^n \sum_1^n w_i w_j \sigma_{ij} W^2$$
$$\left. + \sum_1^m \sum_1^n J_{iW} w_j W g_i \sigma_j \eta_{ij} + \frac{1}{2} \sum_1^m \sum_1^m J_{ij} g_i g_j v_{ij} \right],$$

[23] The analysis would be the same with wage income, provided that investors can issue shares against future income, since we can always redefine wealth as including capitalized future wage income. However, since institutionally this cannot be done, the introduction of wage income will cause systematic effects on the portfolio and consumption decisions.

[24] See Merton [23 and 25]. $J(W, t, X) \equiv \max E_t \{ \int_t^T U(c, s) \, ds + B[W(T), T] \}$ and is called the "derived" utility of wealth function. Substituting from (14) and (15) to eliminate w_i and c in (13) makes (13) a partial differential equation for J, subject to the boundary condition $J(W, T, X) = B(W, T)$. Having solved for J, we then substitute for J and its derivatives in (14) and (15) to find the optimal rules (w_i, c).

subject to $J(W, T, X) = B(W, T)$, where subscripts on the "derived" utility of wealth function, J, denote partial derivatives. The σ_{ij} are the instantaneous covariances between the returns on the ith and jth assets ($\equiv \sigma_i \sigma_j \rho_{ij}$).

The $n + 1$ first-order conditions derived from (13) are

(14) $\qquad 0 = U_c(c, t) - J_W(W, t, X),$

and

(15) $\qquad 0 = J_W(\alpha_i - r) + J_{WW} \sum_1^n w_j W \sigma_{ij} + \sum_1^m J_{jW} g_j \sigma_i \eta_{ji} \qquad\qquad (i = 1, 2, \ldots, n),$

where $c = c(W, t, X)$ and $w_i = w_i(W, t, X)$ are the optimum consumption and portfolio rules as functions of the state variables. Equation (14) is the usual intertemporal envelope condition to equate the marginal utility of current consumption to the marginal utility of wealth (future consumption). The manifest characteristic of (15) is its linearity in the portfolio demands; hence, we can solve explicitly for these functions by matrix inversion,

(16) $\qquad w_i W = A \sum_1^n v_{ij}(\alpha_j - r) + \sum_1^m \sum_1^n H_k \sigma_j g_k \eta_{jk} v_{ij} \qquad\qquad (i = 1, 2, \ldots, n),$

where the v_{ij} are the elements of the inverse of the instantaneous variance-covariance matrix of returns, $\Omega = [\sigma_{ij}]$, $A \equiv -J_W/J_{WW}$, and $H_k \equiv -J_{kW}/J_{WW}$.

Some insight in interpreting (16) can be gained by expressing A and H_k in terms of the utility and consumption functions: namely, by the implicit function theorem applied to (14),

(17) $\qquad A = -U_c \Big/ \left(U_{cc} \dfrac{\partial c}{\partial W} \right) > 0,$

and

(18) $\qquad H_k = -\dfrac{\partial c}{\partial x_k} \Big/ \dfrac{\partial c}{\partial W} \lessgtr 0.$

From (17) and (18), we can interpret the demand function (16) as having two components. The first term, $A\Sigma_1^n v_{ij}(\alpha_j - r)$, is the usual demand function for a risky asset by a single-period mean-variance maximizer, where A is proportional to the reciprocal of the investor's absolute risk aversion.[25] The second term, $\Sigma_1^m \Sigma_1^n H_k \sigma_j g_k \eta_{jk} v_{ij}$, reflects his demand for the asset as a vehicle to hedge against "unfavorable" shifts in the investment opportunity set. An "unfavorable" shift in the opportunity set variable x_k is defined as a change in x_k such that (future) consumption will fall for a given level of (future) wealth. An example of an unfavorable shift would be if $\partial c/\partial x_k < 0$ and x_k increased.

It can be shown, by differentiating (16) with respect to η_{ij}, that all risk-averse utility maximizers will attempt to hedge against such shifts in the sense that if $\partial c/\partial x_k < (>)0$, then, ceteris paribus, they will demand more of the ith asset, the

---

[25] See Merton [26, equation (36)].

more positively (negatively) correlated its return is with changes in x_k. Thus, if the ex post opportunity set is less favorable than was anticipated, the investor will expect to be compensated by a higher level of wealth through the positive correlation of the returns. Similarly, if ex post returns are lower, he will expect a more favorable investment environment.

Although this behavior implies a type of intertemporal consumption "smoothing," it is not the traditional type of maintenance of a constant level of consumption, but rather it reflects an attempt to minimize the (unanticipated) variability in consumption over time. A simple example will illustrate the point. Assume a single risky asset, a riskless asset with return r, and X a scalar (e.g., $X = r$). Further, require that $\alpha = r$. Standard portfolio analysis would show that a risk-averse investor would invest all his wealth in the riskless asset (i.e., $w = 0$). Consider the (instantaneous) variance of his consumption which, by Itô's Lemma,[26] can be written as $[c_x^2 g^2 + c_W^2 w^2 W^2 \sigma^2 + 2c_x c_W w W g \sigma \eta]$, where subscripts denote partial derivatives of the (optimal) consumption function. Simple differentiation will show that this variance is minimized at $wW = -c_x \eta g / \sigma c_W$, which is exactly the demand given by (16), and for $c_x < 0$ and $\eta > 0$, $w > 0$. Thus, an intertemporal investor who currently faces a five per cent interest rate and a possible interest rate of either two or ten per cent next period will have portfolio demands different from a single-period maximizer in the same environment or an intertemporal maximizer facing a constant interest rate of five per cent over time.

While we have derived explicit expressions for the portfolio demands and given some interpretation of their meaning, further analysis at this level of generality is difficult. While some further results could be gained by restricting the class of utility functions (see Merton [25, p. 402]), a more fruitful approach is to add some additional (simplifying) assumptions to restrict the structure of the opportunity set.

6. CONSTANT INVESTMENT OPPORTUNITY SET

The simplest form of the model occurs when the investment opportunity set is constant through time (i.e., α, r, and Ω are constants), and from (6), the distributions for price per share will be log-normal for all assets. This form of the model is examined in detail in Merton [25, p. 384–88], and hence, the main results are presented without proof.

In this case, the demand for the ith asset by the kth investor, (16), reduces to

$$(19) \qquad w_i^k W^k = A^k \sum_1^n v_{ij}(\alpha_j - r) \qquad\qquad (i = 1, 2, \ldots, n),$$

which is the same demand that a one-period[27] risk-averse mean-variance investor would have. If all investors agree on the investment opportunity set

---

[26] Itô's Lemma is the analog to the Fundamental Theorem of the calculus for Itô processes. See Merton [25, p. 375] for a brief description and McKean [22, p. 32] for a formal proof.

[27] Of course, since "one period" is an instant, a meaningful interpretation is that investors behave myopically.

(homogeneous expectations), then the ratio of the demands for risky assets will be independent of preferences, and the same for all investors. Further, we have the following theorem.

THEOREM 1:[28] *Given n risky assets whose returns are log-normally distributed and a riskless asset, then (i) there exists a unique pair of efficient portfolios ("mutual funds") one containing only the riskless asset and the other only risky assets, such that, independent of preferences, wealth distribution, or time horizon, all investors will be indifferent between choosing portfolios from among the original $n + 1$ assets or from these two funds; (ii) the distribution of the return on the risky fund is log-normal; (iii) the proportion of the risky fund's assets invested in the kth asset is*

$$\sum_{1}^{n} v_{kj}(\alpha_j - r) \bigg/ \sum_{1}^{n} \sum_{1}^{n} v_{ij}(\alpha_i - r) \qquad (k = 1, 2, \ldots, n).$$

Theorem 1 is the continuous-time version of the Markowitz-Tobin separation theorem and the holdings of the risky fund correspond to the optimal combination of risky assets (see Sharpe [**39**, p. 69]).

Using the condition that the market portfolio is efficient in equilibrium, it can be shown (see Merton [**26**]) that, for this version of the model, the equilibrium returns will satisfy

$$(20) \qquad \alpha_i - r = \beta_i(\alpha_M - r) \qquad (i = 1, 2, \ldots, n),$$

where $\beta_i \equiv \sigma_{iM}/\sigma_M^2$, σ_{iM} is the covariance of the return on the ith asset with the return on the market portfolio, and α_M is the expected return on the market portfolio. Equation (20) is the continuous-time analog to the security market line of the classical capital asset pricing model.

Hence, the additional assumption of a constant investment opportunity set is a sufficient condition for investors to behave as if they were single-period maximizers and for the equilibrium return relationship specified by the capital asset pricing model to obtain. Except for some singular cases, this assumption is also necessary.

7. GENERALIZED SEPARATION: A THREE-FUND THEOREM

Unfortunately, the assumption of a constant investment opportunity set is not consistent with the facts, since there exists at least one element of the opportunity set which is directly observable: namely, the interest rate, and it is definitely

---

[28] Theorem 1 is stated and proved in a more general form, including the possibility of no riskless asset, in Merton [**25**, p. 384]. The uniqueness of the two funds is ensured by the requirement that one fund hold only the riskless asset and the other only risky assets, and that both funds be efficient. Otherwise, the funds are unique only up to a non-singular, linear transformation. A further requirement is that

$$r < \sum_{1}^{n} \sum_{1}^{n} v_{ij}\alpha_j \bigg/ \sum_{1}^{n} \sum_{1}^{n} v_{ij}.$$

However, since this is a necessary condition for equilibrium, it is assumed to be satisfied. See Merton [**26**] for a complete discussion of this point.

changing stochastically over time. The simplest form of the model consistent with this observation occurs if it is assumed that a single state variable is sufficient to describe changes in the opportunity set. We further assume that this variable is the interest rate (i.e., $\alpha_i = \alpha_i(r)$ and $\sigma_i = \sigma_i(r)$).

The interest rate has always been an important variable in portfolio theory, general capital theory, and to practitioners. It is observable, satisfies the condition of being stochastic over time, and while it is surely not the sole determinant of yields on other assets,[28] it is an important factor. Hence, one should interpret the effects of a changing interest rate in the forthcoming analysis in the way economists have generally done in the past: namely, as a single (instrumental) variable representation of shifts in the investment opportunity set. For example, $\partial c/\partial r$ is the change in consumption due to a change in the opportunity set for a fixed level of wealth.

This assumed, we can write the kth investor's demand function for the ith asset, (16), as

$$(21) \qquad d_i^k = A^k \sum_1^n v_{ij}(\alpha_j - r) + H^k \sum_1^n v_{ij}\sigma_{jr} \qquad\qquad (i = 1, 2, \ldots, n),$$

where $d_i^k \equiv w_i^k W^k$; $H^k \equiv -(\partial c^k/\partial r)/(\partial c^k/\partial W^k)$, and σ_{jr} is the (instantaneous) covariance between the return on the jth asset and changes in the interest rate ($\equiv \rho_{jr}\sigma_j g$). By inspection of (21), the ratio of the demands for risky assets is a function of preferences, and hence, the standard separation theorem does not obtain. However, generalized separation (see [5]) does obtain. In particular, it will be shown that all investors' optimal portfolios can be represented as a linear combination of three mutual funds (portfolios).

Although not necessary for the theorem, it will throw light on the analysis to assume there exists an asset (by convention, the nth one) whose return is perfectly negatively correlated with changes in r, i.e., $\rho_{nr} = -1$. One such asset might be riskless (in terms of default), long-term bonds.[30] In this case, we can re-write the covariance term σ_{jr} as

$$(22) \qquad \sigma_{jr} = \rho_{jr}\sigma_j g,$$
$$= -g(\rho_{jn}\sigma_j\sigma_n)/\sigma_n, \quad \text{because } \rho_{jr} = -\rho_{jn},$$
$$= -g\sigma_{jn}/\sigma_n,$$

where g is the standard deviation of the change in r. From (22), we can write the second term in the demand function (21), $\sum_1^n v_{ij}\sigma_{jr}$, as $-g(\sum_1^n v_{ij}\sigma_{jn})/\sigma_n$ which equals zero for $i \neq n$ and equals $(-g/\sigma_n)$ for $i = n$, because the v_{ij} are the elements of the inverse of the variance-covariance matrix of returns.[31] Hence, we can

[29] The reader should not interpret this statement as implying a causal relationship between interest rates and yields. All that is questioned is whether there exists an implicit functional relationship between the interest rate and other yields.

[30] We only interpret this asset as a long-term bond as a conceptual device. Although long-term bonds will be highly correlated with short rate changes, it is quite likely that they are not perfectly correlated.

[31] I am indebted to Fischer Black for pointing out this simplification.

re-write (21) in the simplified form,

$$(23) \qquad d_i^k = A^k \sum_1^n v_{ij}(\alpha_j - r) \qquad\qquad (i = 1, 2, \ldots, n - 1),$$

$$d_n^k = A^k \sum_1^n v_{nj}(\alpha_j - r) - gH^k/\sigma_n.$$

THEOREM 2 ("*Three Fund*" *Theorem*): *Given n risky assets and a riskless asset satisfying the conditions of this section, then there exist three portfolios ("mutual funds") constructed from these assets, such that* (i) *all risk-averse investors, who behave according to* (8), *will be indifferent between choosing portfolios from among the original n + 1 assets or from these three funds;* (ii) *the proportions of each fund's portfolio invested in the individual assets are purely "technological" (i.e., depend only on the variables in the investment opportunity set for individual assets and not on investor preferences);* and (iii) *the investor's demands for the funds do not require knowledge of the investment opportunity set for the individual assets nor of the asset proportions held by the funds.*

PROOF: Let the first fund hold the same proportions as the risky fund in Theorem 1: namely, $\delta_k = \Sigma_1^n v_{kj}(\alpha_j - r)/\Sigma_1^n \Sigma_1^n v_{ij}(\alpha_j - r)$, for $k = 1, 2, \ldots, n$. Let the second fund hold only the nth asset and the third fund only the riskless asset. Let λ_i^k be the fraction of the kth investor's wealth invested in the ith fund, $i = 1, 2, 3$ ($\Sigma_1^3 \lambda_i^k = 1$). To prove (i), we must show that there exists an allocation (λ_1^k, λ_2^k) which exactly replicates the demand functions, (23), i.e., that

$$(24) \qquad \lambda_1^k \delta_i = (A^k/W^k) \sum_1^n v_{ij}(\alpha_j - r) \qquad\qquad (i = 1, 2, \ldots, n - 1),$$

$$\lambda_1^k \delta_n + \lambda_2^k = (A^k/W^k) \sum_1^n v_{nj}(\alpha_j - r) - gH^k/\sigma_n W^k.$$

From the definition of δ_i, the allocation $\lambda_1^k = (A^k/W^k)\Sigma_1^n \Sigma_1^n v_{ij}(\alpha_j - r)$ and $\lambda_2^k = -gH^k/\sigma_n W^k$ satisfied (24). Part (ii) follows from the choice for the three funds. To prove (iii), we must show that investors will select this allocation, given only the knowledge of the (aggregated) investment opportunity set, i.e., given $(\alpha, \alpha_n, r, \sigma, \sigma_n, \rho, g)$ where α and σ^2 is the expected return and variance on the first fund's portfolio and ρ is its covariance with the return on the second fund. From the definition of δ_i, it is straightforward to show that $(\alpha - r)/\sigma^2 = \Sigma_1^n \Sigma_1^n v_{ij}(\alpha_i - r)$ and $\rho = \sigma(\alpha_n - r)/\sigma_n(\alpha - r)$. The demand functions for the funds will be of the same form as (23) with $n = 2$, and the proportions derived from these equations are λ_1^k and λ_2^k where λ_1^k can be re-written as $A^k(\alpha - r)/\sigma^2 W^k$.   *Q.E.D.*

Theorem 2 is a decentralization theorem which states that if investors believe that professional portfolio managers' estimates of the distribution of returns are at least as good as any the investor might form, then the investment decision can be separated into two parts by the establishment of three financial intermediaries (mutual funds) to hold all individual assets and to issue shares of their own for

purchase by individual investors. Funds one and three provide the "service" to investors of an (instantaneously) efficient, risk-return frontier while fund two allows investors to hedge against unfavorable intertemporal shifts in the frontier. Note that the demand for the second fund by the kth investor, $\lambda_2^k W^k$, will be $\geqq 0$, depending on whether $\partial c^k/\partial r$ is $\geqq 0$, which is consistent with the hedging behavior discussed in the general case of Section 5.

8. THE EQUILIBRIUM YIELD RELATIONSHIP AMONG ASSETS

Given the demand functions (23), we now derive the equilibrium market clearing conditions for the model of Section 7, and from these, derive the equilibrium relationship between the expected return on an individual asset and the expected return on the market.

From (23), the aggregate demand functions, $D_i = \Sigma_1^K d_i^k$, can be written as

$$(25) \qquad D_i = A \sum_1^n v_{ij}(\alpha_j - r) \qquad\qquad (i = 1, 2, \ldots, n - 1),$$

$$D_n = A \sum_1^n v_{nj}(\alpha_j - r) - Hg/\sigma_n,$$

where $A \equiv \Sigma_1^K A^k$ and $H \equiv \Sigma_1^K H^k$. If N_i is the number of shares supplied by the ith firm and if it is assumed that the asset market is *always* in equilibrium, then

$$(26) \qquad N_i = \sum_1^K N_i^k,$$

$$dN_i = \sum_1^K dN_i^k \qquad\qquad (i = 1, 2, \ldots, n + 1).$$

Furthermore, $\Sigma_1^{n+1} N_i P_i = \Sigma_1^{n+1} D_i \equiv M$, where M is the (equilibrium) value of all assets, the market.

The equilibrium dynamics for market value can be written as

$$(27) \qquad dM = \sum_1^{n+1} N_i \, dP_i + \sum_1^{n+1} dN_i(P_i + dP_i)$$

$$= \sum_1^K dW^k$$

$$= \sum_1^{n+1} D_i \, dP_i/P_i + \sum_1^K (y^i - c^i) \, dt.$$

Hence, changes in the value of the market come about by capital gains on current shares outstanding (the first term) and by expansion of the total number of shares outstanding (the second term). To separate the two effects, we use the same technique employed to solve this problem for the individual firm: namely, let P_M be the price per "share" of the market portfolio and let N be the number of shares where $NP_M \equiv M$. Then, $dM = N \, dP_M + dN(P_M + dP_M)$, and P_M and N are defined by

the stochastic differential equations

(28) $$N \, dP_M \equiv \sum_1^{n+1} N_i \, dP_i,$$

$$dN(P_M + dP_M) \equiv \sum_1^{n+1} dN_i(P_i + dP_i),$$

where, by construction, dP_M/P_M is the rate of return on the market (portfolio). Substituting from (27) into (28) and using (11), we have

(29) $$dN(P_M + dP_M) = \sum_1^K (y^i - c^i) \, dt,$$

$$N \, dP_M = \sum_1^{n+1} D_i \, dP_i/P_i.$$

If $w_i \equiv N_i P_i/M = D_i/M$, the percentage contribution of the ith firm to total market value, then, from (6) and (29), the rate of return on the market can be written as

(30) $$\frac{dP_M}{P_M} = \left[\sum_1^n w_j(\alpha_j - r) + r \right] dt + \sum_1^n w_j \sigma_j \, dz_j.$$

Substituting $w_i M$ for D_i in (25), we can solve for the equilibrium expected returns on the individual assets:

(31) $$\alpha_i - r = (M/A) \sum_1^n w_j \sigma_{ij} + (Hg/A\sigma_n)\sigma_{in} \qquad (i = 1, 2, \ldots, n).$$

As with any asset, we can define $\alpha_M (\equiv \Sigma_1^n w_j(\alpha_j - r) + r)$, $\sigma_{iM}(\equiv \Sigma_1^n w_j \sigma_{ij})$, and $\sigma_M^2(\equiv \Sigma_1^n w_j \sigma_{jM})$ as the (instantaneous) expected return, covariance, and variance of the market portfolio. Then (31) can be re-written as

(32) $$\alpha_i - r = (M/A)\sigma_{iM} + (Hg/A\sigma_n)\sigma_{in} \qquad (i = 1, 2, \ldots, n),$$

and multiplying (32) by w_i and summing, we have

(33) $$\alpha_M - r = (M/A)\sigma_M^2 + (Hg/A\sigma_n)\sigma_{Mn}.$$

Noting that the nth asset satisfies (32), we can use it together with (33) to rewrite (32) as

(34) $$\alpha_i - r = \frac{\sigma_i[\rho_{iM} - \rho_{in}\rho_{nM}]}{\sigma_M(1 - \rho_{nM}^2)}(\alpha_M - r) + \frac{\sigma_i[\rho_{in} - \rho_{iM}\rho_{nM}]}{\sigma_n(1 - \rho_{Mn}^2)}(\alpha_n - r)$$

$$(i = 1, 2, \ldots, n - 1).$$

Equation (34) states that, in equilibrium, investors are compensated in terms of expected return, for bearing market (systematic) risk, and for bearing the risk of unfavorable (from the point of view of the aggregate) shifts in the investment opportunity set; and it is a natural generalization of the security market line of the classical capital asset pricing model. Note that if a security has no market risk

(i.e., $\beta_i = 0 = \rho_{iM}$), its expected return will not be equal to the riskless rate as forecast by the usual model.

Under what conditions will the security market plane equation (34) reduce to the (continuous-time) classical security market line, equation (20)? From inspection of the demand equations (21), appropriately aggregated, the conditions are

$$(35a) \qquad H = \sum_1^K - (\partial c^k/\partial r)/(\partial c^k/\partial W^k) \equiv 0$$

or

$$(35b) \qquad \sigma_{ir} \equiv 0 \qquad\qquad\qquad\qquad (i = 1, 2, \ldots, n).$$

There is no obvious reason to believe that (35a) should hold unless $\partial c^k/\partial r \equiv 0$ for each investor, and the only additive utility function for which this is so is the Bernoulli logarithmic one.[32] Condition (35b) could obtain in two ways: $g \equiv 0$, i.e., the interest rate is non-stochastic, which is not so; or $\rho_{ir} \equiv 0$, i.e., all assets' returns are uncorrelated with changes in the interest rate. While this condition is possible, it would not be a true equilibrium state.

Suppose that by a quirk of nature, $\rho_{ir} \equiv 0$ for all available real assets. Then, since the nth asset does not exist, (34) reduces to (20). Consider constructing a "man-made" security (e.g., a long-term bond) which is perfectly negatively correlated with changes in the interest rates, and hence, by assumption, not correlated with any other asset or the market (i.e., $\beta_n = 0$). Since $D_n = 0$, we have, from (25), that $(\alpha_n - r) = Hg\sigma_n \neq 0$, if $g \neq 0$ and $H \neq 0$. Thus, even though security n has a zero beta, investors will pay a premium (relative to the riskless rate) to other investors for creating this security.

An implication of this analysis for the theory of the term structure of interest rates, is that long-term, riskless bonds will not satisfy the expectations hypothesis $(\alpha_n = r)$, even if they have no market risk. The premium charged is not a liquidity premium, and it will be either positive or negative depending on the sign of H. These results are consistent with the "habitat" theory (see [28]), if one interprets habitat as a stronger (or weaker) preference to hedge against changes in future investment opportunities.

9. EMPIRICAL EVIDENCE

Although the model has not been formally tested, we can do some preliminary analysis using the findings of Black, Jensen, and Scholes (BJS) [3] and some later, unpublished work of Scholes [37]. As mentioned earlier, they found that portfolios constructed to have zero covariance with the market (i.e., $\beta = 0$) had average returns that significantly exceeded the riskless rate which suggests that there is (at least) another factor besides the market that systematically affects the returns on securities. They call this second factor the "beta factor" because an individual security's covariance with it is a function of the security's beta. In particular,

---

[32] Hence (20) would be the correct specification for the equilibrium relationships among expected returns in the "growth optimum" model even when the investment opportunity set is not constant through time.

high-beta ($\beta > 1$) stocks had negative correlation and low-beta ($\beta < 1$) stocks had positive correlation. We can summarize the BJS specification and empirical findings as follows:

$$(36) \qquad \alpha_i - r = \beta_i(\alpha_M - r) + \gamma_i(\alpha_0 - r),$$

where α_0 is the expected return on the "zero-beta" portfolio, and

$$(37a) \qquad \alpha_0 > r;$$

$$(37b) \qquad \gamma_i = \gamma_i(\beta_i) \quad \text{with} \quad \gamma_i(1) = 0, \quad \text{and} \quad \partial\gamma_i/\partial\beta_i < 0.$$

While the finding of a second factor is consistent with the a priori specification of our model, it cannot be said that their specific findings are in agreement with the model without some further specification of the effect of a shift in r on the investment opportunity set. However, if a shift in r is an instrumental variable for a shift in capitalization rates generally, then an argument can be made that the two are in agreement.

The plan is to show that qualitative characteristics of the coefficient ($\rho_{in} - \rho_{iM}\rho_{nM})\sigma_i/\sigma_n(1 - \rho_{nM}^2)$ in (34) as a function of β_i would be the same as γ_i in (37b), and that the empirical characteristics of the zero-beta portfolio are similar to those of a portfolio of long term bonds.

If we take the classical security market line, $\alpha_i = r + \beta_i\lambda$, where $\lambda \equiv (\alpha_M - r)$, as a reasonable approximation to the relationship among capitalization rates, α_i, then we can compute the logarithmic elasticity of α_i with respect to r as a function of β_i, to be

$$(38) \qquad \psi(\beta_i) \equiv r(1 + \beta_i\lambda')/(r + \beta_i\lambda),$$

where $\lambda' \equiv \partial\lambda/\partial r$, the change in the slope of the security market line with a change in r. From (27) we have that this elasticity is almost certainly a monotone decreasing function of β_i since $\psi'(\beta_i) \equiv \partial\psi/\partial\beta_i < 0$ if $\psi(1) < 1$.[33]

If we write the value of firm i as $V_i \equiv \overline{X}_i/\alpha_i$ where $\overline{X}_i$ is the "long-run" expected earnings and α_i, the rate at which they are capitalized, then the percentage change in firm value due to a change in r can be written as

$$(39) \qquad \left(\frac{dV_i}{V_i}\right)_r \equiv \left[\frac{\partial\overline{X}_i}{\partial r} \bigg/ \overline{X}_i - \frac{\partial\alpha_i}{\partial r} \bigg/ \alpha_i\right] dr.$$

If we neglect, as second-order, the effect of a shift in r on expected future earnings, then the residual effect on return due to a change in r, after taking out the common market factor, will be a systematic function of β_i:

$$(40) \qquad d\varepsilon(\beta_i) \equiv \left(\frac{dV_i}{V_i}\right)_r - \beta_i\left(\frac{dV_M}{V_M}\right)_r$$

$$= -\psi(\beta_i)\frac{dr}{r} + \beta_i\psi(1)\frac{dr}{r}$$

$$= -\phi(\beta_i)\frac{dr}{r}$$

---

[33] $\psi(1) \geqslant 1$ would imply that $\lambda'/\lambda \geqslant 1/r$ which, for typical values of r, would imply a very large, positive increase in the slope of the security market line. It is contended that such a shift would be highly unlikely.

where $\phi(\beta_i) \equiv \psi(\beta_i) - \beta_i \psi(1)$ satisfies $\phi(1) = 0$ and $\phi'(\beta_i) < 0$. From (40), the correlation coefficient between $d\varepsilon$ and dr, $\rho_{\varepsilon r}$, will satisfy

$$(41) \qquad \rho_{\varepsilon r} \gtreqless 0 \quad \text{as} \quad \beta_i \lesseqgtr 1.$$

From the definition of $d\varepsilon$ in (40), $\rho_{\varepsilon r}$ is the partial correlation coefficient, $\rho_{ir} - \rho_{iM}\rho_{rM}$. By definition the nth asset in (34) is perfectly negatively correlated with changes in r. Hence (41) can be rewritten as

$$(42) \qquad \rho_{in} - \rho_{iM}\rho_{nM} \gtreqless 0 \quad \text{as} \quad \beta_i \lesseqgtr 1.$$

Hence the coefficient of $(\alpha_n - r)$ in (34) could be expected to have the same properties as γ_i in (36) and (37b).

It still remains to be determined whether the zero-beta portfolio is a proxy for our long-term bond portfolio. Since there are no strong theoretical grounds for $(\alpha_n - r)$ to be positive[34] and since the zero-beta portfolio is an empirical construct, we resort to an indirect empirical argument based on the findings of BJS and Scholes.

Since Scholes found the correlation between the market portfolio and the bond portfolio, ρ_{Mn}, to be close to zero and the correlation between the zero-beta portfolio and the bond portfolio to be significantly positive, it then follows from (36) that one would expect to find $(\alpha_n - r)$ significantly positive.

While the analysis of this section can only be called preliminary, the model specification of Section 7 does seem to be more consistent with the data than the capital asset pricing model.[35]

10. CONCLUSION

An intertemporal model of the capital market has been developed which is consistent with both the expected utility maxim and the limited liability of assets. It was shown that the equilibrium relationships among expected returns specified by the classical capital asset pricing model will obtain only under very special additional assumptions. Whether the special form of the general model presented in Sections 7–9 will explain the empirical discrepancies found in the BJS study is an empirical question as yet unanswered. However, whether it does or not, the main purposes were to illustrate how testable specifications can be generated from the model and to induce those who do it best to pursue further empirical testing.

The model is robust in the sense that it can be extended in an obvious way to include effects other than shifts in the investment opportunity set. Two important factors not considered are wage income and many consumption goods whose relative prices are changing over time. In a more complete model the three-fund

---

[34] One could argue that $\alpha_n > r$ on the grounds that current consumption is a normal good and, hence, $\partial c / \partial r < 0$ for most people. Also, the existence of wage income would tend to force $\alpha_n > r$. Finally, in a number of studies of the term structure, investigators have found positive premiums on long-term bonds, implying that $\alpha_n > r$.

[35] M. Scholes is in the process of testing the model of Section 7. D. Rie [31] has also examined the effect of capitalization rate changes on the classical capital asset pricing model.

theorem of Section 7 will generalize to an m-fund theorem. Although there was no discussion of the supply side, given a micro theory of the firm, (1), (2), and (29) could be used to close the model.

Massachusetts Institute of Technology

REFERENCES

[1] ARROW, K. J.: "The Role of Securities in the Optimal Allocation of Risk Bearing," *Review of Economic Studies*, 31 (1964), 91–96.

[2] BLACK, F.: "Capital Market Equilibrium with Restricted Borrowing," *Journal of Business*, 45 (1972), 444–455.

[3] BLACK, F., M. C. JENSEN, AND M. SCHOLES: "The Capital Asset Pricing Model: Some Empirical Tests," in *Studies in the Theory of Capital Markets*, M. C. Jensen, ed. New York: Praeger Publishers, 1972.

[4] BORCH, K.: "A Note on Uncertainty and Indifference Curves," *Review of Economic Studies*, 36 (1969), 1–4.

[5] CASS, D., AND J. E. STIGLITZ: "The Structure of Investor Preferences and Asset Returns, and Separability in Portfolio Allocation: A Contribution to the Pure Theory of Mutual Funds," *Journal of Economic Theory*, 2 (1970), 122–160.

[6] CLARK, P. K.: "A Subordinated Stochastic Process Model with Finite Variance for Speculative Prices," *Econometrica*, 41 (1973), 135–155.

[7] COX, D. A., AND H. D. MILLER: *The Theory of Stochastic Processes*. New York: John Wiley, 1968.

[8] DEBREU, G.: *Theory of Value: An Axiomatic Analysis of Economic Equilibrium*. New York: John Wiley, 1959.

[9] FAMA, E. F.: "Multiperiod Consumption-Investment Decisions," *American Economic Review*, 60 (1970), 163–174.

[10] ———: "Efficient Capital Markets: A Review of Theory and Empirical Work," *Journal of Finance*, 25 (1970), 383–417.

[11] FAMA, E. F., AND M. H. MILLER: *The Theory of Finance*. New York: Holt, Rhinehart, Winston, 1972.

[12] FELDSTEIN, M. S.: "Mean-Variance Analysis in the Theory of Liquidity Preference and Portfolio Selection," *Review of Economic Studies*, 36 (1969), 5–12.

[13] FELLER, W.: *An Introduction to Probability Theory and Its Applications*. Volume 2. New York: John Wiley, 1966.

[14] FRIEND, I., AND M. BLUME: "Measurement of Portfolio Performance under Uncertainty," *American Economic Review*, 60 (1970), 561–575.

[15] HAKANSSON, N. H.: "Capital Growth and the Mean-Variance Approach to Portfolio Selection," *Journal of Financial and Quantitative Analysis*, 6 (1971), 517–557.

[16] ITÔ, K., AND H. P. MCKEAN, JR.: *Diffusion Processes and Their Sample Paths*. New York: Academic Press, 1964.

[17] JENSEN, M. C.: "Capital Markets: Theory and Evidence," *Bell Journal of Economics and Management Science*, 2 (1972), 357–398.

[18] KUSHNER, H. J.: *Stochastic Stability and Control*. New York: Academic Press, 1967.

[19] LINTNER, J.: "The Valuation of Risk Assets and the Selection of Risky Investments in Stock Portfolios and Capital Budgets," *Review of Economics and Statistics*, 47 (1965), 13–37.

[20] ———: "Security Prices, Risk and Maximal Gains from Diversification," *Journal of Finance*, 20 (1965), 587–615.

[21] MARKOWITZ, H.: *Portfolio Selection: Efficient Diversification of Investment*. New York: John Wiley, 1959.

[22] MCKEAN, H. P., JR.: *Stochastic Integrals*. New York: Academic Press, 1969.

[23] MERTON, R. C.: "Lifetime Portfolio Selection under Uncertainty: The Continuous-Time Case," *Review of Economics and Statistics*, 51 (1969), 247–257.

[24] ———: "A Dynamic General Equilibrium Model of the Asset Market and Its Application to the Pricing of the Capital Structure of the Firm," Working Paper 497-70, Sloan School of Management, Massachusetts Institute of Technology, December, 1970.

[25] ———: "Optimum Consumption and Portfolio Rules in a Continuous-time Model," *Journal of Economic Theory*, 3 (1971), 373–413.

[26] ———: "An Analytic Derivation of the Efficient Portfolio Frontier," *Journal of Financial and Quantitative Analysis*, 7 (1972), 1851–1872.

[27] MERTON, R. C., AND P. A. SAMUELSON: "Fallacy of the Asymptotic Log-Normal Approximation in Portfolio Decision Making Over Many Periods," Working Paper 623-72, Sloan School of Management, Massachusetts Institute of Technology, 1972.

[28] MODIGLIANI, F., AND C. R. SUTCH: "Innovations in Interest Rate Policy," *American Economic Review*, 56 (1966), 178–197.

[29] MOSSIN, J.: "Equilibrium in a Capital Asset Market," *Econometrica*, 34 (1966), 768–783.

[30] PRESS, S. J.: "A Compound Events Model for Security Prices," *Journal of Business*, 40 (1967), 317–335.

[31] RIE, D.: "Single Parameter Risk Measures and Multiple Sources of Risk: A Re-Examination of the Data Based on Changes in Determinants of Price Over Time." Working Paper # 14-72, Rodney L. White Center for Financial Research, Wharton School of Finance and Commerce, University of Pennsylvania.

[32] ROLL, R.: "Some Preliminary Evidence on the 'Growth Optimum' Model," Working Paper 3-71-2, Graduate School of Industrial Administration, Carnegie-Mellon University, July, 1971.

[33] SAMUELSON, P. A.: "Proof that Properly Anticipated Prices Fluctuate Randomly," *Industrial Management Review*, 6 (1965), 41–49.

[34] ———: "General Proof that Diversification Pays," *Journal of Financial and Quantitative Analysis*, 2 (1967).

[35] ———: "The Fundamental Approximation Theorem of Portfolio Analysis in Terms of Means, Variances, and Higher Moments," *Review of Economic Studies*, 37 (1970), 537–542.

[36] ———: "The 'Fallacy' of Maximizing the Geometric Mean in Long Sequences of Investing or Gambling," *Proceedings of the National Academy of Sciences*, 68 (1971).

[37] SCHOLES, M.: "The Relationship Between the Returns on Bonds and the Returns on Common Stocks." Mimeograph, Massachusetts Institute of Technology, November, 1971.

[38] SHARPE, W. F.: "Capital Asset Prices: A Theory of Market Equilibrium under Conditions of Risk," *Journal of Finance*, 19 (1964), 425–442.

[39] ———: *Portfolio Theory and Capital Markets*. New York: McGraw-Hill, 1970.

[40] STIGLITZ, J. E.: "A Consumption-Oriented Theory of the Demand for Financial Assets and the Term-Structure of Interest Rates," *Review of Economic Studies*, 37 (1970), 321–351.

[41] ———: "Some Aspects of the Pure Theory of Corporate Finance, Bankruptcies, and Take-Overs," *Bell Journal of Economics and Management Science*, 2 (1972), 458–482.

[42] WELLES, C.: "The Beta Revolution: Learning to Live with Risk," *Institutional Investor* (1971).