Online teaching - Lecture notes
"Financial derivatives and stochastic processes"
M2 Probability et Finance

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October 14, 2021

Notations

We repeatedly use the same notations.

- Interest rate: $(r_u : u \geq 0)$
- Discount factor $D(t,T) = e^{-\int_t^T r_u \mathrm{d}u}$
- (riskless = non defaultable) Zero-coupon bond at time t for the maturity T: $B(t,T)$
- S for the cash-price
- $\mathtt{C}_t(\phi_T, T)$ for the cash-price (prix au comptant) at time t for the cashflow Φ_T delivered at T
- $\mathtt{F}_t(\phi_T, T)$ for the forward price (prix forward) at time t (and paid at T) for the cashflow Φ_T delivered at T
- $\mathtt{Fut}_t(\phi_T, T)$ for the Future price on Φ_T delivered at T

Week 2

1 Black-Scholes formula, extensions, market conventions

1.4 Continuous time market with log-normal asset (hedging risks, dynamic portfolio, vanilla option)

See slides.

1.5 Effective price computation as expectation of discounted cashflows, Black-Scholes formula

1.5.1 Formula

Theorem 1.1 (Black-Scholes Call Formula)**.** *The cash-price at time $t = 0$ of an European Call Option, with payoff $(S-K)^+$ at time T, in the Black-Scholes Model, is $\mathtt{Call}^{\mathtt{BS}}(0, T, S_0, K, \sigma, r)$ with*

$$\mathtt{C}_0((S_T-K)_+, T) = \mathtt{Call}^{\mathtt{BS}}(0, T, S_0, K, \sigma, r) := S_0\mathcal{N}(d_+) - Ke^{-rT}\mathcal{N}(d_-), \tag{1.5.1}$$
$$d_\pm = d_\pm(T, S_0e^{rT}, K) := \frac{1}{\sigma\sqrt{T}}\log\left(\frac{S_0e^{rT}}{K}\right) \pm \frac{1}{2}\sigma\sqrt{T},$$
$$\mathcal{N}(x) := \int_{-\infty}^{x} \frac{e^{-u^2/2}}{\sqrt{2\pi}}\mathrm{d}u.$$

Remark. The forward price of a Call option is

$$\mathtt{F}_0((S_T-K)_+, T) = \frac{\mathtt{C}_0((S_T-K)_+, T)}{e^{-rT}} = \frac{\mathtt{Call}^{\mathtt{BS}}(0, T, S_0, K, \sigma, r)}{e^{-rT}} = S_0e^{rT}\mathcal{N}(d_+) - K\mathcal{N}(d_-).$$

When S does not pay dividend, we have

$$\mathtt{F}_0(S_T, T) = S_0e^{rT},$$

and therefore, the Forward price of the Call option simply writes

$$\mathtt{F}_0((S_T-K)_+, T) = \mathtt{F}_0(S_T, T)\mathcal{N}(d_+) - K\mathcal{N}(d_-)$$

with $d_\pm$ depending only on the ratio $\mathtt{F}_0(S_T, T)/K$. This is simplified form of the Black formula that we will study later.

Proof. $\triangleright$ We start from the Feynman-Kac representation formula giving the interpretation of the PDE as a risk-neutral expectation of discounted payoff, and we linearize the payoff:

$$\mathtt{C}_0((S_T-K)_+, T) = \mathbb{E}_{\mathbb{Q}}\left[e^{-rT}(S_T-K)^+\right]$$

$$
\begin{aligned}
&= \mathbb{E}_{\mathbb{Q}}\left[e^{-rT}S_T\mathbf{1}_{S_T\geq K}\right] - e^{-rT}\mathbb{E}_{\mathbb{Q}}\left[K\mathbf{1}_{S_T\geq K}\right]\\
&= \mathbb{E}_{\mathbb{Q}}\left[e^{-rT}S_T\mathbf{1}_{S_T\geq K}\right] - e^{-rT}K\mathbb{Q}\left(S_T\geq K\right).
\end{aligned}
$$

▷ Under $\mathbb{Q}$, $S_T = S_0e^{(r-\frac{1}{2}\sigma^2)T+\sigma W_T} = S_0e^{(r-\frac{1}{2}\sigma^2)T+\sigma\sqrt{T}Z}$ with $Z = \frac{W_T}{\sqrt{T}} \stackrel{d}{=} \mathcal{N}(0,1)$. Hence the event $\xi := \{S_T \geq K\}$ means

$$
\begin{aligned}
S_0e^{(r-\sigma^2/2)T+\sigma W_T} \geq K &\iff S_0e^{(r-\sigma^2/2)T+\sigma\sqrt{T}Z} \geq K\\
&\iff Z \geq \frac{1}{\sigma\sqrt{T}}\Big(\log(K/S_0e^{rT}) + T\sigma^2/2\Big) = -d_- \ .
\end{aligned}
$$

It readily follows that

$$
\mathbb{Q}\left(S_T\geq K\right) = \mathbb{Q}\left(Z\geq -d_-\right) = \mathbb{Q}\left(Z\leq d_-\right) = \mathcal{N}(d_-)
$$

by symmetry of the standard Gaussian distribution. All in all, we get

$$
\mathtt{C}_0((S_T-K)_+,T) = \mathbb{E}_{\mathbb{Q}}\left[e^{-rT}S_T\mathbf{1}_{\xi}\right] - Ke^{-rT}\mathcal{N}(d_-). \tag{1.5.2}
$$

▷ Now we handle the first term in (1.5.2): there are multiple ways to proceed, like explicit integration, but we prefer to use a smart approach using a change of probability, inspired by the change of numéraire (studied later in the lectures). Define the new probability measure $\mathbb{Q}^S$ by:

$$
\frac{\mathrm{d}\mathbb{Q}^S}{\mathrm{d}\mathbb{Q}} = e^{\sigma W_T-\frac{1}{2}\sigma^2T} = \frac{S_Te^{-rT}}{S_0}.
$$

This is indeed a new probability measure, because $e^{\sigma W_T-\frac{1}{2}\sigma^2T}\geq 0$ and $\mathbb{E}_{\mathbb{Q}}\left[e^{\sigma W_T-\frac{1}{2}\sigma^2T}\right] = 1$. The rules of change of probability ensures that for any function f we have :

$$
\mathbb{E}_{\mathbb{Q}^S}\left[f(W_T)\right] = \mathbb{E}_{\mathbb{Q}}\left[e^{\sigma W_T-\frac{1}{2}\sigma^2T}f(W_T)\right]. \tag{1.5.3}
$$

Then, in particular, we have :

$$
\mathbb{E}_{\mathbb{Q}}\left[e^{-rT}S_T\mathbf{1}_{\xi}\right] = \mathbb{E}_{\mathbb{Q}}\left[e^{-rT}S_0e^{(r-\frac{1}{2}\sigma^2)T+\sigma W_T}\mathbf{1}_{\xi}\right] = S_0\mathbb{E}_{\mathbb{Q}^S}\left[\mathbf{1}_{\xi}\right] = S_0\mathbb{Q}^S(\xi). \tag{1.5.4}
$$

Consequently, this is very similar to the computations we did for the second term in (1.5.2), except that we first need to find the distribution of W_T under $\mathbb{Q}^S$. For this, we make explicit the Laplace transform of W_T under $\mathbb{Q}^S$: for any $\lambda\in\mathbb{R}$, using (1.5.3),

$$
\begin{aligned}
\mathbb{E}_{\mathbb{Q}^S}\left[e^{\lambda W_T}\right] = \mathbb{E}_{\mathbb{Q}}\left[e^{\sigma W_T-\frac{1}{2}\sigma^2T}e^{\lambda W_T}\right] &= \mathbb{E}_{\mathbb{Q}}\left[e^{(\sigma+\lambda)W_T-\frac{1}{2}\sigma^2T}\right]\\
&= e^{\frac{1}{2}(\lambda+\sigma)^2T}e^{-\sigma^2/2} = e^{\frac{\lambda^2T}{2}+\sigma\lambda T}.
\end{aligned}
$$

Then the distribution of W_T under $\mathbb{Q}^S$ is $\mathcal{N}(\sigma T, T)$, which coincides with distribution of $\sigma T + W_T$ under $\mathbb{Q}$. Consequently,

$$
\mathbb{Q}^S\left(\xi\right) = \mathbb{Q}^S\left(S_0e^{(r-\sigma^2/2)T+\sigma W_T}\geq K\right)
$$

$$
\begin{aligned}
&= \mathbb{Q}\left(S_0 e^{(r-\sigma^2/2)T+\sigma(W_T+\sigma T)} \geq K\right) \\
&= \mathbb{Q}\left(S_0 e^{(r+\sigma^2/2)T+\sigma W_T} \geq K\right) \\
&= \dots \\
&= \mathcal{N}(d_+).
\end{aligned}
$$

Plugging the above and (1.5.4) into (1.5.2) gives

$$
\mathtt{C}_0((S_T-K)_+,T) = S_0\mathcal{N}(d_+) - Ke^{-rT}\mathcal{N}(d_-) = c(0,T,S_0,K,\sigma,r),
$$

as announced. □

1.5.2 Parameters

Remark 1.2. *To effectively calculate the price of a Call option we need several parameters :*

- *The strike price K and the maturity T which are both defined in the Call contract.*
- *The spot price of the asset at $t=0$, S_0, and the interest rate r, both being observed in the market.*
- *The asset volatility, which is not be directly available from market data:*
  - *To deal with this issue, we can estimate (statistically) the volatility by using the historical market data : we remark that $\frac{1}{\sqrt{t_{i+1}-t_i}}\log\left(\frac{S_{t_{i+1}}}{S_{t_i}}\right) \overset{d}{=} \mathcal{N}\left(\left(\mu-\frac{1}{2}\sigma^2\right)\sqrt{t_{i+1}-t_i},\sigma^2\right)$ under the historical probability $\mathbb{P}$. The above mean is approximatively equal to 0 in high frequency limit. We can assume then that the historical market data observations are independent and $\sim\mathcal{N}(0,\sigma^2)$: then we can use the variance estimator :*

$$
s_n^2 := \sum_{i=1}^{n}\left(\log\left(\frac{S_{t_{i+1}}}{S_{t_i}}\right)\frac{1}{\sqrt{t_{i+1}-t_i}}\right)^2.
$$

  - *Another way to determine the volatility is to compute the implicit volatility, see later for details: the main idea here is to use the market prices for Call options for various (T,K). And search for the σ that is the most compatible with market data.*

1.5.3 Numerical example

Take $S_0=100$, $r=2\%$, $T=\frac{1}{2}$ (6 months) , $\sigma=25\%$.

K	d_+	d_-	$S\mathcal{N}(d_+)$	$Ke^{-rT}\mathcal{N}(d_-)$	$C_0((S_T-K)_+,T)$
95	0.435	0.258	66.83	56.61	10.21
100	0.145	-0.032	55.76	48.25	7.517
105	-0.131	-0.308	44.79	39.41	5.377

1.5.4 Simple approximation

The next result states some simple approximations of Black-Scholes price for ATM Call options, in order to get rough values with mental calculations.

Proposition 1.3 (Brenner-Subramanian)**.** *The strike at the money forward is given by :* $K^{\mathtt{ATM}} = S_0 e^{rT} = \mathtt{F}_0(S_T, T)$ *(in the no-dividend case). For such strike (corresponding to ATM call) the value of a call is*

$$\mathtt{Call}^{\mathtt{BS}}(T, S_0, K, \sigma, r) \approx 0.4 S_0 \sigma \sqrt{T}.$$

For instance, if $S_0 = 100$, $T = 6$ months , $\sigma = 25\%$, $r = 2\%$ (i.e. $K = 101$), then we get $\mathtt{Call}^{\mathtt{BS}} = 7.07$.

1.5.5 Black-Scholes Put price

Using the Call-Put parity (valid for any model), we can easily deduce the Black-Scholes Put price.

Theorem 1.4 (Black-Scholes Put Formula)**.** *The price at time* 0 *of an European Put Option, with payoff* $(K - S_T)^+$ *at maturity* T*, in the Black-Scholes Model, is* $\mathtt{Put}^{\mathtt{BS}}(0, T, S_0, K, \sigma, r)$ *with*

$$\mathtt{Put}^{\mathtt{BS}}(0, T, S_0, K, \sigma, r) = K e^{-rT} \mathcal{N}(-d_-) - S_0 \mathcal{N}(-d_+),$$

with $d_\pm$ *and* $\mathcal{N}$ *as in Theorem 1.1.*

Proof. We start from

$$\mathtt{Call}^{\mathtt{BS}}(0, T, S_0, K, \sigma, r) - \mathtt{Put}^{\mathtt{BS}}(0, T, S_0, K, \sigma, r) = S_0 - K e^{-rT},$$

which gives

$$\begin{aligned} \mathtt{Put}^{\mathtt{BS}}(0, T, S_0, K, \sigma, r) &= \mathtt{Call}^{\mathtt{BS}}(0, T, S_0, K, \sigma, r) - S_0 + K e^{-rT} \\ &= S_0 \mathcal{N}(d_+) - K e^{-rT} \mathcal{N}(d_-) - S_0 + K e^{-rT} \\ &= -S_0(1 - \mathcal{N}(d_+)) + K e^{-rT}(1 - \mathcal{N}(d_-)) \\ &= -S_0 \mathcal{N}(-d_+) + K e^{-rT} \mathcal{N}(-d_-). \end{aligned}$$

□

1.6 The Greeks - Partial derivatives

To understand how options behave according to various parameters of the model Black-Scholes' price, we calculate sensitivities with respect to these parameters. These sensitivities are called Greeks, because they are labelled using Greek letters.

Most important Greeks are Delta, Gamma, Vega, which are related to first and second order sensitivities w.r.t. the asset price S_0 and volatility σ.

1.6.1 Delta

Definition 1.5. *Delta Δ is the sensitivity of the price to the current value of the underlying asset. It is by far the most important risk factor affecting the price of an option. It is useful for hedging against market variation.*

Proposition 1.6. *The Delta of Call and Put options are :*

$$\begin{aligned}\Delta^{\texttt{Call}^{\text{BS}}} &= \frac{\partial \texttt{Call}^{\text{BS}}}{\partial S_0} = \mathcal{N}(d_+) \in [0,1], \\ \Delta^{\texttt{Put}^{\text{BS}}} &= \frac{\partial \texttt{Put}^{\text{BS}}}{\partial S_0} = \mathcal{N}(d_+) - 1 \in [-1,0].\end{aligned} \tag{1.6.1}$$

For the sake of simplicity, we have omitted the parameters $(t, T, S_0, K, \sigma, r)$ in the arguments of the above functions.

Before we prove the above proposition, we first state a **key relation**, which is very useful when dealing with computations of Greeks, in order to simplify formulas.

Lemma 1.7. *Let $x, y > 0$ and let*

$$d_\pm(x,y) = \frac{1}{\sigma\sqrt{T}} \ln\left(\frac{x}{y}\right) \pm \frac{1}{2}\sigma\sqrt{T}.$$

Then

$$x e^{-\frac{1}{2}d_+^2(x,y)} = y e^{-\frac{1}{2}d_-^2(x,y)}. \tag{1.6.2}$$

Since $\mathcal{N}'(u) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}u^2}$, the above writes

$$x\mathcal{N}'(d_+(x,y)) = y\mathcal{N}'(d_-(x,y)). \tag{1.6.3}$$

Proof. Basic algebra gives

$$\begin{aligned} d_+^2 - d_-^2 &= (d_+ + d_-)(d_+ - d_-) \\ &= \left(\sigma\sqrt{T}\right)\left(\frac{2}{\sigma\sqrt{T}} \log\left(\frac{x}{y}\right)\right), \\ e^{\frac{d_+^2}{2} - \frac{d_-^2}{2}} &= e^{\frac{1}{2}2\log\left(\frac{x}{y}\right)} = \frac{x}{y}. \end{aligned}$$

□

Proof of Proposition 1.6.

$$\begin{aligned}\Delta^{\texttt{Call}^{\text{BS}}} &:= \frac{\partial \texttt{Call}^{\text{BS}}}{\partial S_0} \\ &= \frac{\partial}{\partial S_0}\left(S_0\mathcal{N}(d_+) - Ke^{-rT}\mathcal{N}(d_-)\right)\end{aligned}$$

$$
\begin{aligned}
&= \mathcal{N}(d_+) + S_0 \frac{\partial \mathcal{N}(d_+)}{\partial S_0} - Ke^{-rT} \frac{\partial \mathcal{N}(d_-)}{\partial S_0} \\
&= \mathcal{N}(d_+) + S_0 \mathcal{N}'(d_+) \frac{\partial d_+}{\partial S_0} - Ke^{-rT} \mathcal{N}'(d_-) \frac{\partial d_-}{\partial S_0}.
\end{aligned}
$$

Having in mind the final result, it is tempting to ignore the second and third terms in the above differentiation, as if $d_\pm$ do not depend on S_0: it is of course incorrect, they do depend on S_0. Actually they cancel because of the key relation (1.6.2): indeed the latter implies that

$$
\begin{aligned}
&S_0 \mathcal{N}'(d_+) \frac{\partial d_+}{\partial S_0} - Ke^{-rT} \mathcal{N}'(d_-) \frac{\partial d_-}{\partial S_0} \\
&= S_0 \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} d_+^2 (S_0, Ke^{-rT})} \frac{\partial d_+}{\partial S_0} - Ke^{-rT} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} d_-^2 (S_0, Ke^{-rT})} \frac{\partial d_-}{\partial S_0} = 0
\end{aligned}
$$

because

$$
\frac{\partial d_+}{\partial S_0} = \frac{\partial d_-}{\partial S_0}.
$$

Then :

$$
\Delta^{\mathtt{Call}^{\mathtt{BS}}} = \mathcal{N}(d_+).
$$

For the Put's Delta, we can use the Call-Put Parity :

$$
\Delta^{\mathtt{Put}^{\mathtt{BS}}} = \frac{\partial \mathtt{Put}^{\mathtt{BS}}}{\partial S_0} = \frac{\partial (\mathtt{Call}^{\mathtt{BS}} - S_0 + Ke^{-rT})}{\partial S_0} = \frac{\partial \mathtt{Call}^{\mathtt{BS}}}{\partial S_0} - 1 = \mathcal{N}(d_+) - 1.
$$

□

Remark 1.8. *The delta of a call ($\Delta^{\mathtt{Call}^{\mathtt{BS}}}$) is always positive, the call price is increasing with the asset price: indeed, an increase in the asset price will increase the probability of a positive payoff at expiration resulting in a higher value.*
On the other hand, there is a negative relationship between the put price and the underlying asset price, as an increase in the asset price, will reduce the put's current exercise value $\{Ke^{-rT} - S_0\}$ and therefore the put's price will decrease. This explains a negative $\Delta^{\mathtt{Put}^{\mathtt{BS}}}$.
When the price of the underlying asset changes, put and call option values move in opposite directions, since ($0 \leq \Delta^{\mathtt{Call}^{\mathtt{BS}}} \leq 1$) and ($-1 \leq \Delta^{\mathtt{Put}^{\mathtt{BS}}} \leq 0$). However, the absolute changes in their prices will never exceed those of the underlying asset.

1.6.2 Delta with respect to the Strike

Proposition 1.9. *The Delta w.r.t. the strike for Call and Put options are :*

$$
\begin{aligned}
\Delta_K^{\mathtt{Call}^{\mathtt{BS}}} &= \frac{\partial \mathtt{Call}^{\mathtt{BS}}}{\partial K} = -e^{-rT} \mathcal{N}(d_-) \leq 0, \qquad (1.6.4) \\
\Delta_K^{\mathtt{Put}^{\mathtt{BS}}} &= \frac{\partial \mathtt{Put}^{\mathtt{BS}}}{\partial K} = e^{-rT} (1 - \mathcal{N}(d_-)) \geq 0.
\end{aligned}
$$

Proof. By definition
$$\Delta_K^{\texttt{Call}^{\text{BS}}} = \frac{\partial}{\partial K}(\mathbb{E}_{\mathbb{Q}}\left[e^{-rT}(S_T - K)^+\right]).$$
One could differentiate the Black-Scholes formula as we did for the Delta w.r.t. S_0. Here, we take a different approach, by differentiating the above expectation. Let us apply the Lebesgue differentiation theorem: $K \mapsto (S_T - K)^+$ is almost surely differentiable (since $\mathbb{Q}(S_T = K) = 0$ in the log-normal model) and its derivative $(-1)\mathbf{1}_{S\geq K}$ is bounded uniformly in K by a random variable (equal to 1) which is obviously integrable. Therefore,

$$\begin{aligned}
\frac{\partial}{\partial K}(\mathbb{E}_{\mathbb{Q}}\left[e^{-rT}(S_T - K)^+\right]) &= \mathbb{E}_{\mathbb{Q}}\left[e^{-rT}\frac{\partial}{\partial K}(S_T - K)^+\right] \\
&= \mathbb{E}\left[e^{-rT}(-1)\mathbf{1}_{S\geq K}\right] \\
&= -e^{-rT}\mathbb{Q}\left(S_T \geq K\right).
\end{aligned}$$

And we saw earlier that $\mathbb{Q}\left(S_T \geq K\right) = \mathcal{N}(d_-)$, then we deduce that

$$\Delta_K^{\texttt{Call}^{\text{BS}}} = -e^{-rT}\mathcal{N}(d_-).$$

By using the Call-Put parity, we have :

$$\Delta_K^{\texttt{Put}^{\text{BS}}} = \Delta_K^{\texttt{Call}^{\text{BS}}} + \frac{\partial}{\partial K}(-S_0 + Ke^{-rT}) = e^{-rT}(1 - \mathcal{N}(d_-)).$$

□

Remark 1.10. *The higher the strike price, the less valuable a call option is, since the strike price represents a higher cost of exercising the call and thereby purchasing the stock. This is also a consequence of the No-Arbitrage condition (week 1), and thus this negative sensitivity is valid in any model (not only in Black-Scholes model). In contrast, the higher the exercise price of a put, the higher its price.*
The Black-Scholes formula clearly confirms these relationships:

$$\Delta_K^{\texttt{Call}^{\text{BS}}} = -e^{-rT}\mathcal{N}(d_-) \leq 0\ ,\ \Delta_K^{\texttt{Put}^{\text{BS}}} = e^{-rT}(1 - \mathcal{N}(d_-)) \geq 0.$$

1.6.3 About homogeneity of Call/Put prices with respect to asset price S_0 and Strike K

Observe that, in view of (1.5.1), (1.6.1), (1.6.4), we have the somewhat simple relation:

$$\texttt{Call}^{\text{BS}} = S_0\frac{\texttt{Call}^{\text{BS}}}{\partial S_0} + K\frac{\texttt{Call}^{\text{BS}}}{\partial K}.$$

One may wonder whether the decomposition

$$\texttt{Call}^{\text{BS}} = S_0\mathcal{N}(d_+) + K(-e^{-rT}\mathcal{N}(d_-))$$

of the call price is enough to directly assert the identities (by identification)

$$\frac{\texttt{Call}^{\texttt{BS}}}{\partial S_0} = \mathcal{N}(d_+), \qquad \frac{\texttt{Call}^{\texttt{BS}}}{\partial K} = -e^{-rT}\mathcal{N}(d_-).$$

It would give an alternative (and simple) proof of Propositions 1.6 and 1.9. Actually the relations we are discussing are deeply related to the homogeneity of call price with respect to S and K.

Theorem 1.11 (Euler's Theorem for homogeneous functions)**.** *Assume that f is a α-homogeneous function, i.e.*

$$f(\lambda x, \lambda y) = \lambda^\alpha f(x,y), \qquad \forall \lambda \geq 0, \forall x, \forall y, \tag{1.6.5}$$

and that f is C^1. Then

$$\alpha f(x,y) = x\frac{\partial f}{\partial x}(x,y) + y\frac{\partial f}{\partial y}(x,y). \tag{1.6.6}$$

Conversely, if

$$\alpha f(x,y) = xg(x,y) + yh(x,y)$$

where g and h are C^1 and $(\alpha - 1)$-homogeneous, and if

$$\frac{\partial g}{\partial y}(x,y) = \frac{\partial h}{\partial x}(x,y),$$

then

$$\frac{\partial f}{\partial x}(x,y) = g(x,y) \text{ and } \frac{\partial f}{\partial y}(x,y) = h(x,y).$$

Proof. Differentiate (1.6.5) with respect to λ, it gives

$$\alpha\lambda^{\alpha-1} f(x,y) = x\frac{\partial f}{\partial x}(\lambda x, \lambda y) + y\frac{\partial f}{\partial y}(\lambda x, \lambda y);$$

take $\lambda = 1$ to get (1.6.6). □

We apply the above result to the function $f(S_0, K) = \texttt{Call}^{\texttt{BS}}(0, T, S_0, K)$: from (1.5.1), it is clearly 1-homogeneous w.r.t. (S_0, K), therefore we must have

$$\texttt{Call}^{\texttt{BS}}(0, T, S_0, K) = S_0\frac{\partial \texttt{Call}^{\texttt{BS}}}{\partial S_0}(S_0, K) + K\frac{\partial \texttt{Call}^{\texttt{BS}}}{\partial K}(S_0, K).$$

Set $g(S_0, K) = \mathcal{N}(d_+)$ and $h(S_0, K) = -e^{-rT}\mathcal{N}(d_-)$, so that

$$\texttt{Call}^{\texttt{BS}}(0, T, S_0, K) = S_0\ g(S_0, K) + K\ h(S_0, K).$$

According to the converse statement of Theorem 1.11, we may conclude that

$$\frac{\partial \texttt{Call}^{\texttt{BS}}}{\partial S_0} = \mathcal{N}(d_+) \text{ and } \frac{\partial \texttt{Call}^{\texttt{BS}}}{\partial K} = -e^{-rT}\mathcal{N}(d_-) \tag{1.6.7}$$

upon the two conditions

- g and h are 0-homogeneous and C^1,
- $\frac{\partial g}{\partial K}(S_0, K) = \frac{\partial h}{\partial S_0}(S_0, K)$.

The first condition is satisfied since $d_\pm$ depends (smoothly) on S_0, K only through the ratio S_0/K. The second writes

$$\begin{aligned}
&\mathcal{N}'(d_+)\partial_K d_+ = -e^{-rT}\mathcal{N}'(d_-)\partial_{S_0} d_- \\
\Longleftrightarrow &\mathcal{N}'(d_+)(-\frac{1}{\sigma\sqrt{T}K}) = -e^{-rT}\mathcal{N}'(d_-)(\frac{1}{\sigma\sqrt{T}S_0}) \\
\Longleftrightarrow &S_0 e^{rT}\mathcal{N}'(d_+) = K\mathcal{N}'(d_-).
\end{aligned}$$

This holds owing to the key relation (1.6.3). We have proved (1.6.7) using the homogeneity of call price.

1.6.4 Gamma

Definition 1.12. *Gamma Γ is defined as the second derivative of the price relative to the value of the underlying asset or the first derivative of the delta.*

It measures the speed of delta change when the underlying asset moves.

Proposition 1.13. *The Gamma of Call and Put options are :*

$$\Gamma^{\mathtt{Put}^{\mathtt{BS}}} = \Gamma^{\mathtt{Call}^{\mathtt{BS}}} = \frac{\partial^2 \mathtt{Call}^{\mathtt{BS}}}{\partial S_0^2} = \frac{1}{S_0\sigma\sqrt{T}}\mathcal{N}'(d_+) > 0$$

with

$$\mathcal{N}'(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}.$$

Proof. Direct computations give

$$\Gamma^{\mathtt{Call}^{\mathtt{BS}}} = \frac{\partial^2 \mathtt{Call}^{\mathtt{BS}}}{\partial S_0^2} = \frac{\partial \Delta^{\mathtt{Call}^{\mathtt{BS}}}}{\partial S_0} = \frac{\partial \mathcal{N}(d_+)}{\partial S_0} = \mathcal{N}'(d_+)\frac{\partial d_+}{\partial S_0}$$

and

$$\frac{\partial d_+}{\partial S_0} = \frac{1}{\sigma\sqrt{T}}\frac{1}{S_0}.$$

The call-put parity relation gives

$$\Gamma^{\mathtt{Call}^{\mathtt{BS}}} - \Gamma^{\mathtt{Put}^{\mathtt{BS}}} = \frac{\partial^2 S_0}{\partial S_0^2} = 0,$$

i.e. Call and Put with same (T, K) have the same Gamma. □

Remark 1.14. *Gamma is always positive which means that call/put prices are convex with respect to the asset price, the delta change has the same direction as changes in the underlying asset.*

With gamma being positive, the buyers of the options gain from movements in the price of the underlying assets. For this reason, holders of the options are often referred to as being "long gamma". For sellers of the options, the gamma exposure is exactly opposite to that of buyers of options. Those who sells options can be hurt when gamma is high and the underlying market price moves. Therefore they are often referred to as "short gamma".

1.6.5 Vega

Definition 1.15. *The Vega is the change in the option price with respect to the change in volatility σ.*

As we will see later, it is a measure of the option exposure to changes in (implied) volatility within the option market.

Proposition 1.16. *The Vega of Call and Put options are :*

$$\mathtt{Vega}^{\mathtt{Put}^{\mathtt{BS}}} = \mathtt{Vega}^{\mathtt{Call}^{\mathtt{BS}}} = S_0\sqrt{T}\mathcal{N}'(d_+) > 0.$$

Proof. Direct differentiation gives

$$\mathrm{Vega}^{\mathtt{Call}^{\mathtt{BS}}} = \frac{\partial \mathtt{Call}^{\mathtt{BS}}}{\partial \sigma} = S_0\mathcal{N}'(d_+)\frac{\partial d_+}{\partial \sigma} - Ke^{-rT}\mathcal{N}'(d_-)\frac{\partial d_-}{\partial \sigma}.$$

By using the key formula (1.6.3), we have : $S_0\mathcal{N}'(d_+) = Ke^{-rT}\mathcal{N}'(d_-)$ then

$$\begin{aligned}\mathrm{Vega}^{\mathtt{Call}^{\mathtt{BS}}} &= S_0\mathcal{N}'(d_+)\left(\frac{\partial d_+}{\partial \sigma} - \frac{\partial d_-}{\partial \sigma}\right)\\ &= S\mathcal{N}'(d_+)\left(\frac{\partial \sigma\sqrt{T}}{\partial \sigma}\right) = S\sqrt{T}\mathcal{N}'(d_+).\end{aligned}$$

By applying the Call-Put parity, we get $\mathrm{Vega}^{\mathtt{Put}^{\mathtt{BS}}} = \mathrm{Vega}^{\mathtt{Call}^{\mathtt{BS}}}$. □

Remark 1.17. *When the volatility rises, the option's set of favorable outcomes will also rise. As a result, the chances are higher for the option to be either deeper in-the-money or deeper out-of-the-money at expiration. Since the option bears no downside risk, there is no penalty when the option expires deeper out-of-the-money, but a higher payoff when it expires deeper in-the-money. Due to this antisymmetric payoff structure, the vegas for long options are positive, i.e. for the option buyer, the exposure to changes in implied volatility is positive. By symmetry, the option writer benefits from a decrease in implied volatility and therefore has vega negative exposure.*

1.6.6 Theta

Definition 1.18. *The Theta Θ is the rate of change of the option's price with respect its maturity (or time to maturity) T (or $(T-t)$).*

Proposition 1.19. *The Theta of Call and Put options are :*

$$\Theta^{\texttt{Call}^{\texttt{BS}}} = \frac{\partial \texttt{Call}^{\texttt{BS}}}{\partial T} = \frac{S_0\sigma}{2\sqrt{T}}\mathcal{N}'(d_+) + rKe^{-rT}\mathcal{N}(d_-),$$
$$\Theta^{\texttt{Put}^{\texttt{BS}}} = \frac{\partial \texttt{Put}^{\texttt{BS}}}{\partial T} = \frac{S_0\sigma}{2\sqrt{T}}\mathcal{N}'(d_+) - rKe^{-rT}(1-\mathcal{N}(d_-)).$$

Proof. We get

$$\Theta^{\texttt{Call}^{\texttt{BS}}} = \frac{\partial \texttt{Call}^{\texttt{BS}}}{\partial T} = S_0\mathcal{N}'(d_+)\frac{\partial d_+}{\partial T} - Ke^{-rT}\mathcal{N}'(d_-)\frac{\partial d_-}{\partial T} + rKe^{-rT}\mathcal{N}(d_-).$$

The key relation (1.6.3) gives $S_0\mathcal{N}'(d_+) = Ke^{-rT}\mathcal{N}'(d_-)$ we get

$$\begin{aligned}\Theta^{\texttt{Call}^{\texttt{BS}}} &= S_0\mathcal{N}'(d_+)\left(\frac{\partial d_+}{\partial T} - \frac{\partial d_-}{\partial T}\right) + rKe^{-rT}\mathcal{N}(d_-)\\ &= S_0\mathcal{N}'(d_+)\left(\frac{\partial \sigma\sqrt{T}}{\partial T}\right) + rKe^{-rT}\mathcal{N}(d_-),\\ \Theta^{\texttt{Call}^{\texttt{BS}}} &= \frac{S_0\sigma}{2\sqrt{T}}\mathcal{N}'(d_+) + rKe^{-rT}\mathcal{N}(d_-).\end{aligned}$$

By using the Call-Put parity, we have

$$\begin{aligned}\Theta^{\texttt{Put}^{\texttt{BS}}} &= \frac{\partial \texttt{Put}^{\texttt{BS}}}{\partial T} = \frac{\partial(\texttt{Call}^{\texttt{BS}} - S_0 + Ke^{-rT})}{\partial T}\\ &= \Theta^{\texttt{Call}^{\texttt{BS}}} - rKe^{-rT}\\ &= \frac{S_0\sigma}{2\sqrt{T}}\mathcal{N}'(d_+) - rKe^{-rT}(1-\mathcal{N}(d_-)).\end{aligned}$$

□

1.6.7 Rho

Definition 1.20. *The Rho (ρ) of an option is defined as the rate of change of option price with respect to the interest rate r.*

Proposition 1.21. *The Rho of Call and Put options are :*

$$\rho^{\texttt{Call}^{\texttt{BS}}} = TKe^{-rT}\mathcal{N}(d_-) > 0,$$
$$\rho^{\texttt{Put}^{\texttt{BS}}} = -TKe^{-rT}(1-\mathcal{N}(d_-)) > 0.$$

Proof. Clearly

$$\rho^{\texttt{Call}^{\texttt{BS}}} = \frac{\partial \texttt{Call}^{\texttt{BS}}}{\partial r} = S_0 \mathcal{N}'(d_+) \frac{\partial d_+}{\partial r} - Ke^{-rT} \mathcal{N}'(d_-) \frac{\partial d_-}{\partial r} + TKe^{-rT} \mathcal{N}(d_-).$$

With the same computations as before, we get $S_0 \mathcal{N}'(d_+) = Ke^{-rT} \mathcal{N}'(d_-)$, therefore

$$\begin{aligned} \rho^{\texttt{Call}^{\texttt{BS}}} &= S_0 \mathcal{N}'(d_+) \left(\frac{\partial (d_+ - d_-)}{\partial r} \right) + TKe^{-rT} \mathcal{N}(d_-) \\ &= TKe^{-rT} \mathcal{N}(d_-). \end{aligned}$$

By the Call-Put parity we have

$$\begin{aligned} \rho^{\texttt{Put}^{\texttt{BS}}} &= \rho^{\texttt{Call}^{\texttt{BS}}} + \frac{\partial (Ke^{-rT} - S)}{\partial r} \\ &= TKe^{-rT} \mathcal{N}(d_-) - TKe^{-rT} = -TKe^{-rT} (1 - \mathcal{N}(d_-)). \end{aligned}$$

□

Remark 1.22. *A higher interest rate decreases the present value of the cost of exercising the call at expiration (which is a similar effect to the strike price decreasing) and so increases the call price. The reverse effect holds for the put price. Signs of the rho for call and put prices* $\rho^{\texttt{Call}^{\texttt{BS}}} \geq 0$ *and* $\rho^{\texttt{Put}^{\texttt{BS}}} \leq 0$ *confirm this effect.*

1.6.8 Numerical experiments

We consider a Call option such that $S_0 = 100$, $r = 2\%$, $T = 6$ months , $\sigma = 25\%$.

K	Δ	Delta of the Strike	Γ	Vega	Θ	ρ
95	0.668	-0.595	0.021	25.662	7.548	28.307
100	0.558	-0.482	0.022	27.915	7.944	24.123
105	0.448	-0.375	0.022	27.968	7.780	19.705