Lecture notes
"Financial derivatives and stochastic processes"
M2 Probability et Finance

Chapter 1: Introduction to Financial markets, Black-Scholes formula, market conventions

E. Gobet

September 30, 2021

Notations

We repeatedly use the same notations.

- Interest rate: $(r_u : u \geq 0)$
- Discount factor $D(t,T) = e^{-\int_t^T r_u \mathrm{d}u}$
- (riskless = non defaultable) Zero-coupon bond at time t for the maturity T: $B(t,T)$
- S for the cash-price
- $\mathtt{C}_t(\phi_T, T)$ for the cash-price (prix au comptant) at time t for the cashflow Φ_T delivered at T
- $\mathtt{F}_t(\phi_T, T)$ for the forward price (prix forward) at time t (and paid at T) for the cashflow Φ_T delivered at T
- $\mathtt{Fut}_t(\phi_T, T)$ for the Future price on Φ_T delivered at T

I Black-Scholes formula, extensions, market conventions

I-a Introduction to risk management

See slides.

I-b Model-Free Pricing and no-arbitrage

In the financial markets, vanilla options are contracts that give to their owners the right but not the obligation to sell or buy the underlying stock in the future. Vanilla options are usually traded on organized markets and thus, their prices are public and obey to the supply-demand rule (bid-offer); their observations reveal anticipations from market agents and can be used to infer on the risk models. More complex derivatives (called exotic) are Over-The-Counter products and their prices are usually not publicly available on trading plateforms. Knowing how to fix the prices of these derivatives is of great importance.

All these prices obey to many rules and among them the important rule of **no-arbitrage**. In the market, we can find a special kind of traders whose role is to detect the opportunities of making a riskless profit as soon as it arises. Once the arbitrages are detected, the traders simultaneously enter into transactions in two or more markets making advantage of the differences in prices. Hence, the intervention of the arbitrageurs eliminate the arbitrages, the **no-arbitrage** is therefore a realistic modeling assumption.

I-b.1 Absence of arbitrage opportunity

The notion of arbitrage which is at the origin of the theory of pricing is fundamental to understand. In particular, it is important to verify that when the theory is applied, the conditions of **no-arbitrage** are almost verified in the market in which we are interested.

▷ **Definition of a frictionless market.** A **frictionless** market is a theoretical trading environment where the costs and limits associated with transactions do not exist. The **frictionless** market assumptions are important to consider since actual costs will be associated with real world applications.

Definition I.1 (Frictionless market)**.** *We consider that a market is* ***frictionless*** *if the following assumptions are verified.*

- *There are no transaction, no trade costs, no exchange fees nor liquidity costs.*
- *There are no taxes on transactions and gains.*
- *There is no difference between the ask and the bid price, which means that the spread is zero.*

- *The transactions are instantaneous.*
- *The negotiated assets are very liquid and perfectly divisible.*
- *There are no limit on short sales.*
- *The market participants are price takers and small investors have no impact on the prices. The price is also assumed to be exogenous.*
- *The interest rates for lending or borrowing are the same.*

▷ **Definition of no-arbitrage.** We will introduce the notion of **no-arbitrage**.

Definition I.2 (No-arbitrage)**.** *An arbitrage is a self-financing strategy that is worth zero initially and yields a positive gain without any risk (non-negative almost surely and (strictly)-positive with positive probability).*

On the opposite, the absence of arbitrage opportunity or ***no-arbitrage*** *means that a (almost-sure) profit can not be generated from a zero initial investment.*

We will refer the consequence of this assumption as $\overset{\text{N.A.}}{\Longrightarrow}$.

▷ **Definition of unique-price rule.**

Corollary I.3 (Unique-price principle)**.** *Under the assumption of* ***no-arbitrage****, if the values of two portfolios coincide at a given date, then these two portfolios have the same value at any other intermediate date.*

In a variant of this corollary, we derive $(\forall \omega \in \Omega) V_T^1 \leq V_T^2$, then $V_t^1 \leq V_t^2 \;\; \forall t \leq T$.

Proof. We consider two portfolios of respective values V_1 and V_2. Suppose that $V_t^1(\omega) > V_t^2(\omega)$ in some scenarios $\omega \in A$, and that $V_T^1(\omega) = V_T^2(\omega)$ for all $\omega \in \Omega$. We consider a global Portfolio V and show that an arbitrage can be made using this portfolio, when events in A occur. We consider that short sales as allowed.

- If $\omega \in A$, we construct V by :
  - short selling portfolio (1) at the price V_t^1.
  - buying portfolio (2) at the price V_t^2.
  - investing the cash $V_t^1 - V_t^2$ in the riskless asset, which means that at time T we will have 1 capitalized € multiplied by $V_t^1 - V_t^2$, we will denote the capitalized euro by 1^c. We assume that the institution holding the liquidity does not run any default risk, which means that we will certainly obtain the cash-flow $1^c \times (V_t^1 - V_t^2) > 0$ at time T. At $t = 0$, the value of the portfolio is $V_t = (V_t^1 - V_t^2) - V_t^1 + V_t^2 = 0$ € .

Strategy	Portfolio value at t in €	Portfolio value at T in €
$\omega \in A$	$V_t = (V_t^1 - V_t^2)$[cash] $-V_t^1$[asset 1] $+V_t^2$[asset 2] $= 0$	$V_T = (V_t^1 - V_t^2) \times 1^c > 0$
$\omega \notin A$	$V_t = 0$	$V_T = 0$

Table 1: Arbitrage opportunity made on A, using V

- If $\omega \notin A$, we do not need to make any investment $V_t = 0$ € .

Then we adopt a **static strategy** until we reach T, which means that our position on the assets remains the same.

- If $\omega \notin A$, we have $V_T = 0$ € .
- If $\omega \in A$, we clear out the global position $V_T = -V_T^1 + V_T^2 + (V_t^1 - V_t^2) \times 1^c$ € $= (V_t^1 - V_t^2) \times 1^c$ € > 0 .

An arbitrage is made on A.

$\square$

I-b.2 Impact of no-arbitrage on the prices of derivatives

In the financial market, the prices of many financial products can be deduced without a reference to any model.

$\triangleright$ **Price of Forward contracts.** We assume that the prices of our instruments are expressed in € . We recall that $\mathtt{F}_t(S_T, T)$ stands for the forward price at time t (and paid at T) for the cash-flow S_T delivered at T. This price is different from the cash-price $\mathtt{C}_t(S_T, T)$ delivered at T and for which the payment is immediate. We would like to find a relation between these two prices and the price of a (non-defaultable) Zero-coupon Bond of maturity T, which we will denote by $B(t, T)$. This instrument allows to gain 1 € at T.

Proposition I.4 (Forward Price)**.**

$$\mathtt{C}_t(S_T, T) = B(t, T)\mathtt{F}_t(S_T, T).$$

We notice that this relation will be seen to hold regardless of any choice of modeling, which means that it is model-free.

Proof. A static arbitrage argument allows to compare the price of the forward contract to the spot price of the asset S at the date t. In order to guarantee the holding of the asset S at T, we have two possible strategies that we will detail bellow.

- Strategy (1): Paying at t, the cash-price $\mathtt{C}_t(S_T, T)$ in order to receive S_T at T. $V_t^1 = \mathtt{C}_t(S_T, T)$, $V_T^1 = S_T$.
- Strategy (2): Buying a Forward contract on S between t and T. The price to pay at T is $\mathtt{F}_t(S_T, T)$ € .
  In order to be able to pay this price at T, we need to place in the bank an amount of money that will guarantee to have $\mathtt{F}_t(S_T, T)$ at T. The financial instrument adapted to this kind of situations is, by definition, the (non-defaultable) Zero-coupon bond of maturity T, which price is $B(t, T)$. Thus, the price to pay at t is $V_t^2 = \mathtt{F}_t(S_T, T)B(t, T)$ € , and $V_T^2 = S_T$.

Strategy	Portfolio value at t in €	Portfolio value at T in €
(1)	$V_t^1 = \mathtt{C}_t(S_T, T)$	$V_T^1 = S_T$
(2)	$V_t^2 = \mathtt{F}_t(S_T, T)B(t, T)$	$V_T^2 = S_T$

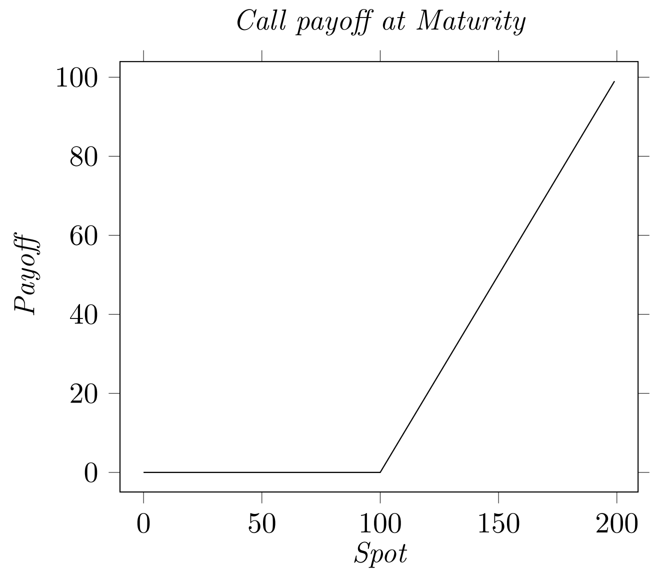
Table 2: Cashflows of the two strategies at times t and T

Since the two portfolios have the same value at T, we conclude from the **no-arbitrage** assumption that $\mathtt{F}_t(S_T, T)B(t, T) = \mathtt{C}_t(S_T, T)$ for all $t \leq T$. □

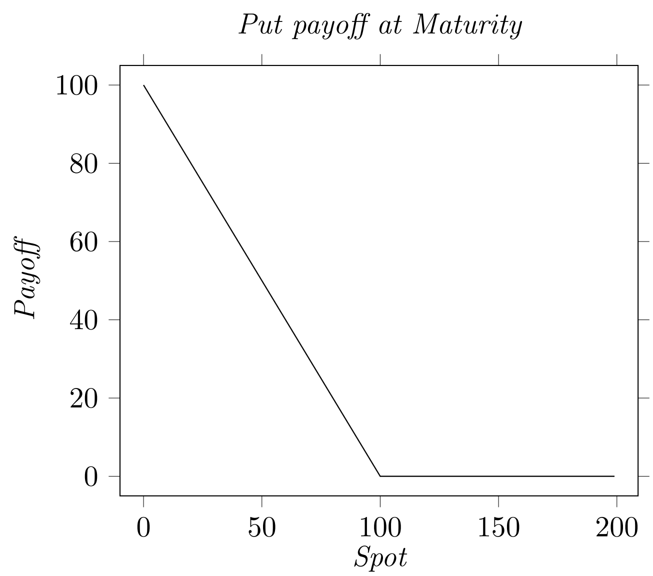
Remark I.5. If S is an asset traded in the market and does not pay dividends between before the maturity T, then $S_t = \mathtt{C}_t(S_T, T)$. In general, if S provides dividends between t and T, then $S_t \geq \mathtt{C}_t(S_T, T)$ and this inequality is strict if the dividends are strictly positive.

▷ **Put-Call Parity.**

Definition I.6 (Call Option)**.** *A Call option on S of strike K and maturity T is a contract giving the right to buy the asset S at price K, at T. This contract allows the protection from the increase of prices of the asset S . The payoff of this option is $(S_T - K)_+ = \max(S_T - K, 0)$.*



Definition I.7 (Put Option)**.** *A put option on S of strike K and maturity T is a contract giving the right to sell the asset S at price K, at T. This contract allows assurance against the fall of prices of the asset S. The payoff of this option is* $(K - S_T)_+ = \max(K - S_T, 0)$.



In the following sections, we denote by $\texttt{Call}_t(T, K)$ and $\texttt{Put}_t(T, K)$ the respective cash-prices at time t of Call and Put contracts on S, with characteristics (T, K):

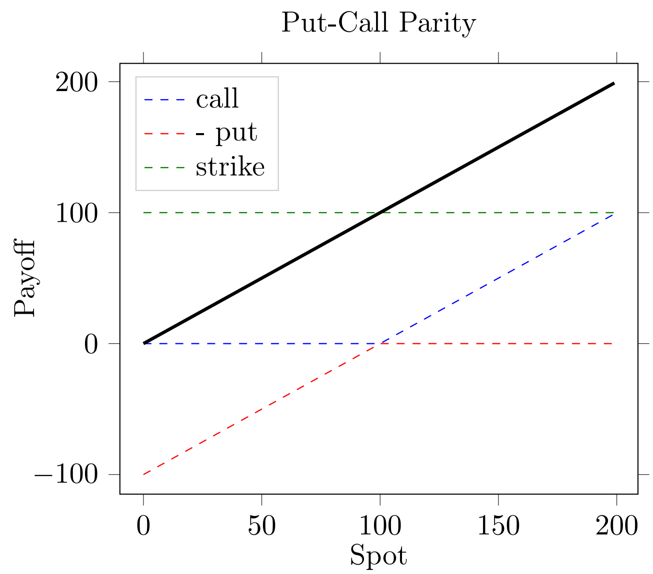
$$\texttt{Call}_t(T, K) := \texttt{C}_t\big((S_T - K)_+, T\big) \qquad \text{and} \qquad \texttt{Put}_t(T, K) := \texttt{C}_t\big((K - S_T)_+, T\big).$$

Proposition I.8 (Put-Call Parity)**.**

$$\mathtt{Call}_t(T,K) - \mathtt{Put}_t(T,K) = \big(\mathtt{F}_t(S_T,T) - K\big)B(t,T)$$

When the asset does not pay dividends between t and T, this expression becomes $\mathtt{Call}_t(T,K) - \mathtt{Put}_t(T,K) = S_t - KB(t,T)$.

This relation can be interpreted as follows: loosely speaking, the holding of a Call and the sale of a put with the same characteristics guarantee to have at time T the value of the asset S and to sell of the exercise price K. This portfolio can also be obtained by buying the share, at t and reimbursing $KB(t,T)$ at t.



Proof. We recall that the **unique-price principle** holds similarly for cash-prices and forward prices.
In order to prove the Put-Call parity, we can use two different investment strategies.

- Strategy (1): At the date t, we buy the forward contract of maturity T on $(S_T - K)_+$ and we sell the forward contract of maturity T on $(K - S_T)_+$.
- Strategy (2): At the date t, we buy the forward contract of maturity T on the asset S, and we sell the forward contract of maturity T on the strike K.

At the date T, the cash-flows of the two strategies are equal and their value is:

$$(S_T - K)_+ - (K - S_T)_+ = S_T - K.$$

Strategy	Portfolio value at t in €	Portfolio value at T in €
(1)	$V_t^1 = \mathtt{F}_t\big((S_T-K)_+,T\big) - \mathtt{F}_t\big((K-S_T)_+,T\big)$	$V_T^1 = S_T - K$
(2)	$V_t^2 = \mathtt{F}_t(S_T,T) - \mathtt{F}_t(K,T)$	$V_T^2 = S_T - K$

Table 3: Cashflows of the two strategies at times t and T

By **no-arbitrage**, the values of the two forward prices of the two portfolios are equal at time t.
We know that the forward prices of $\mathtt{Call}(T,K)$ and $\mathtt{Put}(T,K)$ can be expressed using their cash-prices

$$\mathtt{F}_t\big((S_T-K)_+,T\big) = \frac{\mathtt{Call}_t(T,K)}{B(t,T)}, \qquad \mathtt{F}_t\big((K-S_T)_+,T\big) = \frac{\mathtt{Put}_t(T,K)}{B(t,T)}.$$

Thus,

$$\mathtt{F}_t\big((S_T-K)_+,T\big) - \mathtt{F}_t\big((K-S_T)_+,T\big) = \mathtt{F}_t(S_T,T) - \mathtt{F}_t(K,T) = \mathtt{F}_t(S_T,T) - K.$$

Finally, $\mathtt{Call}_t(T,K) - \mathtt{Put}_t(T,K) = \big(\mathtt{F}_t(S_T,T) - \mathtt{F}_t(K,T)\big)B(t,T) = \big(\mathtt{F}_t(S_T,T) - K\big)B(t,T)$. □

If the underlying asset does not pay out some dividends, then $\mathtt{F}_t(S_T,T) = S_t$ and the call-put parity relation is quite explicit.

In the case of dividends, it is not the case; one possibility to get this value is to regress $\mathtt{Call}_t(T,K) - \mathtt{Put}_t(T,K)$ (obtained from market data) on K, in order to extract the forward price $\mathtt{F}_t(S_T,T)$ (implied by the market).

Alternatively, under some dividend model assumptions, $\mathtt{F}_t(S_T,T)$ can be computed (and then calibrated to the market).

▷ **Size Effect.** One of the most important consequences of the assumption of the frictionless market is the linearity of the pricing. However, this breaks down if one accounts the price impact by the big investors. For instance, if we want to obtain $10^6 S_T$ at T, it does not necessarily imply that we need to purchase 10^6 Forward contracts at price $\mathtt{F}_t(S_T,T)$, it can be much more costly. In fact, the price of a Forward contract should vary with the size of the transaction. Thus, the big investors have a big impact on the price of the contract.

▷ **Model-free bounds of Calls and Puts.** As with forward contracts, the absence of arbitrage makes it possible to establish constraints on the prices of derivatives according to their future characteristics. We will be particularly interested in the properties of the Call and Put options that we have described above, but it will be convenient for the rest to state more general properties regarding the price.

Proposition I.9 (Bounds/properties on call)**.** *We have the following bounds or properties on Call options prices:*

1. *Hedge relation :* $\big(\mathtt{F}_t(S_T,T)-K\big)_+B(t,T) \leq \mathtt{Call}_t(T,K) \leq \mathtt{F}_t(S_T,T)B(t,T) = \mathtt{C}_t(S_T,T)$.

2. *Bull Spread Relation :* $K \mapsto \mathtt{Call}_t(T,K)$ *is non-increasing.*

3. *Butterfly Spread Relation :* $K \mapsto \mathtt{Call}_t(T,K)$ *is convex.*

4. *Calendar Spread Relation :* $T \mapsto \mathtt{Call}_t(T,K)$ *is non-decreasing, when the underlying security pays no dividends and when* $B(t,T) \leq 1$ *(automatic if non-negative interest rate).*

If the first and second derivatives of the Call in strike exist, and that the first derivative in maturity exists as well, the three last properties simply write $\partial_K\mathtt{Call}_t(t,K) \leq 0, \partial_K^2\mathtt{Call}_t(T,K) \geq 0, \partial_T\mathtt{Call}_t(T,K) \geq 0$.

Proof.

1. Let us show that (1): $\big(\mathtt{F}_t(S_T,T)-K\big)_+B(t,T) \leq \mathtt{Call}_t(T,K)$.
   $(S_T-K)_+ \geq 0$ and $(K-S_T)_+ \geq 0 \overset{\textbf{N.A.}}{\Longrightarrow} \mathtt{Call}_t(T,K) \geq 0$ and $\mathtt{Put}_t(T,K) \geq 0$.
   Hence, using the Put-Call parity: $\mathtt{Call}_t(T,K) = \mathtt{Put}_t(T,K) + \big(\mathtt{F}_t(t,T)-K\big)B(t,T) \geq \big(\mathtt{F}_t(t,T)-K\big)B(t,T)$.

   Let us show that (2): $\mathtt{Call}_t(T,K) \leq \mathtt{F}_t(S_T,T)B(t,T) = \mathtt{C}_t(S_T,T)$.
   $(S_T-K)_+ \leq S_T \overset{\textbf{N.A.}}{\Longrightarrow} \mathtt{Call}_t(K,T) \leq \mathtt{C}_t(S_T,T) = \mathtt{F}_t(S_T,T)B(t,T)$.

2. Let us show that the Call price is non-increasing in strike. Let K_1 and K_2 be two exercice prices such as $K_1 > K_2$. For a given $S_T(\omega)$, we have $(S_T-K_1)_+ \leq (S_T-K_2)_+ \overset{\textbf{N.A.}}{\Longrightarrow} \mathtt{Call}_t(T,K_1) \leq \mathtt{Call}_t(T,K_2)$.

3. The payoff of the Call $K \to (S_T-K)_+$ is a convex function of the strike, by definition : $\forall\varepsilon > 0 \ \ \big(S_T-(K-\varepsilon)\big)_+ - 2(S_T-K)_+ + \big(S_T-(K+\varepsilon)\big)_+ \geq 0$. Using the **no-arbitrage** assumption, at time t the cash-prices for the Calls of respective strikes K, $K-\varepsilon$ and $K+\varepsilon$ verify $\mathtt{Call}_t(T,K-\varepsilon) - 2\mathtt{Call}_t(T,K) + \mathtt{Call}_t(T,K+\varepsilon) \geq 0$, i.e. the convexity property.

4. We would like to show that the Call price is non-decreasing in maturity. Let T_1 and T_2 be two maturities such as $T_1 < T_2$: since $\mathtt{F}_{T_1}(S_{T_2},T_2)B(T_1,T_2) = S_{T_1}$ (no dividend), the hedge relation gives $\mathtt{Call}_{T_1}(T_2,K) \geq \big(S_{T_1} - KB(T_1,T_2)\big)_+ \geq (S_{T_1}-K)_+$ since $B(T_1,T_2) \leq 1 \overset{\textbf{N.A.}}{\Longrightarrow} \mathtt{Call}_t(T_2,K) \geq \mathtt{Call}_t(T_1,K)$.

$\square$

I-b.3 Static replication of a payoff

In the market, for each maturity T there are numerous strikes at which call/put option prices are available. Imagine that there is a continuum of such strikes: intuitively it gives a crucial information about every possible future change in the underlying asset, and thus it should define without ambiguity the price of any derivatives written on the asset at time T. This is the meaning of the following Carr formula.
The main interest of the following result is its use in the replication of some payoffs, for example, the log-payoff in Variance payoff using only Call and Put options.

Lemma I.10. *The system of Call and Put options price of maturity T : $\big(\texttt{Call}_t(T,k), \texttt{Put}_t(T,K)\big)_{K\geq 0}$ allow to generate the prices of vanilla options of payoffs $h(S_T)$, where h can be any regular function or a difference of convex functions.*

$$\forall\, x_0, x \geq 0 \quad h(x) = h(x_0) + h'(x_0)(x-x_0) + \int_{x_0}^{+\infty} h''(K)(x-K)_+ \mathrm{d}K + \int_0^{x_0} h''(K)(K-x)_+ \mathrm{d}K.$$

Proof. We consider two cases : $x \geq x_0$ and $x < x_0$.
If $x \geq x_0$

$$\begin{aligned}
\int_{x_0}^{+\infty} h''(K)(x-K)_+ \mathrm{d}K + \int_0^{x_0} h''(K)(K-x)_+ \mathrm{d}K &= \int_{x_0}^{x} h''(K)(x-K) \mathrm{d}K + 0 \\
&= \Big[h'(K)(x-K)\Big]_{K=x_0}^{K=x} + \int_{x_0}^{x} h'(K)\mathrm{d}K \\
&= -h'(x_0)(x-x_0) + h(x) - h(x_0) \\
&= h(x) - h(x_0) - h'(x_0)(x-x_0).
\end{aligned}$$

If $x < x_0$

$$\begin{aligned}
\int_{x_0}^{+\infty} h''(K)(x-K)_+ \mathrm{d}K + \int_0^{x_0} h''(K)(K-x)_+ \mathrm{d}K &= 0 + \int_{x}^{x_0} h''(K)(K-x) \mathrm{d}K \\
&= \Big[h'(K)(K-x)\Big]_{K=x}^{K=x_0} - \int_{x}^{x_0} h'(K)\mathrm{d}K \\
&= h'(x_0)(x_0-x) - (h(x_0) - h(x)) \\
&= h(x) - h(x_0) - h'(x_0)(x-x_0).
\end{aligned}$$

□

By taking $x = S_T$, we get the following result.

Theorem I.11 (Carr formula)**.** *Under the* ***no-arbitrage*** *assumption, the system of Call and Put options price of maturity T : $\big(\texttt{Call}_t(T,k), \texttt{Put}_t(T,K)\big)_{K\geq 0}$ allow to generate the prices of* *vanilla options of payoffs $h(S_T)$, where h can be any regular function or a difference of convex functions. The cash-price at time t of the payoff $h(S_T)$ is*

$$\begin{aligned}\mathtt{C}_t(h(S_T),T) &= h(x_0)B(t,T) + h'(x_0)\big(\mathtt{C}_t(S_T,T) - x_0B(t,T)\big)\\ &\quad + \int_{x_0}^{+\infty} h''(K)\mathtt{Call}_t(T,K)\mathrm{d}K + \int_0^{x_0} h''(K)\mathtt{Put}_t(T,K)\mathrm{d}K.\end{aligned}$$

We choose as a anchor point x_0 the value of the Forward contract on S of maturity T. The price of the payoff S_T is given by $\mathtt{C}_t(S_T,T) = B(t,T)\mathtt{F}_t(S_T,T) = B(t,T)x_0$. The formula above then becomes

$$\begin{aligned}\mathtt{C}_t(h(S_T,T)) &= h\big(\mathtt{F}_t(S_T,T)\big)B(t,T) + \int_{\mathtt{F}_t(S_T,T)}^{+\infty} h''(K)\mathtt{Call}_t(T,K)\mathrm{d}K\\ &\quad + \int_0^{\mathtt{F}_t(S_T,T)} h''(K)\mathtt{Put}_t(T,K)\mathrm{d}K.\end{aligned}$$

This gives a static replication formula for a payoff $h(S_T,T)$, using call/put with the same maturity. This is valid without reference to a specific model (true for Market price data or for synthetic price): this is model-free.

Note that the strikes of call/put are OTM or ATM.

$\triangleright$ **Practical use.** The static model-free replication of the payoff h, requires having a continuum of Calls and Puts in K. However, Call and Put options are typically available for a number of different strikes and maturities. In general, for a given underlying, the Call and Put options are only traded for a finite number of maturities : $\{T_i, i = 1,\dots,n\}$ arranged in ascending order. For a given maturity, these options are only negotiated for a finite number of strikes $\{K_{i,j}, j = 1,\dots,d_i\}$. Hence, we need to approximate the integrals using finite sums in the Carr formula (trapezoidal rule, rectangle , etc), and when the option with payoff h has maturity T_i, the objective is to find the best approximation given a set of Calls/Puts with the same maturity T_i: $\big(\mathtt{Call}_t(T_i,K_{i,j}), \mathtt{Put}_t(T_i,K_{i,j})\big)_{i,j\in\{j=1,\dots,d_i\}}$. Any option of intrinsic value h can be expressed as the limit of a linear combination of Calls and Puts OTM/ATM.

However, this formula of static model-free replication does not allow to price an exotic contract nor path dependent.

I-c The Binomial Asset Pricing Model

I-c.1 The problem

We shall start digging an easy model to price a general payoff - The binomial asset pricing model. Although quite simple, it will introduce some key concepts such as hedging and risk-neutral expectation.

In this model, we will consider only one stock with the initial price at S_0. Let us assume we are only interested in two days, today at time $t = 0$ and maturity $t = T$. We will also consider two positive real numbers u and d with $0 < d < u$ (for up and down), such that at time T the stock price (S_T) can either be uS_0, with probability p, or dS_0, with probability $1 - p$. We have an option with payoff equals to $\Psi(S_T)$ and a riskless asset with a return rate

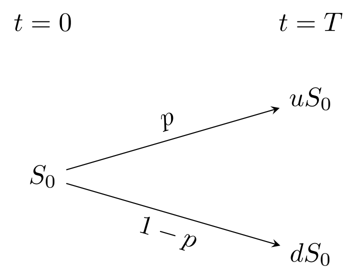


Figure 1: Binomial tree of stock prices with one period

of r, i.e., 1 € at time 0 gives $1 + r$ € at time T.

Therefore we shall write our portfolio as the following

$$V_t = \pi_t^0 + \delta_t \times S_t, \tag{I.1}$$

where

- π_t^0 is the cash amount at the riskless asset,
- δ_t is the number of stocks we've bought or sold,
- S_t is the stock price.

Therefore, the problem of pricing the option can be rewritten as finding δ_0 and π_0^0 such that, almost sure, we have $V_T = \Psi(S_T)$. Doing so, the hedging will be perfect and the hedger will not incure any residual risk in selling the option.

Remark I.12 (Relation between the upper, down and riskless rate)**.** As a consequence of the hypotheses of No-Arbitrage Opportunity, we must have the following relationship between the constants u, d, and r:

$$u > 1 + r > d > 0. \tag{I.2}$$

Proof.

- $d > 0$: we are not considering the possibility of default.
- If $1 + r > u$

  In this case, the stock is less profitable then the risk-less asset, we can create the following strategy:

  - At time 0: Sell $\frac{1}{S_0}$ stocks and put at the bank 1 € in cash, cost of this is zero.
  - At time T: The portfolio will be $(1+r) - \frac{S_T}{S_0} > 0$, since $S_T \in \{dS_0, uS_0\}$, which is against the NA.

- The same for $1 + r < d$
  - At time 0: Buy $\frac{1}{S_0}$ stocks and take a loan of 1 € in cash, cost of this is zero.
  - At time T: The portfolio will be worth $\frac{S_T}{S_0} - (1+r) > 0$, since $S_T \in \{dS_0, uS_0\}$, which is against the NA.

- If $d = 1+r$ and if $p > 0$, so doing the previous strategy, at time T we would have a gain ≥ 0. So we would have with non null probability a chance of earning money without any risk, which is against NA.

- $u = 1 + r$, is the same idea, but in the up side, so impossible.

$\square$

I-c.2 Example

In this small example, we will compute under this model the price of a European Call option, which payoff is $\Psi(S_T) = (S_T - K)_+$. With $S_0 = 4$ € , $u = 2$, $d = \frac{1}{2}$, $r = \frac{1}{4}$, and $K = 5$ € . We show the price evolution at Figure 2.

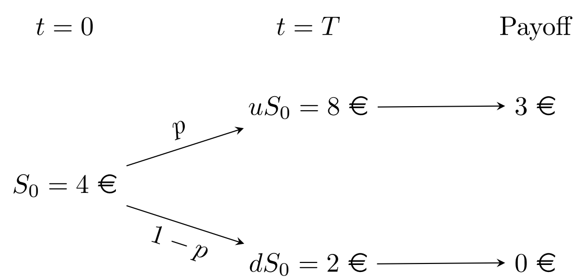


Figure 2: Binomial tree of stock prices with $S_0 = 4$ € , $u = 2$, $d = \frac{1}{2}$, and $K = 5$ € .

We shall replicate this Call Option with $V_0 = 1.2$ € and $\delta_0 = \frac{1}{2}$. Indeed, since the value of π_0^0 is:

$$\pi_0^0 = V_0 - \delta_0 \times S_0 = 1.2 - 0.5 \times 4 = -0.8\text{€} \ ,$$

we would need to take a loan of 80 cents, and now we compute π_T^0:

$$\pi_T^0 = \pi_0^0 \times (1+r) = -0.8 \times (1 + 0.25) = -1\text{€} \ .$$

Therefore at maturity, we would have to pay back one euro to the bank and our V_T would be:

$$V_T = \begin{cases} \pi_T^0 + \delta_0 \times u \times S_0 = 3€ \text{ , if price goes up,} \\ \pi_T^0 + \delta_0 \times d \times S_0 = 0€ \text{ , if price goes down.} \end{cases}$$

This exactly equals the Payoff that we can see in Figure 2.

Remark I.13. A valuable remark that we must see in this example is that we never used the value of the probability p of going up or down to compute the price of the option. <u>The option price does not depend on how likely that the market goes up</u>. We will see later that this is still true for the general case.

I-c.3 One-Period General Case

Let us consider a more general case (still with one period), we know for the statement of the problem the following equations for V_0 and V_T:

$$V_0 = \pi_0^0 + \delta_0 \times S_0, \tag{I.3}$$

$$V_T = \pi_0^0 \times (1+r) + \delta_0 \times S_T. \tag{I.4}$$

To replicate the payoff we need $V_T = \Psi(S_T)$, we shall split this condition into two possibilities: whether the price will go up, having a payoff of Ψ^u, or it will go down with a payoff equals to Ψ^d:

$$\Psi^u = \Psi(uS_0) = \pi_0^0 \times (1+r) + \delta_0 \times S_0 \times u, \tag{I.5}$$

$$\Psi^d = \Psi(dS_0) = \pi_0^0 \times (1+r) + \delta_0 \times S_0 \times d. \tag{I.6}$$

Now, we have a linear system with two variables, δ_0 and π_0^0, solving this system and we find the following answer:

$$\delta_0 = \frac{\Psi^u - \Psi^d}{S_0 \times (u-d)}, \tag{I.7}$$

$$\pi_0^0 = \frac{-d \times \Psi^u + u \times \Psi^d}{(1+r) \times (u-d)}. \tag{I.8}$$

Remark I.14. It is important to notice that we can interpret δ as a discrete derivate concerning the price and to compute the value π_0^0 we didn't take into consideration the probability p. This last remark was also spotted at the example.

Let us finally compute the value of V_0 using what we discovered so far. Using the equations (I.7) and (I.8) in (I.3) we obtain:

$$V_0 = \frac{\Psi^u}{u-d}\left(-\frac{d}{1+r} + 1\right) + \frac{\Psi^d}{u-d}\left(\frac{u}{1+r} - 1\right),$$

which can be rewritten as:

$$V_0 = \frac{\Psi^u}{1+r}\left(\frac{1+r-d}{u-d}\right) + \frac{\Psi^d}{1+r}\left(\frac{u-(1+r)}{u-d}\right). \tag{I.9}$$

We have found a strategy to replicate any payoff in our model. However, can we say that V_0 is the price of the option with payoff $\Psi(S_T)$? Imagine if we had another investment strategy such that $V'_T = \Psi(S_T)$, so by the absence of arbitrage we would have $V'_0 = V_0$, with possible different hedges. Therefore, V_0 must be the option price.

It is interest to name the following variables:

$$\bar{p} = \frac{1+r-d}{u-d}\ ,\ \bar{q} = \frac{u-(1+r)}{u-d} = 1-\bar{p}. \tag{I.10}$$

From (I.12), $\bar{p}$ and $\bar{q}$ are well defined, i.e., the denominator $u-d$ is always greater then 0. Also these variables are smaller then 1, we can therefore see this number as probabilities, such that:

$$\tilde{\mathbb{P}}(S_T = uS_0) = \bar{p}\ ,\ \tilde{\mathbb{P}}(S_T = dS_0) = 1-\bar{p}. \tag{I.11}$$

Putting together (I.9), (I.10) we shall write V_0 as a expected value under the probability as we've defined in (I.11).

$$V_0 = \tilde{\mathbb{E}}\left(\frac{\Psi(S_T)}{1+r}\right). \tag{I.12}$$

A piece of valuable information is that the price of the option is the expected value of the updated payoff under this new probability we've created.

Remark I.15 (Risk Neutral Probability)**.** Taking the expected value of S_T under this new probability we get:

$$\tilde{\mathbb{E}}\left(S_T\right) = uS_0 \times \bar{p} + dS_0 \times (1-\bar{p}) = S_0(1+r).$$

Since the expected value of S_T under $\tilde{\mathbb{P}}$ gives us the same value of a risk-less strategy in cash, we call this probability risk-neutral. In pricing, everything occurs like we were neutral to risk.

I-c.4 Two-Periods Case

Now let us consider a two-periods model. We already know what happens if the maturity was on $t = t_1$, we can now do a backward computation to see what happens in $t = T$. Note that now, we can change our hedging from t_1 to T, so we will introduce δ_1 that refers to the hedging in the second period. Following the same idea than before we will compute the value at the date $t = t_1$, and we have two possible cases:

$$v_{t_1}(uS_0) = \tilde{\mathbb{E}}\left(\frac{\psi(S_T)}{1+r} \mid S_{t_1} = uS_0\right), \tag{I.13}$$

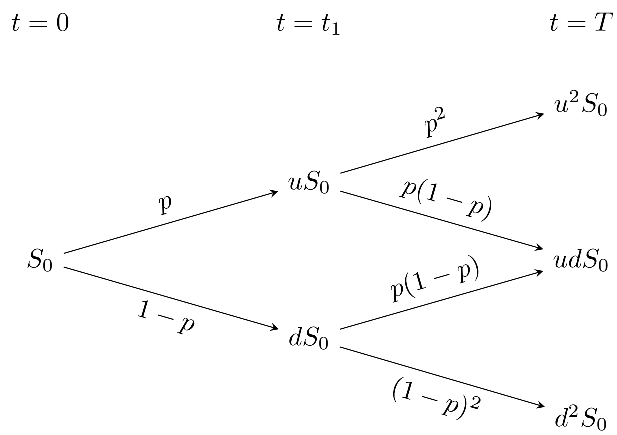


Figure 3: Binomial tree of stock prices with two periods

$$v_{t_1}(dS_0) = \tilde{\mathbb{E}}\left(\frac{\psi(S_T)}{1+r} \mid S_{t_1} = dS_0\right). \tag{I.14}$$

We can compute as well the hedging for this period:

$$\delta_{t_1}(uS_0) = \frac{\Psi\left(u(uS_0)\right) - \Psi\left(d(uS_0)\right)}{(u-d)uS_0}, \tag{I.15}$$

$$\delta_{t_1}(dS_0) = \frac{\Psi\left(u(dS_0)\right) - \Psi\left(d(dS_0)\right)}{(u-d)dS_0}. \tag{I.16}$$

Now it is just a matter of computing the expected value under the risk neutral probability $\tilde{\mathbb{P}}$ for the time $t = 0$:

$$\begin{aligned} V_0 &= \tilde{\mathbb{E}}\left(\frac{v_{t_1}(S_{t_1})}{1+r} \mid S_0\right) \\ &= \tilde{\mathbb{E}}\left(\frac{\tilde{\mathbb{E}}\left(\frac{\Psi(S_T)}{1+r} \mid S_{t_1}\right)}{1+r} \mid S_0\right) \\ &= \tilde{\mathbb{E}}\left(\frac{\Psi(S_T)}{(1+r)^2} \mid S_0\right). \end{aligned}$$

And for δ_0:

$$\delta_0 = \frac{v_{t_1}(uS_0) - v_{t1}(dS_0)}{S_0(u-d)}.$$

I-c.5 n-periods Case

This pattern persists regardless of the number of periods, we can even have a more general formula for the i-th V_i, considering a n-periods binomial model:

$$V_i = \tilde{\mathbb{E}}\left(\frac{\Psi(S_T)}{(1+r)^{n-i}} \mid S_{t_i}\right), \qquad T = t_n. \tag{I.17}$$

This reads as "the price at time t_i is the expectation under the risk-neutral probability of the discounted cash-flow".

This model is quite limited because at each period we can only go up by u or go down by d, it is not realistic at all in practice but it helps much to grasp key concepts in finance: hedging risk with portfolio invested in the risky asset, pricing rules do not depend on historical probability (p, q).