



# Optimal portfolio liquidation

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# Plan

- 1 Introduction
- 2 Almgren and Chriss model
- 3 Naive strategies
- 4 Optimal strategies



# Portfolio liquidation

## Financial problem

- We want to sell a large quantity of a stock (or of several stocks) in one day.
- How to choose a good way to split this large order in time and volume ?



# Strategies (1)

## Naive strategies

2 extreme strategies :

- Sell everything right now → huge transaction cost since we need to “eat” a lot in the order book. However this cost is known.
- Sell regularly in the day small amounts of assets → small transaction costs (volumes are much smaller) but the final profit is unknown because of the daily price fluctuations : Volatility risk.



## Strategies (2)

### Optimization

- We need to optimize between transaction costs and volatility risk.
- To do so, we use the Almgren and Chriss framework which takes into account the market impact phenomenon and emphasizes the importance of having good statistical estimators of market parameters.



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# Trading strategy

## Setup

- We consider we are selling one asset. We have  $X$  shares of this assets at  $t_0 = 0$ .
- We want everything to be sold at  $t = T$ .
- We split  $[0, T]$  into  $N$  intervals of length  $\tau = T/N$  and set  $t_k = k\tau$ ,  $k = 0, \dots, N$ .
- A trading strategy is a vector  $(x_0, \dots, x_N)$ , with  $x_k$  the number of shares we still have at time  $t_k$ .
- $x_0 = X$ ,  $x_N = 0$  and  $n_k = x_{k-1} - x_k$  is the number of assets sold between  $t_{k-1}$  and  $t_k$  (and not exactly at time  $t_{k-1}$  or  $t_k$ ), decided at time  $t_{k-1}$ .



# Price decomposition

## Price components

The price we have access to moves because of :

- The drift → negligible at the intraday level.
- The volatility.
- The market impact.



# Permanent market impact

## Permanent impact component

- Market participants see us selling large quantities.
- Thus they revise their prices down.
- Therefore, the “equilibrium price” of the asset is modified in a permanent way.
- Let  $S_k$  be the equilibrium price at time  $t_k$  :

$$S_k = S_{k-1} + \sigma\tau^{1/2}\xi_k - \tau g(n_k/\tau),$$

with  $\xi_k$  iid standard Gaussian and  $n_k/\tau$  the average trading rate between  $t_{k-1}$  and  $t_k$ .



# Temporary market impact

## Temporary impact component

- It is due to the transaction costs : we are liquidity taker since we consume the liquidity available in the order book.
- If we sell a large amount of shares, our price per share is significantly worse than when selling only one share.
- We assume this effect is temporary and the liquidity comes back after each period.
- Let  $\tilde{S}_k = (\sum n_{k,i} p_i) / n_k$ , with  $n_{k,i}$  the number of shares sold at price  $p_i$  between  $t_{k-1}$  and  $t_k$ . We set

$$\tilde{S}_k = S_{k-1} - h(n_k/\tau).$$

- The term  $h(n_k/\tau)$  does not influence the next equilibrium price  $S_k$ .



# Profit and Loss

## Cost of trading

- The result of the sell of the asset is

$$\sum_{k=1}^N n_k \tilde{S}_k$$

$$= X S_0 + \sum_{k=1}^N (\sigma \tau^{1/2} \xi_k - \tau g(n_k/\tau)) x_k - \sum_{k=1}^N n_k h(n_k/\tau).$$

- The trading cost  $\mathcal{C} = X S_0 - \sum_{k=1}^N n_k \tilde{S}_k$  is equal to

Vol. cost + Perm. Impact cost + Temp. Impact cost.



# Mean-Variance analysis

## Moments

- Consider a static strategy (fully known in  $t_0$ ), which is in fact optimal in this framework. We have

$$\mathbb{E}[\mathcal{C}] = \sum_{k=1}^N \tau x_k g(n_k/\tau) + \sum_{k=1}^N n_k h(n_k/\tau), \quad \text{Var}[\mathcal{C}] = \sigma^2 \sum_{k=1}^N \tau x_k^2.$$

- In order to build optimal trading trajectories, we will look for strategies minimizing

$$\mathbb{E}[\mathcal{C}] + \lambda \text{Var}[\mathcal{C}],$$

with  $\lambda$  a risk aversion parameter.



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# Assumptions (1)

## Permanent impact

- Linear permanent impact :  $g(v) = \gamma v$ .
- If we sell  $n$  shares, the price per share decreases by  $\gamma n$ . Thus

$$S_k = S_0 + \sigma \sum_{j=1}^k \tau^{1/2} \xi_j - \gamma(X - x_k).$$

Moreover, in  $\mathbb{E}[\mathcal{C}]$ , the permanent impact component satisfies

$$\sum_{k=1}^N \tau x_k g(n_k/\tau) = \gamma \sum_{k=1}^N x_k (x_{k-1} - x_k) = \frac{1}{2} \gamma X^2 - \frac{1}{2} \gamma \sum_{k=1}^N n_k^2.$$



## Assumptions (2)

### Temporary impact

- Affine temporary impact :  $h(n_k/\tau) = \varepsilon + \eta(n_k/\tau)$ .
- $\varepsilon$  represents a fixed cost : fees + bid ask spread.
- Let  $\tilde{\eta} = \eta - \frac{1}{2}\gamma\tau$ , we get

$$\mathbb{E}[\mathcal{C}] = \frac{1}{2}\gamma X^2 + \varepsilon X + \frac{\tilde{\eta}}{\tau} \sum_{k=1}^N n_k^2.$$



# Regular liquidation

## Regular strategy

- Take  $n_k = X/N$ ,  $x_k = (N - k)X/N$ ,  $k = 1, \dots, N$ .
- We easily get

$$\mathbb{E}[\mathcal{C}] = \frac{1}{2}\gamma X^2 + \varepsilon X + \tilde{\eta} \frac{X^2}{T},$$

$$\text{Var}[\mathcal{C}] = \frac{\sigma^2}{3} X^2 T \left(1 - \frac{1}{N}\right) \left(1 - \frac{1}{2N}\right).$$

- We can show this strategy has the smallest expectation.  
However the variance can be very big if  $T$  is large.



# Immediate selling

Selling everything at  $t_0$

- Take  $n_1 = X, n_2 = \dots = n_N = 0, x_1 = \dots = x_N = 0$ .
- We get

$$\mathbb{E}[\mathcal{C}] = \varepsilon X + \frac{\eta X^2}{\tau},$$
$$\text{Var}[\mathcal{C}] = 0.$$

- This strategy has the smallest variance. However, if  $\tau$  is small, the expectation can be very large.



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# Optimization (1)

## Optimization program

- The trader wants to minimize

$$U(\mathcal{C}) = \mathbb{E}[\mathcal{C}] + \lambda \text{Var}[\mathcal{C}].$$

- $U(\mathcal{C})$  is equal to

$$\frac{1}{2}\gamma X^2 + \varepsilon X + \frac{\tilde{\eta}}{\tau} \sum_{k=1}^N (x_{k-1} - x_k)^2 + \lambda \sigma^2 \sum_{k=1}^N \tau x_k^2.$$



## Optimization (2)

### Derivation

- For  $j = 1, \dots, N - 1$ ,

$$\frac{\partial U}{\partial x_j} = 2\tau \left( \lambda \sigma^2 x_j - \tilde{\eta} \frac{(x_{j-1} - 2x_j + x_{j+1})}{\tau^2} \right).$$

- Therefore

$$\frac{\partial U}{\partial x_j} = 0 \Leftrightarrow \frac{(x_{j-1} - 2x_j + x_{j+1})}{\tau^2} = \tilde{K} x_j,$$

with  $\tilde{K} = \lambda \sigma^2 / \tilde{\eta}$ .



## Optimization (3)

### Solution

- It is shown that the solution can be written  $x_0 = X$  and for  $j = 1, \dots, N$  :

$$x_j = \frac{\sinh(K(T - t_j))}{\sinh(KT)} X,$$

$$n_j = \frac{2\sinh(K\tau/2)}{\sinh(KT)} \cosh(K(T - j\tau + \tau/2)) X,$$

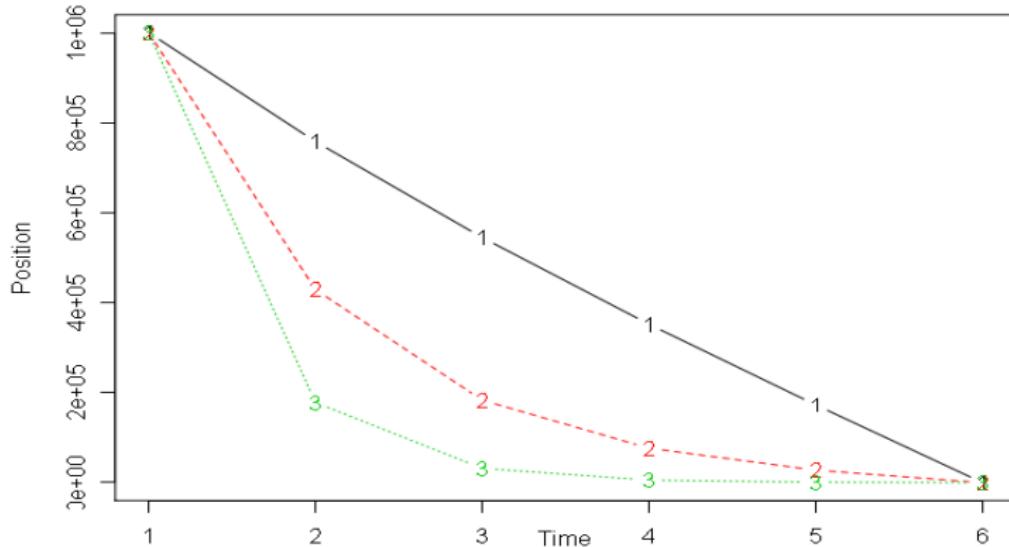
where  $K$  satisfies  $\frac{2}{\tau^2} (\cosh(K\tau) - 1) = \tilde{K}$ .

- If  $\lambda = 0$ , then  $\tilde{K} = K = 0$  and so  $n_j = \tau/T = X/N$ . We retrieve the strategy with minimal expected cost.

# Optimal strategies

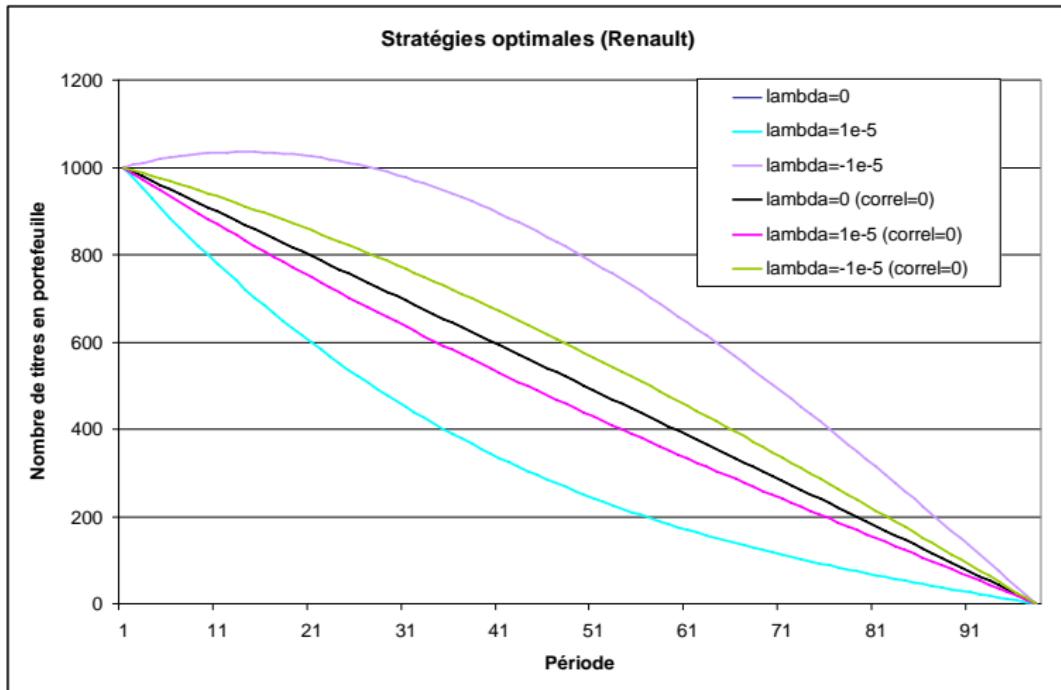
*Optimal Trajectory for a Single-Asset Portfolio*

(green :  $\lambda = 10^{-5}$ , red :  $\lambda = 2 \cdot 10^{-6}$ , black :  $\lambda = 10^{-7}$ )





# Optimal strategies on real data





# Remarks on this approach

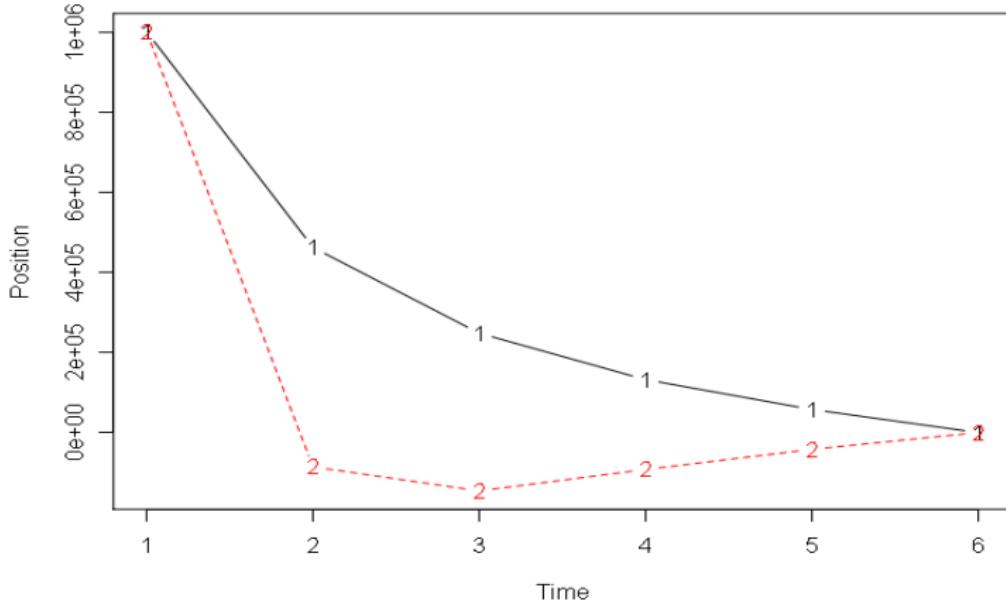
## Remarks

- It is easy to show that the solution is time homogenous : if we compute the optimal strategy in  $t_k$ , we obtain the values between  $t_k$  and  $T$  of the optimal strategy computed in  $t_0$ .
- In this approach, we obtain an efficient frontier of trading.
- The optimal trajectories are very sensitive to the volatility parameter. It is therefore important to obtain accurate volatility estimates.
- The Almgren and Chriss framework can be extended in dimension  $n$  (if we sell several assets). In that case, correlation parameters come into the picture.



# Optimal strategies in dimension 2

*Optimal Trajectories for the liquidation of a Two-Asset Portfolio*





# Important remark

## Trading tactics

- Only the total quantities which have to be executed in each time window are provided by the Almgren-Chriss approach. The way to deal with them inside each window (the trading tactic) is an intricate issue : Should we use market orders only or a combination with limit orders ? When should we trade inside each window ?