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EQUILIBRIUM IN A CAPITAL ASSET MARKET[1]

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This paper investigates the properties of a market for risky assets on the basis of a simple model of general equilibrium of exchange, where individual investors seek to maximize preference functions over expected yield and variance of yield on their portfolios. A theory of market risk premiums is outlined, and it is shown that general equilibrium implies the existence of a so-called "market line," relating per dollar expected yield and standard deviation of yield. The concept of price of risk is discussed in terms of the slope of this line.

1. INTRODUCTION

IN RECENT YEARS several studies have been made of the problem of selecting optimal portfolios of risky assets ([**6**, **8**], and others). In these models the investor is assumed to possess a preference ordering over all possible portfolios and to maximize the value of this preference ordering subject to a budget restraint, taking the prices and probability distributions of yield for the various available assets as given data.

From the point of view of positive economics, such decision rules can, of course, be postulated as implicitly describing the individual's demand schedules for the different assets at varying prices. It would then be a natural next step to enquire into the characteristics of the whole market for such assets when the individual demands are interacting to determine the prices and the allocation of the existing supply of assets among individuals.

These problems have been discussed, among others, by Allais [**1**], Arrow [**2**], Borch [**3**], Sharpe [**7**], and also to some extent by Brownlee and Scott [**5**].

Allais' model represents in certain respects a generalization relative to the model to be discussed here. In particular, Allais does not assume general risk aversion. This generalization requires, on the other hand, certain other assumptions that we shall not need in order to lead to definite results.

Arrow's brief but important paper is also on a very general and even abstract level. He uses a much more general preference structure than we do here and also allows differences in individual perceptions of probability distributions. He then proves that under certain assumptions there exists a competitive equilibrium which is also Pareto optimal.

Borch has investigated the problem with special reference to a reinsurance

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market. He suggests, however, that his analysis can be reversed and extended to a more general market for risky assets. The present paper may be seen as an attempt in that direction. The general approach is different in important respects, however, particularly as concerns the price concept used. Borch's price implies in our terms that the price of a security should depend only on the stochastic nature of the yield, not on the number of securities outstanding. This may be accounted for by the particular characteristics of a reinsurance market, where such a price concept seems more reasonable than is the case for a security market. A rational person will not buy securities on their own merits without considering alternative investments. The failure of Borch's model to possess a Pareto optimal solution appears to be due to this price concept.

Generality has its virtues, but it also means that there will be many questions to which definite answers cannot be given. To obtain definite answers, we must be willing to impose certain restrictive assumptions. This is precisely what our paper attempts to do, and it is believed that this makes it possible to come a long way towards providing a theory of the market risk premium and filling the gap between demand functions and equilibrium properties.

Brownlee and Scott specify equilibrium conditions for a security market very similar to those given here, but are otherwise concerned with entirely different problems. The paper by Sharpe gives a verbal-diagrammatical discussion of the determination of asset prices in quasi-dynamic terms. His general description of the character of the market is similar to the one presented here, however, and his main conclusions are certainly consistent with ours. But his lack of precision in the specification of equilibrium conditions leaves parts of his arguments somewhat indefinite. The present paper may be seen as an attempt to clarify and make precise some of these points.

2. THE EQUILIBRIUM MODEL

Our general approach is one of determining conditions for *equilibrium of exchange* of the assets. Each individual brings to the market his present holdings of the various assets, and an exchange takes place. We want to know what the prices must be in order to satisfy demand schedules and also fulfill the condition that supply and demand be equal for all assets. To answer this question we must first derive relations describing individual demand. Second, we must incorporate these relations in a system describing general equilibrium. Finally, we want to discuss properties of this equilibrium.

We shall assume that there is a large number m of individuals labeled i, $(i=1, 2, ..., m)$. Let us consider the behavior of one individual. He has to select a portfolio of assets, and there are n different assets to choose from, labeled j, $(j=1, 2, ..., n)$. The yield on any asset is assumed to be a random variable whose distribution is known to the individual. Moreover, all individuals are assumed to have identical perceptions of these probability distributions.[3] The yield on a whole portfolio is, of course, also a random variable. The portfolio analyses mentioned earlier assume that, in his choice among all the possible portfolios, the individual is satisfied to be guided by its expected yield and its variance only. This assumption will also be made in the present paper.[4]

It is important to make precise the description of a portfolio in these terms. It is obvious (although the point is rarely made explicit) that the holdings of the various assets must be measured in some kind of units. The Markowitz analysis, for example, starts by picturing the investment alternatives open to the individual as a point set in a mean-variance plane, each point representing a specific investment opportunity. The question is: to what do this expected yield and variance of yield refer? For such a diagram to make sense, they must necessarily refer to some unit common to all assets. An example of such a unit would be one dollar's worth of investment in each asset. Such a choice of units would evidently be of little use for our purposes, since we shall consider the prices of assets as variables to be determined in the market. Consequently, we must select some arbitrary "physical" unit of measurement and define expected yield and variance of yield relative to this unit. If, for example, we select one share as our unit for measuring holdings of Standard Oil stock and say that the expected yield is μ and the variance σ^2, this means expected yield and variance of yield per share; if instead we had chosen a hundred shares as our unit, the relevant expected yield and variance of yield would have been 100 μ, and 10,000 σ^2, respectively.

We shall find it convenient to give an interpretation of the concept of "yield" by assuming discrete market dates with intervals of one time unit. The yield to be considered on any asset on a given market date may then be thought of as the value per unit that the asset will have at the next market date (including possible accrued dividends, interest, or other emoluments). The terms "yield" and "future value" may then be used more or less interchangeably.

We shall, in general, admit stochastic dependence among yields of different assets. But the specification of the stochastic properties poses the problem of identification of "different" assets. It will be necessary to make the convention that two units of assets are of the same kind only if their yields will be identical.

[3] This assumption is not crucial for the analysis, but simplifies it a good deal. It also seems doubtful whether the introduction of subjective probabilities would really be useful for deriving propositions about market behavior. In any case, it may be argued, as Borch [**3**, p. 439] does: "Whether two rational persons on the basis of the same information can arrive at different evaluations of the probability of a specific event, is a question of semantics. That they may act differently on the same information is well known, but this can usually be explained assuming that the two persons attach different utilities to the event."

[4] Acceptance of the von Neumann—Morgenstern axioms (leading to their theorem on measurable utility), together with this assumption, implies a quadratic utility function for yield (see [**4**]). But such a specification is not strictly necessary for the analysis to follow, and so, by the principle of Occam's razor, has not been introduced.

The reason for this convention can be clarified by an example. In many lotteries (in particular national lotteries), several tickets wear the same number. When a number is drawn, all tickets with that number receive identical prizes. Suppose all tickets have mean μ and variance σ^2 of prizes. Then the expected yield on *two* tickets is clearly 2μ, regardless of their numbers. But while the variance on two tickets is $2\sigma^2$ when they have different numbers, it is $4\sigma^2$ when they have identical numbers. If such lottery tickets are part of the available assets, we must therefore identify as many "different" assets as there are different numbers (regardless of the fact that they have identical means and variances). For ordinary assets such as corporate stock, it is of course known that although the yield is random it will be the same on all units of each stock.

We shall denote the expected yield per unit of asset j by μ_j and the covariance between unit yield of assets j and k by σ_{jk}. We shall also need the rather trivial assumption that the covariance matrix for the yield of the risky assets is nonsingular.

An individual's portfolio can now be described as an n-dimensional vector with elements equal to his holdings of each of the n assets. We shall use x_j^i to denote individual i's holdings of assets j (after the exchange), and so his portfolio may be written $(x_1^i, x_2^i, \ldots, x_n^i)$.

One of the purposes of the analysis is to compare the relations between the prices and yields of different assets. To facilitate such comparisons, it will prove useful to have a *riskless asset* as a yardstick. We shall take the riskless asset to be the nth. That it is riskless of course means that $\sigma_{nk} = 0$ for all k. But it may also be suggestive to identify this asset with *money*, and with this in mind we shall write specifically $\mu_n = 1$, i.e., a dollar will (with certainty) be worth a dollar a year from now.

We denote the price per unit of asset j by p_j. Now, general equilibrium conditions are capable of determining relative prices only: we can arbitrarily fix one of the prices and express all others in terms of it. We may therefore proceed by fixing the price of the nth asset as q, i.e., $p_n = q$. This means that we select the nth asset as *numéraire*. We shall return to the implications of this seemingly innocent convention below.

With the above assumptions and conventions, the expected yield on individual i's portfolio can be written:

$$(1) \qquad y_1^i = \sum_{j=1}^{n-1} \mu_j x_j^i + x_n^i \,,$$

and the variance:

$$(2) \qquad y_2^i = \sum_{j=1}^{n-1} \sum_{\alpha=1}^{n-1} \sigma_{j\alpha} x_j^i x_\alpha^i \,.$$

As mentioned earlier, we postulate for each individual a *preference ordering* (utility function) of the form:

$$U^i = f^i(y_1^i, y_2^i) \tag{3}$$

over all possible portfolios, i.e., we postulate that an individual will behave as if he were attempting to maximize U^i. With respect to the form of U^i, we shall assume that it is concave, with the first derivative positive and the second negative. This latter assumption of general risk aversion seems to be generally accepted in the literature on portfolio selection. The investor is constrained, however, to the points that satisfy his *budget equation*:

$$\sum_{j=1}^{n-1} p_j(x_j^i - \bar{x}_j^i) + q(x_n^i - \bar{x}_n^i) = 0\,, \tag{4}$$

where $\bar{x}_j^i$ are the quantities of asset j that he brings to the market; these are given data. The budget equation simply states that his total receipts from the sale of the "old" portfolio should equal total outlays on the "new" portfolio.

Formally, then, we postulate that each individual i behaves as if attempting to maximize (3), subject to (4), (1), and (2). Forming the Lagrangean:

$$V^i = f^i(y_1^i, y_2^i) + \theta^i\left[\sum_{j=1}^{n-1} p_j(x_j^i - \bar{x}_j^i) + g(x_n^i - \bar{x}_n^i)\right],$$

we can then write the first-order conditions for the maxima for all i as:

$$\frac{\partial V^i}{\partial x_j^i} = f_1^i \mu_j + 2 f_2^i \sum_{\alpha=1}^{n-1} \sigma_{j\alpha} x_\alpha^i + \theta^i p_j = 0 \qquad (j=1, \ldots, n-1)\,,$$

$$\frac{\partial V^i}{\partial x_n^i} = f_1^i + \theta^i q = 0\,,$$

$$\frac{\partial V^i}{\partial \theta^i} = \sum_{j=1}^{n-1} p_j(x_j^i - \bar{x}_j^i) + q(x_n^i - \bar{x}_n^i) = 0\,,$$

where f_1^i and f_2^i denote partial derivatives with respect to y_1^i and y_2^i, respectively. Eliminating θ^i, this can be written as:

$$-\frac{f_1^i}{f_2^i} = \frac{2\sum_\alpha \sigma_{j\alpha} x_\alpha{}^i}{\mu_j - p_j/q} \qquad (j=1, \ldots, n-1)\,, \tag{5}$$

$$\sum_{j=1}^{n-1} p_j(x_j^i - \bar{x}_j^i) + q(x_n^i - \bar{x}_n^i) = 0\,. \tag{6}$$

In (5), the $-f_1^i/f_2^i$ is the marginal rate of substitution dy_2^i/dy_1^i between the variance and mean of yield. Equations (5) and (6) constitute, for each individual, n equations describing his demand for the n assets.

To determine general equilibrium, we must also specify equality between demand and supply for each asset. These market clearing conditions can be written:

$$\sum_{i=1}^{m} (x_j^i - \bar{x}_j^i) = 0 \qquad (j=1, \ldots, n) . \tag{7'}$$

As we would suspect, one of these conditions is superfluous. This can be seen by first summing the budget equations over all individuals:

$$\sum_{i=1}^{m} \sum_{j=1}^{n-1} p_j (x_j^i - \bar{x}_j^i) + q \sum_{i=1}^{m} (x_n^i - \bar{x}_n^i) = 0 ,$$

or

$$\sum_{j=1}^{n-1} p_j \sum_{i=1}^{m} (x_j^i - \bar{x}_j^i) + q \sum_{i=1}^{m} (x_n^i - \bar{x}_n^i) = 0 . \tag{8}$$

Suppose that (7′) were satisfied for all j except n. This would mean that the first term on the left of (8) vanishes, so that

$$q \sum_{i=1}^{m} (x_n^i - \bar{x}_n^i) = 0 .$$

Hence also the nth equation of (7′) must hold. We may therefore instead write:

$$\sum_{i=1}^{m} x_j^i = \bar{x}_j \qquad (j=1, \ldots, n-1) , \tag{7}$$

where $\bar{x}_j$ denotes the given total supply of asset j: $\bar{x}_j = \Sigma_{i=1}^m \bar{x}_j^i$.

This essentially completes the equations describing general equilibrium. The system consists of the m equations (4), the $m(n-1)$ equations (5) and (6), and the $(n-1)$ equations (7); altogether $(mn+n-1)$ equations. The unknowns are the mn quantities x_j^i and the $(n-1)$ prices p_j.

We have counted our equations and our unknowns and found them to be equal in number. But we cannot rest with this; our main task has hardly begun. We shall bypass such problems as the existence and uniqueness of a solution to the system and rather concentrate on investigating properties of the equilibrium values of the variables, assuming that they exist.

We may observe, first of all, that the equilibrium allocation of assets represents a Pareto optimum, i.e., it will be impossible by some reallocation to increase one individual's utility without at the same time reducing the utility of one or more other individuals. This should not need any explicit proof, since it is a well known general property of a competitive equilibrium where preferences are concave. We should also mention the problem of nonnegativity of the solution to which we shall return at a later stage.

3. RISK MARGINS

The expected *rate of return* r_j on a unit of a risky asset can be defined by $\mu_j/(1+r_j)=p_j$, i.e., $r_j=(\mu_j/p_j)-1$, $(j=1, \ldots, n-1)$. Similarly, the rate of return of a unit of the riskless asset r_n is defined by $1/(1+r_n)=q$, i.e., $r_n=1/q-1$. With our earlier interpretation of the riskless asset in mind, r_n may be regarded as the *pure rate of interest*.

The natural definition of the pure rate of interest is the rate of return on a riskless asset. In general, we may think of the rate of return of any asset as separated into two parts: the pure rate of interest representing the "price for waiting," and a remainder, a risk margin, representing the "price of risk." When we set the future yield of the riskless asset at 1 and decided to fix its current price at q, we thereby implicitly fixed the pure rate of interest. And to say that the market determines only relative asset prices is seen to be equivalent to saying that the pure rate of interest is not determined in the market for risky assets. Alternatively, we may say that the asset market determines only the risk margins.

The risk margin on asset j, m_j, is defined by

$$m_j=r_j-r_n=\frac{\mu_j-p_j/q}{p_j}.$$

To compare the risk margins of two assets j and k, we write:

$$\frac{m_j}{m_k}=\frac{\mu_j-p_j/q}{\mu_k-p_k/q}\,\frac{p_k}{p_j}.$$

We now make use of the equilibrium conditions. From (5) we have:

$$\frac{\sum_\alpha \sigma_{j\alpha}x_\alpha^i}{\mu_j-p_j/q}=\frac{\sum_\alpha \sigma_{k\alpha}x_\alpha^i}{\mu_k-p_k/q} \qquad (j,\, k=1, \ldots, n-1)\,. \tag{9}$$

Summing over i and using (7), we then get:

$$\frac{\sum_\alpha \sigma_{j\alpha}\bar{x}_\alpha}{\mu_j-p_j/q}=\frac{\sum_\alpha \sigma_{k\alpha}\bar{x}_\alpha}{\mu_k-p_k/q}\,. \tag{10}$$

These equations define relationships between the prices of the risky assets in terms of given parameters only. We can then write:

$$\frac{m_j}{m_k}=\frac{\bar{x}_j\sum_\alpha \sigma_{j\alpha}\bar{x}_\alpha}{\bar{x}_k\sum_\alpha \sigma_{k\alpha}\bar{x}_\alpha}\cdot\frac{p_k\bar{x}_k}{p_j\bar{x}_j}.$$

Now, $\bar{x}_j\,\Sigma_\alpha\sigma_{j\alpha}\bar{x}_\alpha$ is the variance of yield on the total outstanding stock of asset j; $p_j\bar{x}_j$ is similarly the total value, at market prices, of all of asset j. Let us denote these magnitudes by V_j and R_j, respectively. In equilibrium, therefore, the risk margins satisfy:

$$\frac{m_j R_j}{V_j} = \frac{m_k R_k}{V_k} \qquad (j, k=1, ..., n-1), \tag{11}$$

i.e., the risk margins are such that *the ratio between the total risk compensation paid for an asset and the variance of the total stock of the asset is the same for all assets.*

4. COMPOSITION OF EQUILIBRIUM PORTFOLIOS

We can now derive an important property of an individual's equilibrium portfolio.

When (10) is substituted back in (9), the result is:

$$\frac{\sum_\alpha \sigma_{j\alpha} x_\alpha^i}{\sum_\alpha \sigma_{j\alpha} \bar{x}_\alpha} = \frac{\sum_\alpha \sigma_{k\alpha} x_\alpha^i}{\sum_\alpha \sigma_{k\alpha} \bar{x}_\alpha}. \tag{12}$$

Now define for each individual $z_j^i = x_j^i/\bar{x}_j$ ($j=1, ..., n-1$), i.e., z_j^i is the proportion of the outstanding stock of asset j held by individual i. Further, let

$$b_{j\alpha} = \frac{\sigma_{j\alpha}\bar{x}_\alpha}{\sum_\alpha \sigma_{j\alpha}\bar{x}_\alpha},$$

so that $\Sigma_\alpha b_{j\alpha}=1$. Then (12) can be written

$$\sum_\alpha b_{j\alpha} z_\alpha^i = \sum_\alpha b_{k\alpha} z_\alpha^i \qquad (j, k=1, ..., n-1). \tag{13}$$

It is easily proved[5] that these equations imply that the z_j^i are the same for all j (equal to, say, z^i), i.e.,

$$z_j^i = z_k^i = z^i \qquad (j, k=1, ..., n-1). \tag{14}$$

What this means is that in equilibrium, prices must be such that *each individual will hold the same percentage of the total outstanding stock of all risky assets.* This percentage will of course be different for different individuals, but it means that if an individual holds, say, 2 per cent of all the units outstanding of one risky asset, he also holds 2 per cent of the units outstanding of all the other risky assets. Note that we cannot conclude that he also holds the same percentage of the riskless asset; this proportion will depend upon his attitude towards risk, as expressed by his utility function. But the relation nevertheless permits us to summarize the description of an individual's portfolio by stating (a) his holding of the riskless asset, and (b) the percentage z^i held of the outstanding stock of the risky assets. We

[5] Let the common value of the $n-1$ terms $\Sigma_{\alpha=1}^{n-1} b_{j\alpha} z_\alpha^i$ be a^i, and let $c_{j\alpha}$ be the elements of the inverse of the matrix of the $b_{j\alpha}$ (assuming nonsingularity). It is well known that when $\Sigma_\alpha b_{j\alpha}=1$, then also $\Sigma_\alpha c_{j\alpha} = 1$. The solutions for the z_j^i are then: $z_j^i = \Sigma_\alpha c_{j\alpha} a^i = a^i \Sigma_\alpha c_{j\alpha} = a^i$, which proves our proposition.

also observe that if an individual holds any risky assets at all (i.e., if he is not so averse to risk as to place everything in the riskless asset), then he holds some of *every* asset. (The analysis assumes, of course, that all assets are perfectly divisible.)

Looked at from another angle, (13) states that for any two individuals r and s, and any two risky assets j and k, we have $x_j^r/x_k^r = x_j^s/x_k^s$, i.e., the ratio between the holdings of two risky assets is the same for all individuals.

With these properties of equilibrium portfolios, we can return to the problem of nonnegativity of the solution. With risk aversion it follows from (5) that

$$\sum_\alpha \sigma_{j\alpha} x_\alpha^i \Big/ \left(\mu_j - \frac{p_j}{q}\right) > 0 \,.$$

The sum of such positive terms must also be positive, i.e.,

$$\sum_\alpha \sigma_{j\alpha} \bar{x}_\alpha \Big/ \left(\mu_j - \frac{p_j}{q}\right) > 0 \,.$$

But then also, $\Sigma_\alpha \sigma_{j\alpha} x_\alpha^i / \Sigma_\alpha \sigma_{j\alpha} \bar{x}_\alpha > 0$, so that the a^i of footnote 4 is positive, which then implies $z^i > 0$. Hence, negative asset holdings are ruled out.

Our results are not at all unreasonable. At *any* set of prices, it will be rational for investors to diversify. Suppose that before the exchange takes place investors generally come to the conclusion that the holdings they would prefer to have of some asset are small relative to the supply of that asset. This must mean that the price of this asset has been too high in the past. It is then only natural to expect the exchange to result in a fall in this price, and hence in an increase in desired holdings. What the relations of (14) do is simply to give a precise characterization of the ultimate outcome of the equilibrating effects of the market process.

5. THE MARKET LINE

The somewhat diffuse concept of a "price of risk" can be made more precise and meaningful through an analysis of the rate of substitution between expected yield and risk (in equilibrium). Specification of such a rate of substitution would imply the existence of a so-called "market curve." Sharpe illustrates a market curve as a line in a mean-standard deviation plane and characterizes it by saying: "In equilibrium, capital asset prices have adjusted so that the investor, if he follows rational procedures (primarily diversification), is able to attain any desired point along a *capital market line*" (p. 425). He adds that "... some discussions are also consistent with a nonlinear (but monotonic) curve" (p. 425, footnote).

We shall attempt to formulate these ideas in terms of our general equilibrium system.

As we have said earlier, a relation among points in a mean-variance diagram makes sense only when the means and variances refer to some unit common to all assets, for example, a dollar's worth of investment. We therefore had to reject such representations as a starting point for the derivation of general equilibrium conditions. When we study properties of this equilibrium, however, the situation is somewhat different. After equilibrium has been attained, each individual has specific portfolios with specific expected yields and variances of yield. Also, the individual's total wealth, i.e., the value at market prices of his portfolio, has been determined. This wealth, w^i, can be expressed as

$$w^i = \sum_{j=1}^{n-1} p_j x_j^i + q x_n^i = \sum_{j=1}^{n-1} p_j \bar{x}_j^i + q \bar{x}_n^i .$$

(The latter equality follows from (6).) We can now meaningfully define, for each individual, the per dollar expected yield of his equilibrium portfolio, u_1^i, and the per dollar standard deviation of yield of his equilibrium portfolio, u_2^i. These magnitudes are defined in terms of y_1^i and y_2^i by the relations: $u_1^i = y_1^i/w^i$, and $u_2^i = \sqrt{y_2^i}/w^i$. More concretely, we may think of individual i's portfolio as divided into w^i equal "piles," with each asset in equal proportion in all "piles." Each such "pile" has a market value of one dollar; its expected yield is u_1^i, and its standard deviation of yield is u_2^i. We are interested in the relationship between u_1^i and u_2^i.

From (5) we have:

$$\frac{dy_2^i}{dy_1^i} = \frac{2x_j^i \sum_\alpha \sigma_{j\alpha} x_\alpha^i}{(\mu_j - p_j/q) x_j^i} \qquad (j=1, ..., n-1) .$$

But by "corresponding addition,"[6] we also have

$$\frac{dy_2^i}{dy_1^i} = \frac{2\sum_j \sum_\alpha \sigma_{j\alpha} x_j^i x_\alpha^i}{\sum_j (\mu_j - p_j/q) x_j^i} = \frac{2y_2^i}{(y_1^i - x_n^i) - (w^i - q x_n^i)/q} = \frac{2y_2^i}{y_1^i - w^i/q} .$$

We thus have a differential equation in y_1^i and y_2^i, the general solution form of which is given by

$$\lambda^i \sqrt{y_2^i} = y_1^i - w^i/q ,$$

where λ^i is a constant of integration. With this solution, we have

$$\frac{dy_2^i}{dy_1^i} = \frac{2}{\lambda^i} \sqrt{y_2^i} ,$$

and so λ^i can be determined by the condition

$$\frac{2}{\lambda^i} \sqrt{y_2^i} = \frac{2\sum_\alpha \sigma_{j\alpha} x_\alpha^i}{\mu_j - p_j/q} ,$$

[6] If $u = a/b = c/d$, then also $u = (a + c)/(b + d)$.

which gives

$$\lambda^i = \frac{\mu_j - p_j/q}{\sum_\alpha \sigma_{j\alpha} x^i_\alpha} \sqrt{\sum_j \sum_\alpha \sigma_{j\alpha} x^i_j x^i_\alpha}\,.$$

But substituting from (14) $x^i_j = z^i \bar{x}_j$, we end up with

$$\lambda^i = \frac{\mu_j - p_j/q}{\sum_\alpha \sigma_{j\alpha} \bar{x}_\alpha} \sqrt{\sum_j \sum_\alpha \sigma_{j\alpha} \bar{x}_j \bar{x}_\alpha}\,.$$

The important thing to note here is that the righthand side is independent of i, so that we conclude that the λ^i are the same for all i—equal to, say, λ. This means that all points (u^i_1, u^i_2) lie on a straight line, $u_1 = \lambda u_2 + 1/q$, with

$$\lambda = \frac{\mu_j - p_j/q}{\sum_\alpha \sigma_{j\alpha} \bar{x}_\alpha} \sqrt{\sum_j \sum_\alpha \sigma_{j\alpha} \bar{x}_j \bar{x}_\alpha}\,. \tag{15}$$

We note also that according to (10) the factor $(\mu_j - p_j/q)/\sum_\alpha \sigma_{j\alpha}\bar{x}_\alpha$ is the same for all assets, so that the choice of the jth asset as "reference point" is perfectly arbitrary.

We shall analyse λ in detail in the next section, but it may be worth while to give a general appreciation of the results so far, as they are of some interest in themselves.

We have shown, first of all, that a "market line" in the sense discussed above can be derived from the conditions for general equilibrium (if it exists). Second, the fact that the market line is a straight line means that the rate of substitution between per dollar expected yield and per dollar standard deviation of yield is constant, i.e., for any two individuals r and s:

$$\frac{u^r_1 - u^s_1}{u^r_2 - u^s_2} = \lambda\,.$$

Third, these results are independent of any individual characteristics, not only with respect to initial holdings, but also with respect to the individuals' utility functions (except, possibly, that they depend upon the first two moments only of the probability distribution for yield). This is not to say that the *value* of λ is independent of the utility functions, which is clearly not the case, depending as it does upon the prices which in turn cannot be determined without knowledge of the utility functions. But the demonstration of this general property of equilibrium is nevertheless valuable.

The intercept with the u_1-axis, i.e., the point $u_2 = 0$, $u_1 = 1/q$, corresponds to a portfolio consisting entirely of the riskless asset and would be the location for an individual showing an extreme degree of risk aversion. And the further upward along the line an individual is located, the more willing he is to assume risk in order to gain in expected yield.

This concept of the market line as a locus of a finite number of points (u^i_1, u^i_2)

describing individual portfolios should be contrasted with the characterization given by Sharpe and cited earlier. At least with the interpretation we have been able to give to the market line, it is not something along which an individual may or may not choose to place himself. It would be misleading to give the impression that if an individual does not behave rationally he is somehow "off" this line. For the market line is not a construct that can be maintained independently of investor behavior, and it has no meaning as a criterion for testing whether an individual behaves rationally or not. Rather, it is a way of summarizing the result of rational behavior, and nothing more. It describes in a concise fashion the market conditions in general equilibrium, and this equilibrium is defined in terms of conditions implied by the attempts of individuals to maximize their utility functions, i.e., to behave rationally (and this is the only meaning that the term "rational behavior" can have in this context). If one or more individuals do not behave rationally, the whole foundation of the analysis is destroyed, and the concept of equilibrium, and hence also of the market line, becomes meaningless. The only statements that (15) does permit are those involving comparisons of different individuals' equilibrium portfolios with respect to their per dollar yield characteristics.

There is one more property of the market equilibrium that should be made explicit, namely, that it is independent of the definition of assets. More precisely: given society's real investments and their stochastic nature, the existence and slope of the market line is (under assumptions to be specified) independent of the distribution of ownership of these investments among companies.

So far, we have not been very precise about the nature of the various risky assets, although company shares were mentioned as examples of assets. Consider now the possibility of a merger of two companies into a new company. How will such a merger affect market equilibrium?

A detailed analysis of this kind of reorganization would evidently require specification of details of the merger agreements. But the most important results can be derived without this knowledge. We shall, as a matter of fact, consider any reorganization of the original $n-1$ companies into any number $\hat{n}$ of new companies. In the remainder of this section, we shall label the original companies j or k—$j, k=1, ..., n-1$—and retain our earlier notation for the parameters and variables in the original situation. The new companies will be labeled α or β; $\alpha, \beta=1, ..., \hat{n}$, and the corresponding parameters and variables will be distinguished by hats ($\hat{\mu}_\alpha$, $\hat{x}_\alpha^i$, $\hat{w}^i$, etc.). The riskless asset is labeled n in both cases.

We shall make two basic assumptions. The first is that the yield on the securities of a company can be identified with the yield of the real investments that it owns. The second is that the yield on real investments are independent of ownership conditions.

It should be clear that these assumptions imply that we neglect those factors that may account for most real-world mergers, namely, the possibility of reorganization of productive activities so as to improve their yield prospects. Further, it is implicitly taken for granted that the ownership reallocation does not affect investors' perceptions of probability distributions of yield. We are really attempting to compare two entirely different worlds—one with and one without merged companies. There is then no logical reason why there should exist any connections between probability distributions in the two worlds: the μ's and σ's are given data summarizing investors' perceptions when things are organized in a particular way, and would conceivably be different if things were organized differently.

Be that as it may, the immediate results of the assumptions are, first, that the expected yield on total outstanding stock of all companies is the same in both situations, i.e.,

$$\sum_{j=1}^{n-1} \mu_j \bar{x}_j = \sum_{\alpha=1}^{\hat{n}} \hat{\mu}_\alpha \hat{\bar{x}}_\alpha , \tag{16}$$

and, second, that a similar condition holds for the total variance:

$$\sum_{j=1}^{n-1} \sum_{k=1}^{n-1} \sigma_{jk} \bar{x}_j \bar{x}_k = \sum_{\alpha=1}^{\hat{n}} \sum_{\beta=1}^{\hat{n}} \hat{\sigma}_{\alpha\beta} \hat{\bar{x}}_\alpha \hat{\bar{x}}_\beta . \tag{17}$$

From (5) we must have, for each i and any j or α,

$$\frac{\sum_\beta \hat{\sigma}_{\alpha\beta} \hat{x}^i_\beta}{\hat{\mu}_\alpha - \dfrac{\hat{p}_\alpha}{q}} = \frac{\sum_k \sigma_{jk} x^i_k}{\mu_j - \dfrac{p_j}{q}},$$

so that (by summing over i):

$$\frac{\sum_\beta \hat{\sigma}_{\alpha\beta} \hat{\bar{x}}_\beta}{\hat{\mu}_\alpha - \dfrac{\hat{p}_\alpha}{q}} = \frac{\sum_k \sigma_{jk} \bar{x}_k}{\mu_j - \dfrac{p_j}{q}}. \tag{18}$$

This equation corresponds to (10), and it therefore follows that

$$\frac{\hat{x}^i_\alpha}{\hat{\bar{x}}_\alpha} = \frac{x^i_j}{\bar{x}_j} = z^i ,$$

i.e., the proportion held of the outstanding stock of the various risky assets is the same in both situations. Looking now at the expression (15) for λ, we observe that by (18) the first factor is the same in both situations, and that by (17) this also holds for the second factor. Hence we conclude that the market line remains the same.

By corresponding addition on both sides of (18), we also get

$$\frac{\sum_\alpha \sum_\beta \hat{\sigma}_{\alpha\beta} \hat{\bar{x}}_\alpha \hat{\bar{x}}_\beta}{\sum_\alpha \hat{\mu}_\alpha \hat{\bar{x}}_\alpha - \dfrac{1}{q} \sum_\alpha \hat{p}_\alpha \hat{\bar{x}}_\alpha} = \frac{\sum_j \sum_k \sigma_{jk} \bar{x}_j \bar{x}_k}{\sum_j \mu_j \bar{x}_j - \dfrac{1}{q} \sum_j p_j \bar{x}_j}.$$

Therefore, $\Sigma_\alpha \hat{p}_\alpha \hat{\bar{x}}_\alpha = \Sigma_j p_j \bar{x}_j$.

Next we can show that each individual will be located at the same point on the market line in both situations, so that his utility remains the same. This is seen by directly observing the means and variances of yield of portfolios in the two cases:

$$\begin{aligned}\hat{y}_1^i &= \sum_\alpha \hat{\mu}_\alpha \hat{x}_\alpha^i + \hat{x}_n^i = z^i \sum_\alpha \hat{\mu}_\alpha \hat{\bar{x}}_\alpha + \hat{x}_n^i = z^i \sum_j \mu_j \bar{x}_j + \hat{x}_n^i \\ &= \sum_j \mu_j x_j^i + \hat{x}_n^i = y_1^i - x_n^i + \hat{x}_n^i .\end{aligned}$$

But since the budget equations must also hold, we have

$$\sum_\alpha \hat{p}_\alpha \hat{x}_\alpha^i + q\hat{x}_n^i = \sum_j p_j x_j^i + q x_n^i ,$$

$$z^i \sum_\alpha p_\alpha \hat{\bar{x}}_\alpha + q\hat{x}_n^i = z^i \sum_j p_j \bar{x}_j + q x_n^i .$$

Therefore, $\hat{x}_n^i = x_n^i$, and so $\hat{y}_1^i = y_1^i$. Similarly, we find $\hat{y}_2^i = y_2^i$.

In short, then, everything remains essentially the same as before. Investors will just accept the exchange of securities caused by the reorganization of companies, but will not undertake any further adjustment.

The meaning of these results are, then, that when probability distributions are assumed to apply to the real side of the economy, the organization of productive activities is immaterial from the standpoint of valuation. Accordingly, companies may be formed in the way which is the most efficient for carrying out the productive activities (given such phenomena as economies of scale and the like), and that organization will also prove adequate from a "financial markets" point of view.

6. THE PRICE OF RISK

The concept of the "price of risk" can now be explored somewhat more fully in terms of λ, the slope of the market line. The "price of risk" is not a very fortunate choice of terms: "price of risk reduction" might be more satisfactory, since it is the relief of risk for which we must assume individuals are willing to pay. (We would, to make an analogy, certainly hesitate to use the term "price of garbage" for a city sanitation fee.) The price of risk reduction, however, is not only related to the rate of substitution between expected yield and risk, but must indeed be directly identified with it. That is to say, the only sensible meaning we can impute to the "price of risk reduction" is the amount of expected yield that must be sacrificed in order to reduce risk.

We note that when risk is measured, as we have done above, by the value (in dollars, say) of the standard deviation of yield, then the dimension of the price of risk reduction is that of an interest rate. This observation would lead us to try to establish a relation between λ and the risk margins m_j, discussed earlier. These risks margins may, of course, also be looked upon as representing prices of risk reduction, each one, however, referring to the risk aspects of that particular asset. We might then suspect that the equilibrating mechanisms of the market are such that all these risk margins are somehow "averaged" out into an overall market price of risk reduction. And it would certainly be reasonable to conjecture that the larger an asset looms in the market, the larger the weight carried by that asset in the total. Such an interpretation of λ can indeed be given an exact formulation.

Recalling our earlier definitions of m_j, R_j, and V_j, we can write (15) as:

$$\lambda = \frac{p_j \bar{x}_j m_j}{\bar{x}_j \sum_\alpha \sigma_{j\alpha} \bar{x}_\alpha} \sqrt{\sum_j V_j} = \frac{R_j m_j}{V_j} \sqrt{\sum_j V_j}\,.$$

Since this holds for any j, $j=1, \ldots, n-1$, we must also have

$$\lambda = \frac{\sum_j R_j m_j}{\sum_j V_j} \sqrt{\sum_j V_j} = \frac{\sum_j R_j m_j}{\sqrt{\sum_j V_j}}\,. \tag{19}$$

This means that λ is proportional to an arithmetical average of the m_j, the weights for each asset being the outstanding stock of that asset. The factor of proportionality is $\Sigma_j R_j / \sqrt{\Sigma_j V_j}$, the mean-standard deviation ratio for the market as a whole.

Another substitution allows us to write λ in still another fashion, which also throws some light on its composition. We may write (15) as

$$\lambda = \frac{m_k p_k}{\sum_\alpha \sigma_{k\alpha} \bar{x}_\alpha} \sqrt{\sum_j x_j \sum_\alpha \sigma_{j\alpha} \bar{x}_\alpha}$$

$$= \frac{m_k p_k}{\sum_\alpha \sigma_{k\alpha} \bar{x}_\alpha} \sqrt{\sum_j x_j \frac{(\sum_\alpha \sigma_{j\alpha} \bar{x}_\alpha)^2}{\sum \sigma_{j\alpha} \bar{x}_\alpha}}\,.$$

From (10), however, we get

$$(\sum \sigma_{j\alpha} \bar{x}_\alpha)^2 = (m_j p_j)^2 \left(\frac{\sum_\alpha \sigma_{k\alpha} \bar{x}_\alpha}{m_k p_k} \right)^2 .$$

When this is substituted above, the factor $m_k p_k / \Sigma_\alpha \sigma_{k\alpha} \bar{x}_\alpha$ drops out, and we are left with:

$$\lambda = \sqrt{\sum_j \frac{(m_j p_j)^2}{\frac{1}{\bar{x}_j} \sum_\alpha \sigma_{j\alpha} \bar{x}_\alpha}}\,.$$

Now define $s_j^2 = \dfrac{1}{\bar{x}_j} \sum_\alpha \sigma_{j\alpha} \bar{x}_\alpha$; this gives

$$\lambda = \sqrt{\sum_j \left(\frac{m_j p_j}{s_j} \right)^2}\,. \tag{20}$$

This expression is not only simple, but affords an interesting interpretation. Since $s_j^2 = V_j/\bar{x}_j^2$, s_j can be interpreted as the standard deviation of yield per unit of asset j (with the given quantities $\bar{x}_j$); i.e., it measures the risk per unit of asset j. Hence, $m_j p_j$ is clearly the risk compensation per unit of asset j; *$p_j m_j/s_j$ is then the risk compensation per unit of risk on a unit of asset j*, or, to put it differently, the gain in expected yield per unit's increase in the risk on a unit of asset j. The characterization given to λ as a description of equilibrium for the market as a whole was completely analogous. Then (20) specifies λ as the square root of the sum of squares of the individual components $p_j m_j/s_j$; this is a natural result of the properties of the standard deviation as a measure of risk.

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