Homework Submission

CMDA 3605 Mathematical Modeling: Methods and Tools I

Virginia Tech Honor Code Pledge:

"I have neither given nor received unauthorized assistance on this assignment."

Solution to Exercise 1

(a)

$$\dot{S}(t) = -\beta S(t)I(t) + \mu(P - S(t) - I(t)),$$
$$\dot{I}(t) = \beta S(t)I(t) - \gamma I(t)$$

Set equations equal to 0 to find steady states:

$$0 = -\beta S(t)I(t) + \mu(P - S(t) - I(t)),$$
$$0 = \beta S(t)I(t) - \gamma I(t)$$

Solve for second equation first:

$$\beta S(t)I(t) - \gamma I(t) = 0$$
$$\beta \hat{S}\hat{I} - \gamma \hat{I} = 0$$
$$\hat{I}(\beta \hat{S} - \gamma) = 0$$

Case 1: $\hat{I} = 0$. Solve for inside of parentheses, from the equation above we can tell that $\hat{I} = 0$. We will use the calculated value of $\hat{S}$ later:

$$(\beta \hat{S} - \gamma) = 0$$
$$\beta \hat{S} = \gamma$$
$$\hat{S} = \frac{\gamma}{\beta}$$

Now back to the first equation, substitute in $\hat{I} = 0$ back into the first equation to solve for $\hat{S}_1$:

$$-\beta \hat{S} * 0 + \mu(P - \hat{S} - 0) = 0$$
$$\mu(P - \hat{S}) = 0$$
$$\hat{S} = P$$

This proves that $\hat{x_1} = \begin{bmatrix} \hat{S}_1 \\ \hat{I}_1 \end{bmatrix} = \begin{bmatrix} P \\ 0 \end{bmatrix}$ is a steady state of the simplified SIR model.

Case 2: $\hat{S} = \frac{\gamma}{\beta}$ Substitute $\hat{S}$ into the first equation to solve for $\hat{I}_2$:

$$-\beta(\frac{\gamma}{\beta})\hat{I} + \mu(P - \frac{\gamma}{\beta} - \hat{I}) = 0$$
$$-\gamma \hat{I} - \mu(P - \frac{\gamma}{\beta} - \hat{I}) = 0$$
$$-\gamma \hat{I} - \mu P - \mu\frac{\gamma}{\beta} - \mu \hat{I} = 0$$
$$(\gamma + \mu)\hat{I} = \mu(P - \frac{\gamma}{\beta})$$
$$\hat{I} = \frac{\mu(P - \frac{\gamma}{\beta})}{\gamma + \mu}$$

Solution to Exercise 1 cont.

This proves that $\hat{x_2} = \begin{bmatrix} \hat{S}_2 \\ \hat{I}_2 \end{bmatrix} = \begin{bmatrix} \frac{\gamma}{\beta} \\ \frac{\mu(P - \frac{\gamma}{\beta})}{\gamma + \mu} \end{bmatrix}$ is a steady state of the simplified SIR model.

Solution to Exercise 1 cont.

(b)
We need the Jaccobian for the stability analysis of the continuous time nonlinear system:

$$
\begin{aligned}
J(S,I) &= \begin{bmatrix} -\beta I-\mu & -\beta S-\mu \\ \beta I & \beta S-\gamma \end{bmatrix} \\
J(\hat{x_1}) &= \begin{bmatrix} -\beta(0)-\mu & -\beta P-\mu \\ \beta(0) & \beta(P)-\gamma \end{bmatrix} \\
&= \begin{bmatrix} -\mu & -\beta P-\mu \\ 0 & \beta(P)-\gamma \end{bmatrix}
\end{aligned}
$$

The eigenvalues can be read off the diagonal and are equal to:
- $\lambda_1 = -\mu$,
- $\lambda_2 = \beta P - \gamma$

Given that the re susceptibility rate μ will always be greater than 0, The following is always true:

$\lambda_1 \ < \ 0$

If $\lambda_2 \ < \ 0$, the steady state $\hat{x}_1 = \begin{bmatrix} P \\ 0 \end{bmatrix}$ will be asymptotically stable since:
spectral abscissa $\alpha(J[\hat{x}_1]) \ < \ 0$

additionally,

$$
\begin{aligned}
\lambda_2 = \beta P - \gamma \quad (1) \\
\quad (2)
\end{aligned}
$$

and we know, $\lambda_2 \ < \ 0$

$$
\begin{aligned}
\beta P - \gamma \ &< \ 0 \quad (3) \\
\beta P \ &< \ \gamma \quad (4) \\
P \ &< \ \frac{\gamma}{\beta} \quad (5)
\end{aligned}
$$

So if $P \ < \ \frac{\gamma}{\beta}$, $\hat{x}_1$ becomes asymptotically stable.

Solution to Exercise 1 cont.

If $\lambda_2 \ > \ 0$ then the steady state $\hat{x}_1 = \begin{bmatrix} P \\ 0 \end{bmatrix}$ will be unstbale since spectral abscissa $\alpha(J[\hat{x}_1]) \ > \ 0$

additionally,

$$\lambda_2 \ > \ 0 \tag{6}$$

$$\tag{7}$$

$$\begin{aligned} \beta P - \gamma \ &> \ 0 \qquad (8) \\ \beta P \ &> \ \gamma \qquad (9) \\ P \ &> \ \frac{\gamma}{\beta} \qquad (10) \end{aligned}$$

So if $P \ > \ \frac{\gamma}{\beta}$ then $\hat{x}_1$ becomes unstable stable.

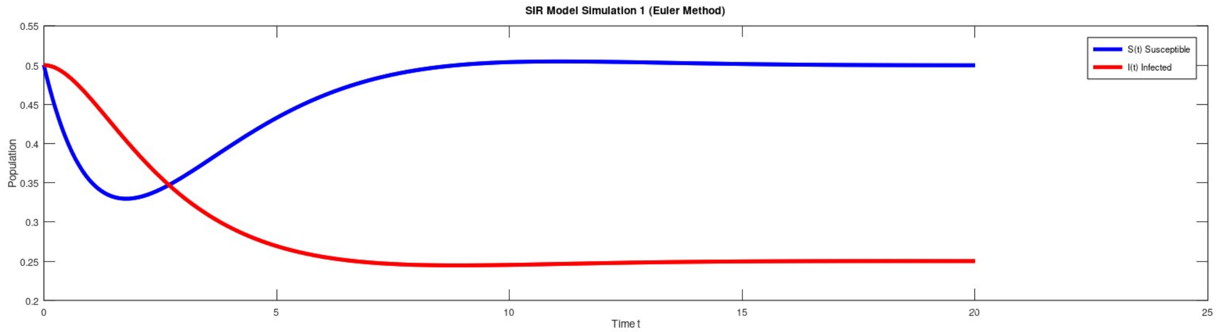
Solution to Exercise 2

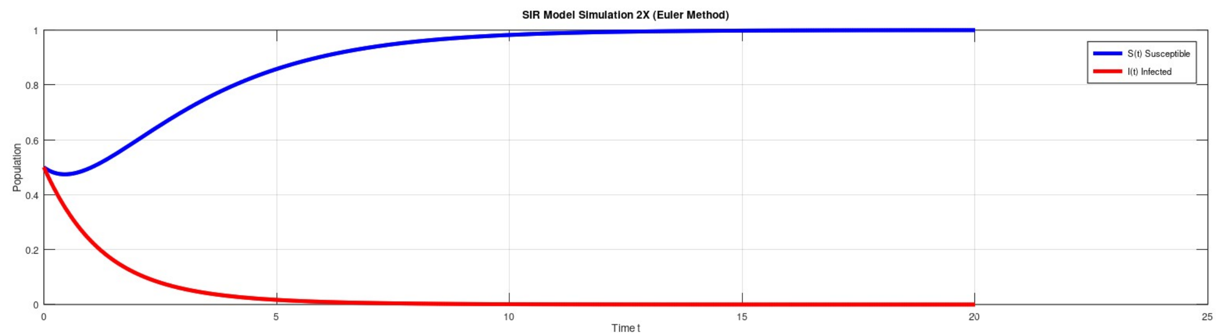
(a)

```
function [t, X] = forward_euler(f, t0, tend, h, x0)
% FORWARD_EULER  Implements the Forward Euler Method for solving
%                (non-linear discrete-time system)
%
% The Forward Euler update rule is:
%           x_{k+1} = x_k + h * f(t_k, x_k)


% number of steps
N = (tend - t0) / h;

% dimension of the state
n = length(x0);

% give outputs empty values
t = zeros(1, N+1);
X = zeros(n, N+1);

% initial conditions
t(1) = t0;
X(:,1) = x0;

% Euler time-stepping loop
for k =  1:N
  X(:, k+1) = X(:,k) + h * f(t(k), X(:, k));
  t(k+1)     = t(k) + h;
end
```

Solution to Exercise 2 cont.

(b)





The First Simulation:

- $\beta = 1$
- $\gamma = 0.5$
- $\mu = 0.5$
- $P = 1$

$$\frac{\gamma}{\beta} = 0.5 \; < \; P \; = \; P \; > \; \frac{\gamma}{\beta}$$

The state in this simulation is converging to the second steady state of the simplified SIR model from Exercise 1.

$$\hat{x}_2 = \begin{bmatrix} \frac{\gamma}{\beta} \\ \frac{\mu(P - \frac{\gamma}{\beta})}{\gamma + \mu} \end{bmatrix}$$

Solution to Exercise 2 cont.

The Second Simulation:

- $\beta = 0.5$
- $\gamma = 1$
- $\mu = 0.5$
- $P = 1$

$$\frac{\gamma}{\beta} = 2 \; > \; P \; = \; P < \; \frac{\gamma}{\beta}$$

The state in this simulation is converging to the first steady of the simplified SIR model from Exercise 1.

$$\hat{x}_1 = \begin{bmatrix} P \\ 0 \end{bmatrix}$$

Theoretical results from Exercise 1:

If $P \; > \; \frac{\gamma}{\beta}$ then $\hat{x}_1$ is an unstable steady state and if $P \; < \; \frac{\gamma}{\beta}$ then $\hat{x}_1$ becomes asymptotically stable.

- In simulation 2: $P \; < \; \frac{\gamma}{\beta}$, and the method converges to $\hat{x}_1$. i.e, $\hat{x}_1$ is an asymptotically stable steady state.

- In simulation 1: $P \; > \; \frac{\gamma}{\beta}$, and the method converges to $\hat{x}_2$. i.e, $\hat{x}_1$ is an unstable steady state.

This confirms the results from Exercise 1:

If the overall population is small enough compared to the recovery rate over the infection rate, then the disease will naturally vanish without additional actions.

Solution to Exercise 3

(a)

With k = 10,   m=1, and d=0 the linear mass-spring-damper system becomes:

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -10 & 0 \end{bmatrix} x(t)$$

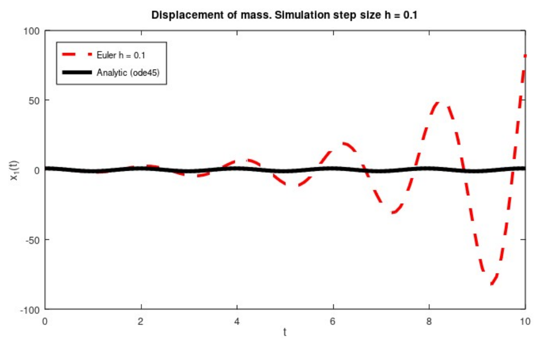
To analyze the stability of the system analytically, we first need eigenvalues:

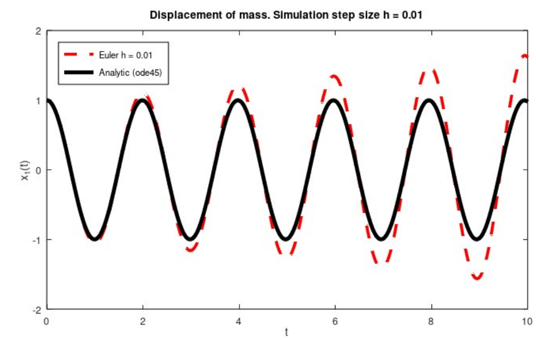
$$\begin{aligned} 0 &= (0-\lambda)(0-\lambda)+10 && (11) \\ &= \lambda^2 + 10 && (12) \\ \lambda^2 &= -10 && (13) \\ \lambda &= \sqrt{-1}\sqrt{10} && (14) \\ \lambda &= \pm\, i\, \sqrt{10} && (15) \end{aligned}$$

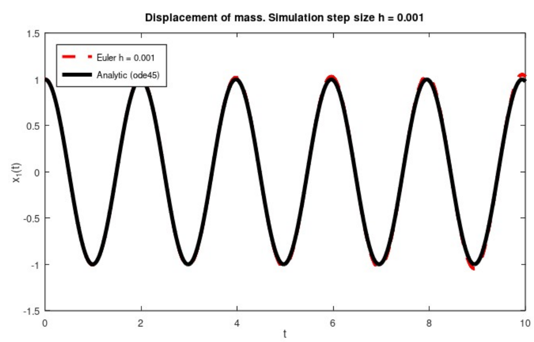
The spectral abscissa(A) = 0, so the system is stable. I suspect the displacement and velocity to oscillate indefinitely with constant amplitude. This is comes from damping = 0.

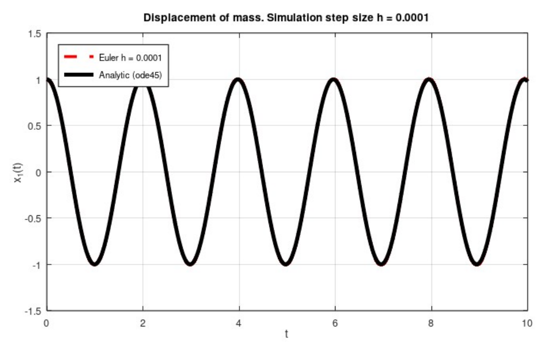
Solution to Exercise 3 cont.

(b)









For the first three step sizes, error in the Euler method can be seen on the plots. However, for the step size h = 0.0001, you can not see any error.

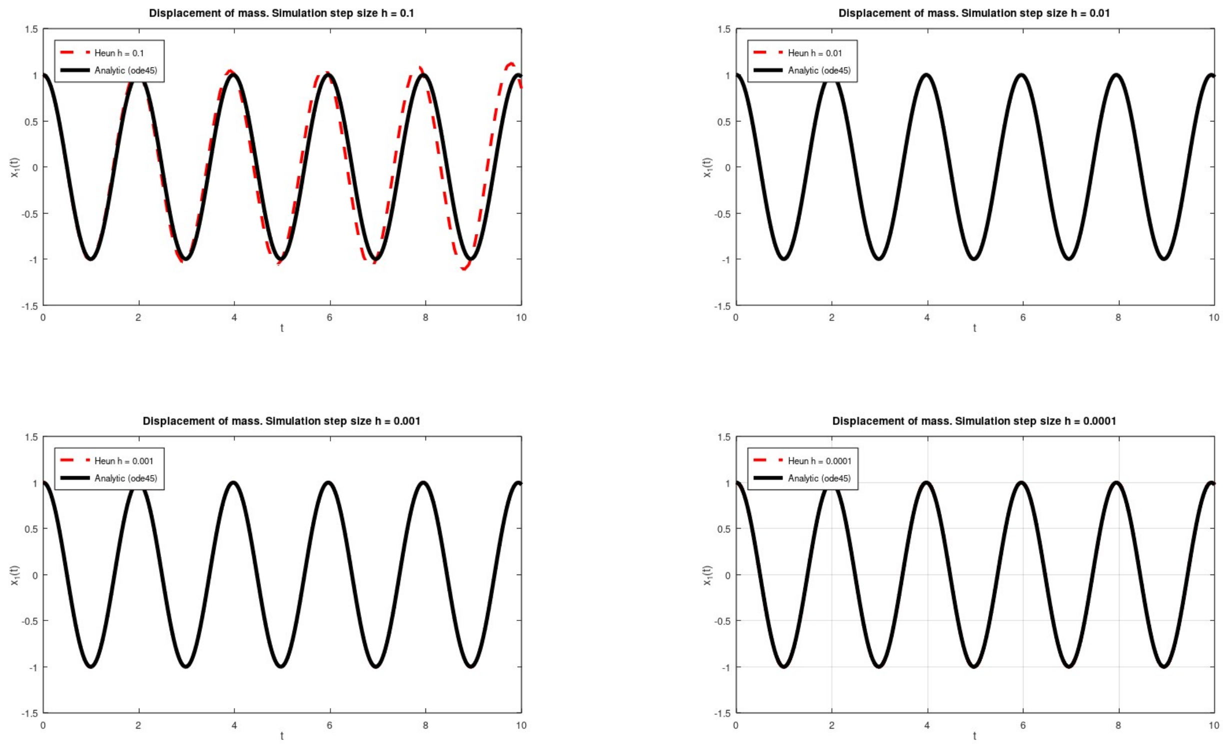
Solution to Exercise 4

(a)

```
function [t, X] = heun(f, t0, tend, h, x0)
% HEUN'S METHOD  Implements Heun's Method for solving dx/dt = f(t, x(t)).

    % number of steps
    N = round((tend - t0)/h);

    % dimension of the state
    n = length(x0);

    % preallocate
    t = zeros(1, N+1);
    X = zeros(n, N+1);

    % initial conditions
    t(1)   = t0;
    X(:,1) = x0;

    % Heun time stepping loop
    for k = 1:N
        tk = t(k);
        xk = X(:,k);

        % K1
        k1      = f(tk, xk);
        x_tilde = xk + h * k1;

        % K2
        k2 = f(tk + h, x_tilde);

        % update
        X(:,k+1) = xk + (h/2) * (k1 + k2);
        t(k+1)   = tk + h;
    end
end
```

Solution to Exercise 4 cont.

(b)



These results differ from the experiments with Euler's method as the error can no longer be seen in the step size h = 0.01. Also, the simulation at step size h = 0.1 is much closer to the true analytical solution than for Euler's method. This method does not over shoot the amplitude as much as Euler's method does in the first couple step sizes (h=0.1, h=0.01).

Bonus Exercise

$$\begin{aligned} w_{k+1} &= w_k + h * f(w_k) \\ &= w_k + hAw_k \\ &= (I + hA)w_k \end{aligned}$$

Substitute known values in:

$$w_{k+1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + h \begin{bmatrix} -10 & 5 \\ 0 & -20 \end{bmatrix}$$

$$w_{k+1} = \begin{bmatrix} 1 - 10h & 5h \\ 0 & 1 - 20h \end{bmatrix} w_k$$

For the Forward Euler Method to be asymptotically stable, the transition matrix of the discretization must have a spectral radius less than 1, i.e., inside the unit circle:

$$\begin{aligned} |1 - 10h| &< 1 \\ = -1 < 1 - 10h &< 1 \end{aligned}$$

Starting with $-1 < 1 - 10h$:

$$\begin{aligned} -2 &< -10h \\ h &< 0.2 \end{aligned}$$

Now for $1 - 10h < 1$:

$$\begin{aligned} -10h &< 0 \\ h &> 0 \end{aligned}$$

Now looking at the other eigenvalue:

$$\begin{aligned} |1 - 20h| &< 1 \\ -1 < 1 - 20h &< 1 \\ -1 < 1 - 20h, 1 - 20h &< 1 \\ -1 &< 1 - 20h \\ -2 &< -20h \\ h &< 0.1 \\ \\ 1 - 20h &< 1 \\ -20h &< 0 \\ h &> 0 \end{aligned}$$

By taking the Union of these intervals, $0 < h < 0.1$