Homework Submission

CMDA 3605 Mathematical Modeling: Methods and Tools I

Virginia Tech Honor Code Pledge:

"I have neither given nor received unauthorized assistance on this assignment."

Solution to Exercise 1

(a) Construction of state transition matrix A:

First column $j = 1$ ($v(1) = 1$):

$$m_{11} = 0, m_{21} = 0, m_{31} = 1, m_{41} = 0$$

Second column $j = 2$ ($v(2) = 2$):

$$m_{12} = 1/3, m_{22} = 0, m_{32} = 1/3, m_{42} = 1/3$$

Third column $j = 3$ ($v(3) = 3$):

$$m_{13} = 1/3, m_{23} = 1/3, m_{33} = 0, m_{43} = 1/3$$

Fourth column $j = 4$ ($v(4) = 0$):

$$m_{14} = 1/4, m_{24} = 1/4, m_{34} = 1/4, m_{44} = 1/4$$

Assemble the matrix:

$$m = \begin{bmatrix} 0 & 1/3 & 1/3 & 1/4 \\ 0 & 0 & 1/3 & 1/4 \\ 1 & 1/3 & 0 & 1/4 \\ 0 & 1/3 & 1/3 & 1/4 \end{bmatrix}$$

$$A = (p) \begin{bmatrix} 0 & 1/3 & 1/3 & 1/4 \\ 0 & 0 & 1/3 & 1/4 \\ 1 & 1/3 & 0 & 1/4 \\ 0 & 1/3 & 1/3 & 1/4 \end{bmatrix} + (1-p) \begin{bmatrix} 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 \end{bmatrix}$$

(b) Given that p = 0.85:

$$A = (0.85)\begin{bmatrix} 0 & 1/3 & 1/3 & 1/4 \\ 0 & 0 & 1/3 & 1/4 \\ 1 & 1/3 & 0 & 1/4 \\ 0 & 1/3 & 1/3 & 1/4 \end{bmatrix} + (1-0.85)\begin{bmatrix} 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 \end{bmatrix}$$

$$A = (0.85)\begin{bmatrix} 0 & 1/3 & 1/3 & 1/4 \\ 0 & 0 & 1/3 & 1/4 \\ 1 & 1/3 & 0 & 1/4 \\ 0 & 1/3 & 1/3 & 1/4 \end{bmatrix} + (0.15)\begin{bmatrix} 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 \end{bmatrix}$$

Since the transition matrix A is a postitive stochastic matrix, A has the simple dominant eigenvalue $\lambda = 1$. The corresponding Markov Chain has the nonzero steady state determined by the eigenvector corresponding to the eigenvalue $\lambda = 1$.

Using MATLAB, the dominant eigenvector, in norm 1,for the eigenvalue $\lambda = 1$ is $v_1 = \begin{bmatrix} 0.2369 \\ 0.1846 \\ 0.3415 \\ 0.2369 \end{bmatrix} = \hat{x}$

The site S_3 is the most important, as it has the highest relative importance weighting of .3415. The next most important sites are S_1 and S_4, both with rates of .2369. The least important cite based on importance weighing is Site S_2. The entries of $\hat{x}$ are the probabilities to land on webpage i in the long run.

Solution to Exercise 2

(a)

```
function [lambda, v, res] = powermethod(A, x0, maxiter, tol)
%POWERMETHOD Computes dominant eigenvalue-eigenvector pair.
%
% INPUTS:
%   A       - matrix of dimensions n x n
%   x0      - starting vector of dimension n (x 1)
%   maxiter - maximum number of iteration steps
%   tol     - tolerance for convergence criterion
%
% OUTPUTS:
%   lambda - dominant eigenvalue
%   v      - dominant eigenvector
%   res    - array of computed residuals

% Normalize the starting vector and save the result in v.
v =  x0 / norm(x0, 2);

% Residual memory.
res = zeros(1, maxiter);

% Iteration.
for k = 1:maxiter
    % Apply the matrix.
    v = A * v;

    % Normalize the iteration vector.
    v = v / norm(v, 2);

    % Compute the eigenvalue estimate.
    lambda = v' * A * v;

    % Convergence criterion.
    res(k) = norm(A * v - lambda * v, 2);
    if res(k) < tol
        break;
    end
end
```

```
Dominant eigenvalue:
1.0000
Corresponding eigenvector (normalized):
    0.6466
    0.5915
    0.4817
>> |
```

Solution to Exercise 2 continued

(b)

```
Top 10 ranked websites and their scores:
 1. https://www.youtube.com/user/VirginiaTech  ->  0.050918
 2. https://www.pinterest.com/imahokie  ->  0.050918
 3. https://www.wvtf.org  ->  0.049633
 4. https://www.vt.edu  ->  0.012252
 5. https://policies.vt.edu  ->  0.011894
 6. https://governance.vt.edu  ->  0.010468
 7. https://ensemble.cms.vt.edu  ->  0.008068
 8. https://www.weremember.vt.edu  ->  0.007717
 9. https://lib.vt.edu  ->  0.007717
10. https://facebook.com/virginiatech  ->  0.007621
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Solution to Exercise 3

(a) We know that $Av = \lambda v$ and $v \neq 0$.
Then:
$(A - \alpha\ \mathbf{1}_n)v = Av - \alpha\ \mathbf{1}_n v = \lambda v - \alpha v = (\lambda - \alpha)v.$

This shows that $\lambda - \alpha$ is an eigenvalue of $A - \alpha\ \mathbf{1}_n$ with eigenvalue v.

(b) Using the proof from part a:
$(A - \mu{-}\mathbf{1}_n)v = (\lambda - \mu)v$

μ is not an eigenvalue of A so $(A\ - \mu{-}\mathbf{1}_n)$ is invertible. Then:
$(A - \mu\mathbf{1}_n)^{-1}(A - \mu\mathbf{1}_n)v = (A - \mu\mathbf{1}_n)^{-1}(\lambda - \mu)v$
$v = (A - \mu\mathbf{1}_n)^{-1}(\lambda - \mu)v$

By multiplying both sides by $\frac{1}{\lambda - \mu}$ we get the following:
$\frac{1}{\lambda-\mu}v = (A - \mu\mathbf{1}_n)^{-1}v$

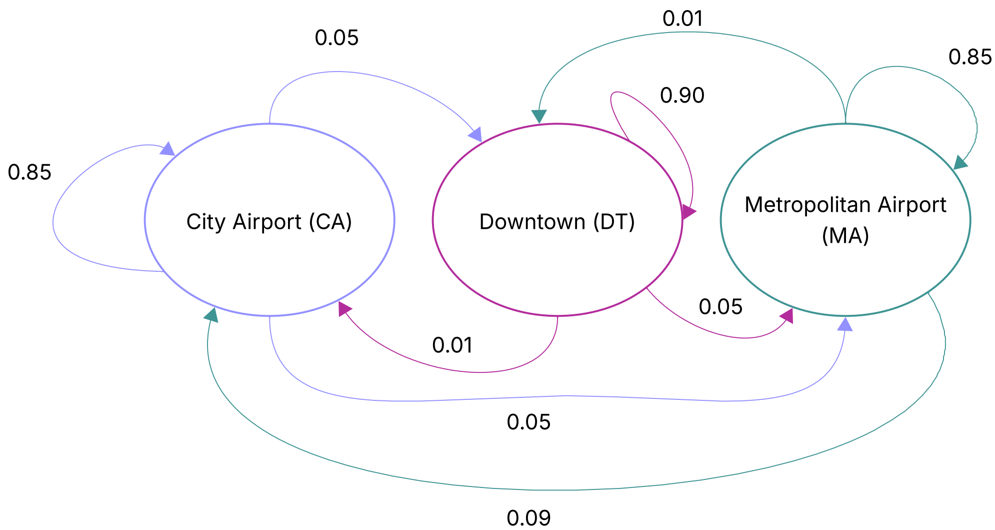
This shows that $\frac{1}{\lambda-\mu}$ is an eigenvalue of $(A - \mu\mathbf{1}_n)^{-1}$ with eigenvalue v.

(c) Again, we start with $Av = \lambda v$ and $v \neq 0$. Then we can multiply on the the left on both sides of the equation by v^T:
$v^T Av = v^T \lambda v$

Multiplying both sides by $\frac{1}{v^T v}$ gives:
$\frac{v^T Av}{v^T v} = \lambda$

Solution to Exercise 4

(a)



Define the state vector:

$$x(k) = \begin{bmatrix} x_{CA}(k) \\ x_{DT}(k) \\ x_{MA}(k) \end{bmatrix}$$

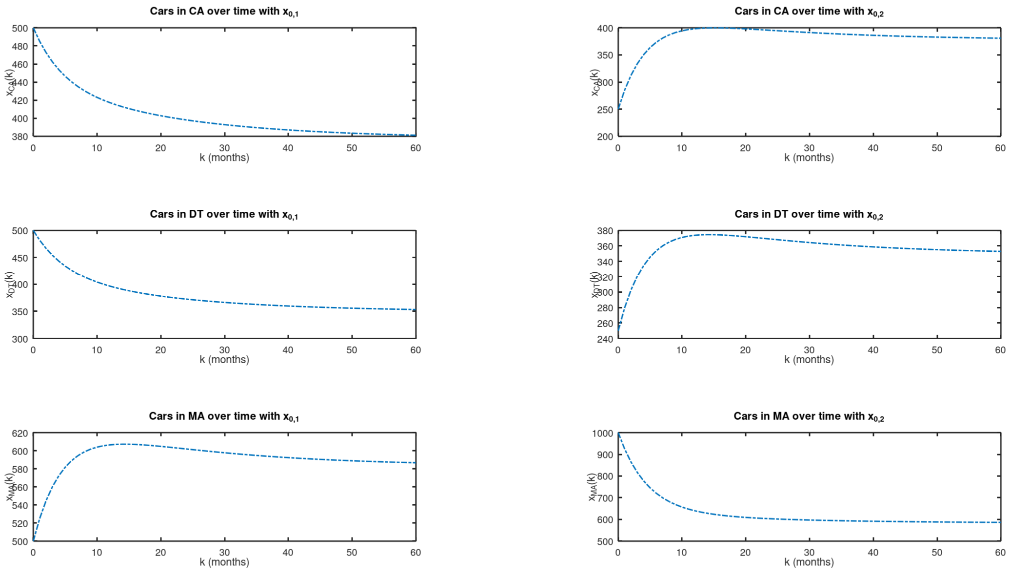
Given the diagram shown above, we can construct matrix A to be:

$$A = \begin{bmatrix} 0.85 & 0.05 & 0.05 \\ 0.01 & 0.90 & 0.05 \\ 0.09 & 0.01 & 0.85 \end{bmatrix}$$

Now construct the nonhomogenous system:

$$x(k+1) = \begin{bmatrix} 0.85 & 0.05 & 0.05 \\ 0.01 & 0.90 & 0.05 \\ 0.09 & 0.01 & 0.85 \end{bmatrix} x(k) + \begin{bmatrix} 10 \\ 2 \\ 50 \end{bmatrix}$$

(b)



(c) The steady state vector that the system approaches is

$$\hat{x} = \begin{bmatrix} 377.46 \\ 349.30 \\ 583.10 \end{bmatrix} = (\mathbf{1_n} - A)^{-1} b$$

This indicates that at the City Airport there will be approximately 378 cars, at downtown there will be approximately 349 cars, and at Metropolitan Airport there will be approximately 583 cars in the long run. I observed that the nonzero steady state reached by both initial conditions looked the same plotted, when visually accounting for the y-axis range.

Looking at the eigenvalues of the transition matrix A in MATLAB you get:

$$\lambda_1 = .9536,\ \lambda_2 = .7973,\ \lambda_3 = .8491$$

Noticing that $S(A) < 1$, you can expect the nonhomogeneous discrete time system to have a unique steady state, as defined above, independent of x_0.

Solution to Exercise 5

(a) $x(k) = \begin{bmatrix} R(k) \\ S(k) \end{bmatrix}$

$$\begin{aligned} R(k+1) &= R(k) + gR(k) + hS(k) + m(k) = (1+g)R(k) + hS(k) + m(k) \\ S(k+1) &= S(k) + rS(k) = (1+r)S(k) \end{aligned}$$

$$x(k+1) = \begin{bmatrix} (1+g) & h \\ 0 & (1+r) \end{bmatrix} + \begin{bmatrix} m(k) \\ 0 \end{bmatrix},\ x(0) = x_0 = \begin{bmatrix} 0 \\ S_0 \end{bmatrix}$$

(b) $x(k+1) = \begin{bmatrix} 1.01 & .02 \\ 0 & 1.005 \end{bmatrix} + \begin{bmatrix} 50 \\ 0 \end{bmatrix}$ $, x_0 = \begin{bmatrix} 0 \\ 4000 \end{bmatrix}$

From MATLAB $x(360) = \begin{bmatrix} 653580 \\ 24090 \end{bmatrix}$ So at the end of the 30 years of working there would be \$653580 in the retirement savings account.

$50(360) = \$18,000$ of the retirement savings account was my own contribution after 30 years(360 months).

(c) Running the same simulation, but with $m(k) = 1.005^k \times 50$ we get $x(360) = \begin{bmatrix} 778100 \\ 24090 \end{bmatrix}$.

Using a for loop to calculate the particular solution $\sum_{j=0}^{k-1} A^j\ b(k-1-j)$ to get $\$299,270$. This is the personal contribution to the retirement account. Personal contributions increase each month by .005 leading to higher retirement savings at 30 years than when we just added \$50 each month. The salary remains unaffected since b(k) does not have a component related to S(k).