Homework Submission

CMDA 3605 Mathematical Modeling: Methods and Tools I

<u>Virginia Tech Honor Code Pledge:</u>

"I have neither given nor received unauthorized assistance on this assignment."

Solution to Exercise 1

(a)

$$
\begin{aligned}
S(k+1) &= S(k) - 0.001\, S(k)I(k) \\
I(k+1) &= I(k) + 0.001\, S(k)I(k) - 0.007\, I(k) \\
R(k+1) &= R(k) + 0.007\, I(k)
\end{aligned}
$$

Infection Rate = $\beta = 0.001$
Recovery Rate = $\alpha = 0.007$
k is hourly.

(b)

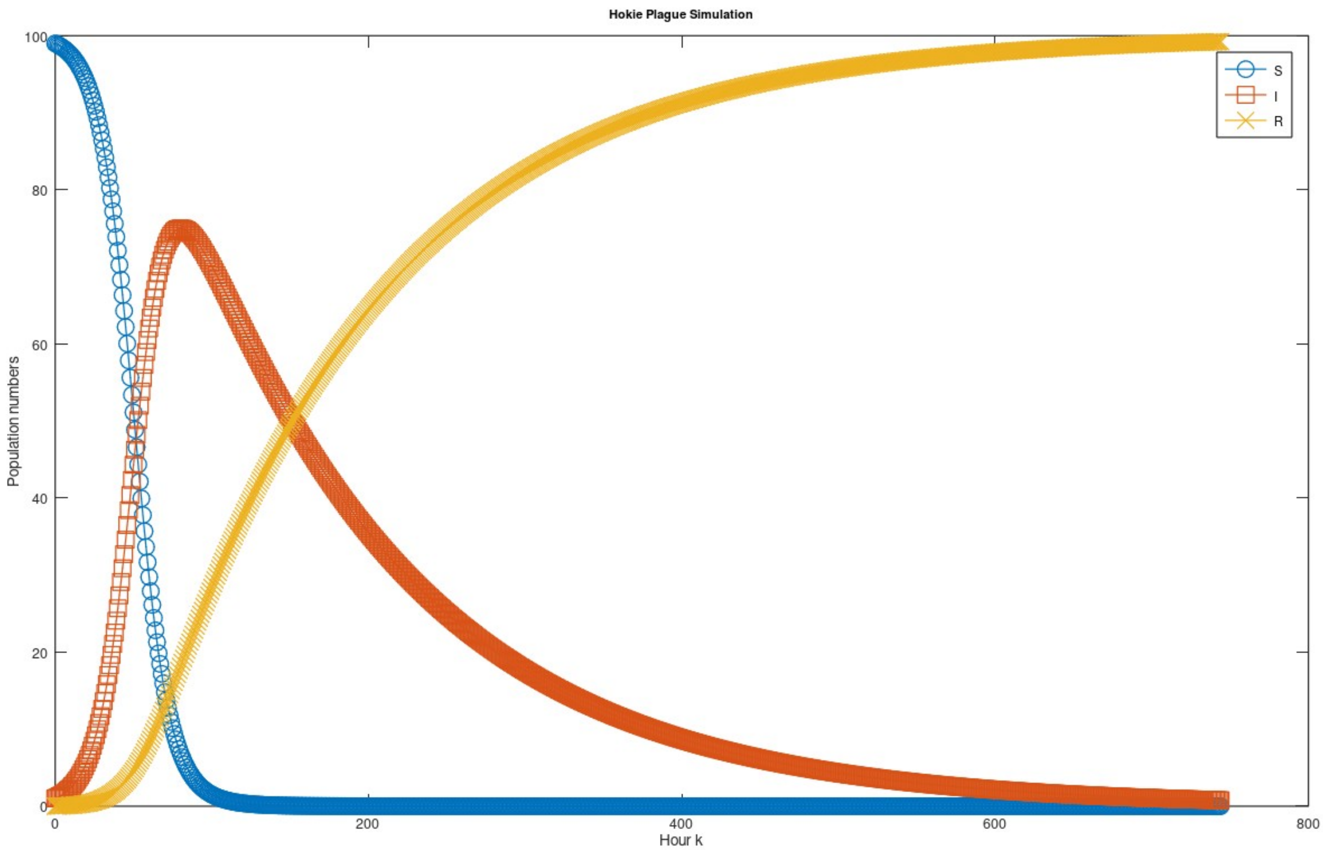
If $P(k) = S(k) + I(k) + R(k)$ is constant over time, then $P(k+1) - P(k) = 0$

$$
\begin{aligned}
P(k+1) &= S(k+1) \ + I(k) \ + R(k) \\
&= ((S - SI\beta) \ + \ (I + SI\beta - \alpha I) \ + \ (R + \alpha I)) \\
&= S + I + R \\
&= P(k)
\end{aligned}
$$

So P(k) is constant over time.

Solution to Exercise 1 Cont.

(c)



At the end of the month everyone is recovered from the Hokie Plague.

Solution to Exercise 2

steady states for nonlinear continous-time systems are where $f(\hat{x}) = 0$

(a) $\dot{x}(t) = \dfrac{e^x - 1}{x + 1}$

$$0 = \frac{e^x - 1}{x + 1}$$

$$\begin{aligned} 0 &= e^x - 1 \\ 1 &= e^x \\ ln(1) &= x \end{aligned}$$

$\hat{x} = 0$

(b) $\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} (x_1 - x_2)(x_2 + 1) \\ (x_1 + 2)(x_2 - 4) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Cases where above is true:

$\dot{x}_1(t): \;\; x_1 = x_2 \;\; OR \;\; x_2 = -1$
$\dot{x}_2(t): \;\; x_1 = -2 \;\; OR \;\; x_2 = 4$

All combinations:

$\hat{x_1} = \begin{bmatrix} -2 \\ -2 \end{bmatrix}, \;\; \hat{x}_2 = \begin{bmatrix} 4 \\ 4 \end{bmatrix}, \;\; \hat{x}_3 = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$

Solution to Exercise 3

(a) $\dot{x}(t) = \dfrac{e^x - 1}{x+1}$ and $\hat{x} = 0$

Linearization of nonlinear continuous-time dynamical system around $\hat{x}$:

$$\dot{u} = f(\hat{x})u(t)$$

$u(t) = x(t) - \hat{x}$

$$\begin{aligned} f(x) &= \frac{e^x(x+1-(e^x-1(1))}{(x+1)^2} \\ &= \frac{e^x x + e^x - e^x + 1}{(x+1)^2} \\ &= \frac{e^x x + 1}{(x+1)^2} \end{aligned}$$

Evaluating stability of $\hat{x} = 0$:

$f(\hat{x}) = f(0) = \dfrac{e^0(0)+1}{1^2} = 1$

Since $|f(\hat{x})| = 1 \ > 0$, the steady state is unstable.

(b) $\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} (x_1 - x_2)(x_2+1) \\ (x_1+2)(x_2-4) \end{bmatrix}$

To compute the linearizations of the dynamical systems we will need the Jaccobian. First we expand $\dot{x}(t)$

$$\dot{x}(t) = \begin{bmatrix} x_1x_2 + x_1 - x_2^2 - x_2 \\ x_1x_2 - 4x_1 + 2x_2 - 8 \end{bmatrix}$$

Then the Jaccobian (J) is:

$$J = \begin{bmatrix} x_2 + 1 & x_1 - 2x_2 - 1 \\ x_4 - 4 & x_1 + 2 \end{bmatrix}$$

Solution to Exercise 3 Cont.

Evaluating stability of steady state $\hat{x}_1 = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$:
Linearization:

$$\dot{u}(t) = J(\begin{bmatrix} -2 \\ -2 \end{bmatrix})u(t)$$

$$J(\hat{x}_1) = \begin{bmatrix} -1 & -2-2(-2)-1 \\ -6 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -6 & 0 \end{bmatrix}$$

Eigenvalues:

$$(-1-\lambda)(0-\lambda)+6$$
$$\lambda^2+\lambda+6$$

using pq formula:

$$\begin{aligned} \lambda_1 &= -1/2 - \sqrt{(1/2)^2-6} \\ &= -1/2 - \sqrt{1/4-6} \\ &= -1/2 - \sqrt{-5.75} \\ &= -1/2 - \sqrt{-1}\sqrt{5.75} \\ &= -1/2 - i\ 2.3979 \end{aligned}$$

$$\lambda_2 = -1/2 + i\ 2.3979$$

Since $\alpha(J(\hat{x}_1)) = -1/2 < 0$ the steady state is a locally asymptotically stable steady state. (α will serve as spectral absessa symbol for this assignment).

Evaluating stability of steady state $\hat{x}_2 = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$:
Linearization:

$$\dot{u}(t) = J(\begin{bmatrix} 4 \\ 4 \end{bmatrix})u(t)$$

$$J(\hat{x}_2) = \begin{bmatrix} 5 & -5 \\ 0 & 6 \end{bmatrix}$$

Eigenvalues:

$\lambda_1 = 5$ and $\lambda_2 = 6$

Since $\alpha(J(\hat{x}_2)) = 6 > 0$ the steady state is unstable.

Solution to Exercise 3 Cont.

Evaluating stability of steady state $\hat{x}_3 = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$:

Linearization:

$$\dot{u}(t) = J(\begin{bmatrix} -2 \\ -1 \end{bmatrix})u(t)$$

$$J(\hat{x}_3) = \begin{bmatrix} 0 & -1 \\ -5 & 0 \end{bmatrix}$$

Eigenvalues:

$$\begin{aligned} 0 &= (-\lambda)(-\lambda) - 5 \\ &= \lambda^2 - 5 \\ \lambda &= \pm\sqrt{5} \end{aligned}$$

$\lambda_1 = 2.23606$ and $\lambda_2 = -2.23606$

Since $\alpha(J(\hat{x}_3)) = 2.23606 > 0$ the steady state is unstable.

Solution to Exercise 4

(a) The state vector can be defined as:

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} q(t) \\ \dot{q}(t) \end{bmatrix}$$

Also, $\dot{x_1} = \dot{q} = x_2$: As a result, $x_2 = \ddot{q} = \frac{-d}{m} x_2 - \frac{1}{m}(e^{kx_1} - 1)$. Using that information, we can construct the standard form of the system:

$$\dot{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ \frac{-d}{m} x_2 - \frac{1}{m}(e^{kx_1} - 1) \end{bmatrix}, x(0) = \begin{bmatrix} q_{0,1} \\ q_{0,2} \end{bmatrix}$$

(b) Compute the steady states:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ \frac{-d}{m} x_2 - \frac{1}{m}(e^{kx_1} - 1) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$x_2 = 0$$

Given that $x_2 = 0$, substitute that back into the second equation to solve for x_1:

$$\begin{aligned} 0 &= \frac{-d}{m}(0) - \frac{1}{m}(e^{kx_1} - 1) \\ -\frac{1}{m}(e^{kx_1} - 1) &= 0 \\ e^{kx_1} - 1 &= 0 \\ e^{kx_1} &= 1 \\ kx_1 &= 0 \\ x_1 &= 0 \end{aligned}$$

This means that the only steady state for the dynamical system is $\hat{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. This means that the displacement of the particle as well as the velocity of the particle are equal to 0. The particle is not moving and has reached equilibrium.

(c) The linearizations of a system about the steady state is determined by the Jacobian:

$$J(x) = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} e^{kx_1} & -\frac{d}{m} \end{bmatrix}$$

From part (b) we calculated that $e^{kx_1} = 1$, so the Jacobian now becomes:

$$J(x) = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{d}{m} \end{bmatrix}$$

Now using the special case of $m = 1, d = 2, k = 2$, the Jacobian becomes:

$$J(x) = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix}$$

Solution to Excerise 4 cont.

Now calculate the eigenvalues:

$$J(x) = \begin{bmatrix} 0-\lambda & 1 \\ -2 & -2-\lambda \end{bmatrix} = 0$$

$$(-\lambda)(-2-\lambda) + 2 = 0$$

$$\lambda^2 + 2\lambda + 2 = 0$$

$$\lambda = \frac{-2 \pm \sqrt{(2)^2 - 4(1)(2)}}{2(1)}$$

$$\lambda = \frac{-2 \pm \sqrt{4-8}}{2(1)}$$

$$\lambda = \frac{-2 \pm \sqrt{-4}}{2(1)}$$

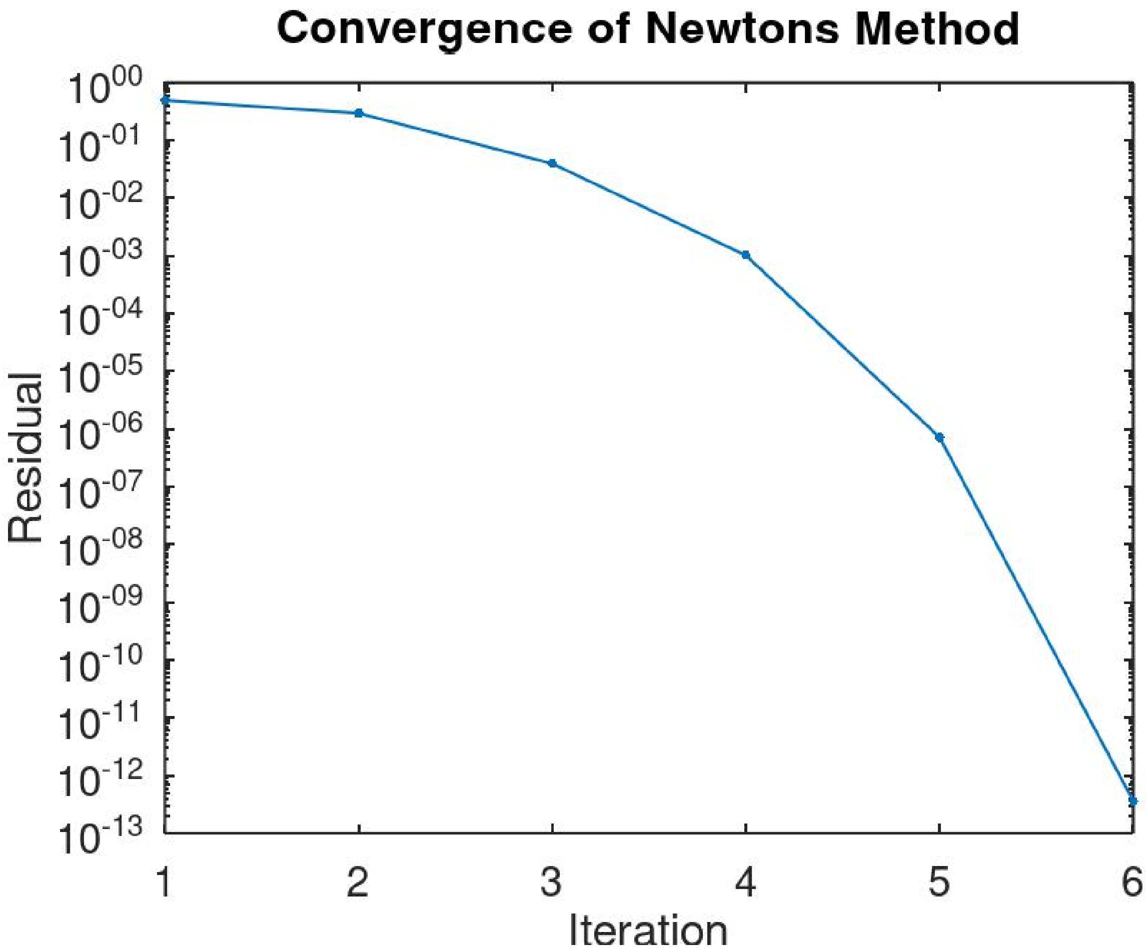
$$\lambda = \frac{-2 \pm 2i}{2}$$

$$\lambda = -1 \pm i$$

Given that the real part of the complex eigenvalues is -1, which is less than 0, the spectral abscissa is -1. Since -1 $<$ 0, the steady state $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is locally asymptotically stable. It is expected that the particle displacement and velocity will oscillate with diminishing magnitude since it is asymptotically stable, until it has reached the steady state.

Solution to Exercise 5

(a) For $x(k+1) = \cos(x(k)^3)$ The initial value that was used is $x_0 = 0.5$. This results in the plot:



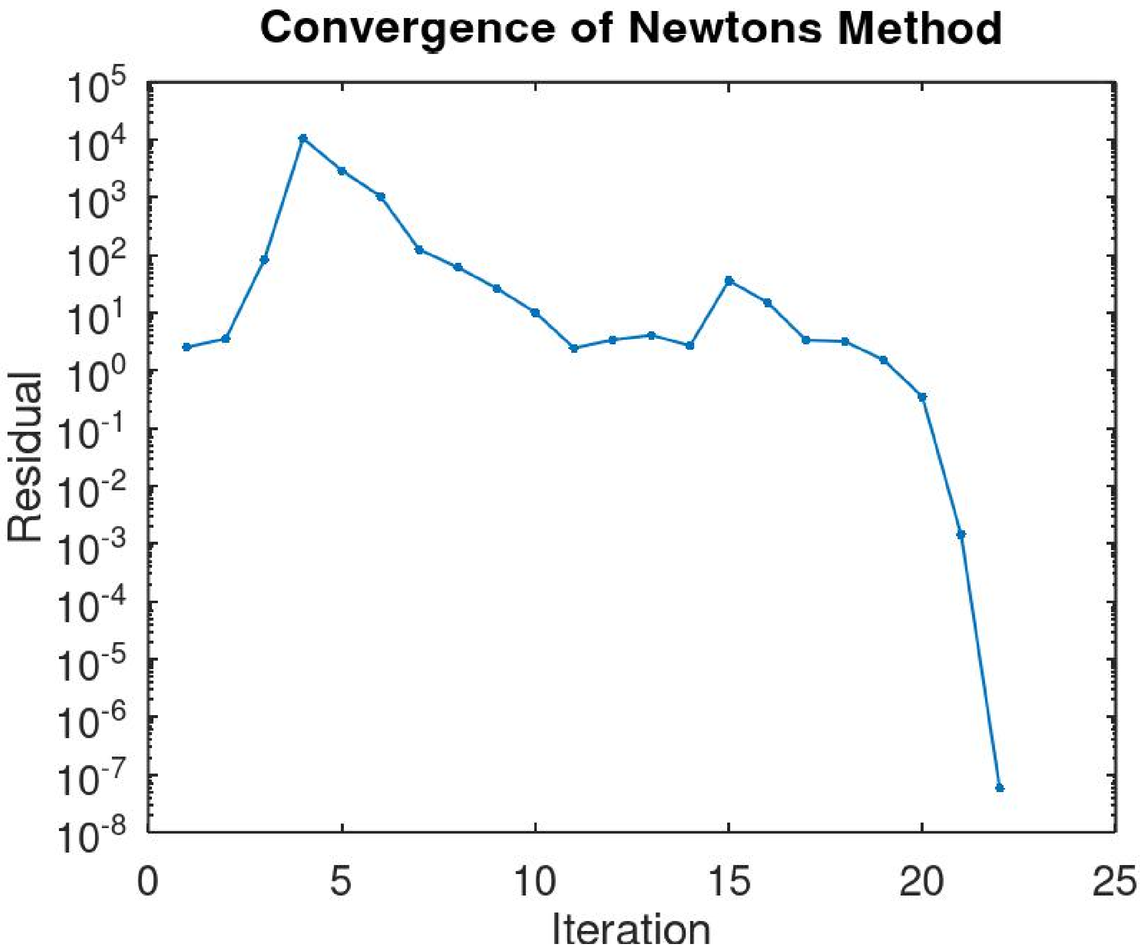
In order to compute the stability of the steady state, which is $\hat{x} \approx 0.8531$, take the derivative of $x(k+1) = \cos(x(k)^3)$:

$$x = -\sin(x^3) * 3x^2$$
$$\dot{x} = -3x^2 \sin(x^3)$$

Now, $f'(\hat{x}) = -3(0.8531)^2 \sin((0.8531)^3) \approx -1.27$. So the spectral radius is ≈ 1.27, which is > 1. This means that the steady state is unstable.

(b) For $\dot{x}(t) = \sin(x(t)) - x(t) + e$ The initial value that was used is $x_0 = 1$. This results in the plot:

Solution to Exercise 5 cont.



In order to compute the stability of the steady state, which is $\hat{x} \approx 2.9291$, take the derivative of $\dot{x}(t) = \sin(x(t)) - x(t) + e$:

$$x = \sin(x) - x$$
$$\dot{x} = \cos(x) - 1$$

Now, $f'(\hat{x}) = \cos(2.9291) - 1 \approx -1.98$. So the spectral abscissa is -1.98, which is < 0. This means that the steady state is locally asymptotically stable.