Homework Submission

CMDA 3605 Mathematical Modeling: Methods and Tools I

<u>Virginia Tech Honor Code Pledge:</u>

"I have neither given nor received unauthorized assistance on this assignment."

Solution to Exercise 1

Given homogeneous continous-time dynamical system:

$$\dot{x}(t) = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} x(t), \quad x(0) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

The solution formula for the system is as follows:

$$\dot{x}(t) = e^{\lambda_1 t} v_1 w_1 + \ e^{\lambda_2 t} v_2 w_2$$

Finding the eigenvalues of A:

$det \begin{bmatrix} (-1-\lambda) & -1 \\ -1 & (-1-\lambda) \end{bmatrix} = 0$

$(-1-\lambda)(-1-\lambda) - 1$
$= \lambda^2 + 2\lambda$
$= \lambda(\lambda + 2)$

so $\lambda_1 = 0$ and $\lambda_2 = -2$

Finding the eigenvectors of A:

for $\lambda_1 = 0$:

$\begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} v = 0$
$v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

for $\lambda_2 = -2$:

$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} v = 0$
$v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Finding the Weights:

$x_0 = VW : \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} w_1 \ + \ \begin{bmatrix} 1 \\ 1 \end{bmatrix} w_2 = \begin{bmatrix} w_1 + w_2 \\ -w_1 + w_2 \end{bmatrix}$

From adding $x_{0,1}$ and $x_{0,2}$ we get $2w_2 = 4$ which means:

$w_2 = 2$ and $w_1 = 1$

Solution to Exercise 1 continued

Since $\alpha(A) = 0 = \lambda_1$ (spectral abssesa) the system is stable as t goes to infinity and the solution formula in the long run can be written as:

$\dot{x}(t) = e^{0t} v_1 w_1$

with the steady state $\hat{x} = w_1 v_W = \begin{bmatrix} 1 \\ -1 \end{bmatrix} (1) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

So as $t \to \infty,\ x(t) \to \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Solution to Exercise 2

(a) $\dot{x}(t) = \begin{bmatrix} 0 & 3 \\ -3 & 0 \end{bmatrix} x(t)$

$$\begin{aligned} \det(A - \lambda I) &= \begin{bmatrix} 0-\lambda & 3 \\ -3 & 0-\lambda \end{bmatrix} \\ &= (0-\lambda)(0-\lambda) + 9 \\ \lambda^2 + 9 &= 0 \\ \lambda &= \pm 3i \end{aligned}$$

The spectral abscissa of the system is equal to 0, because it is the largest real part of the eigenvalue, the system is stable. The system will oscillate due to the fact that there are complex eigenvalues and the oscillations for $x(t)$ will remain bounded as $t \to \infty$.

(b) $\dot{x}(t) = \begin{bmatrix} -0.2 & -2 & 3 \\ 0 & 0.1 & 4 \\ 0 & 0 & -0.5 \end{bmatrix} x(t)$

Since this is an upper triangular matrix, the eigenvalues are the diagonal entries: $\lambda_1 = 0.1$, $\lambda_2 = -0.2$, and $\lambda_3 = -0.5$. The largest eigenvalue is 0.1, so the spectral abscissa is 0.1. Given that the spectral abscissa is greater than 0, the system is unstable because the largest real part is positive. Therefore, $x(t)$ will grow unbounded for $t \to \infty$. Even though the two negative eigenvalues (λ_2, λ_3) will have their terms decay to 0, the term with an eigenvalue of 0.1 causes the system to grow unbounded.

(c) $\dot{x}(t) = \begin{bmatrix} -0.5 & 10 & 0 \\ -10 & -0.5 & 0 \\ 0 & 0 & -0.01 \end{bmatrix} x(t)$

This matrix is block diagonal, so the eigenvalues can be calculated simply from the $\begin{bmatrix} -0.5 & 10 \\ -10 & -0.5 \end{bmatrix}$ portion of the matrix:

$$\begin{aligned} \det(A - \lambda I) &= \begin{bmatrix} -0.5-\lambda & 10 \\ -10 & -0.5-\lambda \end{bmatrix} \\ &= (-0.5-\lambda)(-0.5-\lambda) + 100 \\ \lambda^2 + \lambda + 100.25 &= 0 \\ \lambda &= -0.5 \pm 10i \end{aligned}$$

The last eigenvalue is -0.01, since it is on the remaining diagonal.
After calculating the eigenvalues, we can determine that the spectral abscissa is -0.01, which is less than 0. This is indicative of an asymptotically stable system because the largest real part of the eigenvalues is negative. As $t \to \infty$, the state will oscillate because of the complex eigenvalues until it eventually decays to 0.

Solution to Exercise 3

(a) Using MATLAB to compute eigenvalues (D) and eigenvectors (V):

```
V =

  -0.144699   0.144699  -0.797413  -0.797413
  -0.311782   0.311782   0.123360   0.123360
   0.395324   0.395324   0.583747  -0.583747
   0.851806   0.851806  -0.090306   0.090306

D =

Diagonal Matrix

  -2.7321         0         0         0
        0    2.7321         0         0
        0         0   -0.7321         0
        0         0         0    0.7321
```

The largest eigenvalue is $\lambda_2 = 2.7321$, so the spectral abscissa is 2.731 and is greater than 0. Since the spectral abscissa is greater than 0, the system is unstable.

(b)

$$x(t) = e^{-2.7321t}w_1 \begin{bmatrix} -0.1447 \\ -0.3118 \\ 0.3953 \\ 0.8518 \end{bmatrix} + e^{2.7321t}w_2 \begin{bmatrix} 0.1447 \\ 0.3118 \\ 0.3953 \\ 0.8518 \end{bmatrix} + e^{-0.7321t}w_3 \begin{bmatrix} -0.7974 \\ 0.1234 \\ 0.5837 \\ -0.0903 \end{bmatrix} + e^{0.7321t}w_4 \begin{bmatrix} -0.7974 \\ 0.1234 \\ -0.5837 \\ -0.0903 \end{bmatrix}$$

Looking at the solution to the system, the first term and the third term will eventually decay to 0 because the eigenvalues are negative. Whereas the second and final term cause $x(t)$ to grow unbounded as $t \to \infty$ because those eigenvalues are greater than 0. This means that the second and final term are not in the basis for all non-zero initial conditions x_0 for which the state $x(t)$ will remain bounded as $t \to \infty$. So the basis is: $\beta = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$. Where w_2 and w_4 are equal to 0.

Solution to Exercise 4

System investigation for dynamical system with transition matrix A_1:

$$\lambda_1 = 2i,\ \lambda_2 = -2i \text{ and } \lambda_3 = 0$$
$$\alpha(A_1) = 0$$

Therefore, this system is stable and oscillates indefinitely.
This must be Simulation picture B, since it shows a stable oscillating system and the other oscillation simulation is not stable rather asymptotically stable since that system decays to 0.

System investigation for dynamical system with transition matrix A_2:

$$\lambda_1 = -0.1\ ,\ \lambda_2 = -0.2, \text{ and } \lambda_3 = -1$$
$$\alpha(A_1) = -0.1 < 0$$

Therefore, this system is asymptotically stable and decays to zero as t approaches infinity. This system does not have oscillations since there are no complex eigenvalues. This simulation must be the one shown in simulation A since is shows a system decaying to 0 with no oscillations.

System investigation for dynamical system with transition matrix A_3:

$$\lambda_1 = -0.2\ ,\ \lambda_2 = -2, \text{ and } \lambda_3 = 0$$
$$\alpha(A_1) = 0$$

Therefore, this system is stable with no oscillations. This simulation is pictured in simulation picture D which shows the system reaching a non trivial steady state. This is what happens in homogeneous continuous-time dynamical systems when spectral absessa = 0 and eigenvalues are non-complex.

System investigation for dynamical system with transition matrix A_4:

$$\lambda_1 = -0.1 + 2i\ ,\ \lambda_2 = -0.1 - 2i, \text{ and } \lambda_3 = -1$$
$$\alpha(A_1) = -0.1 < 0$$

Therefore, this system is asymptotically stable with oscillations. This system is pictured in simulation C which shows a system with complex eigenvalues that is decaying to zero. This matches the behavior of the system with transition matrix A_4.

Solution to Exercise 5

(a)

$$e^{tA} = \sum_{j=0}^{\infty} \frac{t^j}{j!} A_j = I_n + \frac{t^2}{2!} A^2 + \frac{t^3}{3!} A^3 + ...$$

Differentiate both sides to understand how the system changes over time:

$$\begin{aligned}\frac{d}{dt}(e^{tA}) &= \frac{d}{dt}\,(\sum_{j=1}^{\infty} \frac{jt^{j-1}}{j!} A^j) \\ &= A \sum_{j=1}^{\infty} \frac{t^{j-1}}{(j-1)!} A^{j-1}\end{aligned}$$

$k = j - 1$:

$$\begin{aligned}\frac{d}{dt}(e^{tA}) &= A \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k \\ &= Ae^{tA}\end{aligned}$$

(b) $A = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}$

Find the eigenvalues and eigenvectors:

$$\begin{aligned}\det(A - \lambda I) &= \begin{bmatrix} -1-\lambda & -1 \\ -1 & -1-\lambda \end{bmatrix} \\ &= (-1-\lambda)(-1-\lambda) - 1 \\ \lambda^2 + 2\lambda &= 0 \\ \lambda(\lambda + 2) &= 0 \\ \lambda_{1,2} &= 0, -2\end{aligned}$$

$\lambda_1 = 0$:

$$\det(A - 0I) = \begin{bmatrix} -1-0 & -1 \\ -1 & -1-0 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}$$

Solution to Excercise 5 cont.

$$\begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$$
$$-x - y = 0$$
$$y = -x$$

Therefore, $v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Then for $\lambda_2 = -2$:

$$\begin{aligned} \det(A - (-2)I) &= \begin{bmatrix} -1+2 & -1 \\ -1 & -1+2 \end{bmatrix} \\ &= \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{aligned}$$
$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$$
$$x - y = 0$$
$$y = x$$

Therefore $v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Now set up $A = Ve^{t\Lambda}V^{-1}$:

$$A = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}$$
$$A = Ve^{t\Lambda}V^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} e^{0t} & 0 \\ 0 & e^{-2t} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}^{-1}$$
$$V^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}^{-1} = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

Now find the matrix exponential e^{tA}:

$$\begin{aligned} e^{tA} &= 1/2 \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} e^{0t} & 1 \\ -1 & e^{-2t} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \\ &= 1/2 \begin{bmatrix} 1 + e^{-2t} & 1 + e^{-2t} \\ e^{-2t} - 1 & 1 + e^{-2t} \end{bmatrix} \end{aligned}$$

(c)

$$\frac{d}{dt}(e^{tA}) = Ae^{tA}$$

Solution to Excercise 5 cont.

Differentiate each entry of the matrix exponential:

$$\frac{d}{dt}e^{tA} = 1/2 \begin{bmatrix} -2e^{-2t} & -2e^{-2t} \\ -2e^{-2t} & -2e^{-2t} \end{bmatrix} = \begin{bmatrix} -e^{-2t} & -e^{-2t} \\ -e^{-2t} & -e^{-2t} \end{bmatrix}$$

Now compute the other side of the equality:

$$\begin{aligned} Ae^{tA} &= \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} 1/2 \begin{bmatrix} 1+e^{-2t} & 1+e^{-2t} \\ e^{-2t}-1 & 1+e^{-2t} \end{bmatrix} \\ &= \begin{bmatrix} -1(\frac{1+e^{-2t}}{2}) + -1(\frac{e^{-2t}-1}{2}) & -1(\frac{1+e^{-2t}}{2}) + -1(\frac{1+e^{-2t}}{2}) \\ -1(\frac{1+e^{-2t}}{2})e^{-2t} + -1(\frac{e^{-2t}-1}{2}) & -1(\frac{1+e^{-2t}}{2}) + -1(\frac{1+e^{-2t}}{2}) \end{bmatrix} \\ &= \begin{bmatrix} -e^{-2t} & -e^{-2t} \\ -e^{-2t} & -e^{-2t} \end{bmatrix} \end{aligned}$$

Therefore, $\frac{d}{dt}e^{tA} = Ae^{tA}$