Homework Submission

CMDA 3605 Mathematical Modeling: Methods and Tools I

Virginia Tech Honor Code Pledge:

"I have neither given nor received unauthorized assistance on this assignment."

Solution to Exercise 1

(a) From MATLAB I got the eigenvalues $\lambda_i = 1, 0.7$, and 0.5 and the corresponding eigenvectors $v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}$, $v_3 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

By using a manipulation of the equation $X_0 = VW$ in MATLAB I found the following Weights corresponding to the eigenvectors: $\begin{bmatrix} 100,000 \\ -20,000 \\ 30,000 \end{bmatrix}$

Then the solution becomes...

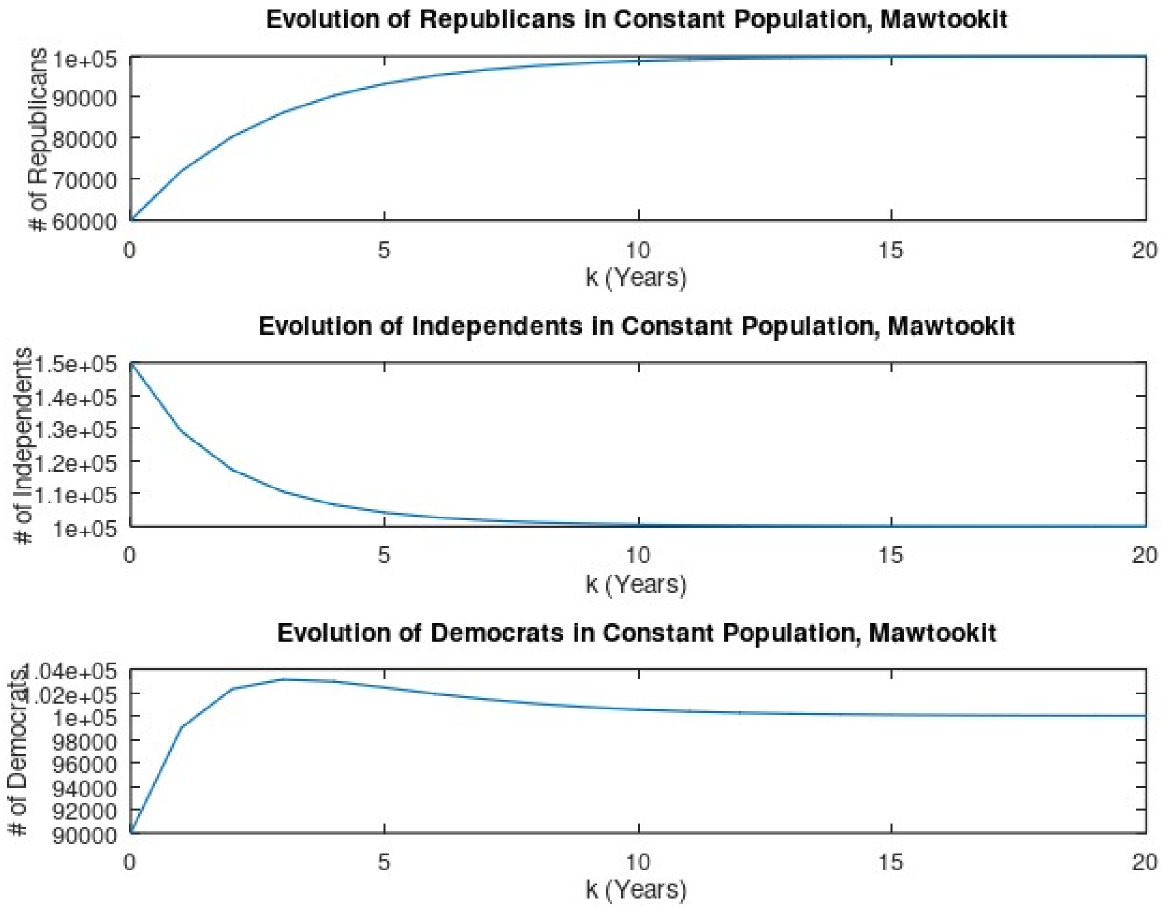
$$x(k) = (1)^k(100,000) \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + (0.7)^k(-20,000) \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix} + (0.5)^k(30,000) \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

As k approaches infinity the second two terms approach 0 since $\lambda_2, \lambda_3 < 1$ and the first term reaches a non-zero constant state $\begin{bmatrix} 100,000 \\ 100,000 \\ 100,000 \end{bmatrix}$ since $\lambda_d = 1$. With the initial condition $x_0 = \begin{bmatrix} 60,000 \\ 150,000 \\ 90,000 \end{bmatrix}$ and the assumption that the total population stays constant at $300,000$.

This means that in the long run there will be 100,000 Republicans, 100,000 Independents, and 100,000 Democrats in Mawtookit.

Solution to Exercise 1 Continued

(b)



Solution to Exercise 2

The dynamical systems representation of the Fibonacci sequence is:

$x(k+1) = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} x(k)$, with $x(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

The characteristic polynomial det $\begin{bmatrix} 0-\lambda & 1 \\ 1 & 1-\lambda \end{bmatrix}$ is $\lambda^2 - \lambda - 1$ since

$(0-\lambda)(1-\lambda) - 1 = \lambda^2 - \lambda - 1$.

Then using the quadratic formula we get:

$$\frac{1 \pm \sqrt{1 - 4(1)(-1)}}{2}$$

which is:

$$\frac{1 \pm \sqrt{5}}{2}$$

So $\lambda_1 \approx 1.618$ and $\lambda_2 \approx -0.618$
and the corresponding eigenvectors can be $v_1 = \begin{bmatrix} 1 \\ 1.618 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 1 \\ -0.618 \end{bmatrix}$
and $X_0 = VW$

$= \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1.618 \end{bmatrix} w_1 + \begin{bmatrix} 1 \\ -0.618 \end{bmatrix} w_2$

we know $w_2 = -w_1$, then
$1.618w_1 + (-0.618)(-w_1) = 1$
$1.618w_1 + 0.618w_1 = 1$
$2.236w_1 = 1$
$w_1 = 1/2.236$
So $w_1 = 1/2.236$ and $w_2 = -1/2.236$

Then the solution formula for the discrete-time system is:

$x(k) = (1.618)^k \begin{bmatrix} 1 \\ 1.618 \end{bmatrix} (\frac{1}{2.236}) + (-0.618)^k \begin{bmatrix} 1 \\ -0.618 \end{bmatrix} \frac{1}{-2.236}$

As k approaches infinity in this solution formula the second term goes to zero as $\lambda_2 < 1$ and the first term grows unbounded as $\lambda_1 > 1$. The rate at which x (k) grows as k increases is the dominant eigenvalue 1. The eigenvector associated with the dominant eigenvalue (v_1) shows the ratio of f(k) to f(k+1) which is $1 : 1.618$. This shows that the ratio in the long run will be the golden ratio.

Solution to Exercise 3

We were given the following discrete-time system

$$x(k+1) = Ax(k), \quad x(0) = x_0, \quad \text{where } A = \begin{bmatrix} 0.8 & 0.6 & 0 & 0 \\ -0.6 & 0.8 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1.3 \end{bmatrix}$$

(a) Since matrix A is a block diagonal matrix, it can be broken into Blocks $B = \begin{bmatrix} 0.8 & 0.6 \\ -0.6 & 0.8 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 0 \\ 0 & 1.3 \end{bmatrix}$ for finding the eigenvalues of a since A is block diagonal.
Finding eigenvalues for B:

$$\begin{aligned} \det \begin{bmatrix} 0.8-\lambda & 0.6 \\ -0.6 & 0.8-\lambda \end{bmatrix} &= (0.8-\lambda)(0.8-\lambda) + 0.36 \\ &= \lambda^2 - 1.6\lambda + 1 \end{aligned}$$

Using the quadratic formula to solve the polynomial gives us:

$$\frac{1.6 \pm \sqrt{(-1.6)^2 - 4}}{2}$$

$$= \frac{1.6 \pm \sqrt{1.44}(i)}{2} = \frac{1.6 \pm 1.2i}{2}$$
$$= 0.8 \pm 0.6i$$

So $\lambda_1 = 0.8 + 0.6i$ and $\lambda_2 = 0.8 - 0.6i$
and $\mid \lambda_1 \mid = \sqrt{0.8^2 + 0.6^2} = 1, \mid \lambda_2 \mid = 1$

The eigenvalues of block D are just $\lambda_3 = 1$ and $\lambda_4 = 1.3$.
$\mid \lambda_3 \mid = 1$ and $\mid \lambda_4 \mid = 1.3$, given that matrix D is a diagonal matrix.

This means that $S(A) = \lambda_4 = 1.3$ making the system Unstable.

(b) The state x(k) will be bounded as long as the absolute value of the dominant/state eigenvalue is less than or equal to one. The solution formula is shown below:

$$x(k) = (0.8 + 0.6i)^k w_1 v_1 + (0.8 - 0.6i)^k w_2 v_2 + (1)^k w_3 v_3 + (1.3)^k w_4 v_4$$

The basis of x_0 for which the above criteria is true is

$$\beta_{x_0} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

Solution to Exercise 3 Continued

(c) Since the eigenvalues in the first two terms of the solution formula are complex, those two components are known to oscillate. Additionally the absolute value of those eigenvalues are equal to one, so the growth rate is not changing as k approaches infinity. This means that the first two compenents oscillate indefinitely, never converging to a steady state. The fourth term would also never converge to a steady state since $\lambda_4 = 1.3 > 1$. The only initial conditions x_0 to lead to convergence to a steady state are those lying in the eigenspace of $\lambda = 1 = \lambda_3$.

This means that the basis x_0 for which the state $x(k)$ will converge to a steady state as k approaches infinity is:

$$\beta_{x_0} = \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

The oscilation can be seen easier by looking at the solution formula using polar form for the complex numbers and $x_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ for the initial value, with, $w_3 = 1$. However, the first two components of the solution formula will oscillate indefinitely unless w and the corresponding row in V is 0. This is possible when the corresponding x_0 row is 0.

Solution to Exercise 4

(a)

$$A = \begin{bmatrix} 0.4 & 0.3 \\ -p & 1.2 \end{bmatrix}$$

$p = 0.325$:

Finding the eigenvalues:

$$\begin{aligned}
\det(A - \lambda I) = \begin{bmatrix} 0.4 - \lambda & 0.3 \\ -0.325 & 1.2 - \lambda \end{bmatrix} &= (0.4 \text{ - } \lambda)(1.2 \text{ - } \lambda) + 0.0975 \\
&= 0.48 - 0.4\lambda - 1.2\lambda + \lambda^2 + 0.0975 \\
&= 0.48 - 1.6\lambda + \lambda^2 + 0.0975 \\
&= \lambda^2 - 1.6\lambda + 0.5775 \\
\lambda^2 - 1.6\lambda + 0.5775 &= 0 \\
&= \frac{1.6 \pm \sqrt{(-1.6)^2 - 4(1)(0.5775)}}{2} \\
&= \frac{1.6 \pm 0.5}{2} \\
(\lambda - 1.05)(\lambda - 0.55) &= 0 \\
\lambda_1 &\approx 1.05, \\
\lambda_2 &\approx 0.55
\end{aligned}$$

Calculating the absolute values of the eigenvalues:

$|\lambda_1| = 1.05$, $|\lambda_2| = 0.55$

Therefore, the dominant eigenvalue, λ_d, is λ_1.

The solution is given by:

$$x(k) = \sum_{i=1}^{2} \lambda_i^k w_i v_i \approx (1.05)^k w_1 v_1 + (0.55)^k w_2 v_2$$

The long term growth rate is 1.05 because the dominant eigen value is 1.05. Whereas the term with $\lambda = 0.55$ will approach 0 for $k \to \infty$.

Solution to Exercise 4 Continued

The eventual ratio of owls to squirrels is characterized by the eigenvector for the dominant eigenvalue, 1.05:

$\lambda_1 \approx 1.05$:

$$\det(A - 1.05I) = \begin{bmatrix} 0.4 - 1.05 & 0.3 \\ -0.325 & 1.2 - 1.05 \end{bmatrix} = \begin{bmatrix} -0.65 & 0.3 \\ -0.325 & 0.15 \end{bmatrix}$$

$$\begin{aligned} -0.65x + 0.3y &= 0 \\ -0.65x &= 0.3y \\ y &= \tfrac{13}{6}\,x \end{aligned}$$

The eigenvector corresponding to $\lambda_1 = 1.05$ is $v_1 = \begin{bmatrix} 1 \\ 13/6 \end{bmatrix}$. This means the ratio of owls to squirrels is 1: 2.17.

(b)

$p = 0.5$:

$$\begin{aligned} A &= \begin{bmatrix} 0.4 & 0.3 \\ 0.5 & 1.2 \end{bmatrix} \\ \mathrm{d}et(\text{A - 1.05 I}) &= \begin{bmatrix} 0.4 - \lambda & 0.3 \\ 0.5 & 1.2 - \lambda \end{bmatrix} \\ &= (0.4 - \lambda)(1.2 - \lambda) + 0.15 \\ &= 0.48 - 1.6\lambda + \lambda^2 + 0.15 \\ \lambda^2 - 1.6\lambda + 0.63 &= 0 \\ \\ &= \frac{1.6 \pm \sqrt{(-1.6)^2 - 4(1)(0.63)}}{2} \\ &= \frac{1.6 \pm 0.2}{2} \\ (\lambda - 0.9)(\lambda - 0.7) &= 0 \\ \lambda_1 &\approx 0.9, \\ \lambda_2 &\approx 0.7 \end{aligned}$$

Calculating the absolute values of the eigenvalues:

$|\lambda_1| \approx 0.9$, $|\lambda_2| \approx 0.7$

Therefore, the dominant eigenvalue, λ_d, is λ_1.

The solution is given by:

Solution to Exercise 4 Continued

$$x(k) = \sum_{i=1}^{2} \lambda_i^k w_i v_i \approx (0.9)^k w_1 v_1 + (0.7)^k w_2 v_2$$

Since both of the absolute values of the eigenvalues are less than 1, this means that $x(k) \to 0$ for $k \to \infty$. . This indicates both the owls and squirrels will eventually perish.

Solution to Exercise 4 Continued

(c)

In order for the population of both owls and squirrels to tend toward constant levels, λ must equal 1:

$$\det(\text{A - }\lambda I) = \begin{bmatrix} 0.4-\lambda & 0.3 \\ -p & 1.2-\lambda \end{bmatrix} = (0.4-\lambda)(1.2-\lambda) + 0.3p$$
$$= 0.48\text{ - }0.4\lambda - 1.2\lambda + \lambda^2 + 0.3p$$

Since λ must equal 1:

$$\begin{aligned} 0.48 - 0.4(1) - 1.2(1) + (1)^2 + 0.3p &= 0 \\ -0.12 + 0.3p &= 0 \\ p &= 0.4 \end{aligned}$$

Substituting in $p = 0.4$:

$$\begin{aligned} 0.48 - 1.6\lambda + \lambda^2 + 0.12 &= 0 \\ \lambda^2 - 1.6 + 0.60 &= 0 \\ (\lambda - 1)(\lambda - 0.6) &= 0 \\ \lambda_1 &\approx 1, \\ \lambda_2 &\approx 0.6 \end{aligned}$$

Therefore, the dominant eigenvalue, λ_d, is $\lambda_1 = 1$.

The corresponding eigenvector for λ_1 is:

$$\begin{aligned} det(A - 1I) &= \begin{bmatrix} 0.4-1 & 0.3 \\ 0.4 & 1.2-1 \end{bmatrix} = \begin{bmatrix} -0.6 & 0.3 \\ 0.4 & 0.2 \end{bmatrix} \\ -0.6x + 0.3y &= 0 \\ y &= 2x \\ v_1 &= \begin{bmatrix} 1 \\ 2 \end{bmatrix} \end{aligned}$$

The relative population size of owls is 1 and for squirrels it is 2.

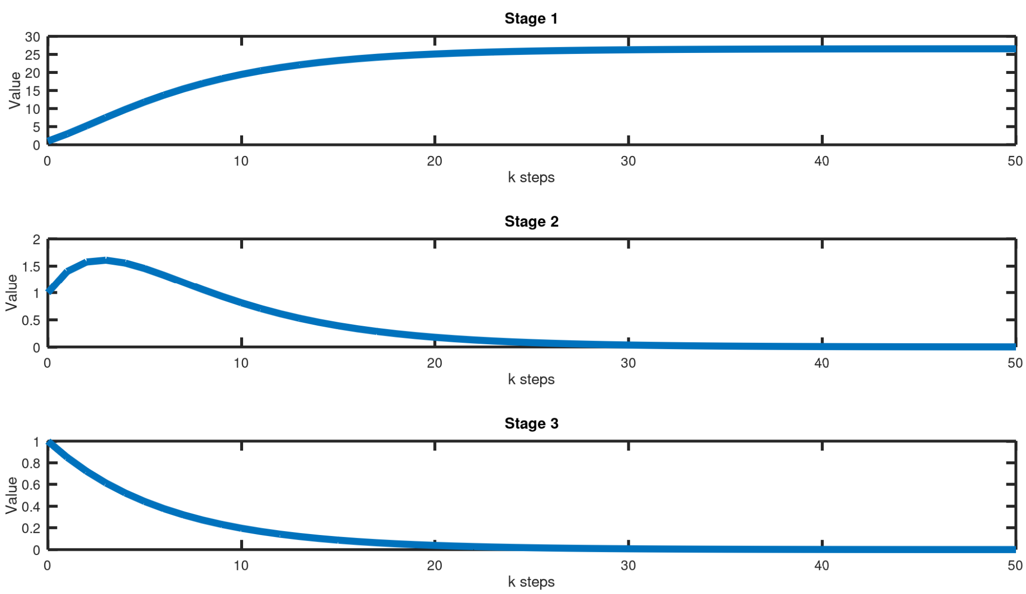
Solution to Exercise 5

(a)

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0.7 & 0.7 \\ 0 & 0 & 0.85 \end{bmatrix}$$

Since A is an upper triangular matrix, the eigenvalues are the diagonal entries: $\lambda_1 = 1, \lambda_2 = 0.7$, and $\lambda_3 = 0.85$. The dynamical system is stable because the absolute value of the dominant eigenvalue, $\lambda_1 = 1$, is equal to 1, which means $S(A) = 1$.

(b)



For stage 1, $x(k) \to 25$ as $k \to \infty$. Since the absolute values of the eigenvalues presented in stage 2 and 3 are < 1, $x(k) \to 0$ as $k \to \infty$. Stage 1 reaches a steady state since the eigenvalue does equal 1 and there are no oscillations beforehand. I expected this because the eigenvalues are not complex and there is only one eigenvalue equal to one while the rest of the components in the solution formula have eigenvalues < 1, causing them to decay. The eigenvectorsfrom MATLAB are:

$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} -.9578 \\ .2873 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} -.9921 \\ .1226 \\ 0.0266 \end{bmatrix}$ Stage one only has the growing component of the solution formula as seen on the graph. Stage two has the growing component and the decaying at rate 0.7 hence the initial growth then decay. The third component of the solution has a negative growing component and both the decaying components of the solution formula hence the decaying in the graph.

Solution to Exercise 5 Continued

(c)

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0.65 & 0.7 \\ 0 & -0.7 & 0.65 \end{bmatrix}$$

$$det(A - \lambda I) = \begin{bmatrix} 1-\lambda & 1 & 1 \\ 0 & 0.65-\lambda & 0.7 \\ 0 & 0.7 & 0.65-\lambda \end{bmatrix}$$

$$= (1-\lambda)\begin{bmatrix} 0.65-\lambda & 0.7 \\ -0.7 & 0.65-\lambda \end{bmatrix}$$

$$\begin{aligned} &= (1-\lambda)((0.65-\lambda)(0.65-\lambda)+0.49) \\ (1-\lambda)((0.65-\lambda)^2+0.49)) &= 0 \\ \lambda_1 &= 1 \end{aligned}$$

Solving for λ_2:

$$\begin{aligned} (0.65-\lambda)^2 + 0.49 &= 0 \\ (0.65-\lambda)^2 &= -0.49 \\ \sqrt{(0.65-\lambda)^2} &= \sqrt{-0.49} \\ 0.65-\lambda &= 0.7i \\ \lambda_{2,3} &= 0.65 \pm 0.7i \end{aligned}$$

Taking the absolute value of the eigenvalues:

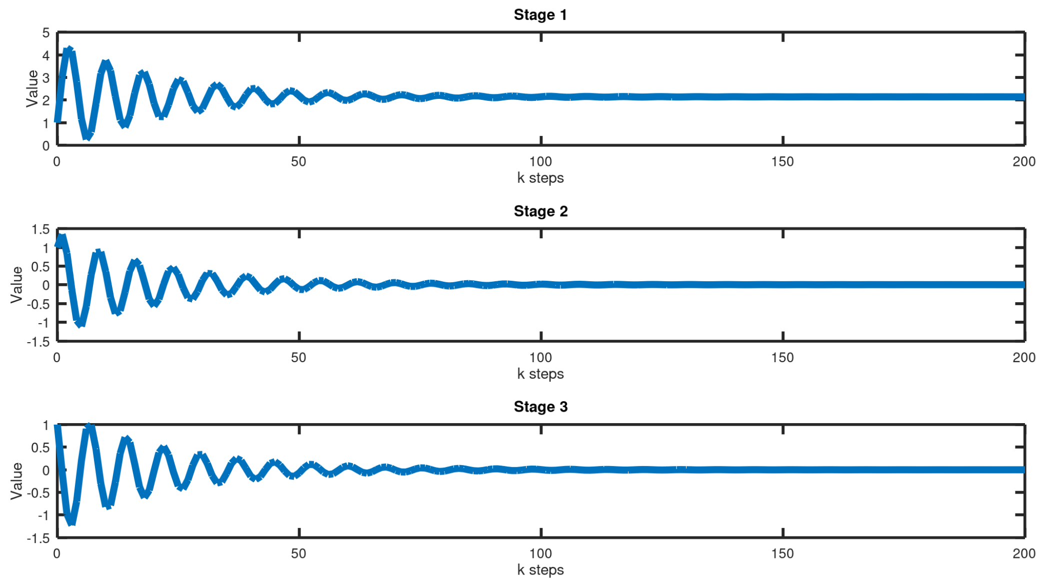
$$|\lambda_1| = S(A) = 1$$

$$\begin{aligned} |\lambda_{2,3}| &= \sqrt{(0.65)^2 + (0.7)^2} \\ |\lambda_{2,3}| &= 0.9952 \end{aligned}$$

Since $0.9952 < 1$, the dominant eigenvalue is 1. Since $S(A) = 1$, the system is stable.

Solution to Exercise 5 Continued

(d)



Since two of the eigenvalues of the system are complex numbers, the graph demonstrates oscillation. Whereas in the previous system presented in part(a), there are no oscillations because there are no complex eigenvalues. The system reaches a steady state at around the value of 2. This was expected since the absolute values of the complex eigenvalues are less than one, so the system will oscillate until the components in the solution formula with λ_2 and λ_3 eventually die out.