Homework Submission

CMDA 3605 Mathematical Modeling: Methods and Tools I

Virginia Tech Honor Code Pledge:

"I have neither given nor received unauthorized assistance on this assignment."

Solution to Exercise 1

(a)

$$x(k+1) = f(x(k)) = Ax(k) + b$$

Taking the Jacobian of the evolution function $f(x)$ yields:

$$J_f(x) = A$$

This is because taking the partial derivative of the entire function simply leaves A since b is just a constant.

(b) In order to show that the $\det(J_g(r, \theta))$ is equal to r, compute the Jacobian given that the first column is the partial derivative with respect to r and the second column is the partial derivative with respect to θ:

$$(J_g(r, \theta)) = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

Now compute the determinant $(\det(J_g(r, \theta)))$ of the Jacobian:

$$\begin{aligned} det(J_g(r, \theta)) &= \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix} \\ &= (\cos\theta)(r\cos\theta) + r\sin^2\theta \\ &= r\cos^2\theta + r\sin^2\theta \\ &= r(\cos^2\theta + \sin^2\theta) \\ &= r(1) \end{aligned}$$

Therefore, $det(J_g(r, \theta)) = r$

Solution to Exercise 2

Part (a)
Considering the nonlinear discrete dynamical system:

$$x(k+1) = ax(k)e^{-rx(k)}$$

Where $a = e$ and $r = 1$. Then:

$$f(x) = exe^{-x} = e^{1-x}x$$

Before writing the linearization, we will need $f'(x)$

$$f'(x) = -e^{1-x}x + e^{1-x}$$

Now looking at the linearization of the first steady state, $\hat{x}_1 = 0$.

$$u(k+1) = f'(\hat{x}_1)u(k)$$

where $u(k)$ is $(x(k) - \hat{x}_1)$.
Now to determine the stability of $\hat{x}_1$ we look at $f'(\hat{x}_1)$

$$f'(\hat{x}_1) = -e^{1-0}(0) + e^1 \approx 2.7$$

Since $|f'(\hat{x}_1| > 1$ the steady state $\hat{x}_1 = 0$ is an unstable steady state and we expect x(k) to drift away from $\hat{x}_1$ for $x_0 \neq \hat{x}_1$
Now looking at the linearization of the second steady state, $\hat{x}_2 = 1$.

$$u(k+1) = f'(\hat{x}_2)u(k)$$

where $u(k)$ is $(x(k) - \hat{x}_2)$.
Now to determine the stability of $\hat{x}_2$ we look at $f'(\hat{x}_2)$

$$f'(\hat{x}_1) = -e^{1-1}(1) + e^{1-1} = -1 + 1 = 0$$

Since $|f'(\hat{x}_2| < 1$ the steady state $\hat{x}_2 = 1$ is an asymptotically stable and we expect x(k) to approach $\hat{x}_2$ for $||x_0 - \hat{x}_2||$ small enough.

Solution to Exercise 2 cont.

Part b

Considering the nonlinear dynamical system

$$f(x(k)) = \begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} \frac{ax_1(k)x_2(k)}{1+x_1(k)} \\ \frac{bx_1(k)x_2(k)}{1+x_2(k)} \end{bmatrix}$$

where $a = 1$ and $b = 2$ Then:

$$f(x) = f(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}) = \begin{bmatrix} \dfrac{x_1x_2}{1+x_1} \\ \\ \dfrac{2x_1x_2}{1+x_2} \end{bmatrix}$$

First we will need the Jacobian of F. Using the Quotient rule we get:

$$J(x) = \begin{bmatrix} \dfrac{x_2(1+x_1) - (x_1x_2)}{(1+x_1)^2} & \dfrac{x_1}{1+x_1} \\ \dfrac{2x_2}{1+x_2} & \dfrac{2x_1(1+x_2) - 2x_1x_2(1)}{(1+x_2)^2} \end{bmatrix}$$

simplified:

$$J(x) = \begin{bmatrix} \dfrac{x_2}{(1+x_1)^2} & \dfrac{x_1}{1+x_1} \\ \dfrac{2x_2}{1+x_2} & \dfrac{2x_1}{(1+x_2)^2} \end{bmatrix}$$

Now looking at the linearization of the first steady state, $\hat{x}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$:

$$u(k+1) = J(\hat{x}_1)u(k)$$

Now to determine the stability of $\hat{x}_1$ we look at $J(\hat{x}_1)$

$$J(\begin{bmatrix} 0 \\ 0 \end{bmatrix}) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Since $S(J(\hat{x}_1)) = 0 < 1$ the steady state $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is an asymptotically stable steady state and we expect x(k) to approach $\hat{x}_1$ for $||x_0 - \hat{x}_2||$ small enough.

Solution to Exercise 2 cont.

Now looking at the linearization of the second steady state, $\hat{x}_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$:

$$u(k+1) = J(\hat{x}_2)u(k)$$

Now to determine the stability of $\hat{x}_2$ we look at $J(\hat{x}_2)$

$$J(\begin{bmatrix} 3 \\ 3 \end{bmatrix}) = \begin{bmatrix} 3/9 & 2/3 \\ 6/4 & 4/16 \end{bmatrix}$$

Then we must find the eigenvalues to determine the spectral radius of the Jacobian at steady state two:

$$(1/3 - \lambda)(1/4 - \lambda) - 1 = \lambda^2 - 7/12\lambda - 11/12$$

Using the Pq formula to find roots of characteristic polynomial:

$$x_1 = 7/24 - \sqrt{(-7/14)^2 + 11/12}$$

$$= 7/24 - 1.0086767912 = -0.7092$$

$$x_2 = 7/24 + 1.0086767912 = 1.292534$$

So th eigenvalues of $J(\hat{x}_2)$ are $\lambda_1 = -0.7092$ and $\lambda_2 = 1.2925$

Since $S(J(\hat{x}_2)) = 1.2925 > 1$ the steady state $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ is an unstable stable steady state and we expect x(k) to drift away from $\hat{x}_2$ for $x_0 \neq \hat{x}_2$

Solution to Exercise 3

$$x(k+1) = f(x(k)) = x(k) - \frac{g(x(k))}{g'(x(k))}$$

The above equation can be written as:

$$f(\hat{x}) = \hat{x} - \frac{g(\hat{x})}{g'(\hat{x})}$$

Now compute the derivative, and utilize the quotient rule on the second term:

$$\begin{aligned} f'(x) &= 1 - \frac{d}{dx}(\frac{g(x)}{g'(x)}) \\ &= 1 - \frac{g'(x)g'(x) - g''(x)g(x)}{(g'(x))^2} \\ &= 1 - \frac{g'(x)^2 - g''(x)g(x)}{(g(x))^2} \\ &= \frac{g(x)g''(x)}{(g''(x))^2} \end{aligned}$$

Now evaluating the equation at $hatx$, given that $g(\hat{x}) = 0$, we can substitute this back into the above equation:

$$\begin{aligned} &= \frac{g(\hat{x})g''(\hat{x})}{(g''(\hat{x}))^2} \\ &= \frac{(0)g''(x)}{(g''(x))^2} \\ &= 0 \end{aligned}$$

Since $|f'(x)| = 0 < 1$, we can conclude that $\hat{x}$ is locally asymptotically stable and indicates that Newton's Method is locally convergent.

Solution to Exercise 4

(a)

```
function [xr, res] = newtonmethod(g, J, x0, maxiter, tol)
x = x0;
res = zeros(maxiter, 1);

for k = 1:maxiter
  gx = g(x);

  res(k) = norm(gx, 2);
  fprintf('Iter %d: residual = %.3e\n', k, res(k));

  if res(k) <tol
    res = res(1:k);
    xr = x;
    return;
  endif



    d = -J(x) \ gx;

    x = x + d;
  end
  xr = x;
  res = res(1:maxiter);
end
```

(b) Using the above Newton's method code with the function:

$$g(x) = \begin{bmatrix} x_1^2 + x_1 x_2^3 - 9 \\ 3x_1^2 x_2 - x_2^3 - 4 \end{bmatrix}$$

where

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

, the root $\hat{x}$ is equal to:

```
x_hat =
   1.3364
   1.7542
```

(c) Using the initial value:

$$x_{0,1} = \begin{bmatrix} -3 \\ -3 \end{bmatrix}$$

Solution to Exercise 4 cont.

The root $\hat{x}$ is equal to:

```
Iter 1: residual = 9.962e+01
Iter 2: residual = 2.807e+01
Iter 3: residual = 6.618e+00
Iter 4: residual = 8.152e-01
Iter 5: residual = 1.180e-02
Iter 6: residual = 3.596e-06
Iter 7: residual = 2.251e-13
 x hat =
  -0.9013
  -2.0866
```

Using the next initial value:

$$x_{0,2} = \begin{bmatrix} -3 \\ 3 \end{bmatrix}$$

The root $\hat{x}$ is equal to:

```
Iter 1: residual = 9.519e+01
Iter 2: residual = 3.117e+01
Iter 3: residual = 1.190e+01
Iter 4: residual = 7.629e+00
Iter 5: residual = 2.735e+01
Iter 6: residual = 7.035e+00
Iter 7: residual = 7.848e-01
Iter 8: residual = 4.407e-03
Iter 9: residual = 4.195e-07
Iter 10: residual = 0.000e+00
 x hat =
  -3.0016
   0.1481
```

Lastly, using the third initial value:

$$x_{0,2} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

Solution to Exercise 4 cont.

The root $\hat{x}$ is equal to:

```
Iter 1: residual = 8.062e+00
Iter 2: residual = 1.738e+00
Iter 3: residual = 1.343e-01
Iter 4: residual = 3.150e-04
Iter 5: residual = 1.826e-09
Iter 6: residual = 2.220e-15
 x hat =
   2.9984
   0.1484
```

Observations: For the first initial value $x_{0,1} = \begin{bmatrix} -3 \\ -3 \end{bmatrix}$, the method converged after 7 iterations according to the output of the code. The residuals appear to converge rapidly as the first iteration starts at $\approx 9.96 * 10^1$ and ends at the last iteration with a value much lower of $\approx 2.25 * 10^{-13}$. For the initial value $x_{0,1} = \begin{bmatrix} -3 \\ -3 \end{bmatrix}$, newton's method converged to the root $\hat{x} = \begin{bmatrix} -0.9013 \\ -2.0866 \end{bmatrix}$.

For the second initial value $x_{0,2} = \begin{bmatrix} -3 \\ 3 \end{bmatrix}$, the residuals decreased rapidly all the way to 0 after 10 iterations. Since the residuals approached 0, this indicates that the method was able to successfully calculate a solution where $g(x) = 0$. For the initial value $x_{0,1} = \begin{bmatrix} -3 \\ 3 \end{bmatrix}$, newton's method converged to the root $\hat{x} = \begin{bmatrix} -3.0016 \\ 0.1481 \end{bmatrix}$.

For the last initial value $x_{0,3} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$, relative to the other two initial values the residuals decreased more rapidly as it only experienced 6 iterations. This indicates that the initial guess was close to the true root, since the iterations converged faster than the other initial values. For the initial value $x_{0,3} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$, newton's method converged to the root $\hat{x} = \begin{bmatrix} 2.9984 \\ 0.1484 \end{bmatrix}$.

Solution to Exercise 5

Part a
Define the nonlinear function

$$f_1(x) = x - g(x) = x - x^3 - 4x^2 + 10$$

with

$$g(x) = x^3 + 4x^2 - 10$$

To verify that $\hat{x}$ is a fixed point of f_1, then $g(\hat{x}) = 0$ we need, that if $\hat{x}$ is a fixed point of f_1, then $f_1(\hat{x}) = \hat{x}$.
Then

$$f_1(\hat{x}) = \hat{x} - g(\hat{x}) = \hat{x}$$

Therefore, $g(\hat{x}) = 0$ must be true.

After running the fixed point iteration, this is output vector shows F1 as $x(k+1) = f_1(x(k))$. $x(0) = 1.5$

```
 Columns 1 through 11:

   1.5000  -0.8750   6.7324  -469.7200  102754555.1874  -1084933870531746352594944.0000  127
3780742545798613145501832505359094183152654933305709884866656.0000  -208271290858102499745717
03354901622173836675970995448130575801767295649567106614767785401768557086338743510717499427
374341691407014825950224181484247938321688218219358704615541966960319844958863360.0000
aN      NaN

 Columns 12 through 22:

     NaN       NaN       NaN       NaN       NaN       NaN       NaN       NaN       NaN       NaN

 Columns 23 through 33:

     NaN       NaN       NaN       NaN       NaN       NaN       NaN       NaN       NaN       NaN
```

This shows that the iteration is not converging, but is diverging. Now consider the linearization of f1 on $\hat{x} \approx 1.365230013414097$

$$u(k+1) = f_1'(\hat{x})u(k)$$

$$u(k) = (f_1(k) - \hat{x})$$

and

$$f_1'(x) = 1 - 3x^2 - 8x$$

now check the stability

$$f_1'(\hat{x}) = 1 - 3(1.36523)^2 - 8(1.36523) \approx -15.513$$

Since $|f_1'(\hat{x})| > 1$ the linearization of f1 on the given root is unstable, and the iteration does not converge with $x(0) = 1.5$

Solution to Exercise 5 cont.

Part b

After running the fixed point iteration, this is output vector shows F2 as $x(k+1) = f_2(x(k))$. $x(0) = 1.5$

```
Columns 1 through 11:

  1.5000   1.2870   1.4025   1.3455   1.3752   1.3601   1.3678   1.3639   1.3659   1.3649   1.3654

Columns 12 through 22:

  1.3651   1.3653   1.3652   1.3652   1.3652   1.3652   1.3652   1.3652   1.3652   1.3652   1.3652

Columns 23 through 33:

  1.3652   1.3652   1.3652   1.3652   1.3652   1.3652   1.3652   1.3652   1.3652   1.3652   1.3652

Columns 34 through 44:

  1.3652   1.3652   1.3652   1.3652   1.3652   1.3652   1.3652   1.3652   1.3652   1.3652   1.3652

Columns 45 through 51:

  1.3652   1.3652   1.3652   1.3652   1.3652   1.3652   1.3652

>> |
```

This shows that the iteration does converge to the desired root.

Now consider the linearization of f2 on $\hat{x} \approx 1.365230013414097$

$$u(k+1) = f_2'(\hat{x})u(k)$$

$$u(k) = (f_2(k) - \hat{x})$$

Solving for f_2'

$$f_2(x) = \frac{1}{2}(10 - x^3)^{\frac{1}{2}}$$

let $z(k) = 10 - x^3$

and $z'(k) = -3x^2$

$$f_2(x) = \frac{1}{2}u(k)^{\frac{1}{2}}$$

$$f_2'(x) = \frac{1}{2} \times \frac{1}{2}u(k)^{\frac{-1}{2}}u'(k)$$

$$f_2'(x) = (1/2)(1/2)(10 - x^3)^{-1/2}(-3x)^2$$

$$f_2'(x) = -\frac{3x^2}{4(10 - x^3)^{1/2}}$$

now check the stability

$$f_2'(\hat{x}) = -\frac{3(1.3652)^2}{4(10 - (1.3652)^3)^{1/2}} \approx -0.51196125503$$

Since $|f_2'(\hat{x})| < 1$ the linearization of the fixed point iteration on f_2 is asymptotically stable. $u(k) \to 0$ and $f_2(k) \to \hat{x}$

Solution to Exercise 5 cont.

Part c
After running the fixed point iteration, this is output vector shows F3 as $x(k+1) = f_3(x(k))$. $x(0) = 1.5$

```
Columns 1 through 11:

  1.5000   1.3733   1.3653   1.3652   1.3652   1.3652   1.3652   1.3652   1.3652   1.3652   1.3652

Columns 12 through 22:

  1.3652   1.3652   1.3652   1.3652   1.3652   1.3652   1.3652   1.3652   1.3652   1.3652   1.3652

Columns 23 through 33:

  1.3652   1.3652   1.3652   1.3652   1.3652   1.3652   1.3652   1.3652   1.3652   1.3652   1.3652

Columns 34 through 44:

  1.3652   1.3652   1.3652   1.3652   1.3652   1.3652   1.3652   1.3652   1.3652   1.3652   1.3652

Columns 45 through 51:

  1.3652   1.3652   1.3652   1.3652   1.3652   1.3652   1.3652
```

This shows that the iteration does converge to the desired root even faster than the previous iteration.

Now consider the linearization of f3 on $\hat{x} \approx 1.365230013414097$

$$u(k+1) = f_3'(\hat{x})u(k)$$

$$u(k) = (f_3(k) - \hat{x})$$

Using the quotient rule we get:

$$f_3'(x) = 1 - \frac{(3x^2+8x)(3x^2+8x) - (x^3+4x^2-10)(6x+8)}{(3x^2+8x)^2}$$

Now we check the stability

$$f_3'(x) = -0\hat{.}0000294$$

Since $|f_3'(\hat{x})| < 1$ the linearization is asymptomatically stable.$u(k) \to 0$ and $f_2(k) \to \hat{x}$