Homework Submission

CMDA 3605 Mathematical Modeling: Methods and Tools I

<u>**Virginia Tech Honor Code Pledge:**</u>

"I have neither given nor received unauthorized assistance on this assignment."

Solution to Exercise 1

(a) $A = \begin{bmatrix} 0.60 & 0.10 & 0.15 \\ 0.15 & 0.90 & 0.15 \\ 0.25 & 0.00 & 0.70 \end{bmatrix}$

(b) A is a stochastic matrix because the columns are probability vectors. From a class theorem, if A is a stochastic matrix, then $\lambda = 1$ is an eigenvalue of A, and all the eigenvalues of A have an absolute value ≤ 1

Additionally, a is a stochastic matric $A \ \epsilon \ IR^{n \times n}$ with the property that A^k is positive for some k is called a regular stochastic matrix.

checking for A^2 :

$$A^2 = \begin{bmatrix} 0.60 & 0.10 & 0.15 \\ 0.15 & 0.90 & 0.15 \\ 0.25 & 0.00 & 0.70 \end{bmatrix} \begin{bmatrix} 0.60 & 0.10 & 0.15 \\ 0.15 & 0.90 & 0.15 \\ 0.25 & 0.00 & 0.70 \end{bmatrix} = \begin{bmatrix} + & + & + \\ + & + & + \\ + & + & + \end{bmatrix}$$

So A is a regular stochastic matrix. From a class theorem, if A is a regular stochastic matrix then $\lambda = 1$ is the simple dominant eigenvalue of A. Moreover, the Markov Chain converges to a non-zero steady state characterized by v corresponding to λ_d

(c) From MATLAB: $\lambda_1 = 1, \ \lambda_2 = 0.75, \ \lambda_3 = 0.45$ and $v_1 = \begin{bmatrix} 1.2 \\ 3.3 \\ 1 \end{bmatrix}, \ v_2 = \begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix}, \ v_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

Becuase A is regularly stochastic with a simple dominant eigenvalue of 1, the Markov Chain in the long run will behave like:

$$x(k) \approx (1)^k \begin{bmatrix} 1.2 \\ 3.3 \\ 1 \end{bmatrix} w_1$$

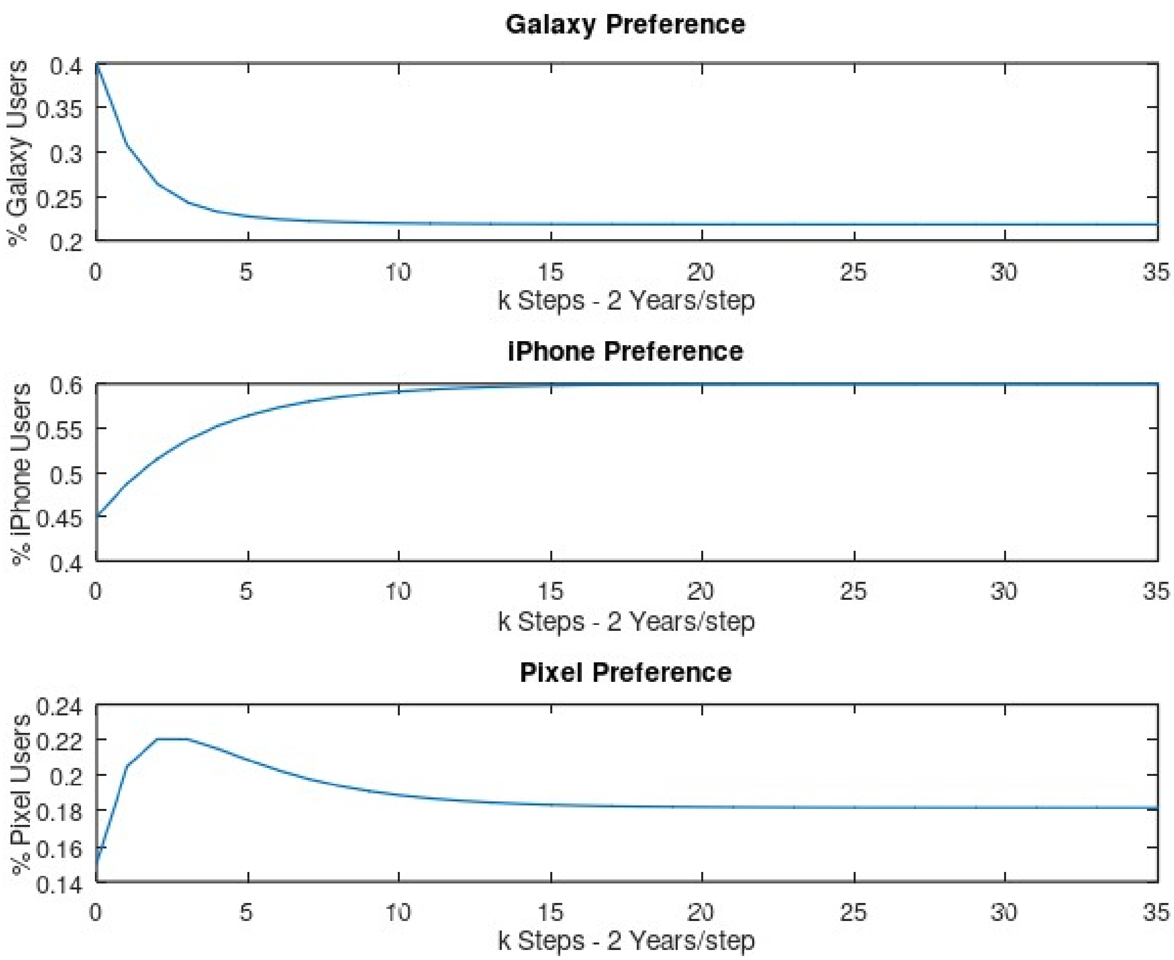
If you normaliz the steady state vector in the long run you get:

$$v_1 = \begin{bmatrix} .218181818 \\ .6 \\ .181818182 \end{bmatrix}$$

So as k approaches infinity, approximately 21.8% of users will prefer Galaxy, 60% prefer iPhone, and 18.1% prefer pixel.

Solution to Exercise 1 continued

(d)



Solution to Exercise 2

(a) The solution to the Markov Chain $x(k) = A^k x(0)$ will look like:

$x(k) \approx (1)^k v_1 w_1$
Since we are given that $\lambda_1 = 1$ and $\mid \lambda_j \mid < 1$.

Since A is doubly stoachastic and the given information, A an eigenvalue $\lambda_1 = 1$, with the eigenvector $v_1 = \begin{bmatrix} 1 \\ 1 \\ \cdot \\ \cdot \\ n \end{bmatrix}$.

Normalizing this vector to make it a probability vector, following the Markov Chain rules gives:

$$\mid\mid v_1 \mid\mid = \begin{bmatrix} 1/n \\ \cdot \\ \cdot \\ 1/n \end{bmatrix}$$

Because all the other eigenvalues satisfy $\mid \lambda_j \mid < 1$, their contributions vanish as k approaches infinity.
Therefore, for any x_0,

$$\lim_{k \to \infty} x(k) = (1) \begin{bmatrix} 1/n \\ \cdot \\ \cdot \\ 1/n \end{bmatrix}$$

With the State vector $\hat{x} = \frac{1}{n}\mathbf{1}_n$

Solution to Exercise 2 Continued

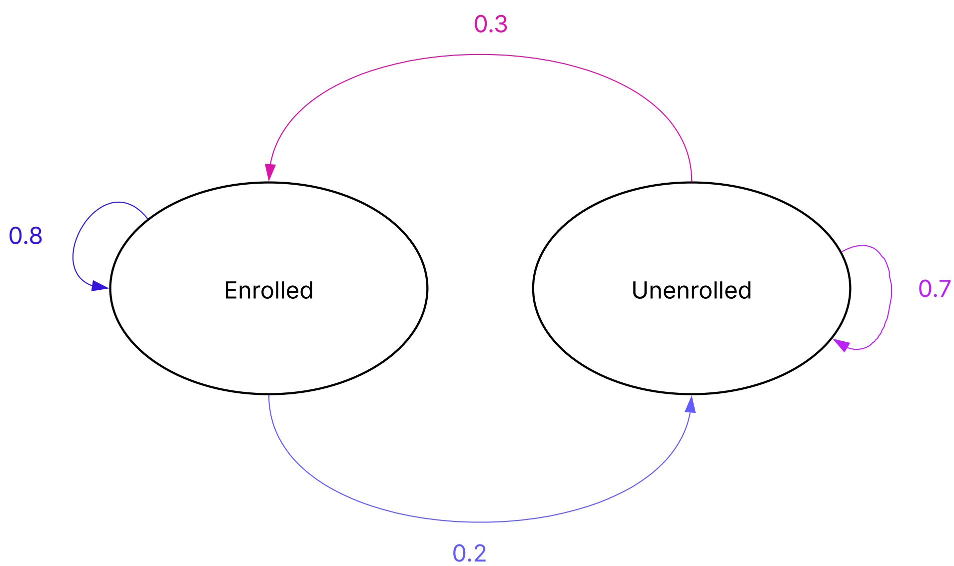
(b)

```
System with x01

x(1) for x01 =
   0.2340
   0.2790
   0.3810
   0.1060
x(2) for x01 =
   0.2584
   0.2641
   0.3139
   0.1636
The steady state system converges to =
   0.2500
   0.2500
   0.2500
   0.2500

System with x02

x(1) for x02 =
   0.1000
   0.1000
   0.1000
   0.7000
x(2) for x02 =
   0.1600
   0.1600
   0.1600
   0.5200
The steady state system converges to =
   0.2500
   0.2500
   0.2500
   0.2500
```

```
System with x03

x(1) for x03 =
   0.2500
   0.2500
   0.2500
   0.2500
x(2) for x03 =
   0.2500
   0.2500
   0.2500
   0.2500
The steady state system converges to =
   0.2500
   0.2500
   0.2500
   0.2500
```

The initial value $x_{0,3}$ converged the fastest. It converged immediately, as the initial value was already at the steady state that the Markov chain converges to. Meanwhile, for the first initial value the system does not converge till step 27 and the second at step 20.

Solution to Exercise 3



Let $x_E(k)$ represent the number of enrolled employees in the plan and $x_U(k)$ represent the number of unenrolled employees. We can define the state vector:

$$x(k) = \begin{bmatrix} x_E(k) \\ x_U(k) \end{bmatrix}$$

Based on the diagram above, define the system with the transition matrix:

$$x(k+1) = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix} x(k)$$

Find the non-zero steady-state vector:

$$\begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix} \begin{bmatrix} x_E(k) \\ x_U(k) \end{bmatrix} = \begin{bmatrix} x_E(k) \\ x_U(k) \end{bmatrix}$$

$$= \begin{bmatrix} 0.8x_E(k) + 0.3x_U(k) \\ 0.2x_E(k) + 0.7xU(k) \end{bmatrix} = \begin{bmatrix} 0.8x_E(k) \\ 0.2x_U(k) \end{bmatrix}$$

$$= \begin{bmatrix} x_E(k) \\ x_U(k) \end{bmatrix}$$

$$-0.2x_E(k) + 0.3x_U(k) = 0$$
$$0.2x_E(k) + 0.3x_U(k) = 0$$
$$x_E(k) = \frac{0.3}{0.2} x_U(k)$$

$$x_U = 0.4$$

Given that $x_E(k) + x_U(k) = 1$, $x_E(k) = 0.6$ since $x_U(k) = 0.4$.

Then the solution:

$$x(k) = \sum_{i=1}^{2} \lambda_i^k w_i v_i \approx (1)^k w_1 \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix}$$

The term with the dominant eigenvalue, $\lambda = 1$, has the nonzero steady state vector that indicates in the longrun, 60% of employees will be enrolled in the program while 40% will not.

Solution to Exercise 4

(a) If $A \in \mathbb{R}^{nxn}$ is a stochastic matrix, then $1_n^T A = 1_n^T$. If $A^k \in \mathbb{R}^{nxn}$ is a stochastic matrix then $1_n^T A = 1_n^T$.

Assuming $1_n^T A = 1_n^T$ then $1_n^T A^{k+1}$ should equal 1_n^T.

$\Rightarrow 1_n^T A^k A$, and by definition $1_n^T A^k = 1_n^T$.

$\Rightarrow 1_n^T A = 1_n^T$ by definition.

This shows that the columns of A^k sum to one, a criteria for a stochastic matrix.

Since A is stochastic, all entries $a_{ij} \geq 0$. This means that the product of nonnegative matrices has nonnegative entries. So $A^k \geq 0$ for all $k \geq 1$.

(b)

```
A =

    0.5000    0.3000
    0.5000    0.7000

A^2 =
    0.4000    0.3600
    0.6000    0.6400
A^3 =
    0.3800    0.3720
    0.6200    0.6280
A to a large k = 100:
    0.3750    0.3750
    0.6250    0.6250
```

For $k = 2, 3$ and $k \to \infty$, all columns of the computed matrices add up to 1. Every time the matrix is multiplied by itself, the resulting vector is still maintained as a probability vector even for large values of k. This indicates that x_k is a probability vector for every k. More observations: The columns for the A^k for very large k do not exactly add up to one, but maybe that is due to MATLAB's rounding. In addition, both columns eventually become the same.

Solution to Exercise 5

(a)
First determine the absolute values of the eigenvalues:

$$|\lambda_1| = 0.3$$

$$|\lambda_2| = 0.9$$

$$|\lambda_3| = \sqrt{(0.6)^2 + (0.8)^2} = 1$$

$$|\lambda_4| = \sqrt{(0.6)^2 + (-0.8)^2} = 1$$

$$|\lambda_5| = 1$$

$$|\lambda_6| = 1.2$$

The dynamical system is unstable because the spectral radius ($\lambda_6 = 1.2$) is greater than 1.

(b) let $x_0 = w_1 v_1 + w_2 v_2$, where $w_1, w_2 \neq 0$.

Then the solution formula follows as:
$x(k) = (0.3)^k v_1 w_1 + (-.9)^k v_2 w_2$

The solution in the long run will behave as below since $\mid \lambda_d \mid= 0.9$
$x(k) \approx (-.9)^k w_1 v_1$

The solution will then converge to 0 as k approached infinity.

(c) let $x_0 = w_1 v_1 + w_3 v_3 + w_4 v_4$, where $w_1, w_3, w_4 \neq 0$.

Then the solution formula follows as:
$x(k) = (0.3)^k v_1 w_1 + (.6 + .8i)^k v_{3w3} + (.6 - .8i)^k w_4 v_4$

The solution in the long run will behave as below since $\mid \lambda_{3,4} \mid= 1 >\mid \lambda_1 \mid= 0.3$
$x(k) \approx (.6 + .8i)^k w_3 v_3 + (.6 - .8i)^k w_4 v$

This solution is stable. It will not grow unboundedly or converge, but oscillate indefintely. This is because the first term vanishes as k grows and other components are complex conjugate eigenvalues with absolute value of 1.

(d) let $x_0 = w_1 v_1 + w_5 v_5 + w_6 v_6$, where $w_1, w_5, w_6 \neq 0$.

Then the solution formula follows as:
$x(k) = (0.3)^k v_1 w_1 + (1)^k v_5 w_5 + (-1.2)^k w_6 v_6$

The solution in the long run will behave as below since $\mid \lambda_d \mid= 1.2$
$x(k) \approx (-1.2)^k w_6 v_6$

This solution will grow unboundedly since $\mid \lambda_d \mid> 1$.