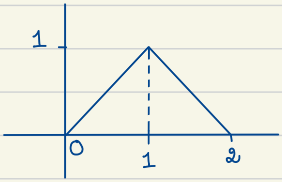
Problem 1

a) Sketch $f_X(x)$



<u>Guess:</u> $E(X) = 1$. The mean is at $x=1$ because it is symmetry.

b) Calculate the standard deviation; σ

$$E(X^2) = \int_0^1 x^2 \cdot x \, dx + \int_1^2 x^2 \cdot (2-x) dx$$

$$= \int_0^1 x^3 \, dx + \int_1^2 2x^2 - x^3 \, dx$$

$$= \frac{1}{4} + \frac{11}{12} = \frac{7}{6}$$

$$E(X) = \int_0^1 x \cdot x \, dx + \int_1^2 x(2-x) dx$$

$$= \int_0^1 x^2 dx + \int_1^2 2x - x^2 \, dx$$

$$= \frac{1}{3} + \frac{2}{3} = 1$$

$$\sigma^2 = E(X^2) - [E(X)]^2$$

$$= \frac{7}{6} - 1^2 = \frac{1}{6}$$

$$\therefore \boxed{\sigma = \sqrt{\frac{1}{6}} \approx 0.408}$$

c) Set up and solve an integer to calculate $P(1/2 \leq X \leq 3/2)$

$$P(1/2 \leq X \leq 3/2) = \int_{1/2}^{1} x\,dx + \int_{1}^{3/2} (2-x)\,dx$$

$$= \frac{3}{8} + \frac{3}{8} = \frac{6}{8}$$

$$= \boxed{\frac{3}{4} = 0.75}$$

Problem 2

$X \sim \text{Uniform}(50, 76)$

a) Calculate the mean and variance of X.

$$\mu = \frac{a+b}{2} = \frac{50+76}{2} = \boxed{63}$$

$$\sigma^2 = \frac{(b-a)^2}{12} = \frac{(76-50)^2}{12} = \boxed{\frac{169}{3}}$$

b) Find $P(65 \leq X \leq 75)$

$$\text{pdf} = \frac{1}{b-a} = \frac{1}{76-50} = \frac{1}{26}$$

$$P(65 \leq X \leq 75) = \frac{1}{26}(10) = \frac{10}{26} = \boxed{\frac{5}{13} \approx 0.385}$$

c) Find the probability density function for Y

$g(y)$: $\quad y = \ln(x-49) \Rightarrow x = e^y + 49 \Rightarrow g(y)$

$g'(y) = e^y$

$x = 50 \Rightarrow y = \ln(50-49) = 0$

$x = 76 \Rightarrow y = \ln(76-49) = \ln(27)$

$$f_Y(y) = f_X(g(y)) \cdot |g'(y)|$$

$$= f_X(e^y + 49) \cdot e^y$$

$$f_X(x) = \frac{1}{b-a} = \frac{1}{26}$$

$$\boxed{f_Y(y) = \left(\frac{1}{26}\right)e^y \quad \text{on} \quad [0, \ln(27)]}$$

d) Vertify the answer from (c) is a valid p.d.f

$$\int_0^{\ln(27)} \frac{1}{26} e^y \, dy = \frac{1}{26} e^y \Big|_0^{27} = \frac{1}{26} e^{\ln(27)} - \frac{1}{26} e^0 = \frac{27}{26} - \frac{1}{26}$$

$$= \boxed{1}$$

Problem 3

$\lambda = \frac{2}{3}$ customer/min

a) $P(X<1) = \int_0^1 f(x)dx = 1 - e^{-\lambda x} = 1 - e^{-(2/3)(1)} = \boxed{0.487}$

b) $P(X>4 \mid X>2) = P(X>2) = e^{-\lambda x} = e^{-(2/3)(2)} = \boxed{0.264}$

c) $\alpha = 4$; $\lambda = \frac{2}{3}$

$$P(X \leq 6 \text{ min}) = \int_0^6 \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} dx$$

$$= \int_0^6 \frac{(2/3)^4}{\Gamma(4)} x^{4-1} e^{-(2/3)x} dx$$

$$= \boxed{0.567}$$