Homework 4

Name : Chhay Lay Heng
Net ID : cxh220032

1. Find a recurrence relation and initial condition

Let a_n denote the number of fruit flies in a jar of week n.

- Since there are 12 flies initially, the initial condition is $a_0 = 12$.
- When every week there are 6 times as many flies, the recurrence relation is $a_n = 6 \cdot a_{n-1}$.

$\therefore$ $\boxed{\begin{array}{l} \text{Recurrence Relation: } a_n = 6 \cdot a_{n-1} \ ; \ \text{where } n \geq 1 \\ \text{Initial Condition: } a_0 = 12 \end{array}}$

2. Find the solution of the recurrence relation $a_n = 3a_{n-1}$; with $a_0 = 2$.

$$a_0 = 2$$
$$a_1 = 3 \cdot a_0 = 3 \cdot 2$$
$$a_2 = 3 \cdot a_1 = 3 \cdot 3 \cdot 2$$
$$a_3 = 3 \cdot a_2 = 3 \cdot 3 \cdot 3 \cdot 2$$
$$\vdots$$

$$\therefore \boxed{a_n = 3^n \cdot 2}$$

3. Find the solution of linear homogeneous recurrence relation $a_n = 7a_{n-1} - 6a_{n-2}$ with $a_0 = -1$ and $a_1 = 4$.

$$a_n - 7a_{n-1} + 6a_{n-2} = 0$$
$$r^2 - 7r + 6 = 0$$
$$(r-1)(r-6) = 0$$
$$r_1 = 1 \quad ; \quad r_2 = 6$$

$$\Rightarrow a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$$
$$= \alpha_1 \cdot 1^n + \alpha_2 \cdot 6^n \qquad (1)$$

$n=0$: $$a_0 = \alpha_1 \cdot 1^0 + \alpha_2 \cdot 6^0$$
$$-1 = \alpha_1 + \alpha_2 \qquad (2)$$

$n=1$: $$a_1 = \alpha_1 \cdot 1^1 + \alpha_2 \cdot 6^1$$
$$4 = \alpha_1 + 6\alpha_2 \qquad (3)$$

$$-1 = \alpha_1 + \alpha_2$$
$$4 = \alpha_1 + 6\alpha_2$$
$$-5 = -5\alpha_2 \quad \Rightarrow \alpha_2 = 1$$

$$-1 = \alpha_1 + 1 \Rightarrow \alpha_1 = -2$$

$$\Rightarrow a_n = -2 \cdot 1^n + 1 \cdot 6^n$$

$$\therefore \boxed{a_n = -2 + 6^n}$$

4. Find $f(81)$

$$f(n) = 2f\left(\frac{n}{3}\right) + 5 \quad ; \quad f(1) = 7$$

$$f(81) = 2f\left(\frac{81}{3}\right) + 5 = 2f(27) + 5$$

$$f(81) = 2\left(2f\left(\frac{27}{3}\right) + 5\right) + 5 = 2\left(2f(9) + 5\right) + 5$$

$$f(81) = 2\left(2\left(2f\left(\frac{9}{3}\right) + 5\right) + 5\right) + 5 = 2\left(2\left(2f(3) + 5\right) + 5\right) + 5$$

$$f(81) = 2\left(2\left(2\left(2f\left(\frac{3}{3}\right) + 5\right) + 5\right) + 5\right) + 5 = 2\left(2\left(2\left(2f(1) + 5\right) + 5\right) + 5\right) + 5$$

$$f(81) = 2\left(2\left(2\left(2 \cdot 7 + 5\right) + 5\right) + 5\right) + 5 = 187$$

$\therefore$ $\boxed{f(81) = 187}$

5. Find the number of elements in $A \cap B \cap C$

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

$$175 = 100 + 100 + 100 - 60 - 50 - 40 + |A \cap B \cap C|$$

$$175 = 300 - 150 + |A \cap B \cap C|$$

$$175 = 150 + |A \cap B \cap C|$$

$\therefore$ $\boxed{|A \cap B \cap C| = 25}$

6. Find number of positive integer set exceeding 1000 are not divisible by either 4 or 6.

Let A the set of positive integer in n that divisable by 4.
Let B the set of positive integer in n that divisable by 6.

$|A| = 1000/4 = 250$
$|B| = 1000/6 = 166$
$|A \cap B| = 1000/12 = 83$

$|A \cup B| = |A| + |B| - |A \cap B| = 250 + 166 - 83 = 333$

$\overline{|A \cup B|} = 1000 - |A \cup B| = 1000 - 333 = \boxed{667}$

7. Find number of onto function

Let $m = 6$ and $n = 4$

$= n^m - C(n,1)(n-1)^m + C(n,2)(n-2)^m - \dots + (-1)^{n-1} C(n,n-1) \cdot 1^m$
$= 4^6 - C(4,1)(4-1)^6 + C(4,2)(4-2)^6 - C(4,3)(4-3)^6$
$= 4^6 - C(4,1)(3)^6 + C(4,2)(2)^6 - C(4,3)(1)^6$
$= 4^6 - 4 \cdot 3^6 + 6 \cdot 2^6 - 4 \cdot 1^6$
$= 4^6 - 4 \cdot 3^6 + 6 \cdot 2^6 - 4$

$= \boxed{1560}$

8. List the derangements of the set $\{1,2,3,4\}$

$\{2,3,4,1\}$; $\{2,1,3,4\}$; $\{2,4,1,3\}$; $\{3,4,1,2\}$; $\{3,4,2,1\}$; $\{3,1,4,2\}$
$\{4,1,2,3\}$; $\{4,3,2,1\}$; $\{4,3,1,2\}$

9. Find the solution to the recurrence relation

$$a_n = 8a_{n-1} + 9a_{n-2}$$
$$a_n - 8a_{n-1} - 9a_{n-2} = 0$$
$$r^2 - 8r - 9 = 0$$
$$(r+1)(r-8) = 0$$
$$r_1 = -1 \quad ; \quad r_2 = 9$$

$$\Rightarrow a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$$
$$= \alpha_1 \cdot (-1)^n + \alpha_2 \cdot (9)^n$$

$n=0$: $a_0 = \alpha_1 (-1)^0 + \alpha_2 \cdot (9)^0$
$$3 = \alpha_1 + \alpha_2$$

$n=1$: $a_1 = \alpha_1 (-1)^1 + \alpha_2 (9)^1$
$$7 = -\alpha_1 + 9\alpha_2$$

$$3 = \alpha_1 + \alpha_2$$
$$7 = -\alpha_1 + 9\alpha_2$$
$$10 = 10\alpha_2 \Rightarrow \alpha_2 = 1$$

$$3 = \alpha_1 + 1 \Rightarrow \alpha_1 = 2$$

$$a_n = 2 \cdot (-1)^n + 1 \cdot 9^n \Rightarrow \boxed{2 \cdot (-1)^n + 9^n}$$

10. Find $f(64)$ with the divide-and-conquer recurrence relation

$f(n) = 3f\left(\frac{n}{4}\right) + \frac{n^2}{8}$; $f(1) = 2$

$f(64) = 3f\left(\frac{64}{4}\right) + \frac{64^2}{8} = 3f(16) + 512$

$f(64) = 3\left(3f\left(\frac{16}{4}\right) + \frac{16^2}{8}\right) + 512 = 3\left(3f(4) + 32\right) + 512$

$f(64) = 3\left(3\left(3f\left(\frac{4}{4}\right) + \frac{4^2}{8}\right) + 32\right) + 512$

$= 3(3(3f(1) + 2) + 32) + 512$

$= 3(3(3(2) + 2) + 32) + 512 = \boxed{680}$

11. Find number of positive integer set exceeding 1000 are not divisible by either 4, 6, 9

Let A the set of positive integer in n that divisable by 4.
Let B the set of positive integer in n that divisable by 6.
Let C the set of positive integer in n that divisable by 9.

$|A| = 1000/4 = 250$
$|B| = 1000/6 = 166$
$|C| = 1000/9 = 111$

$|A \cap B| = 1000/12 = 83$
$|A \cap C| = 1000/36 = 27$
$|B \cap C| = 1000/18 = 55$
$|A \cap B \cap C| = 1000/36 = 27$

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$
$$= 250 + 166 + 111 - 83 - 27 - 55 + 27$$
$$= 389$$

$$\overline{|A \cup B \cup C|} = 1000 - 389 = \boxed{611}$$

12. Find number of ways to assign 6 jobs to four employees

Let $m = 6$ jobs
$n = 4$ employees

$$n^m - C(n,1)(n-1)^m + C(n,2)(n-2)^m - \dots + (-1)^{n-1} C(n, n-1) \cdot 1^m$$
$$= 4^6 - C(4,1)(4-1)^6 + C(4,2)(4-2)^6 - C(4,3)(4-3)^6$$
$$= 4^6 - C(4,1)(3)^6 + C(4,2)(2)^6 - C(4,3)(1)^6$$
$$= 4^6 - 4 \cdot 3^6 + 6 \cdot 2^6 - 4 \cdot 1^6$$
$$= 4^6 - 4 \cdot 3^6 + 6 \cdot 2^6 - 4$$

$$= \boxed{1560}$$

13. Find number of permutations of the digits in string 12345

$$\frac{D_n}{4!} = \left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}\right]$$

$$\therefore \boxed{D_n = 9}$$

14. Find number of student enrolled in either calculus, discrete mathematic, data structure, or programming language.

- $|C| = 507$
- $|DM| = 292$
- $|DS| = 312$
- $|PL| = 344$
- $|C \cap DS| = 14$
- $|C \cap PL| = 213$
- $|DM \cap DS| = 211$
- $|DM \cap PL| = 43$

$$|C \cup DM \cup DS \cup PL| = |C| + |DM| + |DS| + |PL| - |C \cap DS| - |C \cap PL| - |DM \cap DS| - |DS \cap PL| + |C \cap DM \cap DS \cap PL|$$

$$= 507 + 292 + 312 + 344 - 14 - 213 - 211 - 43 + 0$$

$$= \boxed{974}$$

15. a) Set up a recurrence relation

$$\boxed{B_n = 2B_{n-1} - B_{n-2}}$$

b) Find the solution of the relation

$B_0 = 100$

$B_1 = 2 \cdot B_0 = 2 \times 100 = 200$

$B_2 = 2B_1 - B_0 = 2 \times 200 - 100 = 300$

$B_3 = 2B_2 - B_1 = 2 \times 300 - 200 = 400$

$\vdots$

$B_n = 2^n \cdot B_0$

$\therefore \boxed{B_n = 2^n \cdot B_0}$

c) Find the length the colony contain more than 1 million bacteria.

$B_n = 2^n \cdot B_0$

$1000000 = 2^n \cdot 100$

$10000 = 2^n$

$\log_2(10000) = \log_2(2^n)$

$\boxed{n = \log_2(10000) \approx 13.29 \text{ hours}}$