Directions

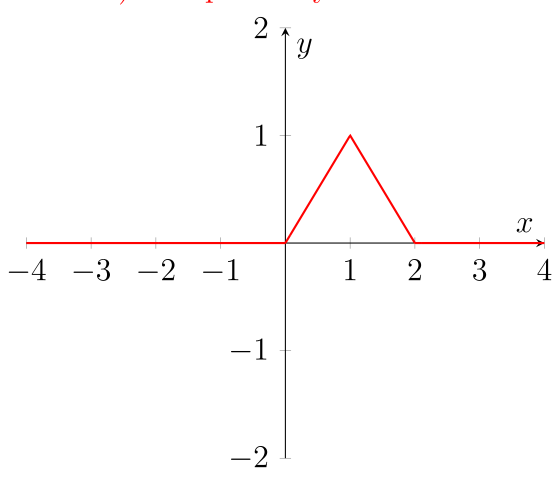
Scan and upload your *handwritten* solutions to eLearning by the end of the day on **Monday, February 27 by 11:59 pm**. Show sufficient work in problems 1 and 2 or no credit may be given. Calculator codes are sufficient for problem 3 (or use formulas/integrals if you prefer). Submit your work in an organized format: solutions should be in question order and please write neatly.

Problem 1 (4 points)

The function $f(x)$ below is a probability density function (pdf) for the continuous random variable X. Answer the following.

$$f(x) = \begin{cases} x & 0 \leq x \leq 1 \\ 2-x & 1 < x \leq 2 \\ 0 & otherwise \end{cases}$$

(a) Neatly sketch $f(x)$. Use the plot to guess the value of $\mathbb{E}(X)$. Explain your guess. Solution: (plot below) The plot is symmetric about $x = 1$, so then $\mathbb{E}(X) = 1$.



(b) Calculate the standard deviation, σ. Solution: Calculate $\sigma = \sqrt{\mathbb{E}(X^2) - [\mathbb{E}(X)]^2}$

Step 1: Calculate expected value: $\mathbb{E}(X) = \int_0^1 x \cdot x \, dx + \int_1^2 x \cdot (2-x) \, dx = 1/3 + 2/3 = 1$

Step 2: Calculate $\mathbb{E}(X^2) = \int_0^1 x^2 \cdot x \, dx + \int_1^2 x^2 \cdot (2-x) \, dx = 1/4 + 11/12 = 7/6$

Step 3: Evaluate $\sigma = \sqrt{7/6 - (1)^2} = \boxed{\sqrt{1/6}}$ or $\sqrt{6}/6 \approx 0.408$

(c) Set up and solve an integral to calculate $P(1/2 \le X < 3/2)$.

Solution (an integral via symmetry is also acceptable):

$$\int_{1/2}^{3/2} f(x)\ dx = \int_{1/2}^{1} x\ dx + \int_{1}^{3/2} (2-x)\ dx$$
$$= \left[\frac{1}{2}x^2\right]_{1/2}^{1} + \left[2x - \frac{1}{2}x^2\right]_{1}^{3/2}$$
$$= \boxed{\frac{3}{4}}$$

Problem 2 (4 points)

The time X it takes to complete an exam is uniformly distributed between 50 and 76 minutes. In other words, $X \sim \text{Uniform}(50, 76)$.

(a) Calculate the mean and variance of X.

Solution: Since X is a uniform r.v., then

$$\mu = \frac{a+b}{2} = \frac{50+76}{2} = \boxed{63}\ min$$
$$\sigma^2 = \frac{(b-a)^2}{12} = \frac{(76-50)^2}{12} = \boxed{\frac{169}{3}} \approx 56.3\ min^2$$

(b) What is the probability that it will take between 65 and 75 minutes to complete the exam?

Solution: The pdf of $f_X(x) = \frac{1}{b-a} = \frac{1}{76-50} = 1/26$. Then,

$$P(65 < X < 75) = \int_{65}^{75} 1/26\ dx$$
$$= \left[\frac{x}{26}\right]_{x=65}^{x=75}$$
$$= \frac{75-65}{26} = \boxed{5/13} \approx 0.385$$

(c) Suppose $y = ln(x-49)$. Find the probability density function for Y.

Solution:

$$\implies g(y) = y = ln(x-49) \implies x = e^y + 49$$
$$\implies g'(y) = e^y$$
$$\implies f_Y(y) = f_X(g(y)) \cdot |g'(y)| \ \text{(definition)}$$
$$= f_X(e^y + 49) \cdot |e^y|$$
$$= \boxed{\frac{e^y}{26}} \text{ on } 0 \le y \le ln(27)$$

(d) Verify that your answer from (c) is a valid p.d.f.
Solution: If $x = 50, y = ln(50-49) = 0$ and if $x = 76, y = ln(76-49) = ln(27)$.

Show $\int_0^{ln(27)} \frac{e^y}{26}\, dy = 1$

$$\implies \int_0^{ln(27)} \frac{e^y}{26}\, dy = \left[\frac{e^y}{26}\right]_{y=0}^{y=ln(27)} = \frac{e^{ln(27)} - e^0}{26} = \frac{27-1}{26} = \frac{26}{26} = 1\checkmark$$

Problem 3 (3 points)

Chipotle can serve at a rate of two customers every three minutes. ($\lambda = 2/3$)

(a) What is the probability that the wait time for the next customer be less than one minute?
Solution: $X \sim$ Exponential$(\lambda = 2/3)$:

$$P(X < 1) = \int_0^1 \frac{2}{3} e^{-\frac{2}{3}x}\, dx = -e^{-2/3}\Big|_0^1 = 1 - e^{-2/3} \approx \boxed{0.487}$$

Alternately, students can use the complement of the survival function.

(b) What is the probability that the wait time for the next customer will exceed 4 minutes, given that they have been waiting for 2 minutes already?
Solution (Memoryless Property): $X \sim$ Exponential$(\lambda = 2/3)$:

$$P(X > 4 | X > 2) = P(X > 2) = e^{-\frac{2}{3}(2)} \approx \boxed{0.264}$$

Alternately, students can integrate the pdf as another approach.

(c) There are 4 people in line. What is the probability that it will take at most 6 minutes to serve all of them?
Solution: $X \sim$ Gamma$(\alpha = 4, \lambda = 2/3)$ and $t = 6$

$$P(T \leq 6) = P(X \geq 4) = 1 - P(X \leq 3) = 1 - \texttt{poissoncdf}(4, 3) \approx \boxed{0.567}$$

using the Gamma-Poisson Formula. Alternately, students can integrate the pdf from 0 to 6 using integration by parts.

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