

## Directions

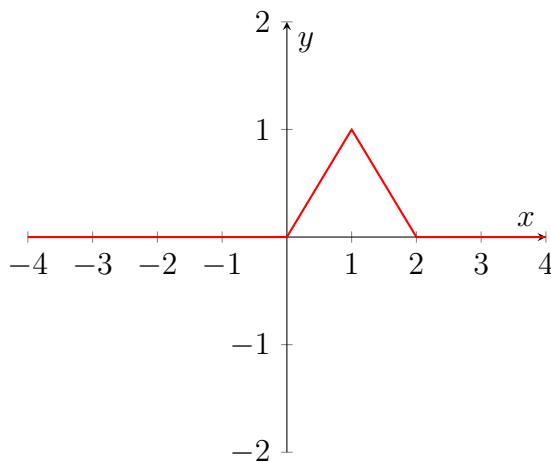
Scan and upload your *handwritten* solutions to eLearning by the end of the day on **Monday, February 27 by 11:59 pm**. Show sufficient work in problems 1 and 2 or no credit may be given. Calculator codes are sufficient for problem 3 (or use formulas/integrals if you prefer). Submit your work in an organized format: solutions should be in question order and please write neatly.

## Problem 1 (4 points)

The function  $f(x)$  below is a probability density function (pdf) for the continuous random variable  $X$ . Answer the following.

$$f(x) = \begin{cases} x & 0 \leq x \leq 1 \\ 2 - x & 1 < x \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

- (a) Neatly sketch  $f(x)$ . Use the plot to guess the value of  $\mathbb{E}(X)$ . Explain your guess. **Solution:** (plot below) The plot is symmetric about  $x = 1$ , so then  $\mathbb{E}(X) = 1$ .



- (b) Calculate the standard deviation,  $\sigma$ . **Solution:** Calculate  $\sigma = \sqrt{\mathbb{E}(X^2) - [\mathbb{E}(X)]^2}$

Step 1: Calculate expected value:  $\mathbb{E}(X) = \int_0^1 x \cdot x \, dx + \int_1^2 x \cdot (2 - x) \, dx = 1/3 + 2/3 = 1$

Step 2: Calculate  $\mathbb{E}(X^2) = \int_0^1 x^2 \cdot x \, dx + \int_1^2 x^2 \cdot (2 - x) \, dx = 1/4 + 11/12 = 7/6$

Step 3: Evaluate  $\sigma = \sqrt{7/6 - (1)^2} = \boxed{\sqrt{1/6}}$  or  $\sqrt{6}/6 \approx 0.408$

- (c) Set up and solve an integral to calculate  $P(1/2 \leq X < 3/2)$ .

**Solution** (an integral via symmetry is also acceptable):

$$\begin{aligned} \int_{1/2}^{3/2} f(x) \, dx &= \int_{1/2}^1 x \, dx + \int_1^{3/2} (2-x) \, dx \\ &= \left[ \frac{1}{2}x^2 \right]_{1/2}^1 + \left[ 2x - \frac{1}{2}x^2 \right]_1^{3/2} \\ &= \boxed{\frac{3}{4}} \end{aligned}$$

## Problem 2 (4 points)

The time  $X$  it takes to complete an exam is uniformly distributed between 50 and 76 minutes. In other words,  $X \sim \text{Uniform}(50, 76)$ .

- (a) Calculate the mean and variance of  $X$ .

**Solution:** Since  $X$  is a uniform r.v., then

$$\begin{aligned} \mu &= \frac{a+b}{2} = \frac{50+76}{2} = \boxed{63} \text{ min} \\ \sigma^2 &= \frac{(b-a)^2}{12} = \frac{(76-50)^2}{12} = \boxed{\frac{169}{3}} \approx 56.3 \text{ min}^2 \end{aligned}$$

- (b) What is the probability that it will take between 65 and 75 minutes to complete the exam?

**Solution:** The pdf of  $f_X(x) = \frac{1}{b-a} = \frac{1}{76-50} = 1/26$ . Then,

$$\begin{aligned} P(65 < X < 75) &= \int_{65}^{75} 1/26 \, dx \\ &= \left[ \frac{x}{26} \right]_{x=65}^{x=75} \\ &= \frac{75-65}{26} = \boxed{5/13} \approx 0.385 \end{aligned}$$

- (c) Suppose  $y = \ln(x - 49)$ . Find the probability density function for  $Y$ .

**Solution:**

$$\begin{aligned} \implies g(y) &= y = \ln(x - 49) \implies x = e^y + 49 \\ \implies g'(y) &= e^y \\ \implies f_Y(y) &= f_X(g(y)) \cdot |g'(y)| \quad (\text{definition}) \\ &= f_X(e^y + 49) \cdot |e^y| \\ &= \boxed{\frac{e^y}{26}} \text{ on } 0 \leq y \leq \ln(27) \end{aligned}$$

- (d) Verify that your answer from (c) is a valid p.d.f.

Solution: If  $x = 50, y = \ln(50 - 49) = 0$  and if  $x = 76, y = \ln(76 - 49) = \ln(27)$ .

Show  $\int_0^{\ln(27)} \frac{e^y}{26} dy = 1$

$$\implies \int_0^{\ln(27)} \frac{e^y}{26} dy = \left[ \frac{e^y}{26} \right]_{y=0}^{y=\ln(27)} = \frac{e^{\ln(27)} - e^0}{26} = \frac{27 - 1}{26} = \frac{26}{26} = 1 \checkmark$$

### Problem 3 (3 points)

Chipotle can serve at a rate of two customers every three minutes. ( $\lambda = 2/3$ )

- (a) What is the probability that the wait time for the next customer be less than one minute?

Solution:  $X \sim \text{Exponential}(\lambda = 2/3)$ :

$$P(X < 1) = \int_0^1 \frac{2}{3} e^{-\frac{2}{3}x} dx = -e^{-2/3} \Big|_0^1 = 1 - e^{-2/3} \approx \boxed{0.487}$$

Alternately, students can use the complement of the survival function.

- (b) What is the probability that the wait time for the next customer will exceed 4 minutes, given that they have been waiting for 2 minutes already?

Solution (Memoryless Property):  $X \sim \text{Exponential}(\lambda = 2/3)$  :

$$P(X > 4 | X > 2) = P(X > 2) = e^{-\frac{2}{3}(2)} \approx \boxed{0.264}$$

Alternately, students can integrate the pdf as another approach.

- (c) There are 4 people in line. What is the probability that it will take at most 6 minutes to serve all of them?

Solution:  $X \sim \text{Gamma}(\alpha = 4, \lambda = 2/3)$  and  $t = 6$

$$P(T \leq 6) = P(X \geq 4) = 1 - P(X \leq 3) = 1 - \text{poissoncdf}(4, 3) \approx \boxed{0.567}$$

using the Gamma-Poisson Formula. Alternately, students can integrate the pdf from 0 to 6 using integration by parts.

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