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Polygon Meshes

Polygons are simple objects that can be easily represented and manipulated. Polygon meshes are collections of polygons used to represent surfaces and objects. Properties of polygons are:

- Polygons are simple to represent and manipulate
- They can be used to approximate smooth surfaces
- each polygon (if planar) has a unique normal vector

The Normal to a Polygon

Computing the normal vector to a planar polygon is simple. With any three adjacent points P_1 , P_2 , and P_3 of the polygon, the normal vector is represented as:

$$\vec{n}=(P_2-P_1)\times(P_3-P_2)$$

If the points for the polygon are listed in a counter-clockwise order, then the above computation produces a normal that has a positive z coordinate from the vantage point. This technique to compute normals will simplify subsequent computations for rendering and shading. Note that if the polygons are not planar, it is then advisable to compute a normal vector for each vertex, as opposed to computing a single normal for the entire polygon. These normal vectors can then be interpolated to produce realistic shading effects.

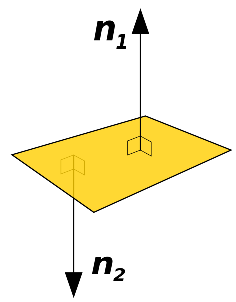

Illustration 1: Outside and inside normal vectors to a planar polygon

Representing Meshes

In a polygon mesh, vertices and outward-facing surface normals are kept and three lists are necessary to efficiently represent them:

- The vertex list which contains all the 3D vertices for the mesh
- The face list which, for each face, lists (in a counter-clockwise order) the vertices of the face, and the normals associated with each vertex
- The normal list which contains all the normals for each face

Here is an example:

Vertex	Point
0	(1.0, 3.1, 0.0)
1	(1.7, 5.0, 0.0)
2	(2.5, 3.8, 0.0)
3	(2.0, 2.2, 0.0)

Face	Vertices	Normals
0	0, 2, 1	0, 0, 0
1	3, 2, 0	1, 1, 1

Normal	Vector
0	(0.0, 0.0, 1.0)
1	(0.0, 0.0, 1.0)

Rotating Polylines

For modeling objects that have a rotational symmetry, one can use a polyline to represent a contour of the object along its axis of symmetry. Once the contour is defined, we proceed by rotating it by a number of degrees, and forming polygons with the preceding polyline and the rotated polyline. These polygons are then stored in vertex, face, and normal lists, to be saved in files for later use. Many different objects can be modeled as polygon meshes with this technique. Here is an example:

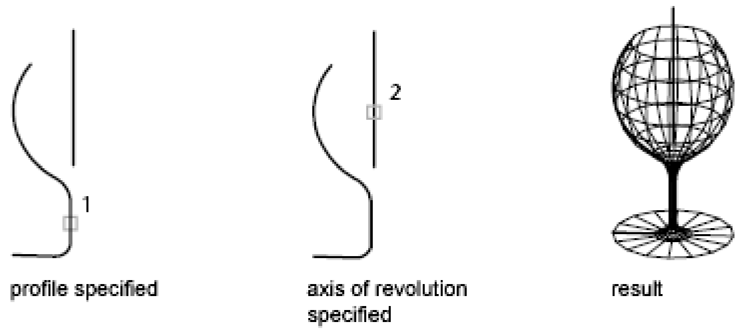

Illustration 2: Creating a wine glass polygon mesh by rotating a profile

Surface Representations

A parametric planar surface is a linear expression $P(u,v)=C+\vec{a}u+\vec{b}v$ that forms a parallelogram in 3D when the parameter values are comprised between 0 and 1. This is the simplest surface we can express with a parametric form. Parametric forms may also be used for non-linear surfaces in the following way:

$$P(u,v)=(X(u,v),Y(u,v),Z(u,v))$$

where $X(u,v)$, $Y(u,v)$, and $Z(u,v)$ are functions in the coordinates of the surface. Surfaces can also be expressed by implicit forms:

$$F(X,Y,Z)=0$$

When the coordinates are on the surface, the implicit equation is zero. Surfaces divide 3D space in two distinct regions that can be labeled as the inside region and the outside region. The implicit form of a surface allows us to easily find these regions:

$F(X,Y,Z)<0$	(X,Y,Z) is inside the surface
$F(X,Y,Z)=0$	(X,Y,Z) is on the surface
$F(X,Y,Z)>0$	(X,Y,Z) is outside the surface

Note that what constitutes the outside and the inside regions often times depends on the application.

Notable Parametric Objects

The torus, sphere, cone, cylinder, circle, and plane are all useful parametric objects. Here is a list of their equations:

The torus ($0 \le (u, v) \le 2\pi$):

$$p(u,v) = \begin{pmatrix} X(u,v) \\ Y(u,v) \\ Z(u,v) \\ 1 \end{pmatrix} = \begin{pmatrix} (c + a\cos(v))\cos(u) \\ (c + a\cos(v))\sin(u) \\ a\sin(v) \\ 1 \end{pmatrix}$$

The sphere ($0 \le u \le 2\pi, 0 \le v \le \pi$):

$$p(u,v) = \begin{pmatrix} X(u,v) \\ Y(u,v) \\ Z(u,v) \\ 1 \end{pmatrix} = \begin{pmatrix} r\cos(v)\sin(u) \\ r\sin(v)\sin(u) \\ r\cos(u) \\ 1 \end{pmatrix}$$

The cone ($0 \le u \le 1, 0 \le v \le 2\pi$):

$$p(u,v) = \begin{pmatrix} X(u,v) \\ Y(u,v) \\ Z(u,v) \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{h(1-u)}{h} r\sin(v) \\ \frac{h(1-u)}{h} r\cos(v) \\ hu \\ 1 \end{pmatrix}$$

The cylinder ($0 \le u \le 1, 0 \le v \le 2\pi$):

$$p(u,v) = \begin{pmatrix} X(u,v) \\ Y(u,v) \\ Z(u,v) \\ 1 \end{pmatrix} = \begin{pmatrix} r\sin(v) \\ r\cos(v) \\ hu \\ 1 \end{pmatrix}$$

The circle ($0 \le u \le 1, 0 \le v \le 2\pi$):

$$p(u,v) = \begin{pmatrix} X(u,v) \\ Y(u,v) \\ Z(u,v) \\ 1 \end{pmatrix} = \begin{pmatrix} ru\cos(v) \\ ru\sin(v) \\ 0 \\ 1 \end{pmatrix}$$

The plane ($0 \leq u \leq 1, 0 \leq v \leq 1$):

$$p(u,v) = \begin{pmatrix} X(u,v) \\ Y(u,v) \\ Z(u,v) \\ 1 \end{pmatrix} = \begin{pmatrix} uw \\ vh \\ 0 \\ 1 \end{pmatrix}$$

Surface Normals

Computing the surface normal $\vec{n}(u,v)$ of a parametric form can be achieved with the following formula:

$$\vec{n}(u_{0,} v_0) = \left(\frac{\partial P}{\partial u} \times \frac{\partial P}{\partial v} \right)$$

evaluated at $(u,v) = (u_{0,} v_0)$. As an example, consider the parametric form of a planar surface $P(u,v) = C + \vec{a}u + \vec{b}v$. It is easy to see that

$$\frac{\partial P}{\partial u} = \vec{a} \quad \frac{\partial P}{\partial v} = \vec{b}$$

and thus $\vec{n}(u,v) = \vec{a} \times \vec{b}$, as expected. In general, we have:

$$\frac{\partial P(u,v)}{\partial u} = \left(\frac{\partial X(u,v)}{\partial u}, \frac{\partial Y(u,v)}{\partial u}, \frac{\partial Z(u,v)}{\partial u} \right)$$

and

$$\frac{\partial P(u,v)}{\partial v} = \left(\frac{\partial X(u,v)}{\partial v}, \frac{\partial Y(u,v)}{\partial v}, \frac{\partial Z(u,v)}{\partial v} \right)$$

In the case when the surface is expressed implicitly, the surface normal is its gradient, defined as:

$$\vec{n}(x_{0,} y_{0,} z_0) = \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \right)$$

evaluated at $(x,y,z) = (x_{0,} y_{0,} z_0)$.

Generic Shapes

The generic shapes used in computer graphics are the sphere, the cylinder, and the cone. The generic sphere has unit radius and is centered at the origin: The implicit form of the sphere is:

$$F(x, y, z)=x^2+y^2+z^2-1$$

and its parametric form is given by:

$$P(u, v)=(\sin u \cos v, \sin u \sin v, \cos u)$$

where $u \in [0, \pi]$ and $v \in [0, 2\pi]$. To verify if an implicit and a parametric form represent the same surface, we test if the following equality holds:

$$F(X(u, v), Y(u, v), Z(u, v))=0$$

for all relevant values of (u, v).

The generic cylinder, which also has a parametric form, has a circular cross-section radius of 1, it axis coincides with the z -axis, and it extends from 0 to 1 in z. The generic cylinder is part of a larger family of surfaces called tapered cylinders, with radius of s at $z=1$ .The implicit form for this surface is given by:

$$F(x, y, z)=x^2+y^2-(1+(s-1)z)^2$$

where $0<z<1$. The parametric form is:

$$P(u, v)=((1+(s-1)v)\cos u, (1+(s-1)v)\sin u, v)$$

The implicit form for the generic cone is similar:

$$F(x, y, z)=x^2+y^2-(1-z)^2$$

where $0<z<1$. The parametric form is given by:

$$P(u, v)=((1-v)\cos u, (1-v)\sin u, v)$$

where $u \in [0, 2\pi]$ and $v \in [0, 1]$.

Normals to Generic Shapes

Surface	Parametric Normal	Implicit Normal
Sphere	$P(u, v)$	(x, y, z)
Tapered Cylinder	$(\cos u, \sin u, 1-s)$	$(x, y, -(s-1)(1+(s-1)z))$
Cylinder	$(\cos u, \sin u, 0)$	$(x, y, 0)$
Cone	$(\cos u, \sin u, 1)$	$(x, y, 1-z)$

Intersections with Rays

The way in which we interact with generic shapes for their rendering is to cast rays that may intersect with them. Let's find what is the intersection of a ray starting at point

$$E = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$$

and in direction

$$\vec{v} = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}$$

with the generic sphere, expressed in its implicit form.

The ray can be expressed as:

$$R(t) = E + \vec{v}t \;=\; (X(t), Y(t), Z(t))^T \;=\; (e_x + v_x t, e_y + v_y t, e_z + v_z t)^T$$

and the generic sphere in implicit form is

$$F(x, y, z) = x^2 + y^2 + z^2 - 1 = 0$$

The point along the ray which also is on the sphere must then satisfy the following equation:

$$F(R(t)) \;=\; F(X(t), Y(t), Z(t)) \;=\; (e_x + v_x t)^2 + (e_y + v_y t)^2 + (e_z + v_z t)^2 - 1 \;=\; 0$$

By substitution, we find

$$3t^2 - 12t + 11 = 0$$

and recognize that it is a quadratic and solve as:

$$t \;=\; \frac{12 \pm \sqrt{12}}{6}$$