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Points and Vectors in Space

A point in n dimensional space represents a position in space and is given by a tuple $\boldsymbol{p}=(p_{1,} p_{2,} \ldots, p_n)^T$ where p_i are scalars. The position of a point is relative to a coordinate system with an origin given by $O=(0,0,\ldots,0)^T$ and unit axes $\vec{u}_1=(1,0,\ldots,0)^T, \vec{u}_2=(0,1,0,\ldots,0)^T, \ldots, \vec{u}_n=(0,0,\ldots,1)^T$. Hence, a 3D point is written as $\boldsymbol{p}=(x,y,z)^T$, and a 2D point, as $\boldsymbol{p}=(x,y)^T$.

A vector in n dimensional space represents a direction and is given by a tuple $\vec{v}=(v_{1,} v_{2,} \ldots, v_n)^T$ where v_i are scalars. A vector is the result of the subtraction of two points. For example the vector $\vec{v}(1,2)^T$ is the result of the subtraction of the two points $\vec{v}(1,2)=\boldsymbol{p}(1,2)^T-\vec{0}^T$. The resulting vector represents the direction and distance between the points. Thus we can write, for any two points $\boldsymbol{p_1}, \boldsymbol{p_2}$: $\vec{v}=\boldsymbol{p_2}-\boldsymbol{p_1}$. It follows directly that $\boldsymbol{p_1}+\vec{v}=\boldsymbol{p_2}$. Adding a vector to a point results in a point.

Vector Addition

Vector addition is defined as $\vec{v}=\vec{u}+\vec{v}=(u_1+v_{1,} u_2+v_{2,} \ldots, u_n+v_n)$ and has the following properties:

1. Associative: $\vec{u}+(\vec{v}+\vec{w})=(\vec{u}+\vec{v})+\vec{w}$
2. Commutative: $\vec{u}+\vec{v}=\vec{v}+\vec{u}$
3. Vector-Point Associative: $P+(\vec{u}+\vec{v})=(P+\vec{u})+\vec{v}$

Scalar Multiplication

Scalar multiplication is defined as $a\vec{v}=(a v_{1,} a v_{2,} \ldots, a v_n)$ and has the following properties:

1. Scalar associative: $(ab)\vec{v}=a(b\vec{v})$
2. Scalar distributive: $(a+b)\vec{v}=a\vec{v}+b\vec{v}$

3. Vector distributive: $a(\vec{u}+\vec{v})=a\vec{u}+a\vec{v}$

Linear Combination

There is one case for which adding points together makes sense. We define a linear combination of points $\boldsymbol{p}_i$ as

$$\boldsymbol{p}=\sum_{i=1}^{n} a_i \boldsymbol{p}_i$$

A linear combination of points for which

$$\sum_{i=1}^{n} a_i=1$$

is called an affine combination of points.

Vector Length

The length of a vector is known as its magnitude and is defined as

$$\|\vec{v}\|=\sqrt{\sum_{i=1}^{n} v_i^2}$$

A unit length vector is a vector of length equal to one. Any vector can be scaled in such a way as to have a length of one:

$$\hat{v}=\frac{1}{\|\vec{v}\|}\vec{v}$$

Vector Dot Product

The dot product between two vectors is defined as

$$\vec{u}\cdot\vec{v}=\sum_{i=1}^{n} u_i v_i$$

and has the following properties:

1. Vector length: $\vec{v}\cdot\vec{v}=\|\vec{v}\|^2$
2. Scalar associative: $(a\vec{u})\cdot(b\vec{v})=ab(\vec{u}\cdot\vec{v})$
3. Commutative $\vec{u}\cdot\vec{v}=\vec{v}\cdot\vec{u}$
4. Addition distributive $\vec{u}\cdot(\vec{v}+\vec{w})=(\vec{u}\cdot\vec{v})+(\vec{u}\cdot\vec{w})$
5. $\vec{u}\cdot\vec{v}=\|\vec{u}\|\|\vec{v}\|\cos\theta$, where θ is the angle between the vectors

This last property is useful in many circumstances. In addition, if both vectors are normalized to unit length, then we have $\hat{u}\cdot\hat{v}=\cos\theta$ and hence $\theta=\cos^{-1}(\hat{u}\cdot\hat{v})$. In addition, we have

1. $\vec{u}\cdot\vec{v}=0\rightarrow\theta=90^o$
2. $\vec{u}\cdot\vec{v}>0\rightarrow\theta<90^o$
3. $\vec{u}\cdot\vec{v}<0\rightarrow\theta>90^o$

3D Cross Product

The cross product between two vectors is defined only for 3D vectors. We compute the cross product in the following way

$$\vec{u}\times\vec{v}=\left(\begin{vmatrix}u_2 & u_3\\ v_2 & v_3\end{vmatrix},\begin{vmatrix}u_3 & u_1\\ v_3 & v_1\end{vmatrix},\begin{vmatrix}u_1 & u_2\\ v_1 & v_2\end{vmatrix}\right)$$

where

$$\begin{vmatrix}a & b\\ c & d\end{vmatrix}=ad-bc$$

The cross product has the following properties:

1. Nilpotent: $\vec{v}\times\vec{v}=\vec{0}=(0,0,0)$
2. Scalar associative: $(a\vec{u})\times(b\vec{v})=(ab)(\vec{u}\times\vec{v})$
3. Anti-symmetric: $\vec{u}\times\vec{v}=-(\vec{v}\times\vec{u})$
4. Addition distributive: $\vec{u}\times(\vec{v}+\vec{w})=(\vec{u}\times\vec{v})+(\vec{u}\times\vec{w})$
5. Dot-cross associative: $\vec{u}\cdot(\vec{v}\times\vec{w})=(\vec{u}\times\vec{v})\cdot\vec{w}$
6. $\|\vec{u}\times\vec{v}\|^2=\|\vec{u}\|^2\|\vec{v}\|^2\sin^2\theta$ where θ is the angle between vectors

The geometric interpretation of the cross product is very useful for a number of processes in computer graphics. Suppose $\vec{u}$ and $\vec{v}$ are not parallel vectors. Then the vector $\vec{w}=\vec{u}\times\vec{v}$ is perpendicular to both $\vec{u}$ and $\vec{v}$. In particular, if both $\vec{u}$ and $\vec{v}$ are orthogonal (perpendicular) and of unit length then, with $\vec{w}=\vec{u}\times\vec{v}$, we have that the vectors $\{\vec{u},\vec{v},\vec{w}\}$ form an orthonormal basis for 3D space. Additionally, for any non parallel vectors $\vec{u}$ and $\vec{v}$, the magnitude of vector $\vec{w}=\vec{u}\times\vec{v}$ represents the area of the parallelogram subtended by vectors $\vec{u}$ and $\vec{v}$.