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Two-Dimensional Clipping Algorithms

The viewport is a region of the window in which graphics are displayed. The transformations from world coordinates, to camera coordinates, to window coordinates, and finally to viewport coordinates usually results in only a part of the scene being visible, and hence the need to clip lines to the viewport in 2D. Suppose a point (x, y). It is inside the viewport if

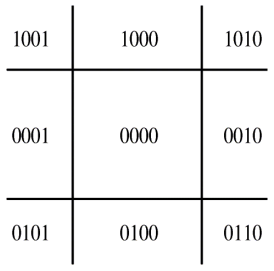
$$X_{min} \leq x \leq X_{max}$$
$$Y_{min} \leq y \leq Y_{max}$$

Clipping Lines

If both endpoints of a line lie inside the viewport, no clipping is necessary, as the line segment is completely visible. If only one endpoint is within the viewport, then we must clip the line at the intersection. If both endpoints are outside the viewport, then the line crosses two boundaries, or is completely outside the viewport.

Cohen-Sutherland's Line Clipping Algorithm

The viewing space is divided into nine encoded regions as shown below:



For each endpoint of a line segment, we assign a 4-bit code following the rules below (the code is formed in the following way: (bit 1, bit 2, bit 3, bit 4):

1. bit 1 is 1 if $x < X_{min}$
2. bit 2 is 1 if $x > X_{max}$
3. bit 3 is 1 if $y < Y_{min}$
4. bit 4 is 1 if $y > Y_{max}$

Hence, if (x_1, y_1) and (x_2, y_2) have code 0000 then both end points are inside. If both endpoints have a 1 in the same bit position, then the line segment is entirely outside. Performing a logical AND on both codes and obtaining something different from 0000 indicates that the line is outside. Otherwise, the line segment must be checked for intersections.

Calculating Intersections

With a vertical boundary, x is either X_{min} or X_{max}. Hence:

$$y = y_1 + \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(X_{min} - x_1)$$

or

$$y = y_1 + \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(X_{max} - x_1)$$

For a horizontal boundary, y is either Y_{min} or Y_{max}. Hence:

$$x = x_1 + (Y_{min} - y_1)\left(\frac{x_2 - x_1}{y_2 - y_1}\right)$$

or

$$x = x_1 + (Y_{max} - y_1)\left(\frac{x_2 - x_1}{y_2 - y_1}\right)$$

The Algorithm

The inputs to the algorithm are the two endpoints of a line segment, and the attributes of the viewport. The output is the clipped line segment:

```
1. if ( $p_1$ and $p_2$ inside)
    1. output $p_1$ and $p_2$ and return
2. else
    1. while (code of $p_1$ != 0000) do
```

```
        1. if ( $p_1$ is to the left)
            2. $p_1$ = intersection of $p_1 p_2$ and $x=X_{min}$
    2. else if ( $p_1$ is to the right)
            1. $p_1$ = intersection of $p_1 p_2$ and $x=X_{max}$
    3. else if ( $p_1$ is below)
            1. $p_1$ = intersection of $p_1 p_2$ and $y=Y_{min}$
    4. else if ( $p_1$ is above)
            1. $p_1$ = intersection of $p_1 p_2$ and $y=Y_{max}$
    5. compute new code for $p_1$
3. repeat these steps for $p_2$
4. output $p_1$ and $p_2$ and return
```

Midpoint Division Method

This technique consists of locating intersections without computing them with a binary search. First, compute the midpoint of the segment, and divide the segment into two as per the midpoint. For these two segments, test if they are totally inside or totally outside of the viewport and either accept or reject the segments accordingly. Repeat for each segment that cannot be either accepted or rejected.

Liang and Barsky's Clipping Algorithm

This algorithm uses the parametric form of line equations, written as:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \end{pmatrix} u = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} u$$

where $u \in [0,1]$. Using this particular form of equations then we can write that when the following inequalities are satisfied

$$X_{min} \leq x_1 + \Delta x u \leq X_{max}$$

$$Y_{min} \leq y_1 + \Delta y u \leq Y_{max}$$

then the line segment lies completely within the window. These inequalities can be written as

$$p_k u \leq q_k$$

for $k=1,2,3,4$.

$k=1$	$p_1=-\Delta x$	$q_1=x_1-X_{min}$	Left boundary
$k=2$	$p_2=\Delta x$	$q_2=X_{max}-x_1$	Right boundary
$k=3$	$p_3=-\Delta y$	$q_3=y_1-Y_{min}$	Bottom boundary
$k=4$	$p_4=\Delta y$	$q_4=Y_{max}-y_1$	Top boundary

If $p_k=0$ then the line segment is parallel to the k^{th} boundary:

1. $p_1=-\Delta x=0$ then the line is vertical and parallel to the left and right boundaries
2. in addition to this , if $q_1<0$ or $q_2<0$, then the line segment is completely outside the window

Hence, if $(p_1=0)\wedge(q_1<0\vee q_2<0)$ the line segment is completely outside. The same rules apply for p_3 and we can write that if $(p_3=0)\wedge(q_3<0\vee q_4<0)$ then the segment is also outside. In general, if $q_k\geq 0$ then the line is inside the k^{th} boundary.

In cases when the line segment is not parallel to any of the view port's borders, then if $p_k<0$ the infinite extension of the line segment proceeds from outside to inside the infinite extension of the k^{th} window boundary. The opposite situation occurs when $p_k>0$.

Consequently, when $p_k\neq 0$, we compute the value of u that yields the intersection of the extended line segment with the extended k^{th} boundary. The value of u is given by equating the initial inequalities:

$$p_k u=q_k$$

and thus, the intersection of the extended line segment with the k^{th} boundary is simply given by:

$$u=\frac{q_k}{p_k}$$

The Algorithm

Computing u_1 :

1. for all $p_k<0$
    1. $r_k=\dfrac{q_k}{p_k}$
2. $u_1=max\{0,r_k\}$

Computing u_2 :

1. for all $p_k>0$
    1. $r_k=\dfrac{q_k}{p_k}$

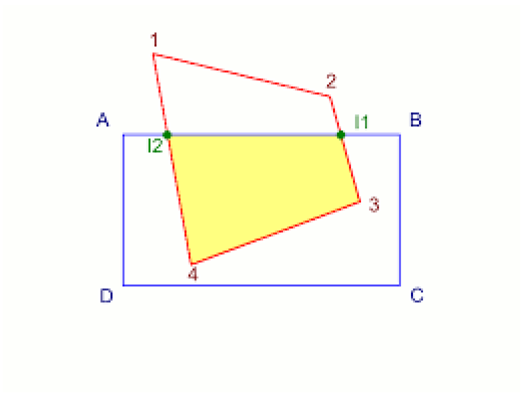
2. $u_2 = min\{1, r_k\}$

The main body of the algorithm is:

```
1. compute $p_k$ and $q_k$, for $k=1,2,3,4$
2. if $[(p_1=0)\wedge(q_1<0\vee q_2<0)]\vee[(p_3=0)\wedge(q_3<0\vee q_4<0)]$
    1. reject the segment
3. else
    1. compute $u_1$
    2. compute $u_2$
4. if $u_1>u_2$
    1. reject the segment
5. else
    1. clipped line segment is $(x_1+\Delta x u_{1,} y_1+\Delta y u_1)$ , $(x_1+\Delta x u_{2,} y_1+\Delta y u_2)$
```

Polygon Clipping

Polygons need to be clipped when objects to render are made with them. Polygons are represented as ordered sets of vertices. An intuitive approach to clip polygons is to trace the vertices that are completely inside the viewport, and get rid of those that are completely outside. For all other vertices, and using their parametric representation, we can determine if they are tracked from the outside toward the inside of the area of the viewport, and conversely. Below is a simple example where the intersections of the vertices are all within the same boundary. All that is needed after clipping is to create a vertex joining the two intersections, and the polygon is clipped. However, this simple method will not work when the intersections are not all with the same boundary. When this is the case, the polygon must be adjusted to include the part of the viewport that finds itself inside it.



Intersection Between Segments

In order to perform this type of polygon clipping, we must first know how to compute intersections when the segments are given in parametric form. Consider:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} u$$

and let's find the intersection between $\vec{p}_1 = (x_{1,} y_1), (x_{2,} y_2)$ and $\vec{p}_2 = (x_{3,} y_3), (x_{4,} y_4)$:

$$\begin{aligned} \vec{p}_1 &= \vec{x}_1 + \Delta \vec{x}_1 u \\ \vec{p}_2 &= \vec{x}_2 + \Delta \vec{x}_2 v \end{aligned}$$

and find (u, v) values such that $\vec{p}_1 = \vec{p}_2$. We can write this equality as:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \end{pmatrix} u + \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} x_4 - x_3 \\ y_4 - y_3 \end{pmatrix} v + \begin{pmatrix} x_3 \\ y_3 \end{pmatrix}$$

and rearrange it in the following way:

$$\begin{pmatrix} x_2 - x_1 & x_3 - x_4 \\ y_2 - y_1 & y_3 - y_4 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} x_3 - x_1 \\ y_3 - y_1 \end{pmatrix}$$

This is then simply a 2 by 2 linear system of equations that need to be solved. If the lines are not parallel, it has a solution for $(u, v)^T$ which yields the intersection point.

Raster Scan Polygon Filling

Rendering polygons implies filling them with color, texture, etc. Here is a simple scan-line algorithm to do just this, with the following steps. The algorithm accepts a list of points describing the polygon and a color to fill it with:

1. Determine the number of scan lines. The first scan line starts at the minimum y value of all points while the last scan line is at the maximum y value of all points.
2. Create a segment list containing all the segments from the polygon, trace the horizontal edges and remove them from the list.
3. For each scan line y do:
   1. Create a list of active segments from the segment list. These are the segments that make an intersection with the scan line y .
   2. For each segment in the active list do:
      1. If segment is not vertical then compute its intersection with the scan line y as $I = \frac{y - b}{m}$, where m is the slope of the segment, and b is the y coordinate of the active segment's first end-point to which we subtract m times the x coordinate of the same active segment's first end-point.

        2. Else compute the intersection I as the x coordinate of the active segment's first end-point.
        3. If the scan line y intersects with the active segment's first endpoint, then
            1. Set y_0 to the y coordinate of the point that immediately precedes the active segment's first point within the active list.
            2. Set y_1 to the y coordinate of the point that immediately follows the active segment's first point (this is the active segment's second point).
            3. If $y_0<y<y_1$ or $y_0>y>y_1$ then add I to the intersection list.
            4. Else add I twice to the intersection list.
        4. Else if the y coordinate of the segment's second point does not intersect with the scan line y then add I to the intersection list.
    3. Sort the intersection list in ascending order.
    4. Trace a horizontal line segment on the scan line y for each contiguous intersection pair from the intersection list.

To implement this algorithm efficiently, it is important to observe that each scan line has a y value one pixel lower than the previous scan line and thus:

- $y_{i+1}=y_i-1$
- slope of segment is $m=\frac{y_{i+1}-y_i}{x_{i+1}-x_i}$
- which implies $x_{i+1}=x_i-\frac{1}{m}$

Adjusting Segment Endpoints

Instead of dealing with the number of intersections where the segments meet on the scan lines, we could also simply perform minor adjustments to the points of the polygons to ensure that each time an intersection is encountered, we either start or stop turning on pixels. On the following page are displayed all the cases (to the left) and how they should be dealt with (to the right). While this technique will slightly modify the appearance of the polygon at low resolutions, it is virtually undetectable with current resolutions and speeds up the scan-line algorithm.

Illustration 1: Cases in need of endpoint adjustments