1. We know that:

$$P(R|A) = 0.99$$

$$P(R|\text{not } A) = 0.1$$

$$P(A) = 0.05, \quad \text{so} \quad P(\text{not } A) = 0.95$$

By Bayes' Rule,

$$P(A|R) = \frac{P(A) \cdot P(R|A)}{P(R)}$$

Moreover, using the Rule of Total Probability,

$$P(R) = P(R|A) \cdot P(A) + P(R|\text{not } A) \cdot P(\text{not } A).$$

$$= 0.99 \cdot 0.05 + 0.1 \cdot 0.95$$

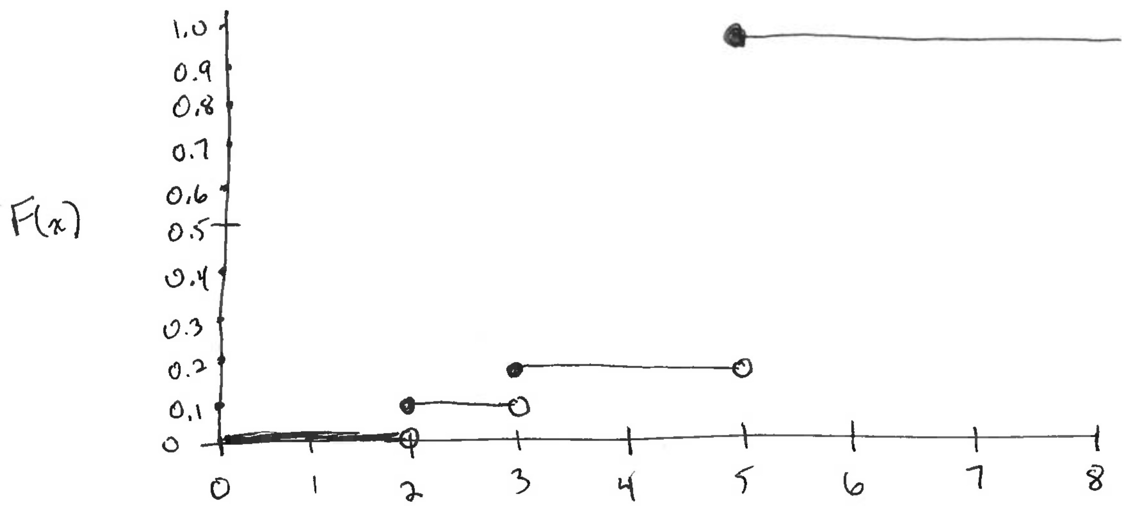
So $$P(A|R) = \frac{0.05 \times 0.99}{0.99 \times 0.05 + 0.1 \times 0.95}$$

(This expression is enough, but the number turns out to be

$P(A|R) \cong 0.343$)

(2) X has PMF $f(x) = \begin{cases} 1/10 & \text{if } x = 2 \\ 1/10 & \text{if } x = 3 \\ 8/10 & \text{if } x = 5 \end{cases}$

The CDF looks like the following picture



$P(2 < X \le 4.5) = P(\{x : x \in (2, 4.5]\})$

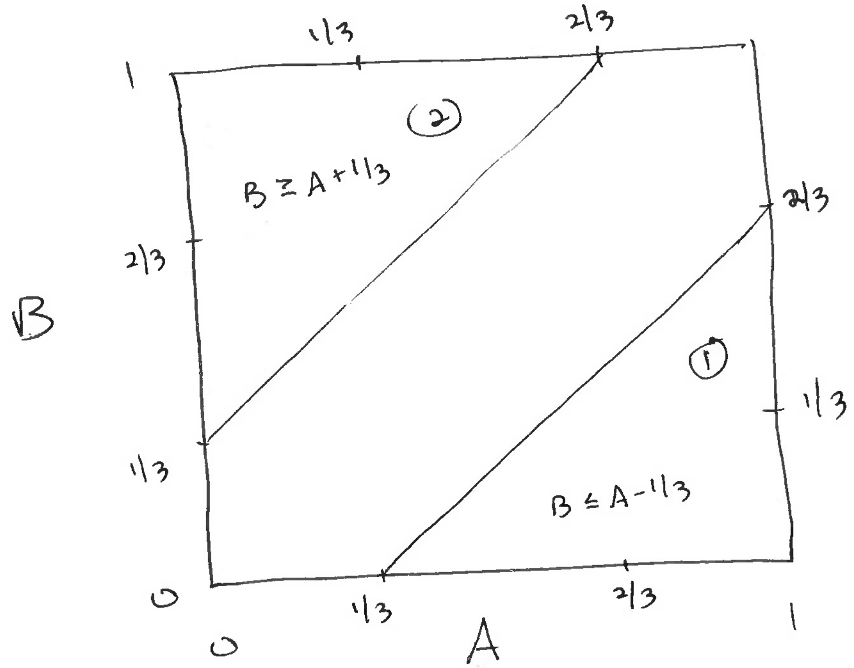
$= P(X = 3)$, since $x = 3$ is the only possible value of x that falls in $(2, 4.5]$.

$= 1/10$

$P(2 \le X \le 4.5) = P(X = 2) + P(X = 3)$

$= \frac{1}{10} + \frac{1}{10} = \frac{2}{10}$

(3) Let A be Alice's delay in hours, and let B be Bob's delay in hours. If all possible pairs of delays are likely, then A and B are uniformly distributed over the following rectangle:



Alice and Bob will miss each other if $|A-B| \geq 1/3$, since each will wait for a maximum of $1/3$ of an hour. And:

$$P(|A-B| \geq 1/3) = P(A-B \geq 1/3) + P(A-B \leq -1/3)$$

$$= P(B \leq A - 1/3) + P(B \geq A + 1/3)$$

$$= \text{Area of triangle 1} + \text{Area of triangle 2}$$

$$= \frac{2}{3} \cdot \frac{2}{3} \cdot \frac{1}{2} + \frac{2}{3} \cdot \frac{2}{3} \cdot \frac{1}{2}$$

$$= \frac{4}{9}$$

So $p(\text{meet}) = 1 - 4/9 = 5/9$

(4) Markov's argument is wrong; we can't calculate

$$E(T) = E\left(\frac{2}{V}\right) = \frac{2}{E(V)}$$

In general, $E(f(X)) \neq f(E(X))$

Instead, to calculate $E(T)$, note that there are two possibilities:

- If he walks, $T = \dfrac{2 \text{ miles}}{5 \text{ miles/hour}} = \dfrac{2}{5}$ hour

- If he rides, $T = \dfrac{2 \text{ miles}}{10 \text{ miles/hour}} = \dfrac{2}{10}$ hour

Therefore, $E(T) = 0.4 \times \dfrac{2}{5} + 0.6 \times \dfrac{2}{10}$

(This expression is enough, but the final value turns out to be $\dfrac{4}{10} \cdot \dfrac{4}{10} + \dfrac{6}{10} \times \dfrac{2}{10} = \dfrac{16}{100} + \dfrac{12}{100}$

$= \dfrac{28}{100}$ hours

≈ 16.8 minutes

(not 15))

5) Part A

Does $P(|W_n - 0| \geq \varepsilon) \to 0$ as $n \to \infty$?

Write: $P(|W_n - 0| \geq \varepsilon)$

$= P(W_n \geq \varepsilon)$ since $W_n \geq 0$

$= P(X_1 \geq \varepsilon, X_2 \geq \varepsilon, \ldots, X_n \geq \varepsilon)$

$= \prod_{i=1}^{N} P(X_i \geq \varepsilon)$ by independence.

Now, $P(X_i \geq \varepsilon) = 1 - P(X_i < \varepsilon)$

$= 1 - F(\varepsilon)$ where F is the CDF of X.

$= 1 - \int_0^{\varepsilon} \lambda e^{-\lambda t}\, dt$

$= 1 - 1 - e^{-\lambda \varepsilon}$ (given integral)

$= e^{-\lambda \varepsilon}$

Putting these together:

$P(W_n \geq \varepsilon) = \prod_{i=1}^{N} e^{-\lambda \varepsilon}$

$= e^{-n \lambda \varepsilon}$

And $\lim_{n \to \infty} e^{-n\lambda\varepsilon} = 0$ for all $\varepsilon > 0$, $\lambda > 0$.

Therefore $W_n \xrightarrow{P} 0$.

5) <u>Part B</u>

$Y_n = \max\{X_1, \ldots, X_N\}$

To find the PDF $f(y)$, first find the CDF.

$$F(y) = P(Y_n \le y) = P(\max\{X_1, \ldots, X_n\} \le y)$$

But $\max\{X_1, \ldots, X_N) \le y \iff X_i \le y$ for all $i = 1, \ldots, N$
(hint)

$$\text{So} \quad P(Y_n \le y) = P(X_1 \le y, X_2 \le y, \ldots, X_n \le y)$$

$$= \prod_{i=1}^{N} P(X_i \le y) \quad \text{By independence.}$$

$$= \prod_{i=1}^{N} [1 - e^{-\lambda y}] \quad \text{By the given integral.}$$

$$= [1 - e^{-\lambda y}]^N$$

$$\text{Therefore } \underbrace{f(y) = F'(y)}_{\text{This is the important thing to recognize}} = \underbrace{N[1 - e^{-\lambda y}]^{N-1}}_{\text{It didn't matter whether you remembered how to take this derivative by hand.}} \lambda e^{-\lambda y}$$