Probability and Stochastic Processes Course Notes

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1 Algebra

Definition 1. *A collection $\mathcal{A}$ of subsets of Ω is called an* **algebra** *if*

Axiom 1.1. $\phi \in \mathcal{A}$.

Axiom 1.2 (Closed under Complement)**.** *If $A \in \mathcal{A}$, then $A^c \in \mathcal{A}$.*

Axiom 1.3 (Finite Additivity)**.** *If $A_1, A_2 \in \mathcal{F}$, then $A_1 \bigcup A_2 \in \mathcal{A}$.*

The last two properties of an **algebra** implies that an algebra is closed under finite intersection and union.

2 σ-algebra

Definition 2 (σ-algebra)**.** *A σ-algebra $\mathcal{F}$on Ω is a collection of subsets in Ω s.t.*

Axiom 2.1. $\phi \in \mathcal{F}$.

Axiom 2.2 (Closed under Complement)**.** *If $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$.*

Axiom 2.3 (Countable Additivity)**.** *If $A_1, A_2, \cdots \in \mathcal{F}$, then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$.*

Note that Axiom 2.3 implies *finite Additivity* (which can be proved rigorously). Another note worthy thing is that this definition is similar to a **topology** defined on Ω.

Definition 3 (sub σ-algebra)**.** *A* **sub σ algebra***, denoted by $\mathcal{G}$, is a collection of sets in Ω s.t.*

Axiom 2.4. $\mathcal{G} \subset \mathcal{F}$

Axiom 2.5. *$\mathcal{G}$ is a σ-algebra.*

3 Product Space on σ-algebra

Let $(\Omega_1, \mathcal{F}_1)$ be a σ-algebra $\mathcal{F}_1$ associated with Ω_1 and $(\Omega_2, \mathcal{F}_2)$ be a σ-algebra $\mathcal{F}_2$ associated with Ω_2. We consider the following operations on $\mathcal{F}_1$, $\mathcal{F}_2$.

Definition 4 (Lifting)**.** *A* **lifting** *of $\mathcal{F}_1$, denoted by $\tilde{\mathcal{F}}_1$, is defined as*

$$\tilde{\mathcal{F}}_1 \triangleq \{A \times \Omega_2 : A \in \mathcal{F}_1\}$$

Clearly, an element in $\tilde{\mathcal{F}}$ is of the form $(A, B), A \subseteq \Omega_1$ and $B \subseteq \Omega_2$. That is, each element of $\tilde{\mathcal{F}}$ is a subset of the product space $\Omega_1 \times \Omega_2$, making $\tilde{\mathcal{F}}$ a collection of subsets in $\Omega_1 \times \Omega_2$.

Proposition 1. *Any lifting $\tilde{\mathcal{F}}$ is a σ-algebra on the product space $\Omega_1 \times \Omega_2$.*

Proof. Given $\Lambda \in \tilde{\mathcal{F}}$, say $\Lambda = (A, \Omega_2)$, where $A \subseteq \Omega_1$. $\Lambda^c = (A^c, \Omega_2) \cup (A, \Omega_2^c) = (A^c, \Omega_2) \cup (A, \phi) = (A^c, \Omega_2) \cup \phi = (A^c, \Omega_2) \in \tilde{\mathcal{F}}$ since $\mathcal{F}_1$ is a σ-algebra. □

4 Borel σ-algebra

Definition 5 (Borel σ-algebra)**.** *A* **Borel $\sigma -$ algebra** *is the smallest σ-algebra on Ω which contains all the open sets in Ω.*

Definition 6 ($\sigma(\mathcal{H})$)**.** *Let $\mathcal{H}$ be a collection of subsets in Ω. A* **$\sigma -$ algebra generated by** *$\mathcal{H}$, denoted by $\sigma(\mathcal{H})$, is the smallest σ-algebra that contains all the elements in $\mathcal{H}$ (i.e. subsets of Ω). In a sense that it contains all the sets in $\mathcal{H}$, all the countable unions of $\mathcal{H}$ and all the complement of sets in $\mathcal{H}$.*

In other words, if $\mathcal{C}$ is a σ-algebra that contains $\mathcal{H}$, then $\sigma(\mathcal{H}) \subseteq \mathcal{C}$, namely

$$\sigma(\mathcal{H}) = \bigcap_{\mathcal{C} \supseteq \mathcal{H}} \mathcal{C}, \qquad \mathcal{C} : \sigma - \text{algebra}$$

Theorem 1 (Borel σ-algebra on $\mathbb{R}$)**.** *The* **Borel $\sigma -$ algebra** *on $\mathbb{R}$, denoted by $\mathcal{B}(\mathbb{R})$ is the σ-algebra generated by the open sets in $\mathbb{R}$. That is, if $\mathcal{O}$ is the collection of open sets in $\mathbb{R}$, then $\mathcal{B}(\mathbb{R}) = \sigma(\mathcal{O})$.*

The proof of Thm.1 is not presented in class. We shall take it as a ground truth.

5 Measurable Maps (Functions)

Definition 7 (Measurable Space)**.** *A* **measurable space** *on Ω is a pair $(\Omega, \mathcal{A})$ such that $\mathcal{A}$ is a σ-algebra on Ω.*

It is clear that for any set Ω, there can be many measurable spaces associated with it since there may be different σ-algebra on Ω.

Definition 8 (Measurable Map)**.** *Given two measurable spaces (X, Σ) and $(Y, \mathcal{T})$, a map $f : X \to Y$ is called a* **Measurable Map** *if for every measurable set in $\mathcal{T}$, its pre-image is also in Σ.*

$$f^{-1}(B) = \{x \in X : f(x) \in B\} \in \Sigma, \forall B \in \mathcal{T}$$

If $f : X \to Y$ is a measurable map, we write $f : (X, \Sigma) \to (Y, \mathcal{T})$ to emphasize the choice of Σ and $\mathcal{T}$.

Definition 9 (Random Variable)**.** *A* **random variable** *X is a* **measurable map** *$X : \Omega \to \mathbb{R}$ between two measurable spaces $(\Omega, \mathcal{F})$ and $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, where Ω is the sample space and $\mathcal{F}$ is a σ-algebra on it. To be more precise, X is a random variable if*

$$f^{-1}(B) = \{\omega \in \Omega : f(\omega) \in B\} \in \mathcal{F}, \forall B \in \mathcal{B}(\mathbb{R})$$

From the above definition, it is clear that X might be a valid RV/mapping in one σ-algebra $\mathcal{F}$ but invalid in another $\mathcal{G}$.

Appendix I: Topology

Given a set X and a collection $\mathcal{T}$ of subsets in X, $\mathcal{T}$ is called a topology on X if

$$
\begin{aligned}
&a)\ \phi \in \mathcal{T} \\
&b)\ \text{If } \forall i \in I, A_i \in \mathcal{T},\ \bigcup_{i\in I} A_i \in \mathcal{T} \\
&c)\ \text{If } A_i \in \mathcal{T}, i = 1, 2, \cdots, n,\ \text{then } \bigcap_{i=1}^{n} A_i \in \mathcal{T}
\end{aligned}
$$

If $\mathcal{T}$ is a topology on X, we call $(X, \mathcal{T})$ a topological space. A member in $\mathcal{T}$ is called an *open set* in X. A subset of X is said to be closed if its complement is in $\mathcal{T}$. Hence, ϕ and X are both closed and open.