Probability and Stochastic Processes Course Notes

Chia-Hao CHang, chchangkh@utexs.edu

Fall 2018

Definition 1 (Riemann-Darboux Integral)**.** A **Riemann – Darboux Integral** is defined when the $U(\alpha, f, \mathcal{P})$ and $L(\alpha, f, \mathcal{P})$ converges to the same value.

Definition 2 (Lebesgue Integral)**.** Given a measurable space (X, Σ, μ), the **Lebsegue Integral** of f on $A \in \Sigma$ is defined as the improper Riemann Integral of

$$\int_A f d\mu = \int_0^\infty f^*(t)dt$$

where

$$f^*(t) = \mu(\{x \in \mathbb{R} : f(x) > t\})$$

Definition 3 (Simple Random Variable)**.** Let $\{E_1, E_2, \cdots, E_n\}$ be a partition of Ω, X is called a **Simple Random Variable** if $X(\omega) = \sum_{i=1}^n c_i \mathbb{1}_{E_i}(\omega)$.

That is, $X(\omega)$ is a discrete random variable mapping sets in Ω to a finite collection of points in $\mathbb{R}$.

Fact 1 (Approximation of Random Variables)**.** *Any (measurable) random variable can be approximated (in a proper sense) by sequence of random variables.*

That is, given a random variable X, we can construct a sequence of random variables $\{Y_1(\omega), Y_2(\omega), \cdots, Y_n(\omega), \cdots\}$ s.t. $\lim_{n\to\infty} ||X(\omega) - Y_n(\omega)|| = 0$, where each $Y_i(\omega)$ is given by

$$Y_n(\omega) = \sum_{i=1}^n c_i^{(n)} \mathbb{1}_{E_i^{(n)}}(\omega)$$

where

$$\{E_1^{(n)}, E_2^{(n)}, \cdots, E_n^{(n)}\}$$

is a partition of Ω. This is an important concept since it inspires our definition about the convergence of random variables.

Definition 4 (Lebsegue Integral of Simple RVs)**.** For a simple random variable $Y(\omega) = \sum_{i=1}^n c_i \mathbb{1}_{E_i}(\omega)$, the Lebsegue integral of Y over the sample space is defined as

$$\int_\Omega Y(\omega)dP(\omega) \triangleq \sum_{i=1}^n c_i P\{E_i\}$$

The tutorial of integration is given as followed. We look for the images/values of $Y(\omega)$ in the range of Y and trace back to its pre-image. Next, we calculate the probability of its pre-image under $(\Omega, \mathcal{F}, P)$. We then multiply the probability with the value of Y and sums them over the entire range.

Recall that **any** random variable X can be approximated by a sequence of random variables Y (with proper definition of approximation). Hence it is intuitive to define the Lebsegue integral of a non-simple random variable X by the limit of the integral values of the sequence of RVS. Formally speaking,

Definition 5 (Lebsegue Integral of non-simple RVs)**.** Let X be any random variable, sequence $\{Y_1, Y_2, \cdots, Y_n, \cdots\}$ is such that

$$\lim_{n\to\infty} ||X(\omega) - Y_n(\omega)|| = 0$$

then the **Lebsegue Integral of X** is defined as

$$\int_\Omega X(\omega)dP(\omega) \triangleq \lim_{n\to\infty} \int_\Omega Y_n(\omega)dP(\omega)$$

Definition 6 (Expectation value)**.** The **expectation value** of a random variable X is defined as

$$\mathbb{E}[X] = \int_\Omega X(\omega)dP(\omega)$$

generally speaking, for a function $g : \mathbb{R} \to \mathbb{R}$, The **expectation value** $\mathbb{E}[g(X)]$ is defined as

$$\mathbb{E}[g(X)] = \int_\Omega g(X(\omega))dP(\omega)$$

One might think of $g(X)$ as a new random variable on $(\Omega, \mathcal{F}, P)$ and use the previous definition to calculate the value of integral.

Remark (Expectation as Probability)**.** *Given $A \in \mathcal{F}$, the probability of event A can also be written as*

$$P\{A\} = P\{\omega \in A\} = \mathbb{E}[\mathbb{1}_A] = \int_\Omega \mathbb{1}\{\omega \in A\}dP(\omega) = \int_A dP(\omega)$$

1 Conditional Expectation

Definition 7 ($\mathcal{F}_X$)**.** Given $(\Omega, \mathcal{F}, P)$ and a random variable $X : \Omega \to \mathbb{R}$. The σ **−algebra induced by X** $\mathcal{F}_X$ is

$$\mathcal{F}_X = \{X^{-1}(B) : \forall B \in \mathcal{B}(\mathbb{R})\}$$

Fact 2. *$\mathcal{F}_X \subseteq \mathcal{F}$. since X is a measurable map.*

Fact 3. *The random variable X is said to be measurable with respect to $\mathcal{F}_X$ (hence, $\mathcal{F}$.)*

Definition 8 (Measurable w.r.t a subset of $\mathcal{F}$)**.** Let $\mathcal{G}$ be a sub-σ algebra of $\mathcal{F}$ and X be a random variable on $(\Omega, \mathcal{F}, P)$. X is said to be **measurable with respect to** $\mathcal{G}$ if

$$X^{-1}(B) \in \mathcal{G}, \quad \forall B \in \mathcal{B}(\mathbb{R})$$

or, equivalently

$$\mathcal{F}_X \subseteq \mathcal{G}$$

1.1 Example

Let Y be s simple random variable with $P_Y(y_i) = q_i$, $i = 1, 2, \cdots, m$. Let $B_i = \{\omega : Y(\omega) = y_i\}$. We observe the following facts

1. $\{B_1, B_2, \cdots, B_m\}$ forms a partition of Ω.
2. $P(B_i) = q_i$, $i = 1, 2, \cdots, m$.
3. $\mathcal{F}_Y = \sigma\{B_1, B_2, \cdots, B_m\}$

Definition 9 (Conditional Expectation)**.** The **conditional expectation of** X **given** Y is a **random variable**

$$\mathbb{E}[X|\mathcal{F}_Y](\omega) \triangleq \sum_{i=1}^{m} \mathbb{E}[X|Y = y_i]\mathbb{1}_{B_i}(\omega)$$

which is a **simple** random variable.

$\mathbb{E}[X|Y = y_i]$ is a **number** given by

$$\mathbb{E}[X|Y = y_i] = \int_{-\infty}^{\infty} f_{X|Y}(x|y_i)dx \quad (X \text{ is continuous})$$
$$\mathbb{E}[X|Y = y_i] = \sum_k P_{X|Y}(x_k|y_i) \quad (X \text{ is discrete})$$

From the above equation, we can observe that

1. $\forall \omega \in B_i$, $\mathbb{E}[X|\mathcal{F}_Y](\omega)$ are all the same ($\mathbb{E}[X|Y = y_i]$ in fact).

The **derived random variable** $Z(\omega) = \mathbb{E}[X|\mathcal{F}_Y](\omega)$ has PMF

$$P\{Z(\omega) = \mathbb{E}[X|Y = y_i]\} = P\{Y = y_i\} = P\{B_i\} = q_i$$

Definition 10. Let X be a random variable over $(\Omega, \mathcal{F}, P)$ and $\mathcal{G}$ is a σ-algebra s.t. $\mathcal{G} \subseteq \mathcal{F}$. We define $Z(\omega) = \mathbb{E}[X|\mathcal{G}]$ to be a **random variable** that is $\mathcal{G}$-measurable s.t.

$$\mathbb{E}[X\mathbb{1}_B] = \mathbb{E}[Z\mathbb{1}_B], \quad \forall B \in \mathcal{G}$$

To be more precise,

$$\mathbb{E}[X\mathbb{1}_B] = \mathbb{E}[\mathbb{E}[X|\mathcal{G}]\mathbb{1}_B], \quad \forall B \in \mathcal{G}$$

If one does not want to be confused by notations, it can be explicitly written as

$$\mathbb{E}[X(\omega)\mathbb{1}_B(\omega)] = \mathbb{E}[\mathbb{E}[X|\mathcal{G}](\omega)\mathbb{1}_B(\omega)], \quad \forall B \in \mathcal{G}$$

Theorem 1 (Existence of conditional expectation)**.** *With the definition of conditional expectation provided, such random variable* $\mathbb{E}[X|\mathcal{G}]$ **always exists**.

It turns out that the existence of conditional expectation (random variable) is a difficult issue to prove. It is far beyond the depth of the course. One might seek for a theoretical probability class for thorough proof.

Definition 11 (Version). Two random variables Z and Z' that are $\mathcal{G}$-measurable are said to be **versions** of each other if

$$\mathbb{E}[Z\mathbb{1}_B] = \mathbb{E}[Z'\mathbb{1}_B], \quad \forall B \in \mathcal{G}$$

for a sub-σ algebra $\mathcal{G} \subseteq \mathcal{F}$

Remark. *Two versions Z and Z' can at most differ by a set of P-measure zero.*

Definition 12 (Equality of Random Variables). We say that two random variables $W(\omega)$ and $R(\omega)$ that are $\mathcal{I}$-measurable are the **same**, i.e. $W(\omega) = R(\omega)$ if and only if

$$\mathbb{E}[W\mathbb{1}_A] = \mathbb{E}[R\mathbb{1}_A], \quad \forall A \in \mathcal{I}$$

That is, two random variables are **equal** in the sense that their "average"(expectation value) are equal **everywhere in the set** in $\mathcal{I}$.

2 Independence

We now formally define the independence of two random variables.

Definition 13 (Independence). Let $(\Omega, \mathcal{F}, P)$ be a probability triplet and $\mathcal{G}_1$, $\mathcal{G}_2$ be sub σ-algebra on $\mathcal{F}$. $\mathcal{G}_1$ and $\mathcal{G}_2$ are said to be **independent** if

$$P\{A \cap B\} = P\{A\}P\{B\}, \quad \forall A \in \mathcal{G}_1, \quad \forall B \in \mathcal{G}_2$$

Definition 14 (Independence of Random Variables). Let $(\Omega, \mathcal{F}, P)$ be a probability triplet and X, Y are two random variables defined on it. $\mathcal{F}_X$ and $\mathcal{F}_Y$ are σ-algebra induced by X and Y, respectively ($\mathcal{F}_X, \mathcal{F}_Y \subseteq \mathcal{F}$ therefore.) X and Y are said to be **independent** if $\mathcal{F}_X$ and $\mathcal{F}_Y$ are independent.