Probability and Stochastic Processes Course Notes

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1 Induced Measure and CDF

Definition 1 (Induced Measure). *Given two measurable spaces* $(\Omega, \mathcal{F}, P)$, $(\Lambda, \mathcal{G})$ *and a measurable map* $f : (\Omega, \mathcal{F}) \to (\Lambda, \mathcal{G})$*, an* **induced measure** $\mu_f(\cdot)$ *is defined as:*

$$\mu_f(C) = P\{f^{-1}(C)\} = P\{\omega \in \Omega : f(\omega) \in C\}, \quad \forall C \in \mathcal{G}$$

The above definition of $\mu_f(\cdot)$ has no ambiguity since $f(\cdot)$ is measurable map. The triplet $(\Lambda, \mathcal{G}, \mu_f)$ forms a (valid) measure.

Definition 2 (Cumulative Distributive Function). *A* **cumulative distributive function** (**CDF**) $F_X(\cdot)$*of a random variable* X *is an induced measure a by measurable map* $X : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R})$ *such that*

$$F_X(u) \triangleq \mu_X((-\infty, u]) = P\{X^{-1}((-\infty, u])\} = P\{\omega : X(\omega) \in (-\infty, u]\}$$

Equivalently, we can say that

$$F_X(u) = P\{X \leq u\}$$

Theorem 1. *Let* $\mu(\cdot)$ *be an induced measure on* $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. $\mu(B)$ *is given by*

$$\mu(B) = P\{\omega : X(\omega) \in B\}, \forall B \in \mathcal{B}(\mathbb{R})$$

Then $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu)$ *is a valid probability triple on* $\mathbb{R}$.

Proposition 1. *A function* F *is CDF of some random variable if and only if*

a. F *is increasing*

b. $\lim_{x\to\infty} F(x) = 1$, $\lim_{x\to-\infty} F(x) = 0$.

c. F *is right continuous.*

Proof. The "only if" side: It is already shown in my notebook.
The "if" side: requires building a probability triplet $(\Omega, \mathcal{F}, P)$. A detailed proof is shown in [1] □

Definition 3 (Discrete Random Variable). *A random variable is* **discrete** *if there is countable or finite set* $\{x_i : i \in I\}$*, where* I *is finite or countable s.t.*

$$P\{\omega : X(\omega) \in \{x_i : i \in I\}\} = 1$$

It is clear that although the sample space Ω might be continuous, the induced random variable might be discrete.

Definition 4 (PMF)**.** *A* **probability mass function PMF***, denoted by $P_X(\cdot)$ is defined by*

$$P_X(x_i) \triangleq P\{\omega : X(\omega) = x_i\} \triangleq P\{X = x_i\}$$

Remark (The CDF of PMF)**.** *The CDF of a RV X is given by*

$$F_X(x) = \sum_{x_i \leq x} P_X(x_i)$$

Definition 5 (Continuous Random Variable)**.** *A random variable X is said to be a* **continuous random variable** *if its CDF is a integration of a function. i.e.*

$$F_X(x) = \int_{-\infty}^{x} f_X(t)dt$$

The function $f_X(\cdot)$ is called the **probability density function pdf**. If f_X is continuous at c, we have

$$f_x(c) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{c}^{c+\varepsilon} f_X(t)dt$$

If A is any Borel set of $\mathbb{R}$, then

$$P\{X \in A\} = \int_A f_X(t)dt$$

Appendix I: Useful Identities

Lemma 1.

$$\bigcap_{n=1}^{\infty} (-\infty, c + \frac{1}{n}] = (-\infty, c]$$

and

$$\bigcup_{n=1}^{\infty} (-\infty, c - \frac{1}{n}] = (-\infty, c)$$

which shows that the union of closed sets might be open while the intersection is still closed.

Lemma 2 (Image as Partition Property)**.** *Let $f : X \to Y$ as a function from X to Y. The inverse images of the range of f forms a partition of Y. i.e.*

$$\{f^{-1}(y) : y \in Range(f)\}$$

is a partition of X.

This is quite useful in probability theory since a RV is a mapping from Ω to $\mathbb{R}$.

Definition 6 (Upper/Lower Bound)**.** *Given a set $A \subseteq \mathbb{R}$, M is called an* **upper bound** *of A if $\forall a \in A$, $a \le M$. Similarly, m is called an* **lower bound** *of A if $\forall a \in A$, $a \ge m$.*

Definition 7 (Least Upper Bound (Supremum))**.** *Given a set $A \subseteq \mathbb{R}$, α is called a* **least upper bound** *of A if*

$$\begin{aligned} &i)\ \alpha \ge x, \forall x \in A \\ &ii)\ \text{If } b < \alpha,\ b \text{ is not an upper bound of } A \end{aligned}$$

α is denoted as $\alpha = \sup A$.

Definition 8 (Greatest Lower Bound (Infimum))**.** *Given a set $A \subseteq \mathbb{R}$, β is called a* **greatest lower bound** *of A if*

$$\begin{aligned} &i)\ \beta \le x, \forall x \in A \\ &ii)\ \text{If } b > \beta,\ b \text{ is not a lower bound of } A \end{aligned}$$

β is denoted as $\beta = \inf A$.

Definition 9 (Supremum of a Function)**.** *Let $f : A \to \mathbb{R}$. The* **supremum** *of f on A, denoted as $\sup\limits_A f$ (if exists), is defined as*

$$\sup_A f = \sup\{f(x) : x \in A\}$$

Similarly, the infimum **inf** is defined as

Definition 10 (Infimum of a Function)**.** *Let $f : A \to \mathbb{R}$. The* **infimum** *of f on A, denoted as $\inf\limits_A f$ (if exists), is defined as*

$$\inf_A f = \inf\{f(x) : x \in A\}$$

There are some useful properties used in the proof related to sup and inf

1. If $\beta < \sup A$, then $\exists x \in A$ s.t. $x > \beta$
2. If $M \ge x, \forall x \in A$, then $M \ge \sup A$. Namely, if M is an **upper bound** of A, then $M \ge \sup A$.
3. If $\beta > \inf A$, then $\exists x \in A$ s.t. $x < \beta$
4. If $m \le x, \forall x \in A$, then $m \le \inf A$. Namely, if m is an **lower bound** of A, then $m \le \inf A$.

Proposition 2. *Let $f, g : A \to \mathbb{R}$. Then*

$$\begin{aligned} \inf_A f + \inf_A g \le \inf_A (f+g) \\ \sup_A f + \sup_A g \ge \sup_A (f+g) \end{aligned}$$

namely

$$\inf_A f + \inf_A g \le \inf_A (f+g) \le sup_A(f+g) \le sup_A f + \sup_A g$$

if $\sup_A f, \sup_A g, \inf_A f, \inf_A g$ exist.

This implies that **summing** two function **nails down** the extreme values.

Lemma 3.

$$\forall \varepsilon > 0,\ p > q - \varepsilon \quad \iff \quad p \ge q$$

References

[1] B. Hajek: Random Processes for Engineers, *Cambridge Press*, 2015