Probability and Stochastic Processes Course Notes

Chia-Hao Chang, chchangkh@utexas.edu

Fall 2018

1 Martingale

1.1 Basics: Definition and Examples of a Martingale

Definition 1 (Martingale)**.** let $\{(Y_n, \mathcal{F}_n) \colon n \geq 1\}$ be a sequence of random variables and sub σ-algebra of $\mathcal{F}$, respectively, s.t. Y_n is $\mathcal{F}_n$-measurable, $\forall n$ and $\{\mathcal{F}_n \colon n \geq 1\}$ is a filtration. $\{Y_n : n \geq 1\}$ is said to be a **martingale** with respect to $\{\mathcal{F}_n \colon n \geq 1\}$ if

(a) $\mathbb{E}[|Y_n|] < \infty$, for all n.

(b) $\mathbb{E}[Y_n|\mathcal{F}_m] = Y_m$, for all $1 \leq m < n$.

Sometimes we say that $\{Y_n \colon n \geq 1\}$ is a martingale adopted to a stochastic process $\{X_n \colon n \geq 1\}$ if $\{Y_n \colon n \geq 1\}$ is a martingale with respect to the sequence of σ-algebras $\mathcal{F}_n \coloneqq \sigma\{X_m \colon 1 \leq m \leq n\}$. If we say $\{Y_n \colon n \geq 1\}$ is a martingale, we usually mean that $\{Y_n \colon n \geq 1\}$ is a martingale adapted to the filtration generated by itself $\mathcal{F}_n \coloneqq \sigma\{Y_i \colon 1 \leq i \leq n\}$. That is to say, if Y_n is a martingale, then the best estimation given the information up to time m ($m < n$) is Y_m.
The following are some examples of martingale.

Example 1 (Random Walk on 1D)**.** Let $\{X_n : n \geq 1\}$ be a sequence of i.i.d. zero mean random variables s.t. $\mathbb{E}[|X_i|] < \infty$. Define $S_0 = 0$ and $S_n = \sum_{i=1}^n X_i$, then $\{S_n : n \geq 0\}$ is a martingale adopted to the process $\{X_n : n \geq 1\}$.

Example 2 (Product of Independent Variables)**.** Let $\{X_n \colon n \geq 0\}$ be a sequence of independent variables with mean 1 and $\mathbb{E}[|X_n|^2] < \infty$. Define $Y_n = \prod_{k=0}^n X_k$. $\{Y_n \colon n \geq 0\}$ is a martingale with respect to the process $\{X_n \colon n \geq 0\}$.

Example 3 (Empty Bin Problem)**.** Suppose there are m balls and N bins in the system. Each ball is dropped independently and randomly into one bins at a time. Define $Y_n \coloneqq$ number of empty bins after dropping n balls and $M_n \coloneqq Y_n(1-\frac{1}{N})^{m-n}, 0 \leq n \leq m$. Then, $\{M_n \colon 0 \leq n \leq m\}$ is a martingale adapted to the process $\{Y_n \colon 0 \leq n \leq m\}$.

Example 4 (Doob's Backward Martingale)**.** Let $(\Omega, \mathcal{F})$ be a measurable space and X be a random variable on it. Let $\{\mathcal{F}_n \colon n \geq 0\}$ be a filtration. Define $Y_n \coloneqq \mathbb{E}[X|\mathcal{F}_n]$. Then $\{Y_n \colon n \geq 0\}$ is martingale with respect to the filtration $\{\mathcal{F}_n \colon n \geq 0\}$.

Example 5 (Gambling)**.** Let $\{X_n \colon n \geq 1\}$ be a sequence of IID random variable s.t. $P(X_i = 1) = P(X_i = -1) = \frac{1}{2}$. Define $Y_0 = a > 0$ and $Y_{n+1} \coloneqq Y_n + b_n(Y_0, Y_1, \cdots, Y_n)X_{n+1}$. $b_n \colon \mathbb{N} \to \mathbb{N}$ is called the *betting strategy*. Then, $\{Y_n \colon n \geq 0\}$ is a martingale adapted to $\{X_n \colon n \geq 1\}$.

Example 6. Let $\{X_1, X_2, \cdots, X_n, \cdots\}$ be a sequence of independent zero-mean random variables. $S_0 \coloneqq 0$, $S_n = \sum_{i=1}^{n} X_i$. $\mathcal{F}_n = \sigma\{X_i : 1 \le i \le n\}$. $Var[X_j] = \mathbb{E}[X_j^2] = \sigma_j^2$. Let

$$A_n = \sum_{i=1}^{n} \sigma_i^2$$
$$M_n = S_n^2 - A_n$$

where $M_0 = 0$. Then $\{(M_n, \mathcal{F}_n) \colon n \ge 0\}$ is a martingale.

1.2 Two Pillars of Martingale: Convergence and Stopping Theorem

Next, we are going to show the two most powerful theorems in martingale theory: *Martingale stopping theorem* and *Martingale Convergence Theorem.*

Theorem 1 (Martingale/Optional Stopping Theorem)**.** *Let $\{(M_n, \mathcal{F}_n) \colon n \ge 0\}$ be a martingale and τ be a stopping time with respect to $\{\mathcal{F}_n \colon n \ge 0\}$. Suppose at least one the following conditions holds:*

(a) $\exists \alpha > 0$ s.t. $|M_n| < \alpha, \quad \forall n$ almost surely.

(b) $\exists \beta > 0$ s.t. $\tau < \beta$ almost surely.

(c) $\exists c > 0$ s.t. $\mathbb{E}[|M_{n+1} - M_n|] \le c$ and $\mathbb{E}[\tau] < \infty$ almost surely.

Then, we have $\mathbb{E}[M_\tau] = \mathbb{E}[M_0]$. (Assume τ is proper i.e., $P(\tau < \infty) = 1$.)

Theorem 2 (Martingale Convergence Theorem in $\mathcal{L}^1$)**.** *Let $\{Y_n \colon n \ge 0\}$ be either a non-negative supermatingale or a bounded submartingale with respect to $\{X_n \colon n \ge 0\}$. Then, almost surely, $\lim_{n\to\infty} Y_n$ exists and is finite.*

We now introduce a stronger version of martingale convergence theorem when the p-th moment exists and is finite.

Theorem 3 (General Martingale Convergence Theorem in $\mathcal{L}^p$ space)**.** *Let $\{X_n \colon n \ge 1\}$ be a martingale with respect to the filtration $\{\mathcal{F}_n \colon n \ge 1\}$ and satisfies $\sup_{n\ge 1} \mathbb{E}[|X_n|^p] < \infty$ for some $p > 1$. Then there exists a random variable X_∞ s.t. $\lim_{n\to\infty} X_n = X_\infty$ almost surely and in $\mathcal{L}^p(\Omega, \mathcal{F}, P)$. Further, $X_n = \mathbb{E}[X_\infty | \mathcal{F}_n]$ almost surely for all $n \in \mathbb{N}$.*

Finally, we close this section by a famous inequality.

Theorem 4 (Azuma-Hoeffding's Inequality)**.** *Let $\{X_n \colon n \ge 0\}$ be a martingale such that $|X_k - X_{k-1}| \le c_k, \quad \forall k \ge 1$. Then, for all $t > 0$ and any $\lambda > 0$*

$$P(|X_t - X_0| \ge \lambda) \le 2 \exp\left(-\frac{\lambda^2}{2\sum_{k=1}^{t} c_k^2}\right)$$