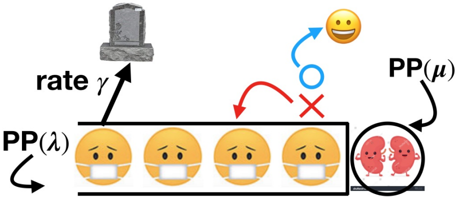
Effects of Patient Heterogeneity on Patients' Choice in a Kidney Transplant System

Chia-Hao Chang, John Hasenbein

Department of ECE and ORIE, The University of Texas at Austin

October 23, 2019

Model



We consider a M/M/1 queue with reneging[1].

1. Patient Arrival $\sim PP(\lambda)$.
2. Kidney Arrival $\sim PP(\mu)$. $\mu \equiv 1$.
   - Kidney quality is **random** and drawn from $\mathcal{X}$.
   - Upon a kidney arrival, its quality $x \in \mathcal{X}$ is revealed to everyone.
3. Each patient has $exp(\gamma)$ lifespan, before transplant.

[1] Su. and Zenios.: Patient Choice in Kidney Allocation: The Role of the Queueing Discipline, *MSOM*, 2004

Literature Review

- Stochastic Models for Transplant Systems:
  1. Queueing Models: Su and Zenios '04.
  2. Stochastic Assignment: Su and Zenios '05.
  3. Mechanism Design: Su and Zenios '04.
- Parameter Sensitivity:
  1. General Framework: Çil et al. '09, Vercraene et al. '18.
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Our Contribution

1. Stepping stones toward understanding a more complex heterogeneous transplant system.
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Patient's Problem

Each patient is faced with a Stochastic Shortest Path problem(SSP)[2], where

1. Two absorbing states: $\{death, transplant\}$.
2. h reward per unit time.
3. Transplant with kidney quality x: Moves to state "transplant" and receive reward x.
4. Die: Moves to state "death" and receives reward nothing.

Patient's Goal

Maximize total reward.

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Patient's Problem(cont'd)

The patient is faced with an MDP, where

$$V(1) = \sup_{a_1 \in [\underline{x}, \bar{x}]} \frac{1}{1+\gamma} \Big\{ h + \bar{F}'(a_1)\mathbb{E}[X | X \geq a_1] + (1 - \bar{F}'(a_1))V(1) \Big\},$$

$$V(k) = \sup_{a_k \in [\underline{x}, a_{k-1}]} \frac{1}{1+\gamma k} \Big\{ h + \bar{F}'(a_{k-1})V(k-1) + (\bar{F}'(a_k) - \bar{F}'(a_{k-1}))\mathbb{E}[X | a_{k-1} > X \geq a_k] + (1 - \bar{F}'(a_k))V(k) + (k-1)\gamma V(k-1) \Big\}.$$

$V(k)$ is the value functions at position k.
$V(k)$ is the (expected) accumulated reward before he leaves the queue.
$\bar{F}'(x) = P(X \geq x)$.

Our Work and Results

Study the effects of **death rates** γ on the value functions.
Non-trivial trade-off because:

1. Higher Death Rates $\rightarrow$ Prone to higher probability of death.
2. Higher Death Rates $\rightarrow$ Move faster to the front of the queue, receive better kidney offers.

Our Main Results

When the support of the kidney qualities $\mathcal{X}$ is bounded, if $\gamma_1 > \gamma_2$, $V_1(k) \leq V_2(k)$ for all k, i.e., the higher the death rate, the lower the value function.

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Road Map of Our Proof

1. When $\mathcal{X}$ is finite, we show

$$V_1(k) < V_2(k), \text{ for all } k \in \mathbb{N},$$

   if $\gamma_1 > \gamma_2$.

2. For a continuous distribution, approximate $V(k)$ by a sequence of r.v.'s with finite support; specifically,

$$V_1^{D_n}(k) \xrightarrow{n\to\infty} V_1^{C}(k), \text{ for all } k \in \mathbb{N}, \text{ if } d(D_n, C) \xrightarrow{n\to\infty} 0,$$

   where $d(\cdot,\cdot)$ is the Kolmogorov metric.

3. In the **limiting argument**

$$V_1^{C}(k) = \lim_{n\to\infty} V_1^{D_n}(k) \le \lim_{n\to\infty} V_2^{D_n}(k) = V_2^{C}(k), \text{ for all } k \in \mathbb{N},$$

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Finite Kidney Type

Consider $\mathcal{X} = \{x_1, \dots, x_n\}$, $x_1 > x_2 > \cdots > x_n$, and $P(X = x_i) = p_i$. $(px)_{\leq i} = \sum_{j \leq i} p_j x_j$. $(px)_{<i} = (px)_{<=i-1}$.

$$V(1) = \max_{a_1 \in \{x_i:\ i=1,2\dots,n\}} \left\{ \frac{1}{1+\gamma} [h + \bar{F}'(a_1)\mathbb{E}[X|X \geq a_1] + (1 - \bar{F}'(a_1))V(1)] \right\}$$
$$= \max_{0 \leq i \leq n} \left\{ \frac{1}{1+\gamma} [h + (px)_{\leq i} + (p_{i+1} + p_{i+2} + \dots p_n)V(1)] \right\}$$

and

$$V(k) = \sup_{a_{k-1} \leq a_k \leq n} \left\{ \frac{1}{1+k\gamma} [h + (p_1 + p_2 + \cdots + p_{a_{k-1}})V(k-1) + \right.$$
$$(p_{a_{k-1}+1} x_{a_{k-1}+1} + p_{a_{k-1}+2} x_{a_{k-1}+2} + \cdots + p_{a_k} x_{a_k}) + (p_{a_k+1} + p_{a_k+2} + \cdots + p_n)V(k)$$
$$\left. + (k-1)\gamma V(k-1)] \right\}.$$

Finite Kidney Type: The Analytical Solution

Lemma

$V(k)$ in the previous page are given by

$$V(k) = \begin{cases} \frac{kh+(px)_{<\ell_{i+1}}}{k\gamma+p_{<\ell_{i+1}}}, & \text{for } k^*_{\ell_i} \leq k < k^*_{\ell_{i+1}}, \\ \frac{kh+(px)_{\leq n}}{k\gamma+1}, & \text{for } k \geq k^*_n, \end{cases}$$

where $k^*_{\ell_i}$ satisfies $V(k^*_{\ell_i} - 1) > x_{\ell_i} > x_{\ell_i+1} > \cdots > x_{\ell_{i+1}-1} > V(k^*_{\ell_i})$ and $1 = \ell_1 < \ell_2 < \ell_3 < \cdots < \ell_m \leq n$.

Intuitively, k^*_α is the position where kidney with quality x_α is accepted.

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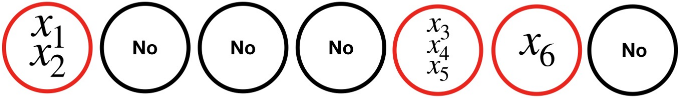
Intuitively, k^*_α is the position where kidney with quality x_α is accepted.

$$\ell_1 = 1,\ \ell_2 = 3,\ \ell_3 = 6$$



$$k^*_{\ell_1} = k^*_1 = k^*_2 = 1,\ k^*_{\ell_2} = k^*_3 = k^*_4 = k^*_5 = 5,\ k^*_{\ell_3} = k^*_6 = 6$$

Property of k^*_α

Lemma

Also let k_j^{*1} and k_j^{*2}, $j = 1, 2, \ldots, n$ be the previously defined positions of in system 1($\gamma = \gamma_1$) and system 2($\gamma = \gamma_2$), respectively. If $\gamma_1 > \gamma_2$, then

$$k_j^{*1} \leq k_j^{*2} \text{ for all } j = 1, 2, \ldots, n.$$

The lemma states that a system where patients are susceptible to higher death rates tend to accept the kidney offers sooner.

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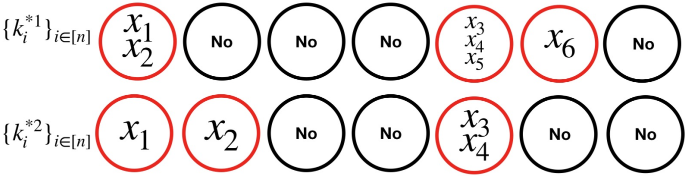

Finite Kidney Type: The Sensitivity Analysis

Theorem

Let the support of the kidney distributions $\mathcal{X}$ be finite. Let $V_1(k)$ and $V_2(k)$ be the value function when $\gamma = \gamma_1$ and $\gamma = \gamma_2$, respectively. If $\gamma_1 > \gamma_2$,

$$V_1(k) < V_2(k), \text{ for all } k.$$

Approximation of Value Functions

Definition (Hypo-Convergence (Rockafeller and Wets '09))

We say $(f^n \colon n \in \mathbb{N})$ *hypo-converges* to $f \colon \mathcal{X} \to \mathbb{R}$ if for any $y \in \mathcal{X}$,

$$\limsup_{n\to\infty} f^n(y^n) \le f(y), \text{ for every } y^n \to y, \text{ and}$$
$$\liminf_{n\to\infty} f^n(y^n) \ge f(y), \text{ for some } y^n \to y.$$

Lemma (Convergence in Maximization (Rockafeller and Wets '09))

If y is a limit point of the maximizers of $(f^n \colon n \in \mathbb{N})$, then y is also a maximizer of f, to which the sequence (f^n) hypo-converges.

$$\lim_{n\to\infty} \arg\max f^n \subseteq \arg\max f.$$

In particular, if f has a unique maximizer, the sequence of maximizers converge to $\arg\max f$.

Approximation of Value functions(cont'd)

Theorem

Let F_c be a continuous distribution with compact support $\mathcal{X} = [\underline{x}, \bar{x}]$. Let $(D_n \colon n \in \mathbb{N})$ be a sequence of discrete distributions whose supports are finite subsets of $\mathcal{X}$ such that $d(D_n, F_c) \to 0$. Denote $V^n(k)$ the value function at position k when the distribution is D_n; denote $V(k)$ the value function at position k when the distribution is F_c. Then,

$$\lim_{n \to \infty} V^n(k) = V(k).$$

Sensitivity on Value functions

Theorem

Let $V_1(k)$ and $V_2(k)$ be the value functions when $\gamma = \gamma_1$ and $\gamma = \gamma_2$, respectively. If $\gamma_1 > \gamma_2$, we have $V_1(k) \le V_2(k)$, for all $k \ge 1$, whenever the support of kidney distributions are bounded subsets of $\mathbb{R}$.

W.L.O.G, let F be a continuous distribution whose support is $\mathcal{X} = [\underline{x}, \bar{x}]$. $\{D_n : n \ge 1\}$: finite distributions whose supports are subsets of $\mathcal{X}$ and $d(F, D_n) \to 0$. We have,

$$V_1^n(k) < V_2^n(k), \text{for all } k \text{ and all } n.$$

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Future Work

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