Game Theory Final Project: Characterizing Voting Equilibria under Different Voting Models

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May 10, 2019

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1 Introduction

In this report, we investigate the equilibria of different voting games. Specifically, we investigated several prevalent voting systems, including the *plurality rule*, *approval voting* and *runoff voting*. Equilibria properties of different voting systems are characterized.

Under plurality rule, each voter can either abstain or cast a single vote for at most one candidate. Under approval voting, each voter can give either one or zero votes independently to each of the candidates. Under Borda rule voting, each voter either abstains or votes. If he chooses to vote, he then casts zero votes to one candidate, one vote to another, two votes to the third and so on, such that each candidate he votes for receives a different number of votes from him. The candidates receiving most votes wins the election in the above three systems. A runoff election is divided into two rounds. A candidate wins outright in the first round if he obtains absolute majority of votes (e.g., more than 50% of the votes). If no such candidate exists, then the two candidates receiving most votes advance to the second round, in which the winner wins the election.

Under strategic voting, voters might not be sincere. Imagine you are in a three-candidate election. Suppose you are the supporter of A, whose approval rating is far ahead of B and C. It is most likely that you will choose to abstain since whether you vote or not does not affect the outcome of the election. On the other hand, if the approval rating are distributed as A: 25%, B: 35% and C: 40% and you prefer candidate A to B to C, it is likely that you will cast a vote for B since your vote is perceived decisive and might be wasted if you vote for A.

So, what differentiates the voting equilibrium and the usual Bayesian-Nash equilibrium is a person decides whether to vote or to abstain and who to vote based on his perception of whether his vote would be *pivotal*, i.e., if his vote would flip the outcome of the election. So instead of making his decision based on (traditional) expected utility maximization, he asks himself *how much utility increase will be claimed if I vote*? That is, in traditional game theory terminology, a user tries to maximize his expected utility *gain* if an additional vote from him is cast.

Nomenclature the set of candidates is represented by $\mathcal{K} \coloneqq \{1, 2, \cdots, K\}$. The types of voters are denoted by $\mathcal{T}$. $r(\cdot)$ is a probability distribution over $\mathcal{T}$ in which $r(t)$ specifies the probability that a voter is type t, for any $t \in \mathcal{T}$. A *voting strategy* $\boldsymbol{\sigma}\colon \mathcal{T} \to \Delta(\mathcal{K})$ is a set of probability distributions on the action set $\mathcal{K}$. $\boldsymbol{\sigma} = \{\sigma(\cdot|t)\colon t \in \mathcal{T}\}$, where $\sigma(i|t)$ specifies the probability that a type-t voter votes for candidate i. A *distributional strategy* $\tau(\cdot, \cdot)$ is a probability distribution over $\mathcal{K} \times \mathcal{T}$ where $\tau(i, t)$ specifies the probability that a player is type t and he votes for i. $(x)^+ = \max\{x, 0\}$.

2 Plurality Rule

2.1 A Model with Costly Voting [1]

2.1.1 Model Description

In [1], a Poisson game $(\mathcal{T}, n, r, \mathcal{K}, U)$ is considered. We refer to "citizen" as people who have the right to vote and "voter" as citizens who actually vote (in equilibrium) in this subsection. If a citizen votes, a cost c is incurred. In this paper [1], we assume $\mathcal{T}$ can be fully characterized by the ordering of the candidates. For example, for a K-candidate election, there are $|\mathcal{T}| = K!$ types of citizens. More specifically, the utility of a citizen of type t derived from having his ℓth-ranked candidate win is λ_ℓ. We assume $1 = \lambda_1 > \lambda_2 > \cdots > \lambda_K = 0$.

2.1.2 Main Results when $K = 3$

Prop. 1 and 2 in [1] are special cases of Prop. 3. However, Prop. 1 is interesting in its own right. We defer Prop. 2 to appendix.

Proposition 1 (Proposition 1 in [1], Duverger's Law Equilibrium in a Three-Candidate Election)**.** *Suppose the voting cost $c \leq \frac{1}{4}$ and fix voter type proportions $r(t) > 0$ for all $t \in \mathcal{T}$. There exists a function $\bar{N}(c)$ such that, for all $n \geq \bar{N}(c)$, the following holds.*

(i) For each pair of candidates $i, j \in \mathcal{K} := \{A, B, C\}$, there exists a unique equilibrium in which exactly two candidates i, j receive a positive expected share of votes.

(ii) In every such two-candidate equilibrium, both candidates win with equal probability, and only two of the six reference type vote, namely those who rank i highest and j lowest and vice versa.

Proposition 1 verifies the famous Duverger's Law when the electorate is large. Essentially, Duverger's Law states that when there are multiple candidates in an election, only two of the candidates will receive a noticeable amount of votes. An interesting result is what (ii) implies: if a citizen ranks his preferred candidates $i > j > k$, the utility difference between whether i or j gets elected is not large enough to incentivize him to vote. The citizens who actually votes are the most "radical" supporters of i (resp. j) since in his opinion j (resp. i) is the last person he would like to see being elected.

2.1.3 Main Results for general K

Proposition 2 (Proposition 3 in [1])**.** *Let things be the same as before and let $\mathcal{R} \subseteq \mathcal{K}$ be a subset of size $|\mathcal{R}| \geq 2$. There exists cost $c_R > 0$ and a function $\bar{N}(c)$ such that, for all $c < c_R$ and $n \geq \bar{N}(c)$, there exists equilibrium in which*

(i) Candidate i receives votes with non-zero probability if and only if $i \in \mathcal{R}$.

(ii) Candidate i wins with probability $\frac{1}{|\mathcal{R}|}$ if and only if $i \in \mathcal{R}$.

(iii) Citizen type t has a probability of voting for i only if t ranks i highest and for all $j \in \mathcal{R} \setminus \{i\}$, j is one of the $|\mathcal{R}| - 1$st-least preferred candidates.

The intuition behind Proposition 2 and Proposition 3 in [1] is that in symmetry equilibrium, there will always be a group of "competitive" candidates, i.e., no *dictator* exists. It might seem weird at first glance since if most of the citizens (e.g. 80% of the citizens) rank A highest, it is likely most A is the only likely winner. The reason we could rule this out is that we only analyze *symmetric* equilibrium in which citizens of the same type *must* take the same strategy. The assumption of symmetric equilibrium is inessential. However, restricting our attention to symmetric equilibrium does simplify our analysis help us establish intuition.

Note that $\nu(c)$ is a decreasing function of c, which means as the cost of voting increases the motivation of voting decreases. This in turn reduces the number of voters. Further, if c gets too high, then $\nu(c) = 0$. That is, no one has incentive to vote in this equilibrium.

2.2 A K-Candidate M-Seat Model under Single Nontransferable Vote[2]

2.2.1 Brief Description of Single Nontransferable Vote (SNTV) System

The SNTV system is where each voter casts a single vote to a candidate, and there are $M \geq 1$ seats indicating the number of admissible winners. The top M candidates with the highest votes win the election.

2.2.2 Model Description

The author of [2] assumes that each of the n voters has a strict preference ranking over the election outcomes. Voter i's type is described by vector u_i=(u_{i1}, ..., u_{iK}), where u_{ij} is the utilities that voter i receives if candidate j is elected. An outcome α of the election is a subset $\alpha \subseteq \mathcal{K}$, where $|\alpha| = M$. An outcome α is preferred to β if and only if $\sum_{j\in\alpha} u_{ij} > \sum_{j\in\beta} u_{ij}$. The voter's knowledge about the expected outcome of the election is summarized in $\boldsymbol{\pi} = (\pi_j : j \in \mathcal{K})$, where π_j is the expected fraction of total votes that j receives.

2.2.3 Main Results

Theorem 1 (Theorem 1 in [2]). *Assuming that the candidates are relabeled such that $\pi_j \geq \pi_{j+1}$. If either (1) $0 < \pi_j < \pi_{M+1}$ or (2) $\pi_1 > \pi_M$, then π is not a limit of rational expectations.*

Corollary 1 (Corollary 1 in [2]). *If π is a limit of rational expectations, then (1) $\pi_1 = \pi_2 = ... = \pi_M$ and (2) $\pi_j \in \{0, \pi_{M+1}\} \forall j > M + 1$*

Intuitively, if all the voters are rational, since $\boldsymbol{\pi}$ is known to everyone, both voting for leading candidates and voting for trailing candidates are both considered "wasting votes". Corollary 1 divides equilibria into (1) Duvergerian equilibrium, where M+1 candidates get non-zero votes, and (2) non-Duvergerian equilibrium, where some of the runner-ups receive the same non-zero votes.

3 Approval Voting

3.1 A Candidate-Positioning Game[3]

3.1.1 Model Description

While in most models, voters' preferences over candidates are exogenously given, in reality, people's preferences over candidates are more likely to depend on the policy the candidate proposes.

In this model, $\mathcal{T} = \{0, 1, \cdots, 100\}$, where $t \in \mathcal{T}$ represents "policy position" (e.g., the tariff imposed) that the candidate favors. r is assumed to be uniform over $\mathcal{T}$. Let $\mathbf{x} \in [0, 100]^K$, where x_i is the policy position of the ith candidate. The utility $U(i|t)$ of a type-t voter if candidate i is elected is $U(i|t) = -(x_i - t)^2$.

This model is a variant of Myerson's voting model. A detailed description may be found in Appendix.

3.1.2 Main Results

Theorem 2 (Theorem 4 in [3]). *Under approval voting, in any positional equilibrium, all likely winners take the position 50.*

In a position game, the equilibrium is much more complex since now the *candidates* can adjust their policies as well. The equilibrium can be thought of as hierarchical- the candidates adjust their policies in order to attract more candidates; on the other hand, once the candidates have chosen their positions, the voters vote to maximize his expected utility gain. More details can be found in Appendix E.

Surprisingly, under approval voting, winners all take the position 50. This means that approval voting can exclude the left-ist and the right-ist.

4 Runoff Election

4.1 Analysis of Runoff Election in [4]

4.1.1 Model Description

In this work [4], a Poisson game $(\mathcal{T}, n, r, \mathcal{K}, U)$ is considered. There are only three candidates: $\mathcal{K} \coloneqq \{A, B, C\}$. The utility of a voter of type t when candidate i is elected is given by $U(i|t)$. When $U(i|t) > U(j|t), \forall j \neq i$, we say t is a i's *supporter*. In this paper [4], we are trying to characterize the voting equilibrium during the *first round election*. Since what voters anticipate will happen in the second round influences the behavior of the voters in the first round, we incorporate the influence of the second-round election on the first-round election by $P(i|i, j)$, the probability that conditional on i and j both advance to the second round, i defeats j, $\forall i, j \in \mathcal{K}$. We assume at the time of the first-round election, $P(i|i, j)$ are given and constant. Since there is no distinction between the candidates, we denote the set of candidates $\mathcal{K} = \{R, S, W\}$, where R is the *front-runner*, S is a *strong opponent* and W is a *weak opponent*.

We define $G_t(i, n, \tau)$ to be the *expected gain* of a voter of type t voting i when the expected share of votes is distributed by τ. Next, we define the *best response correspondence* $\mathcal{B}_t(\tau)$. For a voter of type t, a strategy profile $\boldsymbol{\sigma}$ and distribution r, $\mathcal{B}_t(\tau) \coloneqq \arg\max_{\sigma_t \in \Delta(\mathcal{C})} \sum_{i \in \mathcal{C}} \sigma(i|t) G_t(i, n, \tau)$. That is, $\mathcal{B}_t(\tau)$ is (the set of) the best response that a voter of type t can play when all other players play according to $\boldsymbol{\sigma}$ and the marginal distribution of votes received by each candidates is τ.

4.1.2 Main Results

Definition 1 (Definition 3 in [4], Asymptotically Strictly Perfect Equilibrium). Let $\{\Gamma_n\}_{n=1}^{\infty}$ be a sequence of Poisson games, where $\Gamma_n := (\mathcal{T}, n, r, \mathcal{C}, u)$. An equilibrium profile $\boldsymbol{\sigma}^*$ is said to be *asymptotically strictly perfect* for a sequence if there exists a sequence of Nash equilibrium $\{\sigma_n^*\} \to \boldsymbol{\sigma}^*$ such that for any $\delta > 0$, there exists $\epsilon > 0$ and $N \in \mathbb{N}$ such that for any $n > N$, if $\tau_n \in \Delta(\mathcal{K})$ and $|\tau_n - \tau(\boldsymbol{\sigma}_n^*, r)| < \epsilon$, then $P(t \in \mathcal{T}\colon \sigma_t^* \notin \mathcal{B}_t(\tau_n)) < \delta$.

An intuition behind a *strictly perfect equilibrium* is that it is a *robust* equilibrium. Indeed, in reality, a precise knowledge about τ is unlikely to be possible. However, a close estimation of distribution τ can often be expected (for example, through the polls). A perfect equilibrium guarantees even if there is a slight fluctuation between τ (what people perceive in real world) and $\tau(\boldsymbol{\sigma}_n^*, F)$ (induced by the equilibrium), the equilibrium holds still. In other words, a perfect equilibrium may well predict the "actual" equilibrium in real world.

Proposition 3 (Proposition 1 in [4]). *There always exist three asymptotically strictly perfect Duverger's law equilibria.*

The intuition behind Proposition 3 can be explained as follows. In this first-round election, if a voter expects only two candidates will receive positive share of votes, then exactly one of candidates will win outright. Therefore, his vote will be perceived decisive only when (1) he votes for one of the candidates and (2) a tie between them happens, since the expected number of voters $n \to \infty$. In other words, there is no incentive to vote for any other (non-competitive) candidates as the electorate grows large.

Proposition 4 (Proposition 2 in [4]). *For some distribution of preferences, there exists asymptotically strictly perfect equilibria in which three candidates receive a positive share of the votes. In these equilibria, all voters who prefer the front-runner to the runner-up vote for the front-runner. Some, but not all, of the supporters of the weak opponent vote for the strong opponent, regardless of which opponent is expected to receive more votes.*

The intuition behind Proposition 4 is more complex. For voters who prefer R to S, since R is most likely to win the election, voting for R not only increases R's probability of winning outright but ensures R advancing to the second round. For those who prefer S to R, his problem becomes which of S or W he should vote for. His choice between S and W depends on the relative magnitudes of pivotal probabilities between two events: (1) an additional vote for S makes S win outright in the first round and (2) an additional vote makes S instead of W advance to the second round. On the other hand, when it is more likely that S will advance to the second round against R, we only need to consider the supporter of W since the supporter of S will always vote for S. The supporter of W faces the trade-off between the extent to which he prefers W to S and the likelihood that S beats R in the second round. So, when a voter prefers W to S but the difference is sufficiently close, he will vote for S.

Proposition 5 (Proposition 3 in [4]). *There is no strictly perfect equilibrium other than those characterized in Proposition 3 and 4.*

Somewhat surprisingly, Proposition 5 shows that a commonly believed voting behavior-*pushover tactics* in which a supporter of R votes for the least possible winner of the election to increase R's chances of winning in the second round- doesn't exist. We would like to provide a somewhat different explanation from that in [4]. In real life, if we were to pushover, we must have a "precise estimation" of the probability that R will advance to the second round even without a vote from us. However, when I pushover, I also need to consider the pushover of other supporters of R as well, which might lead to the unpleasant outcome that R fails in the first round. If a coordination between supporters of R is admitted, this might happen. Nonetheless, as the size n of the electorate grows large ($n \to \infty$), such coordination is unlikely to happen.

5 Borda System

5.1 Borda System, Partial Voting, and Manipulation

5.1.1 Model Description

In a Borda *untreated* partial voting, where a voter i only votes for j candidates, then the candidates will receive ($K-1$, $K-2$, ..., $K-j$) votes from i according to the voter's preference.

In a *modified* Borda voting system, a voter's xth preference will always receive 1 point more than the (x+1)th preference regardless of whether the voter has cast that (x+1)th preference, i.e. if a voter only votes for j candidates, the candidates will receive (j, $j-1$, ..., 1) votes.

5.1.2 Main Results

In untreated partial voting, some voters will be incentivized to vote for only their top preference[5]. In untreated Borda voting system, candidates can manipulate the system by introducing other candidates. Suppose candidate B's major opponent is A. Candidate B can increase his chances of winning by introducing a similar candidate C. This device works best when every voter prefers B to C, so that B will increase his votes by the number of voters n. On the other hand, A will only increase his votes by the number of voters that prefer A to B and A to C, which by our construction is strictly less than n[6]. However, it is shown that it is impossible for a single voter to manipulate a Borda System with $|\mathcal{K}|$=3 [7]. Intuitively, no matter how a voter changes his vote, the most he can accomplish is for the winning candidate to tie with another candidate.

6 Conclusion

Several voting systems have been investigated. While voting games are usually complicated, and the analyses require many assumptions, surprisingly the theoretical results match what is observed in real life. Duverger's law is observed in most of the voting systems, and is reflected in real life as two-party systems.

Some lines of future work may include: the comparison of social welfare under different voting rules, how information aggregation shapes the outcome of election, how the other factors (e.g. miscounts) will influence the voting game, etc.

7 Appendix A: Poisson Games and its Applications in Runoff Elections

This section aims to provide some basic knowledge of a Poisson Game. Most of the materials can be found in [8], [9] and Appendix of [4].

In traditional game models, all the information is assumed to be a common knowledge to all the players (e.g., the number of players, the utility, etc.) In large games (e.g., in a voting game), however, it may be more realistic to admit that players have some uncertainty about how many others are in the game. Instead of knowing the exact number of players in the game, each player has a prior knowledge about the distribution of number of players in the game.

In particular, a *game with population uncertainty* can be described by a four-tuple $(\mathcal{T}, Q, \mathcal{C}, U)$, where $\mathcal{T}$ is the set of types of players, $\mathcal{C}$ is the action set which is assumed to be finite. Q is the probability distribution over the set of types $\mathcal{T}$. $U\colon \mathbb{Z}_+^{\mathcal{C}} \times \mathcal{C} \times \mathcal{T} \to \mathbb{R}$ is the utility/payoff function, where if $\mathbf{x} \in \mathbb{Z}_+^{\mathcal{C}}$, $\mathbf{x}$ is a non-negative integer vector with $|\mathcal{C}|$ entries and the cth entry $x(c)$ of $\mathbf{x}$ is the number of players taking action c, for all $c \in \mathcal{C}$; that is $U(\mathbf{x}, b, t)$ is the payoff to a player whose type is t and who chooses action b when the strategy profile is $\mathbf{x}$.

A special instance of games with population uncertainty is the *Poisson game*. A *Poisson game* is described by a five-tuple $(\mathcal{T}, n, r, \mathcal{C}, U)$, where $\mathcal{T}$, $\mathcal{C}$ and U are those defined previously. The population of a Poisson game is a random variable which follows $\textsc{Poisson}(n)$. Conditional on a player is present in the game, the probability that the player has type $t \in S \subseteq \mathcal{T}$ is $r(S)$, independent of any other players in the game. Under this setting, the payoff $U(\mathbf{x}, b, t)$ is defined as the payoff to a player whose type is t and who chooses action b when $\mathbf{x}$ is the action profile of all other players in the game (i.e., for each $c \in \mathcal{C}$, $x(c)$ is the number of players choosing action c, excluding this player when $b = c$).

The *distributional strategy* τ for a Poisson game $(\mathcal{T}, n, r, \mathcal{C}, U)$ is a probability distribution on $\mathcal{C} \times \mathcal{T}$ such that the marginal distribution on $\mathcal{T}$ is the same as r. In particular, for any $c \in \mathcal{C}$ and for any $t \in \mathcal{T}$ $\tau(c, t)$ is the probability that a player is type t and he chooses c as his action. A *strategy profile* σ specifies the strategy that each type of agent adopts. In particular for any $c \in \mathcal{C}$ and for any $t \in T$ $\sigma(c|t)$ is the probability that a player of type t adopts action c. The relation between σ and τ is

$$\sigma(c|t) = \frac{\tau(c, t)}{\tau(t)}$$

where $\tau(t) = \sum_{c \in \mathcal{C}} \tau(c, t)$ $(= r(t))$ is the probability that a player is of type t. Specify either τ or σ can characterize the equilibrium. However, in Poisson games, we use τ more frequently.

Next, given a distributional strategy τ, we have the following property

Theorem 3 (Independent-Actions Property, [9]). *For any $\mathbf{x} \in \mathbb{Z}_+^{\mathcal{C}}$, i.e., $\mathbf{x} = \{x(c)\}_{c \in \mathcal{C}}$, the probability $P(\mathbf{x}|n, \tau)$ that $\mathbf{x}$ is the action profile of the players in the games is given by*

$$P(\mathbf{x}|n, \tau) = \prod_{c \in \mathcal{C}} e^{-n\tau(c)} \frac{(n\tau(c))^{x(c)}}{x(c)!}$$

where $\tau(c) = \sum_{t\in T} \tau(c,t)$. *That is, the number of players chooses action* c *is distributed according* POISSON$(n\tau(c))$*, independent of any other actions.*

The expected payoff to a type$-t$ player choosing action b when τ is the marginal distribution of action profiles is

$$\sum_{\mathbf{x}\in\mathbb{Z}_+^{\mathcal{C}}} P(\mathbf{x}|n,\tau)U(\mathbf{x},b,t)$$

Further, let $G(b,n,\tau)$ denote the set of types for whom choosing action b would yield the maximal payoff. That is,

$$G(b,n,\tau) = \left\{ t\in T\colon b\in \arg\max_{c\in\mathcal{C}} \sum_{\mathbf{x}\in\mathbb{Z}_+^{\mathcal{C}}} P(\mathbf{x}|n,\tau)U(\mathbf{x},c,t) \right\}$$

Finally,

Definition 2 (Equilibrium in Poisson Games, [8])**.** A distributional strategy τ is said to be an *equilibrium* in a Poisson game $(T,n,r,\mathcal{C},U)$ if

$$\tau(b, G(b,n,\tau)) = \tau(b), \text{ for all } b\in\mathcal{C}$$

The following theorem guarantees the equilibrium always exists in a Poisson game.

Theorem 4 (Existence of Equilibrium in a Poisson Game, [8])**.** *For any Poisson game* $(T,n,r,\mathcal{C},U)$ *(where* T *is a compact metric space,* $\mathcal{C}$ *is a finite set, and* u *is continuous and bounded), there exists at least one distributional strategy that is an equilibrium.*

Definition 3 (Magnitude of a Sequence of Vectors [8])**.** Given a distributional strategy τ. Let $\{\mathbf{x}_n\}_{n=1}^{\infty}$ be a sequence of vectors in $\mathbb{Z}_+^{\mathcal{C}}$. The *magnitude* μ of the sequence is defined as

$$\mu \coloneqq \lim_{n\to\infty} \frac{\log(P(\mathbf{x}_n|n,\tau))}{n}$$

That is, the sequence of probabilities $P(\mathbf{x}_n,|\tau,n)$ converges to zero at the rate of $e^{\mu n}$.

An event A is defined as a subset of $\mathbb{Z}_+^{\mathcal{C}}$. Similarly, for a sequence of events $\{A_n\}_{n=1}^{\infty}$, the *magnitude* $\mu(A)$ of the sequence of events is defined as

Definition 4 (Magnitude of a Sequence of Events [8])**.** For a sequence of events $\{A_n\}_{n=1}^{\infty}$, the *magnitude* $\mu(\{A_n\}_{n=1}^{\infty})$ of the sequence of events is defined as

$$\mu(\{A_n\}_{n=1}^{\infty}) \coloneqq \lim_{n\to\infty} \frac{\log(P(A_n|n,\tau))}{n}$$

where for any event A, $P(A|n,\tau) = \sum_{\mathbf{x}\in A} P(\mathbf{x}|n,\tau)$.

That is, the magnitude of events measures the rate at which the probability decays to 0. Notice that if the magnitude $\mu_1 < \mu_2$, where μ_1 are μ_2 are magnitude of events $\{A_n^{(1)}\}_{n=1}^{\infty}$ and $\{A(2)_n\}_{n=1}^{\infty}$, respectively, then $\lim_{n\to\infty} \frac{P(A_n^{(1)}|n,\tau)}{P(A_n^{(2)}|n,\tau)} = 0$. Measuring the magnitude of a sequence of events turn out to be essential when we cast (large) Poisson games as voting models. This is because (1) in real-life election we often encounter a large electorate (2) the pivot probabilities are usually quite small (e.g., the probability that two candidates tie in a 10-million-voters election).

8 Appendix B: Characteristics of Approval Voting

The Model of Approval Voting [10]

Definitions

Definition 5. $\mathcal{P}$ denotes a voter's strict preference, where $a\mathcal{P}b$ means that the voter strictly prefers a to b.

Definition 6 (Definition 1 in [10])**.** Suppose $\mathcal{P}$ partitions the set of all candidates into nonempty subsets A_1, A_2, ..., A_k such that the voter is indifferent to different candidates if and only if the individual candidates are in the same subset. $\mathcal{P}$ is *dichotomous* (or *tricholomous*) if and only if there are exactly 2 (or 3) such subsets. Suppose the voter has $a\mathcal{P}b$ when $a \in A_i$ and $b \in A_j$ if and only if $i < j$. A subset of candidates B is *high* (or *low*) for $\mathcal{P}$ if and only if whenever B contains a candidate in A_j (or A_i), B also contains *every* candidate in A_i (or A_j), for all $i < j$.

Definition 7 (Definition 2 in [10])**.** Strategy S dominates strategy T for a voter if and only if the voter prefers the outcome of S to T in some circumstance, and likes the outcomes of S at least as much as that of T under all possible circumstances.

Definition 8 (Definition 4 and 5 in [10])**.** Assume preference order $\mathcal{P}$. Strategy S is *admissible* for a voting system if and only if S is feasible and there is no other feasible strategy under $\mathcal{P}$ that dominates S. S is *sincere* if and only if it is high for $\mathcal{P}$. The voting system is *strategyproof* if and only if exactly one strategy is admissible.

Main Results

Theorem 5 (Theorem 1 (Dominance) in [10])**.** *Strategy S dominates strategy T if and only if $S \neq T$, $S\backslash T$ is high for $\mathcal{P}$, $T\backslash S$ is low for P, and neither $S\backslash T$ nor $T\backslash S$ is the set of all candidates.*

Theorem 5 says that if a voter considers voting for his second favorite candidate, then he should also vote for his most favorite candidate under approval voting.

Theorem 6 (Theorem 2 (Admissible) in [10])**.** *Assume preference order $\mathcal{P}$. Strategy S is admissible for voting system s if and only if it is feasible, and one or both of the following criteria holds:*

1. *The subset of most-preferred candidates under $\mathcal{P}$ is a subset of S, and S cannot be divided into two nonempty subsets such that one is feasible for s and the other is low for $\mathcal{P}$*

2. *S does not include any candidate in the subset of least-preferred candidates under $\mathcal{P}$, and there is no nonempty subset A of candidates disjoint from S such that $A \cup B$ is feasible for s and A is high.*

Since neither not voting for any candidate nor voting for all candidates satisfy either criterion in Theorem 6, both strategies are inadmissible.

Corollary 2 (Corollary 2 in [10])**.** *Approval voting ensures unique admissible strategy for a voter if and only if his $\mathcal{P}$ is dichotomous.*

In other voting systems, there may be more than one admissible strategy.

Theorem 7 (Theorem 3 in [10])**.** *If $\mathcal{P}$ is dichotomous, then every voting system is sincere for $\mathcal{P}$. If $\mathcal{P}$ is trichotomous, then approval voting is the only sincere system.*

The first part of Theorem 7 simply says that for dichotomous preference, a voter would never vote for a candidate in the least-preferred subset, regardless of the voting system. The second part is intuitive as well; if a voter votes for a second-favorite candidate, there is no harm in also voting for his most favorite candidate in approval voting system. In the plurality rule, however, the voter might opt to vote for his second-favorite instead of his most favorite so as to ensure that his least favorite would not be elected.

9 Appendix C: Deferred Proof in Section 2

Outline of Proofs in Subsection 2.1

Outline of Proof of Proposition 1 in [1]

Without loss of generality, we consider the *AB-equilibrium* in which only candidates A and B receives positive amount of votes.

The first step is to identify that in equilibrium any citizen who votes must be indifferent between voting and not voting (since each citizen can randomize his voting strategy).

In an AB-equilibrium, citizens of type AB, AC, CA prefer A to B. So if they were to vote, they would vote for A. Similar arguments can be applied for citizens of types BA, BC, CB. In a Poisson game, for every candidate i the total votes that he receives follows a Poisson distribution whose mean is denoted by ν_i. Hence, the expected utility gain for candidate A for an AC type citizen is

$$\sum_{x=0}^{\infty}\left[\underbrace{\frac{1}{2}e^{-\nu_A}\frac{\nu_A^x}{x!}e^{-\nu_B}\frac{\nu_B^x}{x!}}_{*}+\underbrace{\frac{1}{2}e^{-\nu_A}\frac{\nu_A^x}{x!}e^{-\nu_B}\frac{\nu_B^{x+1}}{(x+1)!}}_{\star}\right]\times 1 = c \tag{1}$$

where the $*$-term is the probability gain that A and B both receive x votes from the rest of the electorate. The $\frac{1}{2}$-factor appears in $*$ because without his vote, A and B tie and the winner is decided by a fair coin flip. The $\star$-term represents the probability that A is one-vote behind B and thus an additional vote for A will bring A and B to tie. Hence, the left-hand-side is the expected utility gain for voting. In equilibrium, the voter must be indifferent between voting and not voting. The expected utility gain must be equal to the cost of voting c.

Similarly, the expected utility gain for candidate B for an BC type citizen is

$$\sum_{x=0}^{\infty}\left[\frac{1}{2}e^{-\nu_B}\frac{\nu_B^x}{x!}e^{-\nu_A}\frac{\nu_A^x}{x!}+\frac{1}{2}e^{-\nu_B}\frac{\nu_B^x}{x!}e^{-\nu_A}\frac{\nu_A^{x+1}}{(x+1)!}\right]\times 1 = c \tag{2}$$

The authors next show that for any $c \in (0, 0.5)$, there exists a unique pair (ν_A, ν_B) satisfying Eqn.(1) and Eqn.(2). Moreover, $\nu_A = \nu_B$ for any c and the solution $\nu_A(c)$ (or $\nu_B(c)$) is a decreasing function of c. Hence, $\nu_A = \nu_B$ is a necessary condition for AB-equilibrium.

It remains to show that only AC and BC type citizens have incentive to vote, given $\nu_A = \nu_B$. Consider an AC-type citizen, if he were to vote, he must vote for A and his expected utility gain must be equal to the voting cost c. The expected utility gain for candidate A for an AC type citizen can be found by replacing ν_B with ν_C in Eqn.(1), which equals to c as well. However, in an AB-equilibrium, C receives no share of votes, i.e., $\nu_C = 0$. With $\nu_C = 0$ the left-hand-side of Eqn.(1) is strictly greater than the left-hand-side of Eqn.(1) when $\nu_B > 0$. Hence, the expected utility gain for candidate A for an AC type citizen must not equal to c. So an AC-type citizen will not vote. Similarly, an BC-type citizen wouldn't vote either.

For a CA-type citizen, if he votes for A, his expected utility gain would be the left-hand-side of Eqn.(1) multiplied by λ since he ranks A second. But $\lambda < 1$, so his expected gain of voting for A is smaller than c. On the other hand, the pivot probability of C winning the election in an AB-equilibrium is 0. He therefore wouldn't vote for C, either. Similar analysis can be applied for CB-type citizen and he wouldn't vote either.

Finally, because $\nu_A = \nu_B$, it follows both candidate have the same probability of winning the election. This completes the proof.

Outline of Proof of Proposition 2 in [1]

Proposition 6 (Proposition 2 in [1]). *Suppose that* $c \leq \frac{2}{3}(1-\frac{\lambda}{2})$ *and fix voter type proportions* $r(t) > 0$ *for all* $t \in \mathcal{T}$. *There exists a function* $\bar{N}(c)$ *such that for all* $n \geq \bar{N}(c)$,

(i) There exists equilibria in which all three candidate receive a positive expected number of votes.

(ii) In every such equilibrium, all candidates win with probability $\frac{1}{3}$ *and the expected number of votes received by each candidate (denoted by* $\nu_i(c)$, *for* $i \in \{A, B, C\}$*) is the same, i.e.,* $\nu_A(c) = \nu_B(c) = \nu_C(c) := \nu(c)$. *Moreover,* $\nu(c)$ *is a decreasing function of* c. *Furthermore, in every such equilibrium, every voter votes for his highest-ranked candidate.*

Firstly note that since N is sufficiently large, no citizen votes almost surely. If a citizen were to vote, he was indifferent between voting and not voting.

The first step is show the *existence* of such ABC-equilibrium with the stipulated property. In the ABC-equilibrium, every candidate receives a positive share of votes. By assumption, in ABC-equilibrium if a citizen were to vote, he voted for his highest-ranked candidate. In particular, for each $i \in \{A, B, C\}$

$$P(i \text{ receives } n \text{ votes}) = e^{-(\nu_{ij}+\nu_{ik})}\frac{(\nu_{ij} + \nu_{ik})^n}{n!}$$

for $j \neq i$ and $k \neq i$. ν_{ij} is the expected number of votes that i received from ij-type citizens.

Next, the author consider 6 pivotal events that a citizen of type AB would face if he were to vote for A. With the help of these pivotal events, the expected utility gain $EB_{AB}(\nu_A, \nu_B, \nu_C)$ of voting for A for a citizen of type-AB is

$$EB_{AB}(\nu_A,\nu_B,\nu_C) = [F_2(;1,1;\nu_A\nu_B\nu_C)\frac{2-\lambda}{3} + F_2(;2,2;\nu_A\nu_B\nu_C)\nu_B\nu_C\frac{2-\lambda}{6}]e^{-(\nu_A+\nu_B+\nu_C)} + \text{PIV}_{AB,A}\frac{1-\lambda}{2} + \text{PIV}_{AC,A}\frac{1}{2}$$

The expected utility gain for other types (e.g. AC, CB, etc.) of citizens follows a similar formula as the above one. In an ABC-equilibrium, we have $EB_{ij}(\nu_A,\nu_B,\nu_C) = c$ for all $i,j \in \{A,B,C\}$. By solving the set of equations, we have established the existence of ABC-equilibrium with the desired properties.

The next step is show that if ν_A, ν_B and ν_C satisfy the set of equations: $EB_{ij}(\nu_A,\nu_B,\nu_C) = c$ for all $i,j \in \{A,B,C\}$, then $\nu_A = \nu_B = \nu_C$. Essentially, this part of the proof proceeds by comparing any two different $EB_{ij} = EB_{i'j'}$. By canceling out the identical terms, the authors were able to show that (1) the pivot probabilities $\text{PIV}_{ij,i} = \text{PIV}_{ij,k}$ for any distinct i,j,k combinations; (2) $EB_{ij} = EB_{ji}$ for all i,j.

Next, the authors proceeds by contradiction to show the ordering between ν_i's. The authors show that $\text{PIV}_{AB,A} < \text{PIV}_{BA,B}$ if $\nu_A > \nu_B$, which contradicts $\text{PIV}_{AB,A} = \text{PIV}_{BA,B}$. Thus, it has to be $\nu_A \leq \nu_B$. Similarly, we can show $\nu_i \leq \nu_j$ for all i,j, in which case $\nu_A = \nu_B = \nu_C$.

The next step is to show that for any $c < \frac{2}{3}(1-\frac{\lambda}{2})$, the equilibrium is unique. First note that the equilibrium is solely characterized by ν, the expected votes received by each candidate. The problem now boils down to whether there is a unique solution ν for each c.

By replacing $\nu_A = \nu_B = \nu_C = \nu$ in $EB_{ij}(\nu_A,\nu_B,\nu_C)$, the expected utility gain can now be represented as a function of ν which we denote by $f(\nu)$.

The authors first show that $f(\nu)$ is strictly decreasing on $[0,\infty)$. Next, the authors show that $\lim_{\nu\to 0} f(\nu) = \frac{2}{3}$ and $\lim_{\nu\to\infty} f(\nu) = 0$. In this way, the equation $f(\nu) = c$ has a unique solution for $c \in [0,\frac{2}{3}]$. In particular, for $c \in [0,\frac{2}{3}(1-\frac{\lambda}{2})]$, $f(\nu) = c$ has a unique solution, implying that the equilibrium is unique.

In remains to show that everyone votes *sincerely*, i.e., everyone votes for his most preferred candidate. We defer the proof to the next subsection in which a more general result can be obtained.

Outline of Proof of Proposition 3 in [1]

The proof is similar to that of Proposition 1 and 2. Consider first a strategy profile in which only candidates in relevant group $\mathcal{R}$ receives votes. Further, we assume under such strategy profile, each candidate receives the same expected votes ν.

The next step is consider the expected utility gain for a voter of type t if he votes for his ℓth preferred candidate. Recall if the ℓth candidate of a voter is elected, his utility is λ_ℓ. The expected utility gain for a voter is comprised of three pivotal events: (1) none of the candidates in $\mathcal{R}$ receives any vote (2) there are multiple candidates (including the one our voter votes) tie for the first place (3) the candidate for whom the voter votes is one behind the candidates tying for the first place. We denote this expected utility gain by $g(\nu)$.

Again, the authors show that g is strictly decreasing in ν and evaluated the values $g(0)$ and $\lim_{\nu\to\infty} g(\nu)$. With a similar argument, we can establish the uniqueness of the equilibrium.

To show that each voter votes for his most-preferred candidate, the authors proceed by contradiction. It is argued that if a voter of type t votes for his ℓth preferred candidate ($\ell \geq 2$) instead of voting his most preferred one, he could unilaterally increase his expected utility gain by switching to his most-preferred candidate. Note this part also completes the rest of the proof of Proposition 2.

10 Appendix D: Myerson's Voting Model in [3]

Myerson's Model in [3]

Model Description

The voting model is described by $(\mathcal{K}, V, \mathcal{T}, r)$, where V is the set of possible ballots. For example, under plurality rule, $V = \{\mathbf{e}_j : 1 \leq j \leq K\} \cup \{\mathbf{0}\}$, where $\mathbf{e}_j$ means voting for the jth candidate; under approval voting, $V = \{0,1\}^K$, where a ballot $\mathbf{v} \in V$ has its ℓth entry v_ℓ being 1 if and only if he votes for the ℓth candidate. $\mathcal{T} \subseteq \mathbb{R}^k$ is assumed to be finite. For any $\mathbf{t} \in \mathcal{T}$, t_j is the utility that a voter of type $\mathbf{t}$ receives if candidate j is elected. In this model, the voters are assumed to be *non-atomic* in which case the each voter has infinitesimal influence on the outcome of the election. One might think of such setting as an approximation when the number of voters tends to infinity. Let $\mathbf{p} = \{p_{ij}\}_{ij \in H}$ be a pivot probability vector perceived by a voter, where H is the set of all unordered pairs of candidates and p_{ij} is the (perceived) probability that i and j will tie for the first place. In this model, it is assumed that a tie with more than three candidates has negligible probability compared to that of two-candidate ties. We assume that when submitting a ballot $\mathbf{v}$ the probability of flipping the outcome from j to i is $p_{ij}(v_i - v_j)^+$. Let $G(\mathbf{p}, \mathbf{v}, \mathbf{t})$ be the expected utility gain by a voter of type $\mathbf{t}$ submitting ballot $\mathbf{v}$ when $\mathbf{p}$ is his vector of perceived probabilities.

$$G(\mathbf{p}, \mathbf{v}, \mathbf{t}) = \sum_{ij \in H} p_{ij}(v_i - v_j)(t_i - t_j)$$

Given an election result τ, the predicted score $S_i(\tau)$ (i.e., the expected vote received per-capita) of candidate i is given by

$$S_i(\tau) = \mathbb{E}_{\mathbf{v} \sim \tau(\cdot, \mathcal{T})}[v_i] = \sum_{\mathbf{v} \in V} v_i \tau(\mathbf{v}, \mathcal{T})$$

Also, given any election result τ a *likely winner* is a candidate with the highest predicted scores. In particular, $W(\tau)$ denotes the set of likely winners.

$$W(\tau) := \arg\max_{j \in \mathcal{K}} S_j(\tau)$$

Next, we consider the relation between a pivot probability vector $\mathbf{p}$ and the outcome τ of the election. We say $\mathbf{p}$ *justifies* τ if

$$\tau \in \{\mu \text{ is a probability distribution over } \mathcal{K} \times \mathcal{T} \colon \text{if } \mu(\mathbf{v}, \mathbf{t}) > 0, G(\mathbf{p}, \mathbf{v}, \mathbf{t}) = \max_{\mathbf{w} \in V} G(\mathbf{p}, \mathbf{w}, \mathbf{t})\}$$

That is to say, if pivot probability $\mathbf{p}$ justifies τ, then the resulting equilibrium formed by voters who (strategically) cast his vote $\mathbf{v}$ according to $\mathbf{p}$ is τ.

Given an election result τ, if the predicted score $S_i(\tau) < S_j(\tau)$, then it should be the case that for any other candidate h, the pivot probability between i and h is much smaller than that between j and h. Formally speaking, given an election result τ and any $0 < \epsilon < 1$, we say that the pivot probability $\mathbf{p}$ satisfies the *ordering condition* for ϵ if, for every distinct candidates i, j, h, if $S_i(\tau) < S_j(\tau)$, then $p_{ih} \leq \epsilon p_{jh}$.

Definition 9 (Voting Equilibrium in [3])**.** An election result τ is a *voting equilibrium* if for every $\epsilon > 0$, there exists some vector $\mathbf{p}$ of pivot probabilities that justifies τ and that satisfies the ordering condition for ϵ.

Theorem 8 (Theorem 1 in [3])**.** *In every electorate situation, the set of voting equilibria is nonempty.*

We omit the proof here since the proof requires some additional knowledge in real analysis (in particular, the fixed-point theory). Interested readers may refer to [3] for details.

11 Appendix E: Deferred Proofs in Section 3

Supplementary Materials in Section 3.1

In a candidate-positioning game, there are two groups competing for equilibrium- the candidates and the voters.

Candidates adjust their policy position in order to attract more voters to be elected; in the mean while, once the policy positions have been placed, a equilibrium will be formed among the voters. This equilibrium, in turn, incentivizes the candidates to modify their policy position in order to gain more votes. The complicated interaction between the candidates and votes makes the equilibrium analysis much harder to handle. We begin by defining (a proper) notion of equilibrium.

Definition 10 (Positional Equilibrium in [3])**.** Under any electoral system V (e.g. plurality rule or approval voting), for any vector $\mathbf{x} \in [0, 100]^K$, we let $\mathcal{E}(\mathbf{x})$ denote the set of voting equilibria for the electoral system that would exist after the candidates chose the positions listed in $\mathbf{x}$. A pair $(\mathbf{x}, \tau)$ is said to be a *positional equilibrium* under V if the following conditions are satisfied

(a) $\tau \in \mathcal{E}(\mathbf{x})$.

(b) For each candidate $i \in W(\tau)$, for every position y_i there exists a voting equilibrium $\eta \in \mathcal{E}(\mathbf{x}_{-i}, y_i)$ such that either $W(\tau) \subseteq W(\eta)$ or $i \notin W(\eta)$.

(c) For each candidate $j \notin W(\tau)$, for every position y_j, there exists a voting equilibrium $\eta \in \mathcal{E}(\mathbf{x}_{-j}, y_j)$ such that $j \notin W(\eta)$.

Condition (a) simply states that τ has to be an equilibrium induced by the policy positions the candidates choose. The second condition asserts that for any likely winner, a unilateral deviation from him can either lead to a new equilibrium in which every likely winner is still a likely winner or a new equilibrium in which he is eliminated; in other words, condition (b) asserts that each likely winner can not better off by unilateral deviation from the equilibrium. Condition (c) asserts that a deviation by a candidate who is not a likely winner can lead to another equilibrium in which he is still not a likely winner. That is, (c) and (b) jointly asserts that each candidate can not better off by a unilateral deviation of policy in equilibrium.

Outline of Proof of Theorem 4 in [3]

The proof is based on the following lemma in [3].

Lemma 1 (Lemma in [3]). *Under approval voting, for any* $\mathbf{x}$ *with at least one entry* $x_i = 50$*, if* $\tau \in \mathcal{E}(\mathbf{x})$*, then only candidates positioned at 50 can be likely winners in* τ*.*

The lemma shows that if any of the candidates takes position 50, then all other candidate must also take position 50 in equilibrium.

The second step is to show that in equilibrium, *at least one* the candidate i chooses his position $x_i = 50$. Indeed, the authors also argue that if a candidate is not positioned at 50, he could better off by moving to 50. This completes the proof.

12 Appendix F: Deferred Proofs in Section 4

To begin with, we first introduce a few notations. We denote by γ_{ij} the expected fractions of voters who prefer $i \succ j \succ k$. The utility $U(i, j|t)$ of a second-round election being i against j is thus given by $U(i, j|t) = P(i|ij)U(i|t) + P(j|ij)U(j|t)$. Also, given σ and r, the marginal distribution $\tau(\sigma, r)$ of votes received by each candidate is $\tau(\sigma, r) = \{\int_{\mathcal{T}} \sigma(i|t)dr(t) \colon i \in \mathcal{K}\}$.

Next, we define the possible pivotal events in a runoff election. A ballot is *threshold pivotal*, denoted by $\texttt{piv}_{i/ij}$, if candidate i lacks one vote to become a majority in the first round. $\texttt{piv}_{ij/i}$ is the event that any ballot against i prevents an outright of i in the first round. $\texttt{piv}_{ki/kj}$ refers to an event when k ranks first but does not become an outright in the first round. An additional vote for i allows i to advance to the second round.

Outline of Proof of Proposition 1 in [4]

It suffices to show that for any $j \in \{R, S\}$, as $n \to \infty$, there exists an equilibrium in which j receives no share of votes. (Since there is no distinguish between A, B and C, we can choose any 2 out of $\{A, B, C\}$ to be set of R and S) Let i be the candidate other than j and R. Our goal is to show that, in equilibrium, any one who prefers R to i votes for R and vice versa.

The authors show the following two conditions will be implied

(a) $U(R|t) > U(i|t)$ implies that $\lim_{n\to\infty} G_t(R, n\tau) > \lim_{n\to\infty} G_t(j, n\tau)$.

(b) $U(i|t) > U(R|t)$ implies that $\lim_{n\to\infty} G_t(i, n\tau) > \lim_{n\to\infty} G_t(j, n\tau)$.

That is, (a) implies that if a voter of type t prefers R to i, then he voting j is a (strictly) dominated strategy. On the other hand, (b) implies that if a voter prefers i to R, then voting

j is still a (strictly) dominated strategy. Recall we assume that voters being indifferent between any two candidates have measure 0. This shows the existence of Duverger's Law equilibrium.

The next step is to show this equilibrium is strictly perfect. The proof first assumes a small deviation from the equilibrium in which case candidate j receives a positive amount of share of votes.

The deviated equilibrium results in a small deviation of magnitude $\mu(\text{PIV}_{R/Ri}) > \mu(\text{PIV}_{i/Ri}) >$any other magnitude. Then magnitude relations imply a similar consequence in the previous proof of the existence of the Duverger's Law equilibrium. That is, under such magnitude relations, a voter who prefers i to R vote for R and vice versa. That is, j receives no share of votes, which leads to a contradiction.

Outline of Proof of Proposition 2 in [4]

The first step is to shown the existence of such equilibrium. To this end, we rely on Lemma 4 in [4], which characterize the possible largest magnitude of all pivotal events. Follow a similar argument as we prove Proposition 1, it is shown that voters with $U(R|t) > U(i|t)$ vote for R. The analysis of the case $U(R|t) < U(i|t)$ is divided into two cases which correspond to the first and the second scenario in Lemma 4 in [4].

In both cases, a voter of type t ($U(R|t) < U(i|t)$) will vote for S only if $U(R, S|t) > U(R, W|t)$. However, whether this inequality holds or not is independent of which candidate W or S will receive more votes but merely depends on the prior knowledge (e.g., $P(R|RS)$, $U(R|t)$, $U(S|t)$ etc.) of the game. So, each S and W receives a positive amount of shares of votes which depends on the whether $U(R, S|t) > U(R, W|t)$.

The final step is to show this equilibrium is strictly perfect. Essentially, the central idea of the proof is to note that the magnitude of an event is *continuous* in the space $\Delta(\mathcal{K})$. Thus, if we deviate τ by a small amount ϵ (i.e., $\tau \in \Delta(\mathcal{K})$: $|\tau - \tau(\boldsymbol{\sigma}^*, r)| < \epsilon$), the relative order of magnitude would not change. In this case, the strategy profile $\boldsymbol{\sigma}^*$ is still optimal. This completes the proof.

Outline of Proof of Proposition 3 in [4]

Recall in Lemma 4 of [4], the relative magnitudes can be partitioned into three cases. We've established equilibria under the first two. It thus suffices to show that the third case in Lemma can not induce any strictly perfect equilibrium.

The proof is based on reaching a contradiction. The y first show that $\boldsymbol{\sigma}^*$ can be a equilibrium other than those mentioned before only if $\tau(\boldsymbol{\sigma}^*, r)$ implies the third case in Lemma 4. When the magnitude order is that of case 3 in Lemma 4, it is shown that $\tau(\boldsymbol{\sigma}^*, r) > 0$. However, under the magnitude ordering of case 3, people who vote for R turn out to form a measure-0 set, i.e., $\tau_R(\boldsymbol{\sigma}^*, r) = 0$. Thus, R receives no share of votes in equilibrium, which contradicts $\tau_R(\boldsymbol{\sigma}^*, r) > 0$. Thus no any other equilibrium can exist and the proof is complete.

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