Probability and Stochastic Processes Course Notes

Chia-Hao CHang, chchangkh@utexs.edu

Fall 2018

1 Probability Measure

A probability space is a triplet $(\Omega, \mathcal{F}, P)$, where

$$\Omega = \text{Sample Space} = \{\text{collection of outcomes}\}$$

$\mathcal{F}$ is a σ-algebra, meaning it satisfies the following axioms

Axiom 1.1. $\Omega \in \mathcal{F}$

Axiom 1.2. *If $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$.*

Axiom 1.3 (Closed under Countable Union)**.** *If $A, B \in \mathcal{F}$, then $A \cup B \in \mathcal{F}$. If $A_1, A_2, \cdots \in \mathcal{F}$, then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$.*

$$\mathcal{F} = \text{Events} = \{\text{collection of subsets } of\ \Omega\}$$

We might think of $\mathcal{F}$ as the power set of Ω.

$(\Omega, \mathcal{F})$ is called a *measurable space*. P is a probability measure.

$$P : \mathcal{F} \to [0, 1]$$

2 Countable v.s. Uncountable

Examples of countable sets: $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$. Examples of uncountable sets : $[0, 1]$, $\mathbb{R}$.

Remark (Countable Union [1])**.** *Countable union of countable sets is countable.*

Remark (Subset of Countable Sets [1])**.** *Subsets of a countable set are countable.*

3 Cantor's Diagonal Proof

Proves a subset of [0,1) is uncountable, which implies $[0, 1)$ is uncountable.

4 Non-Measurable Sets (Vitali Sets)

The goal of this section is to show that there exists non-measurable sets (namely, sets with no ”lengths”).

Axiom 4.1. *$\forall\ A \subseteq [0,1)$, $P\{A\} \geq 0$, $P\{[0,1]\} = 1$, $P\{\phi\} = 0$.*

Axiom 4.2. *If $A \cap B = \phi$, $P\{A \cup B\} = P\{A\} + P\{B\}$*

Axiom 4.3. *$A \oplus x \triangleq \{a + x : a \in A\}$.*
$P\{A\} = P\{A \oplus x\}$, $\forall x$.

Target: Show that: $\exists B \subseteq [0,1)$ s.t. $P\{B\}$ does not exist.

Proof. Firstly, construct $S_\alpha \triangleq \{r + \alpha : r \in \mathbb{Q}\}$. It is clear that $S_\alpha = S_\beta$ if $\alpha - \beta \in \mathbb{Q}$.

Remark. *S_α is countable. Since there is an one-one and onto projection between $\mathbb{Q}$ and S_α.*

Remark. *For two different α, β, either*

$$K_\alpha = K_\beta, \text{if } \alpha - \beta \in \mathbb{Q} \tag{1}$$

or

$$K_\alpha \cap K_\beta = \phi \tag{2}$$

Remark. *There is a subset of $\{S_\alpha\}_{\alpha \in [0,1)}$, $\{K_\alpha\} \in$ s.t. $\bigcup\limits_\alpha K_\alpha = [0,1)$.*

Remark. *$\{K_\alpha\}$ is uncountable.*

This can be easily verified by the fact that each K_α is countable. If $\{K_\alpha\}$ is countable, $\bigcup\limits_\alpha K_\alpha = [0,1)$ would be countable, which constitutes a contradiction. ($[0,1)$ is uncountable.)

Remark. *$\{K_\alpha\}$ is a partition of $[0,1)$.*

By the Axiom of Choice, we construct $\mathcal{B}$ as follows:
From each K_α, we select an element from it and put it into $\mathcal{B}$.
Clearly, $\mathcal{B}$ is not countable. Next, consider $\mathbb{Q} \triangleq \{r_0, r_1, r_2, \cdots\}$, namely, $\{r_0, r_1, \cdots\}$ is the sequence of rational numbers. Consider a collection of sets $\{\mathcal{B}_1, \mathcal{B}_2, \cdots, \mathcal{B}_n \cdots\}$, where

$$\mathcal{B}_i \triangleq \mathcal{B} \oplus r_i$$

Lemma 1. *$\{\mathcal{B}_n\}$ is a disjoint set.*

This can be seen by:
Say, $x \in \mathcal{B}_n \cap \mathcal{B}_m$. Then, $x = r_n + y_n = r_m + y_m$, where $y_n, y_m \in \mathcal{B}$. This implies that $|y_n - y_m| \in \mathbb{Q}$, which contradicts that elements in $\mathcal{B}$ differ by a irrational number.

Lemma 2. *$\bigcup\limits_{n=0}^{\infty} \mathcal{B}_n = [0,1)$.*

Since $\bigcup_\alpha K_\alpha = [0,1)$. Given $s \in [0,1)$, $s = \gamma + r_\gamma$, where $\gamma \in [0,1)$ and $r_\gamma \in \mathbb{Q}$. We can always find a element in $\mathcal{B}$, say, $b_\gamma \in \mathcal{B}$ s.t. $\gamma - b_\gamma \in \mathbb{Q}$. Hence, s differ by a rational amount to b_γ. Implying $s \in \mathcal{B}_{b_\gamma}$.

$\{\mathcal{B}_n : n = 1, 2, 3, \cdots\}$ forms a partition of $[0,1)$. Thus,

$$\sum_{m=1}^{\infty} P\{\mathcal{B}_m\}$$
$$= \sum_{m=1}^{\infty} P\{\mathcal{B}\}$$

The summation must either be 0 or ∞, which contradicts the assumption of probability measure. This completes the proof. $\square$

Appendix I: Group view of Vitali Set

In fact, if we view $[0,1)$ as the group with binary operation as $\oplus$, namely

$$x \oplus y = (x + y) \text{mod } 1$$

Then, $\mathbb{Q}$ is the *normal subgroup* of $[0,1)$. i.e.

$$\mathbb{Q} \triangleleft [0,1)$$

Thus, $[0,1)/\mathbb{Q} \cap [0,1)$ defines a *Quotient Group* and $[0,1)/\mathbb{Q}$ is a partition of $[0,1)$.

References

[1] W. Rudin: Principles of Mathematical Analysis, *McGraw-Hill*, 1976.