Queens College, CUNY, Department of Computer Science

Computational Finance

CSCI 365 / 765

Spring 2018

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Final Spring 2018

Solutions added

ImpliedVolatilityPrivate $= D-$

BinomialModel::FairValue(int n, Derivative* p_derivative, double S, double t0, double &V) $= D-$

For the entire course

- **<u>NOTE</u>: It is the policy of the Computer Science Department to issue a failing grade to any student who either gives or receives help on any test.**
- This is an **open-book** test.
- **Any problem to which you give two or more (different) answers receives the grade of zero automatically.**
- This is a **take home exam.**
  Please submit your solution via email, as a file attachment, to `Sateesh.Mane@qc.cuny.edu`.
  The file name should have either of the formats:

  `StudentId_first_last_CS365_final_May2018`

  `StudentId_first_last_CS765_final_May2018`

  Acceptable file types are txt, doc/docx, pdf (also cpp, with text in comment blocks).
- **Send one cpp file with all your code, one docx/pdf/txt file with non-code answers.**
- **In all questions where you are asked to submit programming code, programs which display any of the following behaviors will receive an automatic F:**
  1. Programs which do not compile successfully (compiler warnings which are not fatal are excluded, e.g. use of deprecated features).
  2. Array out of bounds, dereferencing of uninitialized variables (including null pointers).
  3. Operations which yield NAN or infinity, e.g. divide by zero, square root of negative number, etc. *Infinite loops.*
  4. Programs which do NOT implement the public interface stated in the question.
- **In addition, note the following:**
  1. Programs which compile and run successfully but have memory leaks will receive a poor grade (but not F).
  2. All debugging and/or output statements (e.g. `cout` or `printf`) will be commented out.
  3. Program performance will be tested solely on function return values and the values of output variable(s) in the function arguments.
  4. In other words, program performance will be tested solely via the public interface presented to the calling application. (I will write the calling application.)

1 Question 1 (no code)

- **You do NOT need to submit programming code for this question.**
- The time today is $t_0 = 0$.
- The par yields and bootstrapped spot rates of the yield curve are tabulated below.

t	y (%)	r (%)
0.5	1	0.99751
1	1.62383	1.61980
1.5	1.98875	1.98461
2	2.24766	2.24432

- You are given a newly issued bond with a maturity of $T = 2$ years and face value $F = 100$.
- The bond pays 8 coupons, with a **quarterly frequency** at times $t_i = 0.25$, 0.5, 0.75, 1.0, 1.25, 1.5, 1.75 and 2.0, where $i = 1, \dots, 8$.
- The amounts of the annualized coupon rates are as follows:
  1. **The (annualized) coupon rates are the 8 digits of your student id.**
  2. For example if your student id is 23054617, the coupons are $(2, 3, 0, 5, 4, 6, 1, 7)$.
  3. It is possible for some coupon amounts to be zero.
  4. Recall there is a factor of 4 for quarterly coupons.
  5. Hence in this case the bond fair value B is, with discount factors d_{t_i},

$$B = \frac{2}{4}\, d_{0.25} + \frac{3}{4}\, d_{0.5} + \frac{0}{4}\, d_{0.75} + \frac{5}{4} d_{1.0} + \frac{4}{4}\, d_{1.25} + \frac{6}{4}\, d_{1.5} + \frac{1}{4}\, d_{1.75} + \left(F + \frac{7}{4}\right) d_{2.0}\,. \tag{1.1}$$

- The discount factors d_{t_i} are obtained via **cfr (constant forward rate) interpolation of the yield curve.**
- **Calculate the fair value of the bond in eq. (1.1).** Call the answer B_{FV}. State your answer to 2 decimal places.
- **Calculate the yield y of the bond in eq. (1.1), if the bond market price is B_{FV}.** Call the answer y_1. State the value of y_1 (in percent) to 2 decimal places.
- Recall for quarterly coupons the bond fair value is given by

$$B_{FV} = \left[\sum_i \frac{c_i/4}{(1+\frac{1}{4}y)^{4t_i}}\right] + \frac{F}{(1+\frac{1}{4}y)^8}\,. \tag{1.2}$$

- Define a constant coupon, say c.
- **Find the value of c such that**

$$B_{FV} = \frac{c}{4}\, d_{0.25} + \frac{c}{4}\, d_{0.5} + \frac{c}{4}\, d_{0.75} + \frac{c}{4}\, d_{1.0} + \frac{c}{4}\, d_{1.25} + \frac{c}{4}\, d_{1.5} + \frac{c}{4}\, d_{1.75} + \left(F + \frac{c}{4}\right) d_{2.0}\,. \tag{1.3}$$

- State the value of c to 2 decimal places.

- **The interpolated spot rates and discount factors are the same for all students.**
- **Interpolation is required at four tenors $t = 0.25$, 0.75, 1.25 and 1.75.**
- **For $t = 0.25$, the spot rate $r_{0.25}$ is the same as $r_{0.5}$ (first point in the yield curve).**
- **The interpolation fraction is $\lambda = 0.5$ in all cases for $t > 0.25$.**
- **The cfr interpolated spot rates are as follows:**

$$
\begin{aligned}
r_{0.25} &= r_{0.5} && = 0.99751\,, \\
r_{0.75} &= \frac{0.5 \times r_{0.5} \times 0.5 + 0.5 \times r_{1.0} \times 1.0}{0.75} && \simeq 1.41237\,, \\
r_{1.25} &= \frac{0.5 \times r_{1.0} \times 1.0 + 0.5 \times r_{1.5} \times 1.5}{1.25} && \simeq 1.83869\,, \\
r_{1.75} &= \frac{0.5 \times r_{1.5} \times 1.5 + 0.5 \times r_{2.0} \times 2.0}{1.75} && \simeq 2.13302\,,
\end{aligned}
$$

- **In all cases the discount factor is $d_t = e^{-0.01\, r_t t}$ (where r_t is a percentage).**
- **The table of tenors, spot rates and discount factors is**

t	r_t	d_t
0.25	**0.99751**	**0.997509**
0.5	**0.99751**	**0.995025**
0.75	**1.41237**	**0.989463**
1.0	**1.61980**	**0.983932**
1.25	**1.83869**	**0.977279**
1.5	**1.98461**	**0.970670**
1.75	**2.13302**	**0.963360**
2.0	**2.24432**	**0.956106**

- **Experience has shown that the numbers calculated by students vary slightly depending on computer and compiler.**
- **However, for the number of decimal places required in the solution, the differences aer not significant.**

- **The bond fair value depends on the student id.**
- **For the id 23054617, the answer is (see eq. (1.1)):**

$$
\begin{aligned}
B_{FV} &= \frac{2}{4}\,d_{0.25} + \frac{3}{4}\,d_{0.5} + \frac{0}{4}\,d_{0.75} + \frac{5}{4}d_{1.0} + \frac{4}{4}\,d_{1.25} + \frac{6}{4}\,d_{1.5} + \frac{1}{4}\,d_{1.75} + \Bigl(F + \frac{7}{4}\Bigr)\,d_{2.0} \\
&\simeq 102.43\,.
\end{aligned}
$$

- **The yield therefore also depends on the student id.**
  1. **The yield is obtained by iteration.**
  2. ***You were actively encouraged to use the code written earlier in the semester, for homework.***
  3. **For the id 23054617, the value of the yield is:**

$$y_1 \simeq 2.23808 \simeq 2.24\%$$

  4. **This value is required as an input to answer Question 6.**
- **The fixed coupon c can be calculated without interpolation. Note the following:**

$$
\begin{aligned}
B_{FV} &= \frac{c}{4}\,\bigl(d_{0.25} + d_{0.5} + d_{0.75} + d_{1.0} + d_{1.25} + d_{1.5} + d_{1.75} + d_{2.0}\bigr) + d_{2.0}\,F\,, \\
c &= \frac{4(B_{FV} - d_{2.0}\,F)}{d_{0.25} + d_{0.5} + d_{0.75} + d_{1.0} + d_{1.25} + d_{1.5} + d_{1.75} + d_{2.0}}\,.
\end{aligned}
$$

- **The value of the fixed coupon therefore depends on the student id.**
- **For the id 23054617, the answer is:**

$$c \simeq 3.4837 \simeq 3.48\,.$$

2 Question 2 (no code)

2.1 Case 1

- In this question, the stock does not pay dividends.
- The prices of an American call C and American put P with the same strike K and expiration T (and on the same stock S) satisfy the following arbitrage bounds

$$C - P \geq S - K \,, \qquad C - P \leq S - \mathrm{PV}(K)\,. \tag{2.1}$$

- Suppose that at time t_0 (today), $S_0 = 100.5$, $K = 100$ and $e^{-r(T-t_0)} = 0.99$.
- All symbols have their usual meanings.
- An American call and put both trade today at $C = P = 10$.
- **Formulate an arbitrage strategy to take advantage of the above mispricing.**

2.2 Case 2

- In this question, the stock does not pay dividends.
- The prices of two American calls C_1 and C_2 with strikes K_1 and K_2, respectively (and $K_1 < K_2$), with the same expiration T (and on the same stock S) satisfy the following arbitrage inequality

$$C_1 - C_2 \leq K_2 - K_1\,. \tag{2.2}$$

- Suppose that at time t_0 (today), $S_0 = 100.5$, $K_1 = 100$, $K_2 = 105$ and $e^{-r(T-t_0)} = 0.99$.
- All symbols have their usual meanings.
- The American calls have prices today of $C_1 = 11$ and $C_2 = 5$.
- **Formulate an arbitrage strategy to take advantage of the above mispricing.**

2.3 Case 3

- In this question, the stock does not pay dividends.
- The prices of two American puts P_1 and P_2 with strikes K_1 and K_2, respectively (and $K_1 < K_2$), with the same expiration T (and on the same stock S) satisfy the following arbitrage inequality

$$P_2 - P_1 \leq K_2 - K_1\,. \tag{2.3}$$

- Suppose that at time t_0 (today), $S_0 = 100.5$, $K_1 = 100$, $K_2 = 105$ and $e^{-r(T-t_0)} = 0.99$.
- All symbols have their usual meanings.
- The American puts have prices today of $P_1 = 5$ and $P_2 = 11$.
- **Formulate an arbitrage strategy to take advantage of the above mispricing.**

Case 1

- **The data:** $C-P=0$ **also** $S_0-K=100.5-100=0.5$ **and** $S_0-\mathbf{PV}(K)=100.5-99=1.5$.
  1. **Hence** $C-P<S-\mathbf{PV}(K)$ **which satisfies the rational option pricing inequality.**
  2. **Also** $C-P<S-K$ **which violates the rational option pricing inequality.**
  3. **Hence we formulate an arbitrage strategy to take advantage of the fact that** $C-P<S-K$.
- **The strategy is:**
  1. **At time** t_0, **go long one call, short one put and short one share of stock.**
  2. **Our initial cash amount is** $-C+P+S_0=-10=10+100.5=100.5$.
  3. **This is a positive number so we save the cash in a bank and earn interest (at the risk-free rate).**
- **What happens next?**
- **Because we are** ***short the put***, **we cannot exercise it.**
- **We must wait to see if the holder of the put exercises it.**
- **Early exercise of the put:**
  1. **If the holder of the put exercises the put early, at any time** $t_0<t<T$, **we pay cash equal to the strike** $K=100$ **and receive stock worth** S_t.
  2. **We use the stock to close our short stock position.**
  3. **Our cash in the bank at time** t **is** $100.5\,e^{r(t-t_0)}>100$, **hence we have enough cash to pay the strike.**
  4. **We now have positive cash** $=100.5\,e^{r(t-t_0)}-100$ **and long one call. The payoff from the call is never negative (because we are long the call). This is sufficient to guarantee a profit.**
  5. **Anyway we wait till expiration.**
     (a) **If the call expires in the money, we exercise the call, pay the strike** $(=100)$ **and receive one share of stock worth** $S_T\geq 100$. **We sell the stock immediately to repay of loan of** 100 **for the strike. Our total cash at the expiration time** T **is** $(S_T-100)+(100.5\,e^{r(t-t_0)}-100)\,e^{r(T-t)}$ **which is** >0.
     (b) **If the call expires out of the money, it is worthless. Our total cash at the expiration time** T **is** $(100.5\,e^{r(t-t_0)}-100)\,e^{r(T-t)}$, **which is** >0.
  6. **Hence in all cases we have positive cash at expiration.**

- **Exercise of the put on the expiration date:**
  1. **This is exactly the same as the above except that** $t = T$**.**
  2. **Then by definition** $S_T \leq K$ **at expiration, so the call will be out of the money at expiration.**
  3. **Hence the call expires worthless.**
  4. **Our total cash at expiration is** $100.5\, e^{r(T-t_0)} - 100\, e^{r(T-t)}$ **which is** > 0**.**
  5. **Hence we have positive cash at expiration.**
- **Put not exercised.**
  1. **This means the time reached expiration and the put expired out of the money.**
  2. **Therefore at expiration the call is in the money, so** $S_T \geq K$**.**
  3. **We exercise the call and pay the strike** $(= 100)$**. Our cash in the bank is** $100.5\, e^{r(T-t_0)} = 100.5/0.99 \simeq 101.5152$**. Hence we have enough cash to pay the strike.**
  4. **We receive one share of stock, which we use to close our short stock position.**
  5. **Our total cash at the expiration time** T **is** $100.5\, e^{r(T-t_0)} - 100 = 100/5/0.99 - 100 \simeq 1.5152$ **which is** > 0**.**
  6. **Hence we have positive cash at expiration.**
- **Hence in all cases we have positive cash at the expiration time = arbitrage.**

Case 2

- This is an American call spread with $C(K_1) - C(K_2) > K_2 - K_1$.
- I forgot that I displayed the solution in the worked examples in Lecture 8a (Sec 8.2).
- Hence all students should have aced this question.
- The arbitrage strategy is:
  1. At the time t_0, go short the low strike call $C(K_1)$ and long the high strike call $C(K_2)$.
  2. Our initial cash is $C(K_1) - C(K_2) = 11 - 5 = 6$.
  3. This is a positive number so we save the cash in a bank and earn interest (at the risk-free rate).
- What happens next?
- Because we are *short the low strike call*, we cannot exercise it.
- We must wait to see if the holder of the low strike call exercises it.
- Early exercise of the low strike call:
  1. Suppose the holder of the low strike call exercises it at any time $t_0 < t < T$.
  2. Our cash in the bank is $6\,e^{r(t-t_0)}$. This is greater than $K_2 - K_1 = 5$.
  3. Suppose $K_1 \le S_t \le K_2$ (stock price between option strikes).
     (a) We receive cash $= K_1 = 100$ from the holder of the low strike call. Hence we have cash greater than $K_2 = 105$.
     (b) By definition $S_t \le 105$. Hence we have enough cash to buy the stock (pay S_t) and deliver the stock to the holder of the low strike call.
     (c) Hence we have positive cash and long the high strike call.
     (d) This always has positive value and is sufficient to guarantee a profit.
     (e) Anyway we wait till expiration.
        i. If the high strike call expires in the money, we exercise the call, pay the strike $(= 105)$ and receive one share of stock worth $S_T \ge 105$. We sell the stock immediately to repay of loan of 105 for the strike. Our total cash at the expiration time T is $(S_T - 105) + (6\,e^{r(t-t_0)} + 100 - S_t)\,e^{r(T-t)}$ which is > 0.
        ii. If the call expires out of the money, it is worthless. Our total cash at the expiration time T is $(6\,e^{r(t-t_0)} + 100 - S_t)\,e^{r(T-t)}$ which is > 0.
     (f) Hence in all cases we have positive cash at expiration.

- **Suppose $S_t > K_2$ (stock price above high strike).**

  1. **We exercise the high strike call at the time t.**
  2. **We are long the high strike call hence we can do this.**
  3. **We receive one share of stock from the exercise of the high strike call, which we deliver to the holder of the low strike call.**
  4. **We also receive cash of $K_1 = 100$ from the holder of the low strike call.**
  5. **Our cash in the bank is $6\,e^{r(t-t_0)}$. This is greater than $K_2 - K_1 = 5$.**
  6. **Hence we have total cash greater than $K_2 = 105$. Hence we have enough cash to pay the strike price K_2 to exercise the high strike call.**
  7. **After all the trades are completed, we have cash left over in the amount of $6\,e^{r(t-t_0)} + 100 - 105 = 6\,e^{r(t-t_0)} - 5$. This is a positive amount of cash.**

- **No early exercise of the low strike call.**

  1. **Suppose at expiration $S_T < K_1$. Both calls expire out of the money and are worthless. Our cash in the bank is $6\,e^{r(T-t_0)}$. This is a positive number.**
  2. **Suppose at expiration $K_1 \le S_T \le K_2$. The analysis is the same as above, except now $t = T$ so we do not "wait until expiration" to see what happens to the high strike option. Our total cash at the expiration time T is $6\,e^{r(T-t_0)} + 100 - S_T$ which is > 0.**
  3. **Suppose at expiration $S_T > K_2$. The analysis is the same as above, except now $t = T$. We exercise the high strike call. Our total cash at the expiration time T is $6\,e^{r(T-t_0)} + 100 - 105 = 6\,e^{r(T-t_0)} - 5$. This is a positive amount of cash.**

- **Hence in all cases we have positive cash at the expiration time = arbitrage.**

Case 3

- **This is an American put spread with $P(K_2) - P(K_1) > K_2 - K_1$.**
- **I forgot that I displayed the solution in the worked examples in Lecture 8a (Sec 8.4).**
- **Hence all students should have aced this question.**
- **The arbitrage strategy is:**
  1. **At the time t_0, go short the high strike put $P(K_2)$ and long the low strike put $P(K_1)$.**
  2. **Our initial cash is $P(K_2) - P(K_1) = 11 - 5 = 6$.**
  3. **This is a positive number so we save the cash in a bank and earn interest (at the risk-free rate).**
- **What happens next?**
- **Because we are *short the high strike put*, we cannot exercise it.**
- **We must wait to see if the holder of the high strike put exercises it.**
- **Early exercise of the high strike put:**
  1. **Suppose the holder of the high strike put exercises it at any time $t_0 < t < T$.**
  2. **Our cash in the bank is $6\,e^{r(t-t_0)}$. This is greater than $K_2 - K_1 = 5$.**
  3. **Suppose $K_1 \le S_t \le K_2$ (stock price between option strikes).**
     (a) **We receive one share of stock worth S_t from the holder of the high strike put. By definition $S_t \ge 100$. We sell the stock immediately and have total cash worth $S_t + 6\,e^{r(t-t_0)}$. By definition this is greater than 105.**
     (b) **Hence we have enough cash to pay the high strike of 105 to the holder of the high strike put.**
     (c) **Hence we have positive cash and long the low strike put.**
     (d) **This always has positive value and is sufficient to guarantee a profit.**
     (e) **Anyway we wait till expiration.**
        i. **If the low strike put expires in the money, we exercise the put, receive the strike ($= 100$) and use the money to buy one share of stock worth $S_T < 100$. We deliver the stock as part of our exercise of the low strike put. Our total cash at the expiration time T is $(100 - S_T) + (6\,e^{r(t-t_0)} + S_t - 105)\,e^{r(T-t)}$ which is > 0.**
        ii. **If the put expires out of the money, it is worthless. Our total cash at the expiration time T is $(6\,e^{r(t-t_0)} + S_t - 105)\,e^{r(T-t)}$ which is > 0.**
     (f) **Hence in all cases we have positive cash at expiration.**

- **Suppose $S_t < K_1$ (stock price below low strike).**
  1. **We exercise the low strike put at the time t.**
  2. **We are long the low strike put hence we can do this.**
  3. **We receive one share of stock from the holder of the high strike put, which we deliver to the writer of the low strike put.**
  4. **We also receive cash of $K_1 = 100$ from the writer of the low strike put.**
  5. **Our cash in the bank is $6\,e^{r(t-t_0)}$. This is greater than $K_2 - K_1 = 5$.**
  6. **Hence we have total cash greater than $K_2 = 105$. Hence we have enough cash to pay the strike price K_2 to the holder of the high strike put.**
  7. **After all the trades are completed, we have cash left over in the amount of $6\,e^{r(t-t_0)} + 100 - 105 = 6\,e^{r(t-t_0)} - 5$. This is a positive amount of cash.**

- **No early exercise of the high strike put.**
  1. **Suppose at expiration $S_T > K_2$. Both puts expire out of the money and are worthless. Our cash in the bank is $6\,e^{r(T-t_0)}$. This is a positive number.**
  2. **Suppose at expiration $K_1 \le S_T \le K_2$. The analysis is the same as above, except now $t = T$ so we do not "wait until expiration" to see what happens to the low strike put. Our total cash at the expiration time T is $6\,e^{r(T-t_0)} + S_T - 105$ which is > 0.**
  3. **Suppose at expiration $S_T < K_1$. The analysis is the same as above, except now $t = T$. We exercise the low strike put. Our total cash at the expiration time T is $6\,e^{r(T-t_0)} + 100 - 105 = 6\,e^{r(T-t_0)} - 5$. This is a positive amount of cash.**

- **Hence in all cases we have positive cash at the expiration time = arbitrage.**

3 Question 3 (no code)

- A **butterfly spread** consists of three options on the same stock.
- All three options have the same expiration time T.
- The options have strike prices K_1, K_2 and K_3, which are are equally spaced.
- Hence K_2 is located at the midpoint of K_1 and K_3, so $K_2 = (K_1 + K_3)/2$.
- A butterfly spread can be created using three call options or three put options.
- The spread consists of long one option at K_1, short two options at K_2, long one option at K_3.

3.1 European option butterfly spreads

- Suppose the market price of a stock is S at the current time t. The stock does not pay dividends. The interest rate is $r > 0$ (a constant).
- For a European call and put c and p with the same strike K and expiration T, the put-call parity relation in this case is
$$c - p = S - Ke^{-r(T-t)}\,. \tag{3.1}$$
- Let c_i and p_i, $i = 1, 2, 3$, be the values of European calls and puts with strikes K_1, K_2 and K_3, where $K_2 = (K_1 + K_3)/2$.
- Use eq. (3.1) to prove the following relation:
$$c_1 - 2c_2 + c_3 = p_1 - 2p_2 + p_3\,. \tag{3.2}$$
- The values of a European call butterfly spread and a European put butterfly spread are equal.
- The corresponding relation is not necessarily true for American options.

3.2 American calls

- A butterfly spread consists of three American calls C_1, C_2 and C_3, with strikes K_1, K_2 and K_3, as described above.
- The butterfly spread consists long C_1, short $2 \times C_2$, long C_3:

$$B_{\text{call}}(S,t) = C_1 - 2C_2 + C_3\,. \tag{3.3}$$

- **Draw a graph of the intrinsic value of the butterfly spread $B_{\text{call}}(S,t)$.**
- **Show that $B_{\text{call}}(S,t) < 0$ if**

$$C_2(S,t) > \frac{C_1(S,t) + C_3(S,t)}{2}\,. \tag{3.4}$$

- **Formulate an arbitrage trade if $C_2(S,t) > (C_1(S,t) + C_3(S,t))/2$.**
- **Therefore deduce the following inequality must be true at any time $t \le T$:**

$$C_2(S,t) \le \frac{C_1(S,t) + C_3(S,t)}{2}\,. \tag{3.5}$$

- The mathematical expression is that the value of a call option is a **convex function** of the strike.

3.3 American puts

- A butterfly spread consists of three American puts P_1, P_2 and P_3, with strikes K_1, K_2 and K_3, as described above.
- The butterfly spread consists long P_1, short $2 \times P_2$, long P_3:

$$B_{\text{put}}(S,t) = P_1 - 2P_2 + P_3\,. \tag{3.6}$$

- **Draw a graph of the intrinsic value of the butterfly spread $B_{\text{put}}(S,t)$.**
- **Show that $B_{\text{put}}(S,t) < 0$ if**

$$P_2(S,t) > \frac{P_1(S,t) + P_3(P,t)}{2}\,. \tag{3.7}$$

- **Formulate an arbitrage trade if $P_2(S,t) > (P_1(S,t) + P_3(S,t))/2$.**
- **Therefore deduce the following inequality must be true at any time $t \le T$:**

$$P_2(S,t) \le \frac{P_1(S,t) + P_3(S,t)}{2}\,. \tag{3.8}$$

- The mathematical expression is that the value of a put option is a **convex function** of the strike.

European option butterfly spread

- **Add them up!**
- **Using eq.** (3.1) **and the relation** $K_1 - 2K_2 - K_3 = 0$ **yields**

$$
\begin{aligned}
(c_1 - p_1) - 2(c_2 - p_2) + (c_3 - p_3) &= (S - K_1 e^{-r(T-t)}) - 2(S - K_2 e^{-r(T-t)}) + (S - K_3 e^{-r(T-t)}) \\
&= (S - 2S + S) + (K_1 - 2K_2 + K_3)\, e^{-r(T-t)} \\
&= 0 + 0 = 0\,.
\end{aligned}
$$

- **Rearrange terms to obtain the claimed result:**

$$c_1 - 2c_2 + c_3 = p_1 - 2p_2 + p_3\,.$$

American call butterfly spread

- A graph of the intrinsic value of an American call butterfly spread is displayed in Fig. 1.

  1. The difference between "intrinsic value" and "terminal payoff" is that American options can be exercised at any time t, hence on the horizontal axis the stock price is S_t not S_T.
  2. Otherwise the graph looks the same as the terminal payoff.
  3. There are other types of derivatives where the graph of the intrinsic value depends on the time t.

- If eq. (3.4) is true, them rearrange terms in eq. (3.4) to obtain

$$
\begin{aligned}
\frac{C_1(S,t) + C_3(S,t)}{2} &< C_2(S,t) \\
C_1(S,t) + C_3(S,t) &< 2C_2(S,t) \\
C_1(S,t) - 2C_2(S,t) + C_3(S,t) &< 0 \\
\implies \qquad B_{\rm call}(S,t) = C_1 - 2C_2 + C_3 &< 0\,.
\end{aligned}
$$

- Arbitrage strategy.

  1. If $C_2(S,t) > (C_1(S,t) + C_3(S,t))/2$ then we proved $B_{\rm call}(S,t) < 0$.
  2. Therefore go long the butterfly spread.
  3. Since its cost is ***negative***, it means we ***receive cash*** to go long the butterfly spread.
  4. We save the cash in a bank.
  5. Furthermore, the payoff of a long butterfly spread is always ≥ 0 (see Fig. 1).
  6. Hence we have positive cash and we are long a derivative whose payoff is always ≥ 0.
  7. This is sufficient to prove a guaranteed profit = arbitrage.
  8. We simply wait until the spread is sufficiently in the money so that we wish to exercise.
  9. Else the minimum that we obtain from the butterfly spread is zero, but we have positive cash in the bank anyway.

American put butterfly spread

- **A graph of the intrinsic value of an American put butterfly spread is displayed in Fig. 2.**
- **It looks the same as Fig. 1.**
  1. **The difference between "intrinsic value" and "terminal payoff" is that American options can be exercised at any time t, hence on the horizontal axis the stock price is S_t not S_T.**
  2. **Otherwise the graph looks the same as the terminal payoff.**
  3. **There are other types of derivatives where the graph of the intrinsic value depends on the time t.**
- **If eq. (3.7) is true, them rearrange terms in eq. (3.7) to obtain**

$$\begin{aligned} \frac{P_1(S,t)+P_3(S,t)}{2} &< P_2(S,t) \\ P_1(S,t)+P_3(S,t) &< 2P_2(S,t) \\ P_1(S,t)-2P_2(S,t)+P_3(S,t) &< 0 \\ \implies \qquad B_{\rm put}(S,t) = P_1 - 2P_2 + P_3 &< 0\,. \end{aligned}$$

- **Arbitrage strategy.**
  1. **If $P_2(S,t) > (P_1(S,t)+P_3(S,t))/2$ then we proved $B_{\rm put}(S,t) < 0$.**
  2. **Therefore go long the butterfly spread.**
  3. **Since its cost is *negative*, it means we *receive cash* to go long the butterfly spread.**
  4. **We save the cash in a bank.**
  5. **Furthermore, the payoff of a long butterfly spread is always ≥ 0 (see Fig. 2).**
  6. **Hence we have positive cash and we are long a derivative whose payoff is always ≥ 0.**
  7. **This is sufficient to prove a guaranteed profit = arbitrage.**
  8. **We simply wait until the spread is sufficiently in the money so that we wish to exercise.**
  9. **Else the minimum that we obtain from the butterfly spread is zero, but we have positive cash in the bank anyway.**

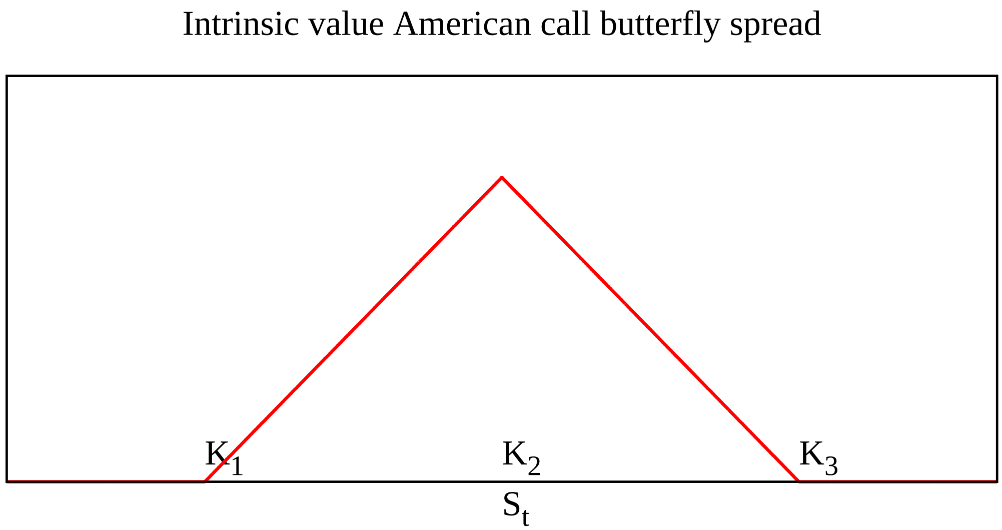

Figure 1: Graph of the intrinsic value of a butterfly spread of American calls.

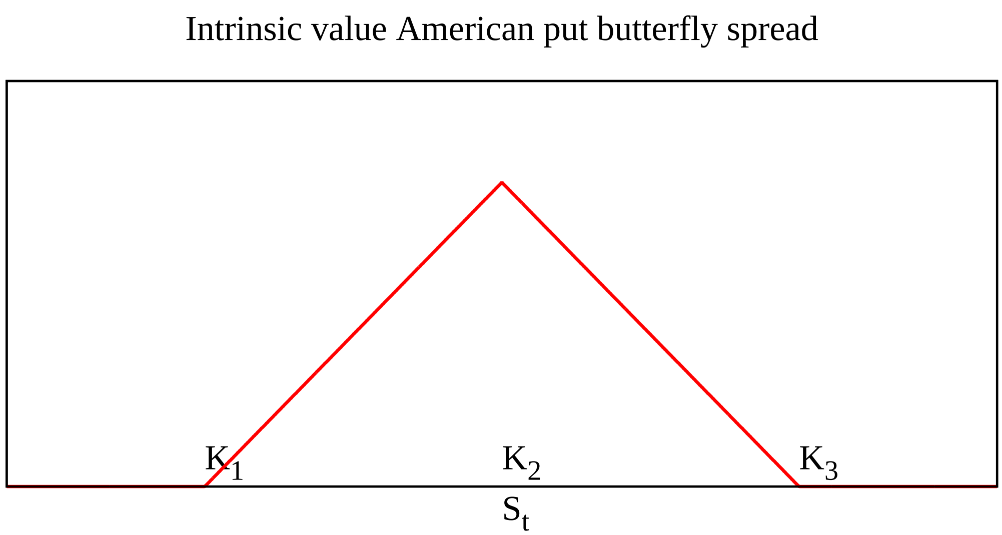

Figure 2: Graph of the intrinsic value of a butterfly spread of American puts.

4 Question 4 (submit code)

- In this question we shall employ the classes `BinomialTree` and `Derivative`, etc. introduced in Homework 9.
- **Write a class `Option` which inherits from `Derivative`.**
  1. The option has a strike K and expiration T.
  2. The option can be a put/call (`bool isCall`).
  3. The option can be American/European (`bool isAmerican`).
  4. Make all the data members public so I can set them in my calling application.
- **Submit your code for all of the above classes (including the `Database` class, etc.**
- Denote the fair values of a European call and put option by c and p, respectively.
- Denote the fair values of an American call and put option by C and P, respectively.
- **Your code will be tested by a calling application with random input values for $(S_0, K, r, q, \sigma, T, t_0)$.**
  1. **The option fair values will be calculated using the `BinomialTree` class.**
  2. Your code will be tested to see if it satisfies put–call parity for European options (with a tolerance of 10^{-6}):

$$\left|(c-p) - (Se^{-q(T-t_0)} - Ke^{-r(T-t_0)})\right| \leq 10^{-6}\,. \tag{4.1}$$

  3. Your code will be tested to see if it satisfies the following inequalities for American options (with a tolerance of 10^{-6}):

$$\begin{aligned} C - P &\geq (Se^{-q(T-t_0)} - K) - 10^{-6}\,, \\ C - P &\leq (S - Ke^{-r(T-t_0)}) + 10^{-6}\,. \end{aligned} \tag{4.2}$$

  4. Other inequalities will also be checked:

$$\begin{aligned} 0 \leq c &\leq Se^{-q(T-t_0)} + 10^{-6}\,, \\ 0 \leq C &\leq S + 10^{-6}\,, \\ 0 \leq p &\leq Ke^{-r(T-t_0)} + 10^{-6}\,, \\ 0 \leq P &\leq K + 10^{-6}\,. \end{aligned} \tag{4.3}$$

  5. ***I may perform other tests. Your code must pass all the rational option pricing inequalities in Lecture 7 (for a stock with a continuous dividend yield).***
  6. There will be no tests with discrete dividends.

The results depend on student code.
Nothing to display.

5 Question 5 (submit code)

- A **straddle** is an option spread with expiration time T and strike price K and a terminal payoff of $|S_T - K|$.
- A European straddle is therefore equal to a long European call plus a long European put, both with the same expiration T and the same strike K.
- An American straddle can be exercised at any time $t_0 \le t \le T$, and pays $|S_t - K|$ if exercised.
- An American straddle is therefore **cheaper** than a long American call plus a long American put, both with the same expiration T and the same strike K. This is because the American options can be exercised individually, whereas when the straddle is exercised, the entire package terminates.
- **Write a class `Straddle` which inherits from `Derivative`.**
  1. The straddle has a strike K and expiration T.
  2. The straddle can be American/European (`bool isAmerican`).
  3. Make all the data members public so I can set them in my calling application.
- **Your code will be tested by a calling application with random input values for $(S_0, K, r, q, \sigma, T, t_0)$.**
  1. **The fair value of a straddle will be calculated using the `BinomialTree` class.**
  2. Denote the fair value of a straddle by Z.
  3. Your code for the fair value of a European straddle will be tested to see if it satisfies the following equality (with a tolerance of 10^{-6}):
  $$|Z_{\text{Eur}} - (c + p)| \le 10^{-6} \,. \tag{5.1}$$
  4. Your code for the fair value of an American straddle will be tested to see if it satisfies the following inequalities (with a tolerance of 10^{-6}):
  $$\begin{aligned} Z_{\text{Am}} &\ge |S_0 - K| - 10^{-6} \,, \\ Z_{\text{Am}} &\le (C + P) + 10^{-6} \,. \end{aligned} \tag{5.2}$$

The results depend on student code.
Nothing to display.

6 Question 6 (submit code)

- A **binary option** (also known as a **digital option**) is an option with expiration time T and strike price K and the following terminal payoff:
  1. A binary call option pays \$1 if $S_T \geq K$ and zero otherwise.
  2. A binary put option pays \$1 if $S_T < K$ and zero otherwise.
- **Write a class `BinaryOption` which inherits from `Derivative`.**
- We shall consider only European binary call options below so we can ignore the early exercise valuation tests.
- You are given the following input values:
  $\boldsymbol{S_0 = 90}$, $K = 100$, $q = 0.02$, $T = 1$, $t_0 = 0$.
- **Set the risk free rate to the value of y_1 from Question 1.**
  (Note that r is a decimal value, so for example if $y_1 = 5.12\%$ then set $r = 0.0512$.)

$$r = y_1 \qquad \text{(decimal)}\,. \tag{6.1}$$

- There are also Black–Scholes–Merton formulas for the fair values of binary options

$$c_{\text{bin}} = e^{-r(T-t_0)}\,N(d_2)\,, \qquad p_{\text{bin}} = e^{-r(T-t_0)}\,N(-d_2)\,. \tag{6.2}$$

- **Write functions to compute the formulas in eq. (6.2).**
- **Submit you code as part of your solution to this question.**
- Use $\boldsymbol{n = 1000}$ steps in the binomial model.
- **Fill the following table.**

σ	$c_{\text{binary}}^{\text{binomial}}$	$p_{\text{binary}}^{\text{binomial}}$	$c_{\text{binary}}^{\text{BSM}}$	$p_{\text{binary}}^{\text{BSM}}$
0.1	3 d.p.	3 d.p.	3 d.p.	3 d.p.
0.2	3 d.p.	3 d.p.	3 d.p.	3 d.p.
0.3	3 d.p.	3 d.p.	3 d.p.	3 d.p.
0.4	3 d.p.	3 d.p.	3 d.p.	3 d.p.
0.5	3 d.p.	3 d.p.	3 d.p.	3 d.p.
0.6	3 d.p.	3 d.p.	3 d.p.	3 d.p.
0.7	3 d.p.	3 d.p.	3 d.p.	3 d.p.
0.8	3 d.p.	3 d.p.	3 d.p.	3 d.p.
0.9	3 d.p.	3 d.p.	3 d.p.	3 d.p.
1.0	3 d.p.	3 d.p.	3 d.p.	3 d.p.

- **Plot a graph of the binary call option fair value as a function of the volatility, using (i) the binomial model and (ii) the Black–Scholes–Merton formulas in eq. (6.2), for $0.01 \leq \sigma \leq 1.0$ in steps of 0.01.**

- **Plot a graph of the binary put option fair value as a function of the volatility, using (i) the binomial model and (ii) the Black–Scholes–Merton formulas in eq. (6.2), for $0.01 \le \sigma \le 1.0$ in steps of 0.01.**
- *Do not be concerned if the graphs look choppy (not smooth curves) for the binomial model.*
- If you have done your work correctly, the binary option fair values will NOT be monotonic functions of the volatility. The fair value of the binary call will exhibit a peak and the fair value of the binary put will exhibit a dip.
- Because of the peak/dip, implied volatility is not a useful concept for binary options. For a given market price, there can be two solutions for the implied volatility, i.e. not a unique value.

- **The valuations depend on the value of the yield y_1 from Question 1.**
- **The option fair values are also calculated using student code.**
- **For the id 23054617, the value of the risk-free rate is:**

$$r = 0.01\, y_1 \simeq 0.0224\,.$$

- **Using $r = 0.0224$ yields the following results.**

σ	$c_{\text{binary}}^{\text{binomial}}$	$p_{\text{binary}}^{\text{binomial}}$	$c_{\text{binary}}^{\text{BSM}}$	$p_{\text{binary}}^{\text{BSM}}$
0.1	**0.139**	**0.839**	**0.137**	**0.841**
0.2	**0.260**	**0.718**	**0.263**	**0.715**
0.3	**0.305**	**0.673**	**0.304**	**0.674**
0.4	**0.309**	**0.669**	**0.317**	**0.661**
0.5	**0.313**	**0.665**	**0.317**	**0.661**
0.6	**0.318**	**0.660**	**0.312**	**0.666**
0.7	**0.300**	**0.678**	**0.303**	**0.675**
0.8	**0.283**	**0.695**	**0.292**	**0.686**
0.9	**0.287**	**0.691**	**0.280**	**0.698**
1.0	**0.271**	**0.707**	**0.267**	**0.711**

- **A graph of the binary call option fair value as a function of the volatility, using (i) the binomial model and (ii) the Black–Scholes–Merton formulas in eq. (6.2), is displayed in Fig. 3. The valuations using a binomial model and the Black–Scholes–Merton formula are plotted as the solid and dotted curves, respectively.**
- **A graph of the binary put option fair value as a function of the volatility, using (i) the binomial model and (ii) the Black–Scholes–Merton formulas in eq. (6.2), is displayed in Fig. 4. The valuations using a binomial model and the Black–Scholes–Merton formula are plotted as the solid and dotted curves, respectively.**
- **The question does not ask for this, but I plotted both sets of fair values in one graph in Fig. 5. The valuations using a binomial model and the Black–Scholes–Merton formula are plotted as the solid and dotted curves, respectively.**

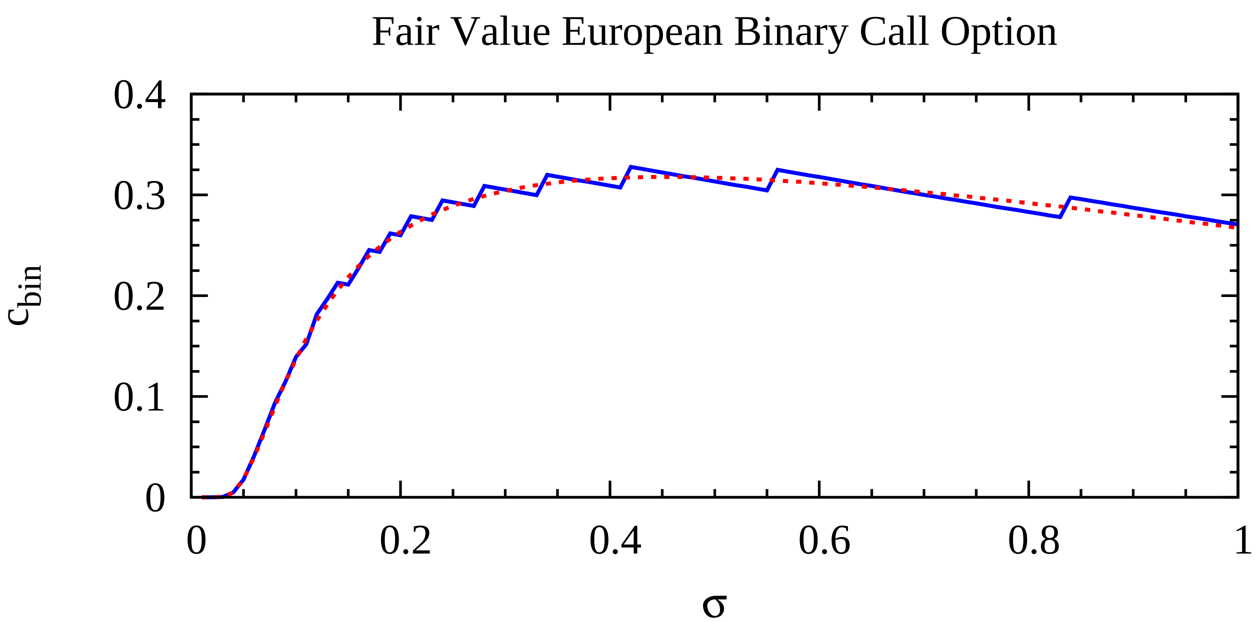

Figure 3: Fair value of European binary call option plotted against the volatility. Solid = binomial model. Dotted = BSM formula.

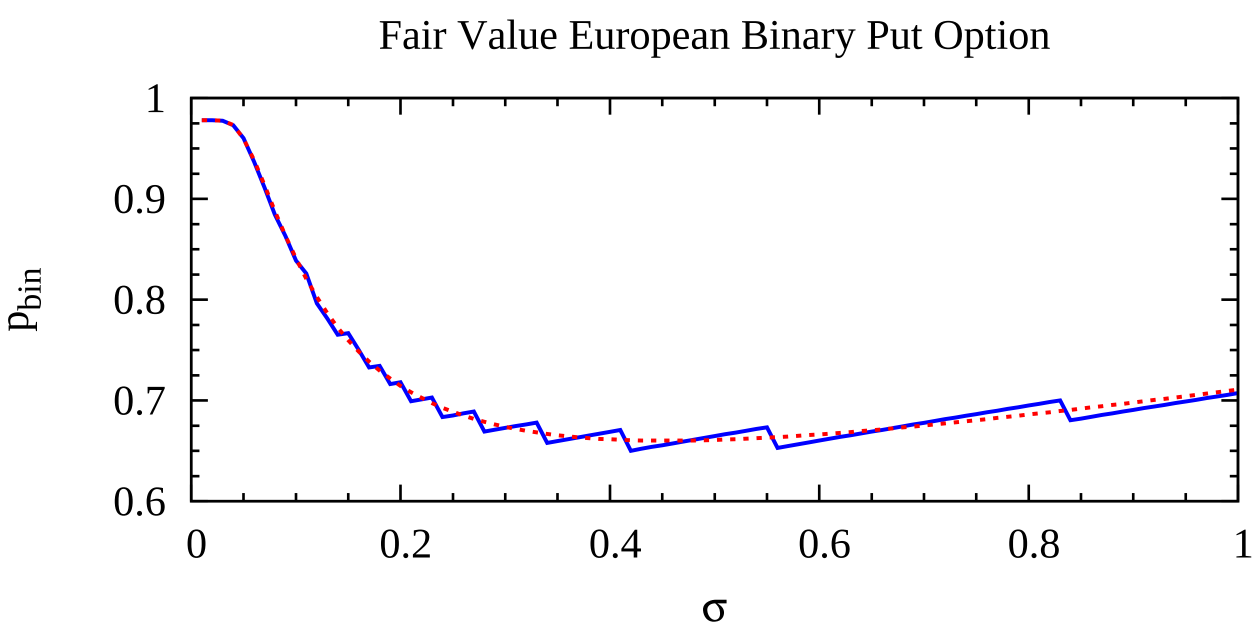

Figure 4: Fair value of European binary put option plotted against the volatility. Solid = binomial model. Dotted = BSM formula.

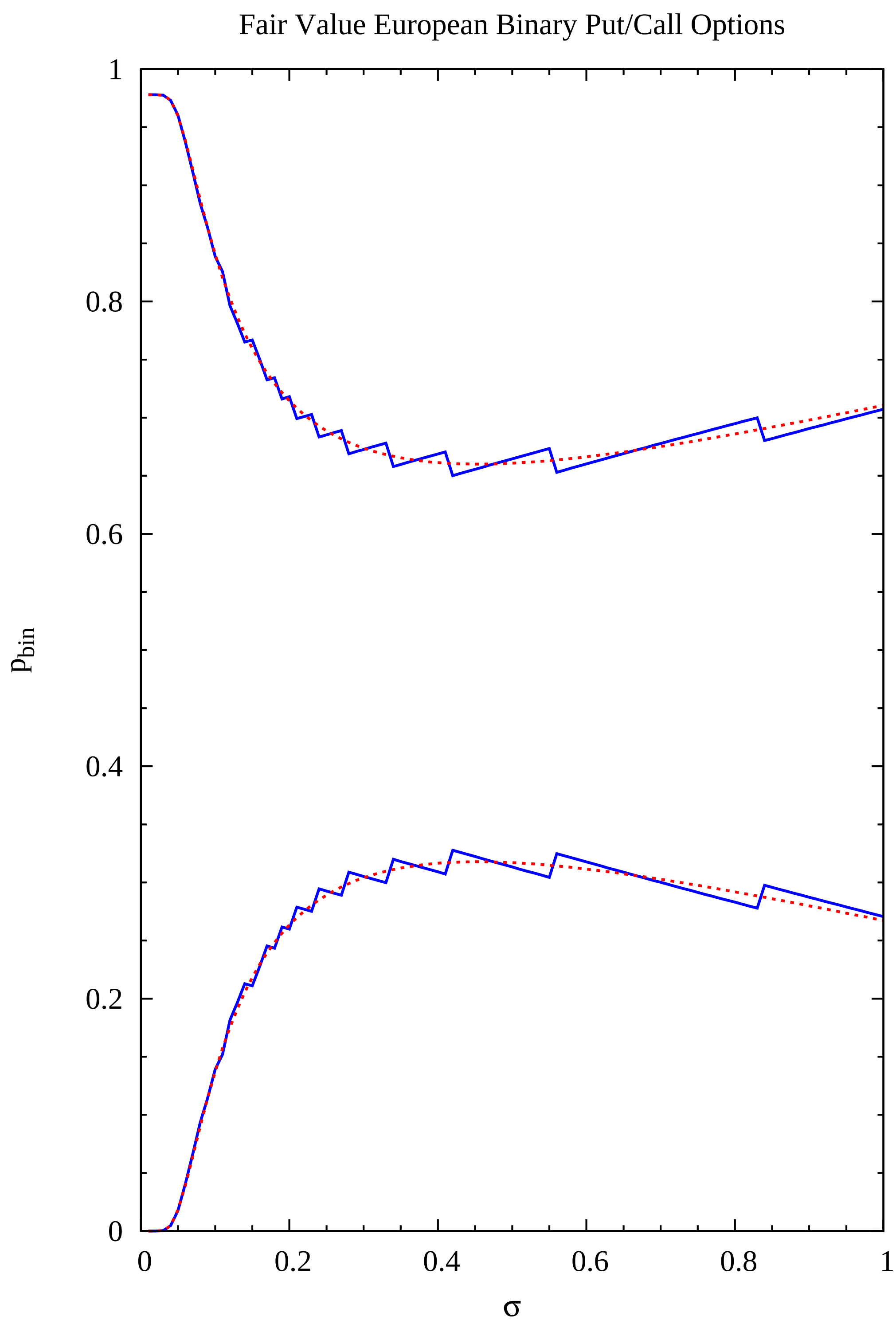

Figure 5: Union of graphs in Figs. 3 and 4.

7 Question 7 (submit code)

- **This question will carry more weight than the others, maybe double weight.**
- A **convertible bond** is an equity derivative with the following characteristics.
  1. It has a strike K and expiration T and can be American or European.
  2. The terminal payoff is
  $$\text{terminal payoff} = \max(S_T, K)\,. \tag{7.1}$$
  3. This is similar to the payoff in Midterm 2 Question 6(a).
  4. What this means is that if $S_T \geq K$ at expiration, the holder receives one share of stock and if $S_T < K$ at expiration, the holder receives cash in the amount K.
  5. Prior to expiration, if a convertible bond is exercised, the intrisic value at time t is S_t:
  $$\text{intrinsic value at time } t = S_t\,. \tag{7.2}$$
  6. We shall add one more feature, essentially a knockout barrier.
  7. Many convertible bonds are **callable.**
  8. This means that there is a threshold B and if $S \geq B$ at any time, the convertible bond terminates. The value of the convertible bond if called in this way is $V = S$.
  9. Hence in addition to early exercise, we add one more feature to the valuation tests, which is $V = S_t$ if $S_t \geq B$ for $t_0 \leq t < T$.
- In more detail, a convertible bond is intermediate between a bond and an option.
  1. A convertible bond also pays coupons.
  2. We shall ignore coupons and consider only a zero coupon convertible bond.
  3. Zero coupon convertible bonds do exist.
- **Prove that $V \geq \mathrm{PV}(K)$ for a zero coupon convertible bond.**
- Convertible bonds are issued by companies, and when an investor exercises a convertible bond, the company prints new shares of stock and delivers them to the investor.
- In that sense, convertible bonds are different from exchange listed options and are more similar to warrants.
- Convertible bonds usually have much longer expiration times than exchange listed options, extending many years (30 years is not uncommon).
- One reason investors buy convertible bonds is because they can perform gamma trading with a longer time horizon than is possible with exchange listed options.
- In this question, we shall perform **gamma trading** with a convertible bond.
- **See next page.**

- **Write a class `ConvertibleBond` which inherits from `Derivative`.**
- In this question, we shall employ a callable American convertible bond.
- Denote the fair value of the convertible bond by U.
- Denote the market price of the convertible bond by M.
- You are given the following inputs: $K = 100$, $B = 130$, $T = 5.0$, $r = 0.05$ and $q = 0$.
- Use $\boldsymbol{n = 1000}$ steps in the binomial model.
- A graph of the fair value of a convertible bond with the above parameters in displayed in Fig. 6. The fair value was calculated with a volatility of $\sigma = 0.35$.
- **Plot a graph of the fair value of a convertible bond with the above parameters using a volatility of $\sigma = 0.5$.**
- If you have done your work correctly, the fair value will be higher than that displayed in Fig. 6. (Except $V = S$ when $S \geq B$ and also at $S = 0$.)
- **(Bonus) Explain why the fair value at $S \to 0$ is the same in both graphs, independent of the value of the volatility.**

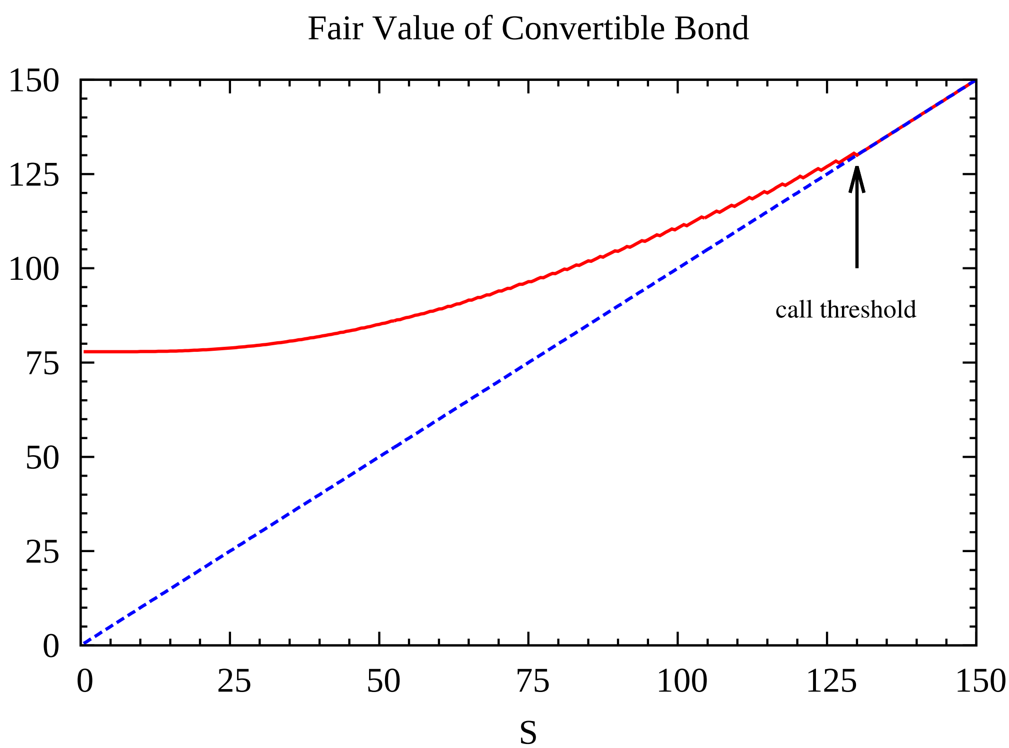

Figure 6: Graph of the fair value of a convertible bond. The intrinsic value is shown as a dashed line. The call threshold is also indicated.

- We require two more values.

  1. **Take the first 4 digits of your student id. Define:**

$$\delta S_1 = \frac{\text{first 4 digits of your student id}}{10^4}\,. \tag{7.3}$$

  2. **Take the last 4 digits of your student id. Define (note the minus sign):**

$$\delta S_2 = -\frac{\text{last 4 digits of your student id}}{10^4}\,. \tag{7.4}$$

  3. For example if your student id is 23054617, then

$$\delta S_1 = 0.2305\,, \qquad \delta S_2 = -0.4617\,. \tag{7.5}$$

  4. Note that δS_2 is **negative.**

- **See next page.**

- We shall perform **gamma trading** using a convertible bond.

1. **Day 0:**
   (a) The time is $t_0 = 0$.
   (b) The stock price is $S_0 = 60$.
   (c) The convertible bond market price is $M_0 = 90$.
   (d) **Calculate the implied volatility of the convertible bond (4 decimal places).**
   (e) Denote the answer by σ_0.
   (f) **Calculate the Delta of the convertible bond (4 decimal places) using a finite difference numerical derivative as follows. The volatility is $\sigma = \sigma_0$.**

$$\Delta_0 = \frac{U(S_0+1) - U(S_0-1)}{2} \qquad (t_0 = 0,\ \sigma = \sigma_0)\,. \tag{7.6}$$

   (g) We start trading with zero cash and zero stock and options.
   (h) We create a portfolio of **long one convertible bond U and short sale of Δ_0 shares of stock.**
   (i) The money in the bank on day 0 is therefore

$$\text{Money}_0 = \Delta_0 S_0 - M_0\,. \tag{7.7}$$

   (j) **Calculate the value of Money_0 to 2 decimal places.** (It may be negative.)
   (k) **See next page.**

2. **Day 1:**
   (a) The time is $t_0 = 0.01$.
   (b) **The stock price has changed to:**
   $$S_1 = S_0 + \delta S_1 \,. \tag{7.8}$$
   (c) **The convertible bond market price has changed to:**
   $$M_1 = 90.2 \,. \tag{7.9}$$
   (d) **Calculate the implied volatility of the convertible bond (4 decimal places).**
   (e) Denote the answer by σ_1.
   (f) **Calculate the Delta of the convertible bond (4 decimal places) using a finite difference numerical derivative as follows. The volatility is $\sigma = \sigma_1$.**
   $$\Delta_1 = \frac{U(S_1+1) - U(S_1-1)}{2} \qquad (t_0 = 0.01,\ \sigma = \sigma_1) \,. \tag{7.10}$$
   (g) **We rebalance the hedge to short Δ_1 shares of stock.**
   (h) Hence we sell $\Delta_1 - \Delta_0$ shares of stock, at the new stock price S_1.
   (i) The money in the bank on day 1 is therefore:
   $$\text{Money}_1 = \text{Money}_0 + (\Delta_1 - \Delta_0) S_1 \,. \tag{7.11}$$
   (j) **Calculate the value of Money_1 to 2 decimal places.**
   (k) The value of Money_1 may be greater or less that Money_0, it will be different for each of you.
   (l) **See next page.**

3. **Day 2:**
   (a) The time is $t_0 = \mathbf{0.02}$.
   (b) **The stock price has changed to:**
   $$S_2 = S_1 + \delta S_2\,. \tag{7.12}$$
   (c) Note that this is a negative change in the stock price.
   (d) **The convertible bond market price has changed to:**
   $$M_2 = 90.15\,. \tag{7.13}$$
   (e) **Calculate the implied volatility of the convertible bond (4 decimal places).**
   (f) Denote the answer by σ_2.
   (g) **Calculate the Delta of the convertible bond (4 decimal places) using a finite difference numerical derivative as follows. The volatility is $\sigma = \sigma_2$.**
   $$\Delta_2 = \frac{U(S_2+1) - U(S_2-1)}{2} \qquad (t_0 = 0.02,\ \sigma = \sigma_2)\,. \tag{7.14}$$
   (h) **We rebalance the hedge to short Δ_2 shares of stock.**
   (i) Hence we sell $\Delta_2 - \Delta_1$ shares of stock, at the new stock price S_2.
   (j) (Note that $\Delta_2 - \Delta_1$ is a negative number, so we are really buying stock at the new stock price S_2. We let the mathematics take care of the minus signs.)
   (k) **The money in the bank on day 2 is therefore:**
   $$\text{Money}_2 = \text{Money}_1 + (\Delta_2 - \Delta_1)S_2\,. \tag{7.15}$$
   (l) **Calculate the value of Money_2 to 2 decimal places.**
   (m) Once again, the value of Money_2 may be greater or less than Money_1.
   (n) **See next page.**

- **We close out our gamma trading portfolio at the end of day 2.**

  1. We sell our convertible bond at the price M_2.
  2. We buy Δ_2 shares of stock at the stock price S_2 to close out our short stock position.
  3. Hence there is only cash in the bank. The convertible bond and the stock are gone.
  4. **Your total profit is therefore:**

$$\text{Profit} = \text{Money}_2 + M_2 - \Delta_2 S_2 \,. \tag{7.16}$$

  5. **Calculate the value of the profit to 2 decimal places.**
  6. This is your profit from gamma trading.
  7. If you have done your work correctly, you should obtain a positive profit.
  8. **Note that I neglected interest rate compounding to calculate the profit/loss.** For an expiration time of 5 years and a trading interval of two days, the effects of interest rate compounding are negligible and merely make the calculations complicated.
  9. **Notice that the stock price was always $S < K$, yet the gamma trading strategy yielded a profit.**
  10. This is what options traders do.
  11. Options traders buy and sell volatility.
  12. Options traders do not care if an option (or convertible bond) is out of the money its whole life.

- **Prove that $V \geq \mathbf{PV}(K)$ for a zero coupon convertible bond.**

  1. **Since $V(S_T) \geq K$ at the expiration time T, for all values of S_T, the fair value at t_0 must be not less than $\mathbf{PV}(K)$, i.e. $V \geq \mathbf{PV}(K)$.**
  2. **If you prefer, construct an arbitrage argument. Suppose $V < \mathbf{PV}(K)$ at time t_0. Say the price is V_0, where $V_0 < Ke^{-r(T-t_0)}$. Go long the convertible bond at time t_0, borrow cash V_0 from a bank. Wait until expiration. The loan amount to repay is $V_0 e^{r(T-t_0)}$ and by definition $V_0 e^{r(T-t_0)} < K$. The terminal payoff at expiration is either cash K (if $S_T < K$) or stock S_T, if $S_T \geq K$. Sell the stock immediately to receive cash whose value is $\geq K$. Hence for all values of S_T, we have enough cash to repay the loan plus extra cash left over = guaranteed profit = arbitrage.**

- **Plot a graph of the fair value of a convertible bond with the above parameters using a volatility of $\sigma = 0.5$.**

- **The graph is plotted in Fig. 7, showing the fair values for both volatilities $\sigma = 0.35$ (solid curve, same as in Fig. 6) and $\sigma = 0.5$ (dotted curve). The fair value for $\sigma = 0.5$ is higher than that for $\sigma = 0.35$, except at $S = 0$ and for $S \geq B$, in which case the fair values are equal.**

- **(Bonus) Explain why the fair value at $S \to 0$ is the same in both graphs, independent of the value of the volatility.**

  1. **For Geometric Brownian Motion, the changes in the stock price S are proportional to S.**
  2. **Hence for $S = 0$ the changes in the value of S are zero, independent of the value of the volatility.**
  3. **Hence if $S = 0$ at time t_0, <u>the value of S remains zero at all future times.</u>**
  4. **Hence $S_T = 0$ at expiration and so $V(S_T) = K$, independent of the value of the volatility.**
  5. **Hence at time t_0, the fair value is $V = \mathbf{PV}(K)$, independent of the value of the volatility.**
  6. **All student solutions which expressed the above basic idea were accepted. Fancy mathematics from students was not required.**

- **Gamma trading: the results depend on student code, also the student id.**

- **For the id 23054617, the results are tabulated below.**

day	t_0	S	Market	implied vol	Delta	cash
0	0	60	90	0.3736	0.4805	−61.17
1	0.01	60.23	90.2	0.3744	0.4814	−61.12
2	0.02	59.77	90.15	0.3843	0.4805	−61.17

- **The final profit is**

$$\mathbf{profit} = 0.2615\,.$$

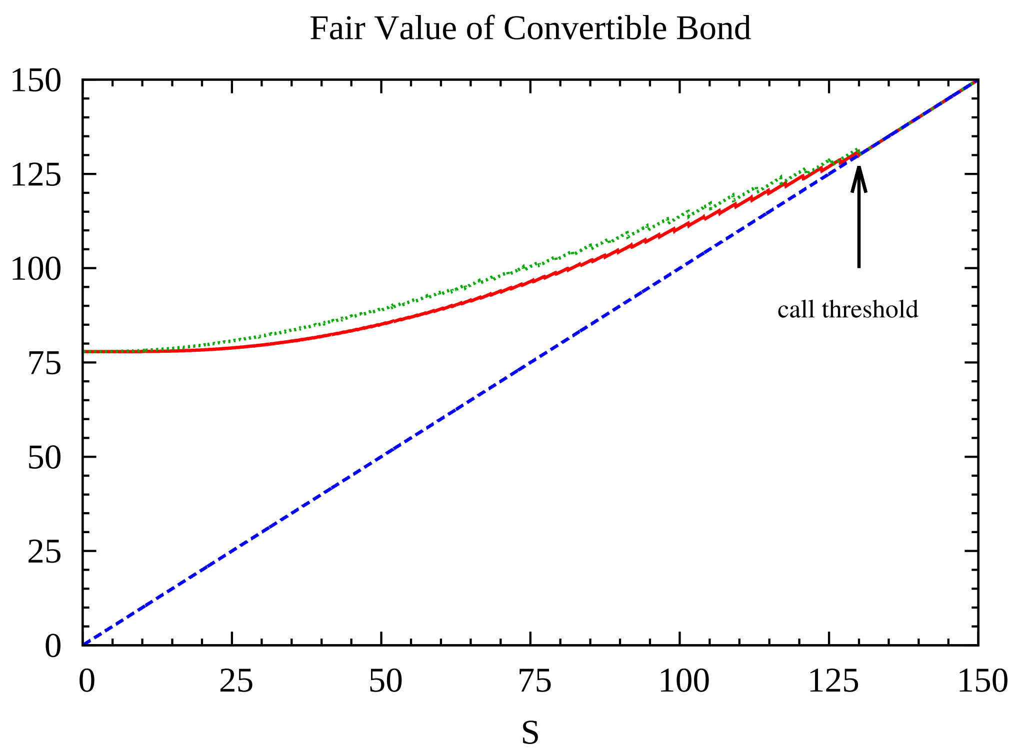

Figure 7: Graph of the fair value of a convertible bond. The fair value for a volatility of $\sigma = 0.35$ is plotted as the solid curve (same as in Fig. 6). The fair value for a volatility of $\sigma = 0.5$ is plotted as the dotted curve. The intrinsic value is shown as a dashed line. The call threshold is also indicated.