SC-HW2

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due on 03/20

1.

Weibull($\alpha = 1, \beta = 1.5$):

$$f(x; \alpha, \beta) = 1.5\sqrt{x}\exp(-x^{1.5}), x > 0.$$

$$F(x) = \int_0^\infty f(x)dx = 1 - \exp(-x^{1.5}), x \geq 0.$$

$$E(X) = \alpha\Gamma(1 + 1/\beta) = 0.9027453$$

$$Var(X) = \alpha^2[\Gamma(1 + 2/\beta) - (\Gamma(1 + 1/\beta))^2] = 0.3756903$$

```
a=1
b=1.5
true.mean = a*gamma(1+1/b)
true.var = a^2*(gamma(1+2/b)-(gamma(1+1/b)^2))
kable(t(c(true.mean,true.var)))
```

0.9027453	0.3756903

(a) (Both mean and variance are unknown)

By W.L.L.N and Binomial expansion,

$$\frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^k \xrightarrow{p} E(X - E(X))^k$$

Then,we have

$$\frac{\frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^4}{(\frac{1}{n}\sum_{j=1}^{n}(X_j - \bar{X})^2)^2} \xrightarrow{p} E(\frac{X - E(X)}{\sqrt{Var(X)}})^4$$

Do the simplify:

$$\frac{\frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^4}{(\frac{1}{n}\sum_{j=1}^{n}(X_j - \bar{X})^2)^2} = \frac{1}{n}\sum_{i=1}^{n}(\frac{nY_i^2}{\sum_{j=1}^{n}Y_j^2})^2$$

, where $Y_i = X_i - \bar{X}$

Thus,the MC estimator is

$$\hat{\theta} = \frac{1}{n}\sum_{i=1}^{n}(\frac{nY_i^2}{\sum_{j=1}^{n}Y_j^2})^2 = \frac{1}{n}\sum_{i=1}^{n}W_i$$

,and the MC s.e is

$$\text{root}(\frac{1}{n(n-1)}\sum_{i=1}^{n}(W_i - \hat{\theta})^2)$$

Set n=100000:

```
set.seed(1)
n = 100000
mc_samples <- rweibull(n,shape = 1.5,scale = 1)
mc_cent <- (mc_samples-mean(mc_samples))^2
normalize_mc <- (n*mc_cent/(sum(mc_cent)))^2
mc.mean <- mean(normalize_mc)
mc.sd <- sqrt(var(normalize_mc)/n)
kable(data.frame("mc.mean"=mc.mean,"mc.se"=mc.sd))
```

mc.mean	mc.se
4.363301	0.0957054

(a1)

Let $T(X) = X^3$,

$$\begin{aligned} E(T(X)) &= \int_0^\infty T(x) \times f(x)dx \\ &= \int_1^0 T(x) \times 1(-du), \text{let } u = \exp(-x^{1.5}), du = -f(x)dx \\ &= E_U(T((\ln(1/U))^{1/1.5})), U \sim U(0,1). \end{aligned}$$

Then,applied Monte Carlo estimator,

$$\text{The estimator of this expectation is } \hat{\theta} = \frac{1}{n}\sum_{i=1}^{n}(\ln(1/U_i))^2, U_i \overset{i.i.d}{\sim} U(0,1).$$

and its s.e.,

$$\text{root }(\frac{1}{n(n-1)}\sum_{i=1}^{n}((\ln 1/U_i)^2 - \hat{\theta})^2)$$

```
set.seed(2)
n = 100000
mc_samples <- 0
for (i in 1:n){
  u <- runif(n=1)
  mc_samples[i] <- (log(1/u))^2
}
```

```r
mc_mean <- mean(mc_samples)
mc_sd  <- sqrt(var(mc_samples)/n)
kable(data.frame("mc.mean"=mc_mean,"mc.se"=mc_sd))
```

mc.mean	mc.se
1.996292	0.0139745

(b)

Let $l(x;\theta) = \log f(x;\theta)$,

$$\frac{\partial^2 l}{\partial \alpha^2}|_{(\alpha=1,\beta=1.5)} = 1.5[2.5x^{1.5} - 1]$$
$$\frac{\partial^2 l}{\partial \beta^2}|_{(\alpha=1,\beta=1.5)} = -[1/1.5^2 + x^{1.5}(\log x)^2]$$
$$\frac{\partial^2 l}{\partial \beta \partial \alpha}|_{(\alpha=1,\beta=1.5)} = \frac{\partial^2 l}{\partial \alpha \partial \beta}|_{(\alpha=1,\beta=1.5)} = -[1 - x^{1.5} - 1.5x^{1.5}\log x]$$

接著，利用 (a1) 的概念分別估計這三個的 Monte Carlo estimation(Fisher information 記得加上負號):

```r
set.seed(3)
n = 100000
mc_samples11 <- 0
mc_samples12 <- 0
mc_samples22 <- 0
for (i in 1:n){
  u <- (log(1/runif(n = 1)))^(1/1.5)
  mc_samples11[i] <- -(1.5-1.5*2.5*u^(1.5))
  mc_samples12[i] <- 1-u^1.5-1.5*u^1.5*log(u)
  mc_samples22[i] <- 1/1.5^2+u^1.5*(log(u))^2
}
Fisher_11 <- mean(mc_samples11)
Fisher_12 <- Fisher_21 <- mean(mc_samples12)
Fisher_22 <- mean(mc_samples22)
sd_11 <- sqrt(var(mc_samples11)/n)
sd_12 <- sd_21 <- sqrt(var(mc_samples12)/n)
sd_22 <- sqrt(var(mc_samples22)/n)
mc_Fisher_matrix <- matrix(c(Fisher_11,Fisher_21,Fisher_12,Fisher_22),2,2)
mc_Fisher_sd <- matrix(c(sd_11,sd_21,sd_12,sd_22),2,2)
```

- M.C. Fisher information matrix:

```
kable(mc_Fisher_matrix)
```

2.2645080	-0.4330478
-0.4330478	0.8142425

- M.C. s.e. :

```
kable(mc_Fisher_sd)
```

0.0119162	0.0078956
0.0078956	0.0031148

(c)

I choose the proposal 2+Exp(1) distribution:

$$
\begin{aligned}
E(X|X>2) &= \int_2^\infty xf(x)dx = \int_2^\infty [xf(x)/h(x)]h(x)dx \\
&= \int_2^\infty [yf(y)/h(y)]h(y)dy \\
&= E_Y[Y\frac{f(Y)}{h(Y)}], \text{where } Y \sim h(.) : \text{the p.d.f. of 2+Exp(1)}
\end{aligned}
$$

```
set.seed(4)
a=2
y = a+rexp(n = 100000,rate = 1)
weight = dweibull(y,1.5,1)/dexp(y-a,rate = 1)
mc_samples = y*weight
kable(data.frame("mc.mean"=mean(mc_samples),"mc.se"=sqrt(var(mc_samples)/n)))
```

mc.mean	mc.se
0.1433181	0.0002546

另外，使用 identical function $I_{(X_i>2)}$，得到 M.C. estimation，與 importance sampling method 做個比較:

$$
\hat{E}(X|X>2) = \frac{1}{n}\sum_{i=1}^{n} I_{(X_i>2)}X_i
$$

```
set.seed(5)
x <- rweibull(n = 100000,shape = 1.5,scale = 1)
ident <- as.numeric(x>2)
kable(data.frame("mc.mean"=mean(x*ident),"mc.se"=sqrt(var(x*ident)/n)))
```

mc.mean	mc.se
0.1463586	0.0018555

Accroding to the comparison between MC s.e.,MC estimator by importance sampling method is better.

2.

(a)

The joint pdf f(x,y) is derived by $\frac{\partial F(x,y)}{\partial x \partial y}$.

$$\frac{\partial F(x,y)}{\partial x} = F(x,y) \times [\frac{1}{x^2}(\Phi\{u(x,y)\} + \phi\{u(x,y)\}) - \frac{1}{xy}\phi(1-u(x,y))]$$

, where $u(x,y) = \frac{1}{2} - \log{(x/y)}$.

$$\frac{\partial F(x,y)}{\partial y} = F(x,y) \times [\frac{1}{y^2}(\Phi\{1-u(x,y)\} + \phi\{1-u(x,y)\}) - \frac{1}{xy}\phi(u(x,y))]$$

By the above,we have

$$\frac{\partial^2 F(x,y)}{\partial x \partial y} = \frac{F(x,y)}{xy}[(\underline{\frac{1}{x}(\Phi\{u\} + \phi\{u\}) - \frac{1}{y}\phi\{1-u\}})(\underline{\frac{1}{y}(\Phi\{1-u\} + \phi\{1-u\}) - \frac{1}{x}\phi\{u\}}$$

$$+\underline{\frac{1}{x}(1-u)\phi\{u\} + \frac{1}{y}u\phi\{1-u\}}]$$

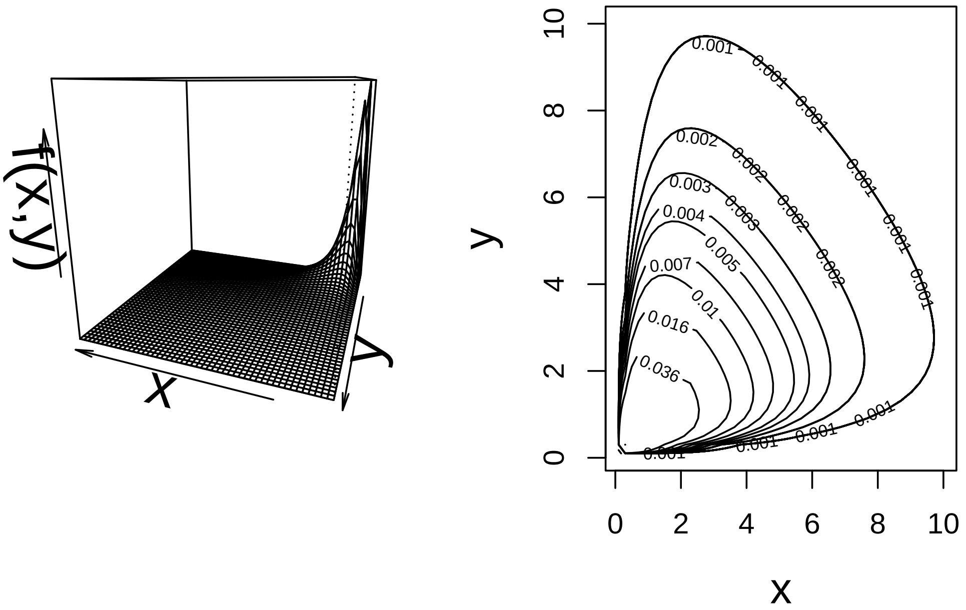
$$\overset{\text{simplify}}{=} \frac{F(x,y)}{xy}[k_1(x,y)k_2(x,y) + k_3(x,y)], \text{where } k_1, k_2, k_3 \text{ are the corresponding bottom lines above.}$$

$$= f(x,y) : \text{joint p.d.f }, x, y > 0$$

Next,draw the density plot and contour plot:

```r
f <- function(x){
  u <- 1/2-log(x[1]/x[2])
  V <- 1/x[1]*pnorm(u)+1/x[2]*pnorm(1-u)
  F_cdf <- exp(-V)
  k1 <- 1/x[1]*(pnorm(u)+dnorm(u))-1/x[2]*dnorm(1-u)
  k2 <- 1/x[2]*(pnorm(1-u)+dnorm(1-u))-1/x[1]*dnorm(u)
  k3 <- 1/x[1]*((1-u)*dnorm(u))+1/x[2]*(u*dnorm(1-u))
  k <- k1*k2+k3
  return(F_cdf*k/(x[1]*x[2]))
} # joint pdf
grid1 <- seq(0.1,10,length=50)
xy = expand.grid(x=grid1, y=grid1)
z = apply(xy,1,f)
```

```r
par(mfcol=c(1,2))
persp(grid1, grid1, matrix(z,50,50), theta= -160,
      zlab="f(x,y)", xlab="x", ylab="y",cex.lab=2)
contour(grid1, grid1, matrix(z,50,50), levels= round(quantile(z, 0.04*(1:25)),3),
        xlab="x", ylab="y", cex.lab=1.5)
```



As we see,the density is larger when (x,y) is close to (0,0). Next,I choose the Exp(rate=1) as the proposal distribution:

- Alogrithm(MH,independent sampler):

1. Set the jump proposal:Exp(1)
2. For t=1,2,...,($x^{(0)} = 0.5, y^{(0)} = 0.5$)

   (2a) draw $x^* \sim Exp(1)$,its pdf is $q(x) = exp(-x), x > 0$

   (2b) calculate $r = \dfrac{f(x^*)}{f(x^{(t-1)})} \dfrac{q(x^{(t-1)})}{q(x^*)}$

   (2c) set $x^{(t)} = x^*$ with probability min{1,r},otherwise $x^{(t-1)}$.
3. Repeat step 1,2 until n.iter=10000 pairs of samples (x,y) are generated.

```r
#proposal:exp(rate=1)
MH1 <- function(init, n.iter){
  xx <- matrix(0,n.iter,2)
  count <- 0
  xx[1,] <- init
  joint_exp <- function(uv){dexp(uv[1],rate = 1)*dexp(uv[2],rate = 1)}
```

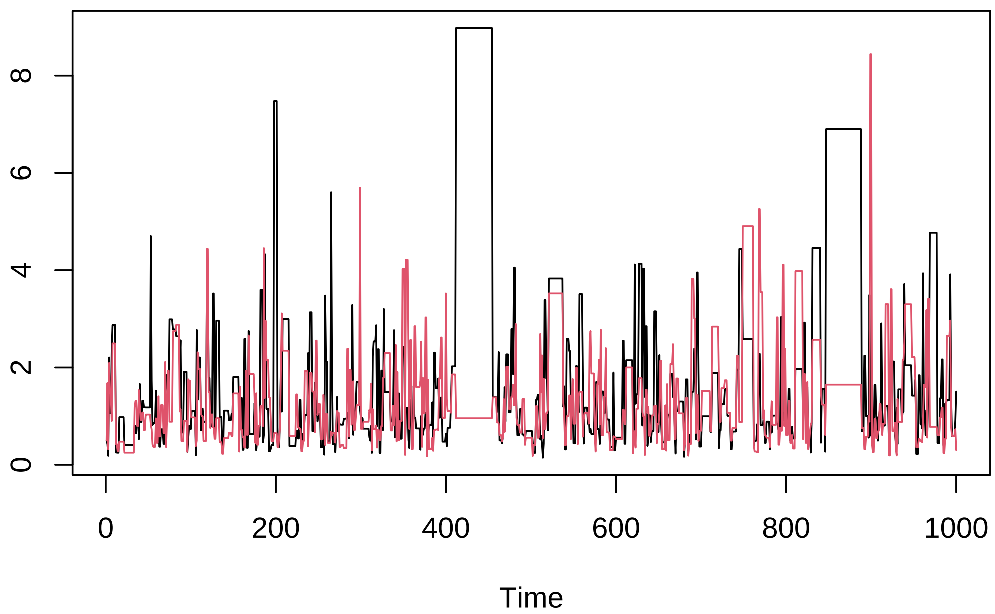
```
  for (i in 2:n.iter){
    uv <- rexp(2,rate = 1)
    r <- (f(uv)/f(xx[i-1,]))*(joint_exp(xx[i-1,])/joint_exp(uv))
    U <- runif(1)
    if (U< min(r,1)) {
      xx[i,] <- uv
      count <- count +1
    }
    else {xx[i,] <- xx[i-1,]}
  }
  return(list(mc=xx, r.mean=count/n.iter))
}
```

```
set.seed(1019)
mc1<- MH1(c(0.5,0.5),10000)
cat("Acceptance rate:",mc1$r.mean)
```

```
## Acceptance rate: 0.4067
```

- Check series:

```
ts.plot(mc1$mc[1:1000,1:2], col=1:2)
```

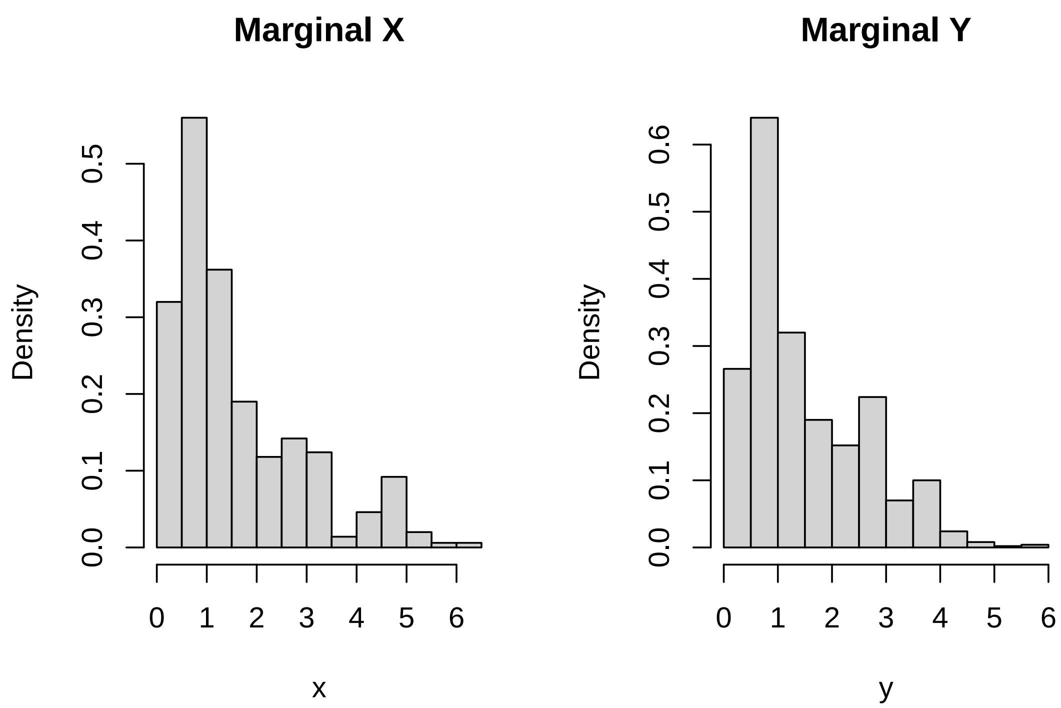


From the above, observe the first 1000 iteration, I think that the chains tend to a fairly stable structure after 1000.So,select the last 1000 pair samples (X,Y):

```
new_mc1 <- mc1$mc[9001:10000,]
```

Next,let's see the marginal historgram of new_mc1:

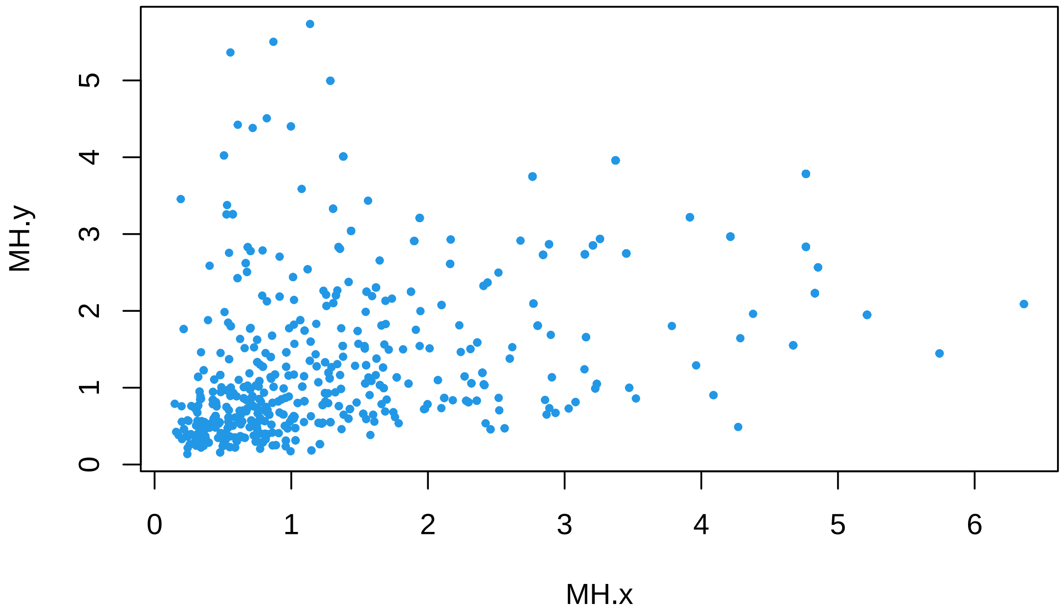
```
par(mfcol=c(1,2))
hist(new_mc1[,1],freq = F,main="Marginal X",
     xlab = "x")
hist(new_mc1[,2],freq = F,main="Marginal Y",
     xlab = "y")
```



As we expected,the margin density of both are large when it is close to 0.

- Scatter plot:

```
plot(new_mc1[,1], new_mc1[,2],
     pch=19, col=4, cex=0.5,
     xlab="MH.x",ylab="MH.y")
```



As we see,this scatter plot is similar to the population plot.

(b)

Accroding to hint,we should obtain the MC estimator of $E(h(X,Y))$ by using the 1000 samples of (X,Y) from 2.(a):

```
mc.samples <- 0
x <- new_mc1[,1]
y <- new_mc1[,2]
for (i in 1:1000){
  if ({y[i] <= 5-x[i]} & {x[i] <5}){
    mc.samples[i] <- 1
  }
  else{
    mc.samples[i] <- 0
  }
}
mc.mean <- mean(mc.samples)
mc.se <- sqrt(var(mc.samples)/1000)
kable(data.frame("mc.mean"=mc.mean,"mc.se"=mc.se))
```

mc.mean	mc.se
0.8	0.0126554

From the above,we obtain the MC estimation of $E(h(X,Y))$ is $1-\hat{\tau} = 0.8$ with M.C. s.e.$(\hat{\tau}) = 0.0126554$. Thus,the desired probability is estimated to be :

```
kable(1-mc.mean,col.names = "")
```

0.2