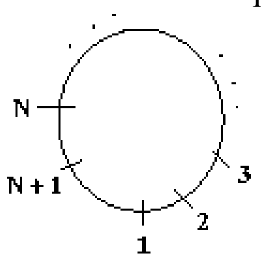
Thomas Algorithm for Periodic Tridiagonal Systems

Figure: Domain for solving the steady heat equation in a ring.



The boundary value problem we considered in the example above employed Dirichlet boundary conditions, but often periodic boundary conditions are required. This could be a case for example, where an infinite domain is simulated or the physics of the problem dictates it, as in solving an elliptic problem on a ring (see figure). In this case, despite the sparsity of the matrix resuling from the discretization and its almost tri-diagonal form everywhere, the bandwidth is actually equal to the rank of the matrix in the form shown below:

$$\begin{bmatrix} a_1 & c_1 & & & b_1 \\ b_2 & a_2 & c_2 & 0 & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & c_N \\ c_{N+1} & 0 & & b_{N+1} & a_{N+1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \\ x_{N+1} \end{bmatrix} = \begin{bmatrix} q \end{bmatrix}$$

where we assume that b_1 and c_{N+1} are coefficients corresponding to the periodic boundary conditions, (e.g. equal to = 1 in the example above).

We can solve this system by first ``condensing'' the matrix, that is eliminating the last row and the last column to arrive at:

$$\underbrace{\begin{bmatrix} a_1 & c_1 & & & \\ b_2 & a_2 & c_2 & 0 & \\ & \ddots & \ddots & & \\ & & & & c_{N-1} \\ 0 & & & b_N & a_N \end{bmatrix}}_{[A^c]} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} = [q] - \begin{bmatrix} b_1 \\ 0 \\ \vdots \\ 0 \\ c_N \end{bmatrix} x_{N+1}$$

Now we use the linear proprerty and propose a superposition of the form

$$x = x^{(1)} + x^{(2)} \cdot x_{N+1},$$

where $x^{(1)}$ and $x^{(2)}$ are solutions of the tridiagonal ``condensed'' system with N unknowns, i.e.,

$$\begin{aligned} [A^c]\left[x^{(1)}\right] &= [q] \\ [A^c]\left[x^{(2)}\right] &= \begin{bmatrix} -b_1 \\ 0 \\ \vdots \\ 0 \\ -c_N \end{bmatrix} \end{aligned}$$

We finally compute x_{N+1} from the last equation in the original system by back substitution, i.e.

$$c_{N+1} \,(x^{(1)}{}_1 + x_{N+1}\, x^{(2)}{}_1) + b_{N+1}\, (x^{(1)}{}_N + x_{N+1}\, x^{(2)}{}_N\,) + a_{N+1}\, x_{N+1} = q_{N+1}$$

and we solve for x_{N+1}:

$$x_{N+1} = \frac{q_{N+1} - c_{N+1}x_1^{(1)} - b_{N+1}x_N^{(1)}}{a_{N+1} + c_{N+1}x_1^{(2)} + b_{N+1}x_N^{(2)}}.$$

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