SPF Office Simulation

Simulation Final Project

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5th April 2023

MA424 Modeling in Operations Research

1 Find applicants' expected waiting times.

1.1 Build simulation model

In order to simulate the expected waiting times of applicants, define time variable t. Since the arrival time of applicants are unique to them and that time is the only required data when calculating the total time they spent when leaving, the arrival time is the only data that need to be stored. In the R code, q_sf, q_sc, q_sw are the three vectors used to store the arrival time of applicants in queues S_F, S_C, S_W respectively. Meanwhile, define t_1 as the time the next applicant arrives. Define t_2, t_3, t_4 as the time the next applicant be served in the queues S_F, S_C, S_W respectively. Therefore, the event list $EL = t_1, t_2, t_3, t_4$. In the R code, ta, tc, tr are three vectors storing the simulating results of output variables T_a, T_c, T_r. The average of numbers in ta, tc, tr are respectively estimators of $E[T_a]$, $E[T_c]$, $E[T_r]$. The followings use U to represent any uniform(0,1) random variable.

There are four cases in total excluding the stopping case, and each one represent an event. When $t > T$ and there are no applicants in any queues, the process will end. The pseudocode and the events with brief explanation are as follows:

Initialize

1. Set S_F, S_C, S_W to empty queues.
2. Set $t_2 = t_3 = t_4 = \infty$.
3. Generate the time the first applicant enters: $t = t_1 = -\frac{1}{\lambda}log(U)$.

Case 1: t_1 is the minimum among t_1, t_2, t_3, t_4. Applicant entering the office. This happens in working hours according to Poisson process with rate λ per hour.

1. Is $t \leq T$? If so, do the following steps. Otherwise set $t_1 = \infty$.
2. Is the applicant using ACS? Generate U. If $U < p$ the applicant enters the queue for S_F (do step 3a). Otherwise the applicant enters the queue for S_C (do step 3b).

3a. Is the queue for S_F empty? If so, generate U and $t_2 = t - \frac{1}{\mu_F}log(U)$. Otherwise do not change t_2. Then store t as the time the applicant arrives.

3b. Is the queue for S_C empty? If so, generate U and $t_3 = t - \frac{1}{\mu_C}log(U)$. Otherwise do not change t_3. Then store t as the time the applicant arrives.

4. Generate the time the first applicant enters: generate U and $t_1 += -\frac{1}{\lambda}log(U)$.

Case 2: t_2 is the minimum among t_1, t_2, t_3, t_4. Applicant finish their submission in S_F. Then the applicants completed their application and they leave the office.

1. The first applicant of the queue for S_F leaves the queue. Set $c =$ the time this applicant enters the office.
2. Store the total time the applicant spent in the office by appending the total time $t - c$ on the tail of array T_a and T_r.
3. Is the queue for S_F empty? If no, generate the time the next applicant completes: generate U and $t_2 += -\frac{1}{\mu_F} log(U)$. Otherwise set $t_2 = \infty$.

Case 3: t_3 is the minimum among $t_1,\ t_2,\ t_3,\ t_4$. Applicant finish the application checking in S_C. The the applicants go to S_F or S_W.

1. The first applicant of the queue for S_C leaves the queue. Set $c =$ the time this applicant enters the office.
2. Does this applicant require interview? Generate U. If $U < q$ the applicant enters the queue for S_F (do step 3a). Otherwise the applicant enters the queue for S_W (do step 3b).

3a. Is the queue for S_F empty? If so, generate U and $t_2 = t - \frac{1}{\mu_F} log(U)$. Otherwise do not change t_2. Then store c to the queue for S_F.

3b. Is the queue for S_W empty? If so, generate U and $t_4 = t - \frac{1}{\mu_W} log(U)$. Otherwise do not change t_4. Then store c to the queue for S_W.

4. Is the queue for S_C empty? If no, generate the time the next applicant completes: generate U and $t_3 += -\frac{1}{\mu_C} log(U)$. Otherwise set $t_3 = \infty$.

Case 4: t_4 is the minimum among $t_1,\ t_2,\ t_3,\ t_4$. Applicant finish the interview in S_W. Then the applicants either be refused and leave or go to S_F.

1. The first applicant of the queue for S_W leaves the queue. Set $c =$ the time this applicant enters the office.
2. Does the applicant pass the interview? Generate U. If $U < q$ the applicant enters the queue for S_F (do step 3a). Otherwise, calculate the time that customer spent (do step 3b).

3a. Is the queue for S_F empty? If so, generate U and $t_2 = t - \frac{1}{\mu_F} log(U)$. Otherwise do not change t_2. Then store c to the queue for S_F.

3b. Store the total time the applicant spent in the office by appending the total time $t - c$ on the tail of array T_a and T_r.

4. Is the queue for S_W empty? If no, generate the time the next applicant completes: generate U and $t_4 += -\frac{1}{\mu_W} log(U)$. Otherwise set $t_4 = \infty$.

The R code is in document **SPF1-1.ipynb**. After $K = 500$ iterations, the result of the simulation is as follow: $E[T_a] = 1.4971$, $E[T_c] = 1.5577$, $E[T_r] = 1.0524$.

1.2 Point estimation

According to the central limit theorem, when k is big enough, the estimate of T_a approximately distributes as normal. Let c be a positive number, and d is the estimator of the standard deviation $(\frac{\sigma}{\sqrt{k}})$. Since the confident level is 95%, the following equation satisfies.

$$Pr\{|\overline{T_a} - E[T_a]| > c \cdot d\} = 2(1 - \Phi(c)) = 0.05$$

According to the requirement, the value of $c \cdot d = \frac{1}{6} = 0.1667$. Therefore, $c = \Phi^{-1}(1 - 0.05 \div 2) = 1.96$, $d = 0.0850$. Let S_a be the standard deviation of all simulated T_a stored in array variable ta. To find the minimum iterations, set the $K_{min} = 100$ for the central limit theorem and find the first k that satisfies $\frac{S_a}{\sqrt{k}} < d = 0.0850$. The R code is in document **SPF1-2.ipynb**, and it can be seen that the minimum k equals 219. Besides, the estimation of mean value of T_a as known as $\bar{T_a} = 1.5048$ by then.

1.3 Interval estimate

The normal distribution can be used to calculate the interval of the mean value. Hence, with probability $(1 - \alpha)$ the value $E[T_a]$ will lie within the region $\bar{T_a} \pm z_{\frac{\alpha}{2}} \frac{S_a}{\sqrt{k}}$. It can be seen from the last section that $z_{0.05} = -1.65$, $\frac{S_a}{\sqrt{k}} = d = 0.0850$. Therefore, $E[T_a]$ lies in $\bar{T_a} \pm z_{0.05} \frac{S_a}{\sqrt{k}} = 1.5048 \pm (-1.65 \cdot 0.0850) = 1.5048 \mp 0.1403$ at a 90% confidence. After calculation, it can be seen that it is 90% confident that $E[T_a] \in [1.3645, 1.6451]$.

1.4 Results of applications

Theoretically, the analytical answer of the proportion of applicants complete their is application is $1 - (1 - p) \cdot (1 - q) \cdot (1 - r) = 1 - 0.5 \cdot 0.4 \cdot 0.6 = 0.88$, equals 88.000000%. Similarly, the analytical answer of the proportion that the applications are rejected is $(1 - p) \cdot (1 - q) \cdot (1 - r) = 0.5 \cdot 0.4 \cdot 0.6 = 0.12$, equals 12.000000%. In the section 1.2, when K equals 219, the simulated complete rate and reject rate are respectively 87.9728% and 12.0272%. Furthermore, in the section 1.1, it was mentioned $K = 500$, and the simulated complete rate and reject rate are respectively 87.999874% and 12.000126%, which is really close to the analytical answer.

2 Improve the estimator of $E[T_a]$

The variance of the estimator can be reduced by introducing a control variable. Suggest that a control variable Z represents the arrival time between two continuously entered applicants, the mean value of the control variable μ_Z is equal to $\frac{1}{\lambda} = 0.125$. When Z is increasing, the number of applicants who enter the office will fall, which would lead to the decreasing of the number of people in the queue and the total waiting time including $T_a,\ T_c,\ T_r$. Therefore, the control variable Z is highly correlated and have a negative correlation with the waiting time T_a, which satisfies the requirement of control variables.

Hence, the controlled estimator $T_a + c(Z - \frac{1}{\lambda})$, which is an estimator of $E[T_a] + c(E[Z] - \frac{1}{\lambda})$, will reduce variance with an appropriate value of c. In order to minimize the variance, the optimal value of c is: $c^* = \frac{Cov(T_a,Z)}{Var(Z)}$. The covariance $Cov(T_a, Z)$ can be estimated by simulating enough (for example: 100) values of T_a and Z and using the sample covariance. And $Var(Z)$ can be calculated by either simulation or using the variance of the exponential distribution, which is $\frac{1}{\lambda^2} = \frac{1}{64}$ theoretically.

By using control variable Z, the variance of the estimator of $E[T_a]$ will decrease and be improved.

3 Distributions of servers S_C and S_W

3.1 Service time simulation

By using the rejection method, with the exponential($\frac{1}{2}$) distribution $g(x) = \frac{1}{2}e^{-\frac{x}{2}},\ x > 0$, the maximum of $\frac{f(x)}{g(x)} = (1 + x)e^{-\frac{x}{2}}$ is less or equal than $\frac{f(1)}{g(1)} = 2e^{-\frac{1}{2}}$ because its derivative equals 0 when $x = 1$. Let $c = 2e^{-\frac{1}{2}}$, then $\frac{f(x)}{cg(x)} = \frac{1+x}{2}e^{\frac{1-x}{2}}$. Thus, the distribution for the service time of server S_C can be generated as follows.

1. Generate $Y \sim exp(\frac{1}{2}),\ U \sim uniform(0, 1)$.
2. if $U \leq \frac{1+Y}{2}e^{\frac{1-Y}{2}}$, set $X = Y$. Otherwise, return to step 1.

By using the inverse transform method, the distribution for the service time of server S_W can be generated as follows. Let $x = F^{-1}(u),\ x > 0$, then the following equations are satisfied.

$$
\begin{aligned}
u = F(x) &= 1 - e^{-3x^2} \\
e^{-3x^2} &= 1 - u \\
-3x^2 &= ln(1-u) \\
x^2 &= -\frac{1}{3}ln(1-u) \\
x &= \sqrt{-\frac{1}{3}ln(1-u)}
\end{aligned}
$$

Since U and $1-U$ have the same distribution, the service time of server S_W can be generated as $x = \sqrt{-\frac{1}{3}ln(U)}$. The R language functions are provided in document **SPF3.ipynb**.

3.2 Histogram of the distributions

In the R code, arrays Tsc and Tsw store the generated service time of server S_C and S_W respectively. After 10000 iterations, the histogram of Tsc is shown in figure 1. Since most values are within $[0, 15]$, using breaks as 15 will make every range has an approximately 1 length. Therefore, the first bar will have an around $f(0) * K = 5000$ value and the constant the densities multiply is exactly K.

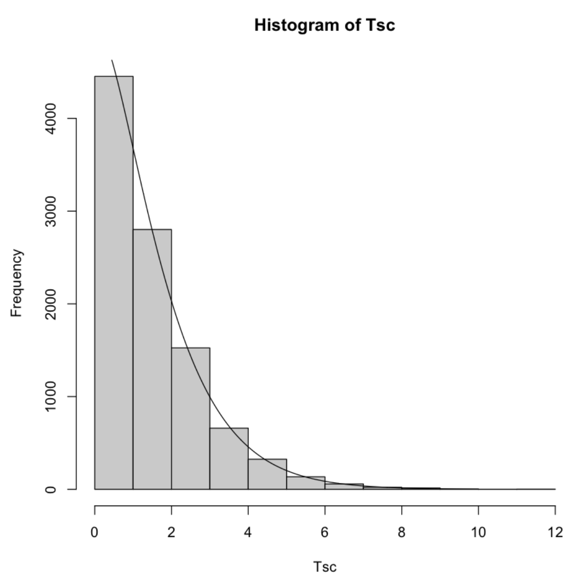

Figure 1: The Histogram of the service time of server S_C (in minutes)

The histogram of Tsw is shown in figure 2. Since most values are within $[0, 1.5]$, using breaks as 15 will make every range has an approximately 0.1 length. Therefore, the constant that multiplies to the density will be $K * 0.1 = 1000$. Furthermore, in both histograms, if the lengths of each range multiply a constant, the constants that densities multiply shall multiply the same value. That is why a variable n is introduced to control the densities of both histogram in the R code. When a specific n is chosen, the constants that densities multiply in figure 1 and 2 will be respectively $\frac{K}{n}$ and $\frac{K}{10n}$.

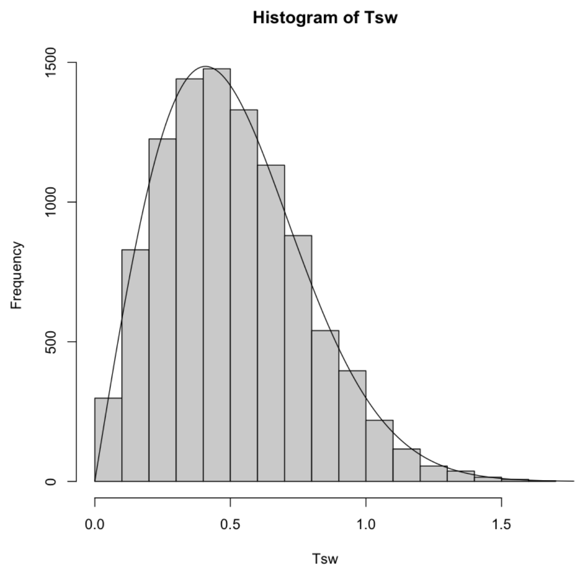

Figure 2: The Histogram of the service time of server S_W (in hours)

It is significant that the histogram is highly similar to the densities of the distributions, which is the evidence of the correctness of the program. The R language codes and related histograms are provided in document **SPF3.ipynb**.