Digital Communication Circuit Laboratory

Lab. 5

Coordinate Rotation Digital Computer (CORDIC)

I. Purpose

In this lab., we will learn the principle of CORDIC operation to implement the conversion from Cartesian coordinate to polar coordinate. In other words, we can obtain the phase and magnitude of a complex value.

II. Principle

In digital signal processing or communication algorithms, we often need the phase or magnitude of a complex value. Given $Z = X + jY$, define

$$\measuredangle Z = \tan^{-1}\left(\frac{Y}{X}\right) \tag{1}$$

and

$$|Z| = \sqrt{X^2 + Y^2}. \tag{2}$$

Because the tangent and arctangent functions have $\pi/2$ symmetry as shown in Fig. 1, in hardware implementation, we usually consider the case in the first quadrant and then extend the result to the remaining quadrants according to the sign values of X and Y.

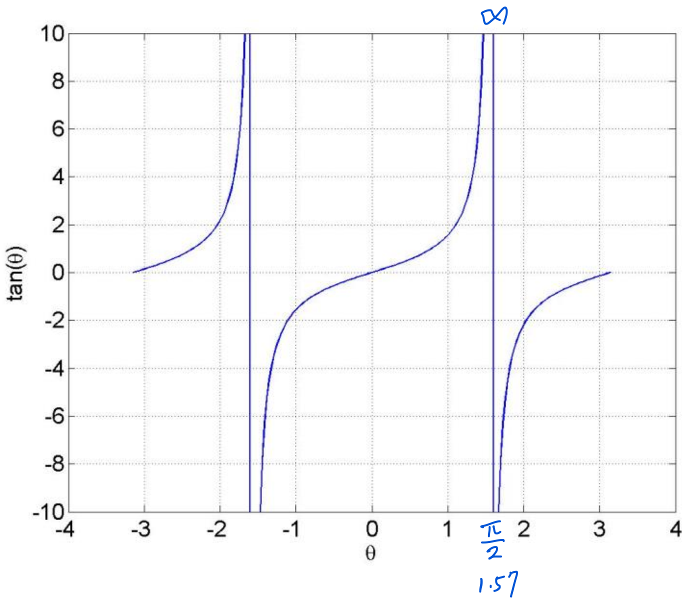

Fig. 1 The tangent function.

The CORDIC (COordinate Rotation DIgital Computer) operation is a good mean to convert values between the Cartesian coordinate and the polar coordinate. It is also widely used in QR decomposition, matrix triangularization, singular value decomposition, and Eigen-value decomposition.

The basic concept of CORDIC is to partition one rotation angle into the sum of several elementary angles. These elementary angles, denoted by $\theta_e(i)$ for $i = 0,1,...$, are special angles, which can be accomplished by shift-and-add,

$$\theta = \sum_{i=0}^{N-1} \mu_i \theta_e(i), \tag{3}$$

$$\theta_e(i) = tan^{-1}(\frac{1}{2^i}) \tag{4}$$

where $\mu_i \in \{+1, -1\}$, and it determines the counter-clockwise rotation or clockwise rotation. At the ith micro-rotation step, vector $(X(i), Y(i))$ is converted to be

$$\begin{bmatrix} X(i+1) \\ Y(i+1) \end{bmatrix} = \begin{bmatrix} 1 & -\mu_i 2^{-i} \\ \mu_i 2^{-i} & 1 \end{bmatrix} \begin{bmatrix} X(i) \\ Y(i) \end{bmatrix}. \tag{5}$$

A. Phase in polar coordinate

If $Z = X + jY$ and the phase of Z is desired, we only need to rotate vector (X, Y) back to the x-axis and calculate the summation of all these angles, as shown in Fig. 2.

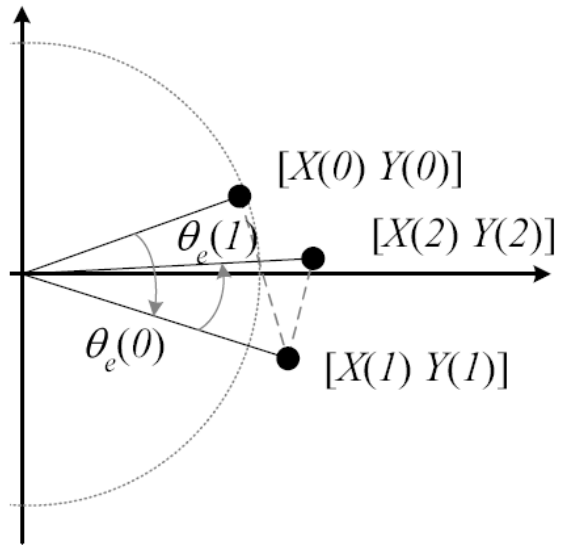

Fig. 2 CORDIC operation.

The procedure can be summarized as the following steps. Assume that $X > 0$.

1. Initialization

   $X(0) = X,\ Y(0) = Y,\ \hat{\theta}(0) = 0$

2. Determine direction

   $$\mu_i = -sgn\big(Y(i)\big). \tag{6}$$

3. Perform micro-rotation

$$\begin{bmatrix} X(i+1) \\ Y(i+1) \end{bmatrix} = \begin{bmatrix} 1 & -\mu_i 2^{-i} \\ \mu_i 2^{-i} & 1 \end{bmatrix} \begin{bmatrix} X(i) \\ Y(i) \end{bmatrix} \circ \tag{7}$$

4. Phase accumulation

$$\hat{\theta}(i+1) = \hat{\theta}(i) - \mu_i \tan^{-1}(2^{-i}) = \hat{\theta}(i) - \mu_i \theta_e(i) \tag{8}$$

5. Repeat step 2 to step 4 until $Y(i) \approx 0$.

In each micro-rotation steps, the length of vector $(X(i), Y(i))$ is changed [3]. However, if only the phase of vector (X, Y) is desired, it is not necessary to scale down the length of vector $(X(i+1), Y(i+1))$. The hardware architecture of the micro-rotation and phase accumulation is shown in Fig. 3.

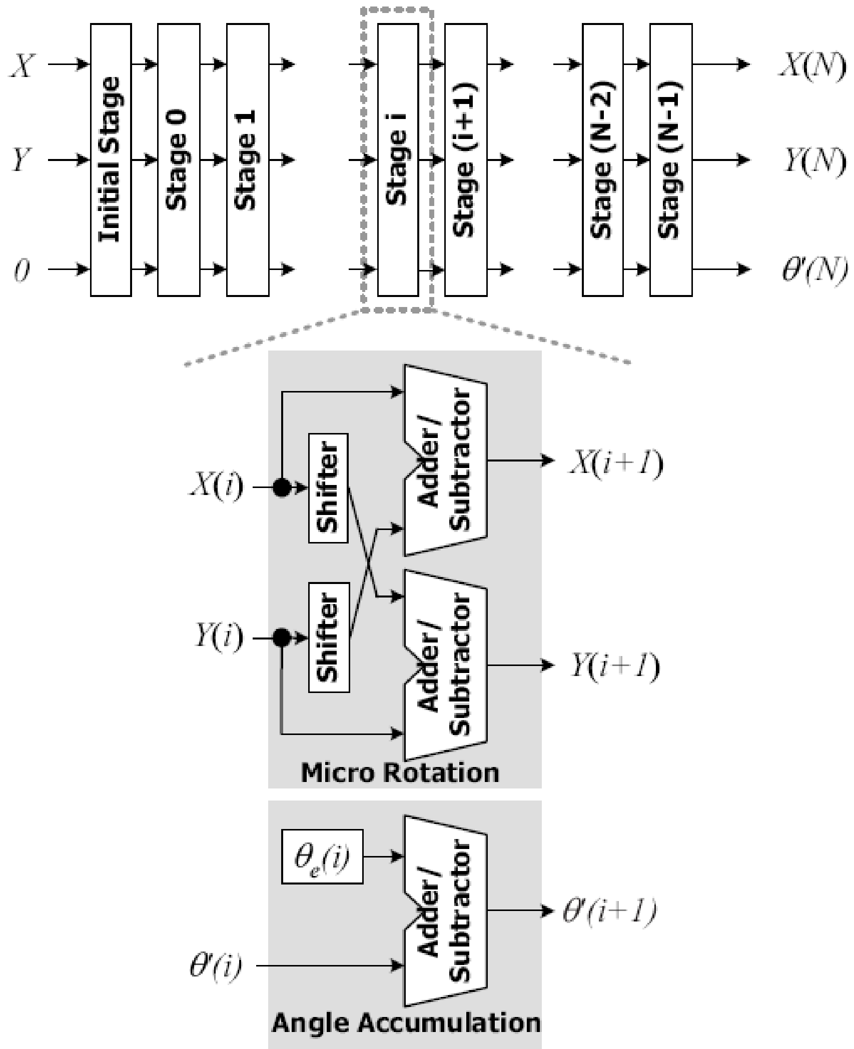

Fig. 3 Hardware architecture of CORDIC.

The convergence range of CORDIC can be computed as $\pm\sum_{i=0}^{N-1}\theta_e(i)$, which is $\pm 99^{\circ}$ approximately. Hence, we need to convert the vector in the second and the third quadrants into the first and the fourth quadrant. The initial stage in Fig. 3 deals with this operation. Subsequently, each micro-rotation is processed by each stage.

B. Magnitude in polar coordinate

From Fig. 2, if vector (X, Y) is rotated back to the x-axis with unchanged magnitude, the magnitude of vector (X, Y) is equal to the horizontal component along the x-axis. Observing Eq. (7), we can see that the vector length is increased by $\sqrt{1+2^{-2i}}$. Hence if $Y(N)=0$ after micro-rotations for N times, then

$$X(N)=\sqrt{X^2+Y^2}\prod_{i=0}^{N-1}\sqrt{1+2^{-2i}}. \tag{9}$$

As a result,

$$\sqrt{X^2+Y^2}=\frac{X(N)}{\prod_{i=0}^{N-1}\sqrt{1+2^{-2i}}}=S(N)X(N) \tag{10}$$

Usually, the number of micro-rotations N is determined according to the error tolerance. Consequently, $S(N)$ is a constant and can be represented by shift-and-add operation. The divider is not necessary for the scaling stage.

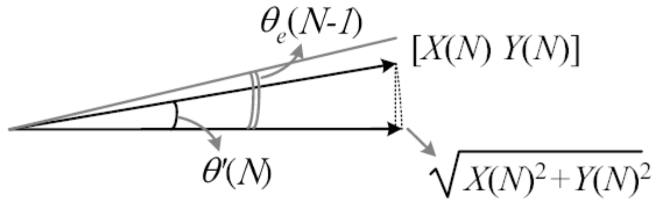

Fig. 4 The approximation error of magnitude function by using finite micro-rotations in CORDIC operation.

The error of using finite micro-rotations to approximate the magnitude function can be analyzed in the following.

$$\frac{|X+jY|-S(N)X(N)}{|X+jY|}=\frac{S(N)\sqrt{X(N)^2+Y(N)^2}-S(N)X(N)}{S(N)\sqrt{X(N)^2+Y(N)^2}}$$

$$=1-\frac{X(N)}{\sqrt{X(N)^2+Y(N)^2}}=1-\cos(\theta'(N)) \tag{11}$$

where the relationship for $X(N)$, $Y(N)$ and $\phi(N)$ is shown in Fig. 4. The residual phase $\phi(N)$ must be less than the rotation angle at the previous step, namely $\theta'(N)<\theta_e(N-1)=\tan^{-1}(\frac{1}{2^{N-1}})$. Therefore,

$$\frac{|X+jY|-S(N)X(N)}{|X+jY|}<1-\cos\big(\theta_e(N-1)\big)=1-\frac{1}{\sqrt{1+2^{-2(N-1)}}}. \tag{12}$$

From Eq. (12), we can derive the number of micro-rotations N if the CORDIC operation is adopted to approximate the magnitude function.

III. Procedures

1. Although we do not perform vector scaling when using CORDIC to obtain the phase of a complex number, we still need to know the increase in magnitude after infinite micro-rotations for reserving sufficient dynamic range during implementation. Find out the scaling factor $S(N)$ for ≥ 30 .
2. Assume that $X = \sin(\alpha)$, $Y = \cos(\alpha)$, where $\alpha = \frac{(4n+\beta)}{24}\pi$ for $n = 0,1,\ldots,11$ and $\beta = mod(I,3) + 1$, where I is the last digit in your student ID. Both X and Y are quantized into 12 bits including the sign bit and the 10-bit fractional part. According to Q1, determine the word-length of $X(i)$ and $Y(i)$ at all the stages if they use the same format (Hint: Consider the possible growth of the input signal.)
3. Determine the number of required micro rotations and also determine the related elementary angles with proper word-length so that the average phase error of $\tan^{-1}(\frac{Y}{X})$ obtained by the CORDIC operation will be less than $a \times 2^{-8}$ radian. For even-numbered student, $a = 1$. For odd-numbered student $a = 0.5$.
4. Given 0.5% error tolerance of the magnitude function approximated by the CORDIC operation, determine the number of the required micro-rotations.
5. Design the shift-and-add operation for the scaling factor $S(N)$ with an error less than $a \times 2^{-8}$. (Using CSD)
6. Design the complete architecture of the CORDIC block for the arctangent function $\measuredangle Z = \tan^{-1}(\frac{Y}{X})$ including the initial stage so that we can obtain the correct results when the 12 input sets enter. Note that you need to use the number of micro-rotations that you have derived in Q3.
7. Design the complete architecture of the CORDIC block for the magnitude function $|Z| = \sqrt{X^2 + Y^2}$ including the initial stage and the scaling stage so that we can obtain the correct results when 12 input sets enter. Note that you need to use the number of micro-rotations that you have derived in Q4 and Q5.

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8. Implement your design in Q6 by Verilog and verify it with the 12 input sets. Compare the Verilog results and the Matlab result.
9. Synthesize your design of the arctangent function and find out the critical path.
10. Implement your design in Q7 by Verilog and verify it with the 12 input sets. Compare the Verilog results and the Matlab result. (You can reuse your modules in Q8.)

11. Synthesize your design of the magnitude function and find out the critical path.
12. Download your design in Q8 and Q10 into the FPGA board and measure the results (before 12/06).

IV. Results

1. Please show how you calculate the scaling factor, write down the N value that you use and the result of $S(N)$.
2. Write down the word-length of $X(i)$ and $Y(i)$ that you use. Please explain it.
3. Please draw a figure to denote the average phase errors of 12 input pairs (X, Y) versus different numbers of micro-rotations N and draw a figure to show the resulted phase errors of 12 input pairs versus the word-length of quantized elementary angles. Explain how you determine it. Also list a table of the elementary angles (both in floating-point representation and binary fixed-point representation).
4. Please show how you decide the number of micro-rotations for the magnitude function with error tolerance of 0.5%.
5. Write down the power-of-2 expression for the scaling factor $S(N)$. Depict your design for the shift-and-add block. (Using CSD)
6. Depict your design of the complete CORDIC architecture for the arctangent function. Mark the word-length in the block diagram.
7. Depict your design of the complete CORDIC architecture for the magnitude function. Mark the word-length in the block diagram.

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8. Print the timing diagram of the behavior simulation result and post-route simulation result of your arctangent function. Show the error between the Verilog output and Matlab output by figures.
9. List the timing report of the arctangent function and show the critical path in your block diagram.
10. Print the timing diagram of the behavior simulation result and post-route simulation result of your magnitude function. Show the error between the Verilog output and Matlab output by figures.
11. List the timing report of the magnitude function and show the critical path in your block diagram.
12. Show your measurement results of Q8.and Q10 (40%). Please paste your measurement results and show the error between measurement results and post-route simulation results by Matlab figure.

六、參考資料:

[1] R. Lyons, “Another contender in the arctangent race," IEEE Signal Processing Magazine,

pp. 109-110, Jan. 2004.
[2] Y. H. Hu, “CORDIC-based VLSI architectures for digital signal processing," IEEE Signal Processing Magazine pp. 16-35, July 1992.
[3] Y. H. Hu, “The quantization effects of the CORDIC algorithm," IEEE Trans. Signal Processing, Vol. 40, No. 4, pp. 834-844, Apr. 1992.