Problem 1 (Support vector machine)

(a) $\phi(x_1) = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$, $\phi(x_2) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\phi(x_3) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$\phi(x_4) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

(b) $K(x_1, x_1) = \begin{bmatrix} -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = 3$

$K(x_2, x_2) = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 1$

$K(x_3, x_3) = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 3$

$K(x_4, x_4) = 0$

$K(x_1, x_2) = K(x_2, x_1) = \begin{bmatrix} -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 1$

$K(x_1, x_3) = K(x_3, x_1) = \begin{bmatrix} -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 1$

$K(x_1, x_4) = K(x_4, x_1) = 0$

$K(x_2, x_3) = K(x_3, x_2) = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 1$

$K(x_2, x_4) = K(x_4, x_2) = 0$

$K(x_3, x_4) = K(x_4, x_3) = 0$

(c)

Lagrangain function

$$L(w, b, \lambda_i) = \frac{1}{2}\|w\|^2 - \sum_{i=1}^{n} \lambda_i y_i (w^T x_i + b) + \sum_{i=1}^{n} \lambda_i$$

(d)

$$Ln(\lambda_i, \alpha) = f(\lambda_i) - \alpha g(\lambda_i)$$

$$= \sum_{i=1}^{4} \lambda_i - \frac{1}{2} \sum_{i=1}^{4} \sum_{j=1}^{4} \lambda_i \lambda_j y_i y_j x_i^T x_j - \alpha \sum_{i=1}^{4} \lambda_i y_i$$

$$\begin{aligned}
=&(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4) - \frac{1}{2}\Big(\lambda_1 \lambda_1 y_1 y_1 \overbrace{x_1^T x_1}^{=3} + \\
&\lambda_2 \lambda_2 \overbrace{y_2 y_2}^{=1} \overbrace{x_2^T x_2}^{=1} + \lambda_3 \lambda_3 y_3 y_3 \overbrace{x_3^T x_3}^{=3} + \\
&\lambda_4 \lambda_4 y_4 \cancel{y_4 x_4^T x_4} + 2\lambda_1 \lambda_2 y_1 y_2 \overbrace{x_1^T x_2}^{=1} \\
&+ 2\lambda_1 \lambda_3 y_1 y_3 \overbrace{x_1^T x_3}^{=1} + 2\lambda_1 \lambda_4 y_1 \cancel{y_4 x_1^T x_4} \\
&+ 2\lambda_2 \lambda_3 y_2 y_3 \overbrace{x_2^T x_3}^{=1} + 2\lambda_2 \lambda_4 y_2 y_4 \cancel{x_2^T x_4} \\
&+ 2\lambda_3 \lambda_4 y_3 y_4 \cancel{x_3^T x_4}\Big) - \alpha(\lambda_1 y_1 + \lambda_2 y_2 \\
&+ \lambda_3 y_3 + \lambda_4 y_4)
\end{aligned}$$

$$= \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 - \frac{1}{2}(3\lambda_1^2 + \lambda_2^2 + 3\lambda_3^2$$

$$- 2\lambda_1\lambda_2 + 2\lambda_1\lambda_3 - 2\lambda_2\lambda_3) - \alpha(\lambda_1 - \lambda_2 + \lambda_3 + \lambda_4)$$

$$\frac{\partial Ln}{\partial \lambda_1} = 1 - 3\lambda_1 - \lambda_2 + \lambda_3 - \alpha = 0$$

$$\frac{\partial Ln}{\partial \lambda_2} = 1 - \lambda_2 - \lambda_1 - \lambda_3 - \alpha = 0$$

$$\frac{\partial L_n}{\partial \lambda_3} = 1 - 3\lambda_3 + \lambda_1 - \lambda_2 - \alpha = 0$$

$$\frac{\partial L_n}{\partial \lambda_4} = 1 - \alpha = 0$$

$$\alpha = 1$$

$$\begin{cases} -3\lambda_1 - \lambda_2 + \lambda_3 = 0 \\ -\lambda_2 - \lambda_1 - \lambda_3 = 0 \\ 3\lambda_3 + \lambda_1 - \lambda_2 = 0 \end{cases} \Rightarrow \begin{cases} \lambda_1 = 1 \\ \lambda_2 = 4 \\ \lambda_3 = 1 \\ \lambda_4 = 2 \end{cases}$$

(e)

$$w = \sum_{i=1}^{4} \lambda_i y_i \phi(x_i) = 1 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} + 4(-1) \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$+ 1 \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ 2 \end{bmatrix}$$

$$b = 1 - w^T \phi(x_i) = 1 - \begin{bmatrix} 0 & -2 & -2 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

$$= 1$$

(f)