In [ ]:

```
import numpy as np
import matplotlib.pyplot as plt
import scipy.linalg
```

MLE

In [ ]:

```
def define_dataset(a, b, N, k, sigma):
    X = np.linspace(a, b, N)

    Phi = vander(X, k)

    theta_true = np.ones((k, ))

    Y = Phi @ theta_true
    gaussian_noise = np.random.normal(0, 1, Y.shape)
    Y = Y + sigma * gaussian_noise #Noisy Y

    D = (X, Y)

    return D

#Generalized Vandermonde Matrix
def vander(X, k):
    N = len(X)

    phi = np.zeros((N, k))

    for j in range(k):
        phi[:,j] = X**j

    return phi
```

In [ ]:

```
#Degree of polynomial
k_chosen = (int) (input("Choose the degree of the polynomial: "))

#Input dataset
a = (int) (input("Choose an interval [a, b] --a: "))
b = (int) (input("Choose an interval [a, b] --b: "))

sigma = (float) (input("Choose the variance of the noise: "))

N = 100 #number of datapoints

X, Y = define_dataset(a, b, N, k_chosen, sigma)
D = (X, Y)
```

Pretend not to know the correct value of k. The task is to try guess it and use it to approximate the true solution $theta_{true}$ by MLE and MAP.

In [ ]:

```
def f_MLE(X, Y):
    return lambda theta: 0.5 * (np.linalg.norm((vander(X, len(theta)) @ theta) - Y))**
```

```python
def grad_f_MLE(X, Y):
    return lambda theta: vander(X, len(theta)).T @ ((vander(X, len(theta)) @ theta) -

def f_MAP(X, Y, lamda):
    return lambda theta: 0.5 * (np.linalg.norm((vander(X, len(theta)) @ theta) - Y))**

def grad_f_MAP(X, Y, lamda):
    return lambda theta: vander(X, len(theta)).T @ ((vander(X, len(theta)) @ theta) -
```

In [ ]:

```python
def GD(grad_f, x0, kmax=100, tolf=1e-6, tolx=1e-6, alpha=1e-3):
    k = 0
    dim_m, dim_n = kmax+1, x0.shape[0]
    x = np.empty((dim_m, dim_n))

    x[k]=x0

    k+=1
    x[k]=x[k-1]-alpha*grad_f(x[k-1])

    cond1 = (np.linalg.norm(grad_f(x[k])) > tolf * grad_f(x[k-1])).all()
    cond2 = (np.linalg.norm(x[k] - x[k-1]) > tolx * np.linalg.norm(x[k-1])).all()
    conditions = cond1 and cond2

    while (conditions and k < kmax):
        k = k+1

        x[k] = x[k-1]-alpha*grad_f(x[k-1])

        cond1 = np.linalg.norm(grad_f(x[k])) > tolf * grad_f(x[k-1]).all()
        cond2 = np.linalg.norm(x[k] - x[k-1]) > tolx * np.linalg.norm(x[k-1]).all()
        conditions = cond1 and cond2

    x = x[:k+1]

    return x[-1]

def SGD(grad_l, w0, D, batch_size, n_epochs, lamda=0, alpha=1e-3):
    X, Y = D
    N = X.shape[0]

    n_batch_per_epoch = N//batch_size
    tot_batch = n_batch_per_epoch * n_epochs

    w = np.array(w0)
    w_vector = np.zeros((tot_batch, len(w0)))

    for epoch in range(n_epochs):
        X_shuffle, Y_shuffle = shuffle_data(X, Y)

        for b in range (n_batch_per_epoch):
            n = b*batch_size
            m = (b+1)*batch_size

            Mx = X_shuffle[n:m]
            My = Y_shuffle[n:m]

            if lamda==0:
```

```python
                gradient=grad_l(Mx, My)
            else:
                gradient=grad_l(Mx, My, lamda)

            w=w-alpha*gradient(w)
            w_vector[epoch*n_batch_per_epoch + b, :] = w

    return w_vector[-1]

def shuffle_data(X, Y):
    N = X.shape[0]
    indexes = np.arange(N)
    np.random.shuffle(indexes)

    X_shuffle = X[indexes]
    Y_shuffle = Y[indexes]

    return X_shuffle, Y_shuffle
```

In [ ]:

```python
def MLE(D, k_trial, mod):
    X, Y = D
    Phi_trial = vander(X, k_trial)

    if mod[0]=='N':
        #Normal equation
        A = Phi_trial.T @ Phi_trial
        b = Phi_trial.T @ Y
        try:
            L = scipy.linalg.cholesky(A, lower = True)
            y = scipy.linalg.solve_triangular(L, b, lower = True)
            theta_mle = scipy.linalg.solve_triangular(L.T, y)
        except:
            theta_mle = np.linalg.solve(Phi_trial.T @ Phi_trial, Phi_trial.T @ Y)

    elif mod[0]=='G':
        #Gradient Descent
        theta_mle = GD(grad_f_MLE(X, Y), np.zeros((k_trial,)))

    else:
        #Stochastic Gradient Descent
        theta_mle = SGD(grad_f_MLE, np.zeros((k_trial, )), D, batch_size = 5, n_epochs

    return theta_mle
```

In [ ]:

```python
theta_mle_normal = MLE(D, k_chosen, 'Normal equation')
theta_mle_gd = MLE(D, k_chosen, 'GD')
theta_mle_sgd = MLE(D, k_chosen, 'SGD')

print("Theta, MLE - Normal equation: ", theta_mle_normal)
print("Theta, MLE - GD: ", theta_mle_gd)
print("Theta, MLE - SGD: ", theta_mle_sgd)
```

```
Theta, MLE - Normal equation:  [1.00433531 1.23625381 0.37101921 1.41183883]
Theta, MLE - GD:  [1.07493231 1.05713449 0.92306977 0.80851575]
Theta, MLE - SGD:  [1.05750436 0.69124807 0.52724733 0.42996496]
```

In [ ]:

```python
def split_data(X, Y, percentage_train):
```

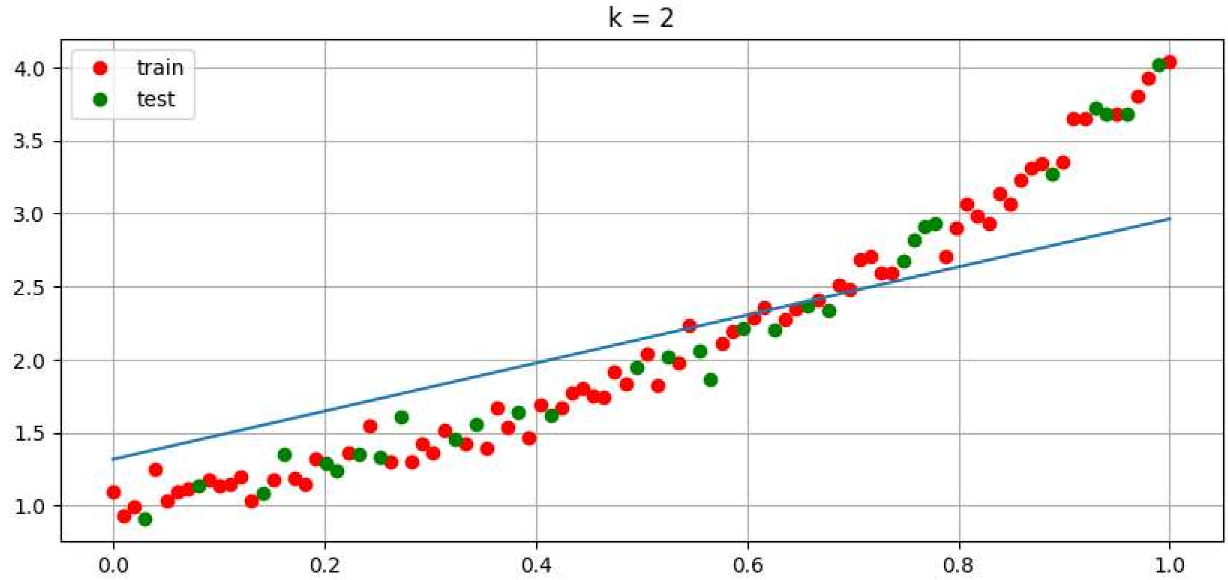
```python
    N = len(X)
    Ntrain = int(percentage_train*N/100)

    idx = np.arange(N)
    np.random.shuffle(idx)

    train_idx = idx[:Ntrain]
    test_idx = idx[Ntrain:]

    Xtrain = X[train_idx]
    Ytrain = Y[train_idx]

    Xtest = X[test_idx]
    Ytest = Y[test_idx]

    return (Xtrain, Ytrain), (Xtest, Ytest)
```

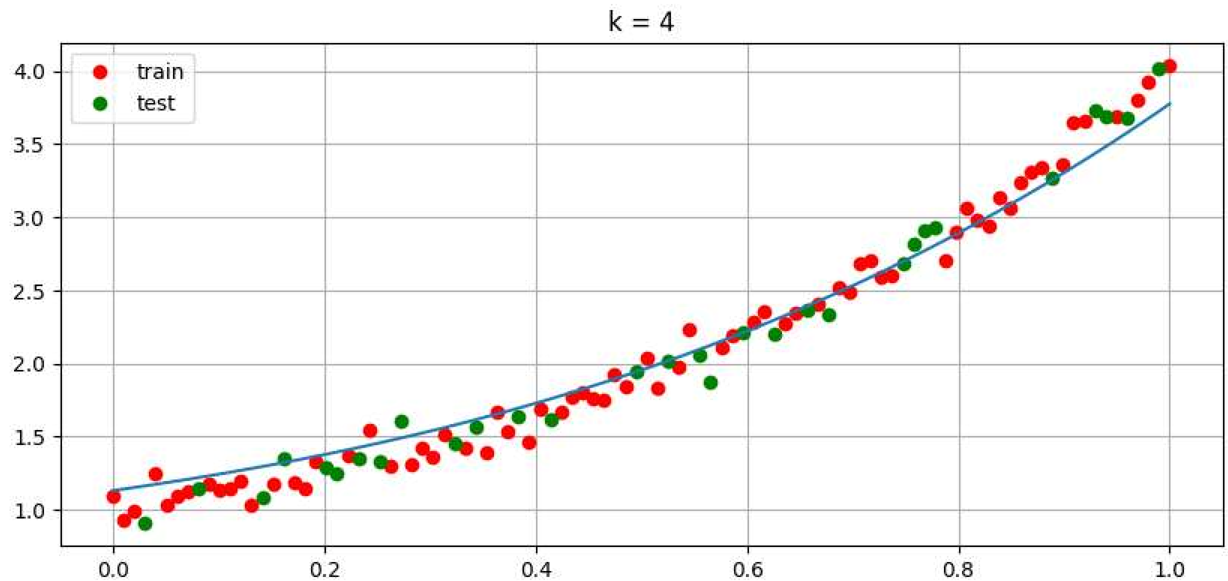
In [ ]:
```python
def polynomial_regression(X, k, theta):
    Phi = vander(X, k)
    return Phi @ theta

def error(D, k, theta):
    X, Y = D
    N = len(Y)

    f_theta = polynomial_regression(X, k, theta)

    return ((np.linalg.norm(f_theta - Y))**2) / N
```

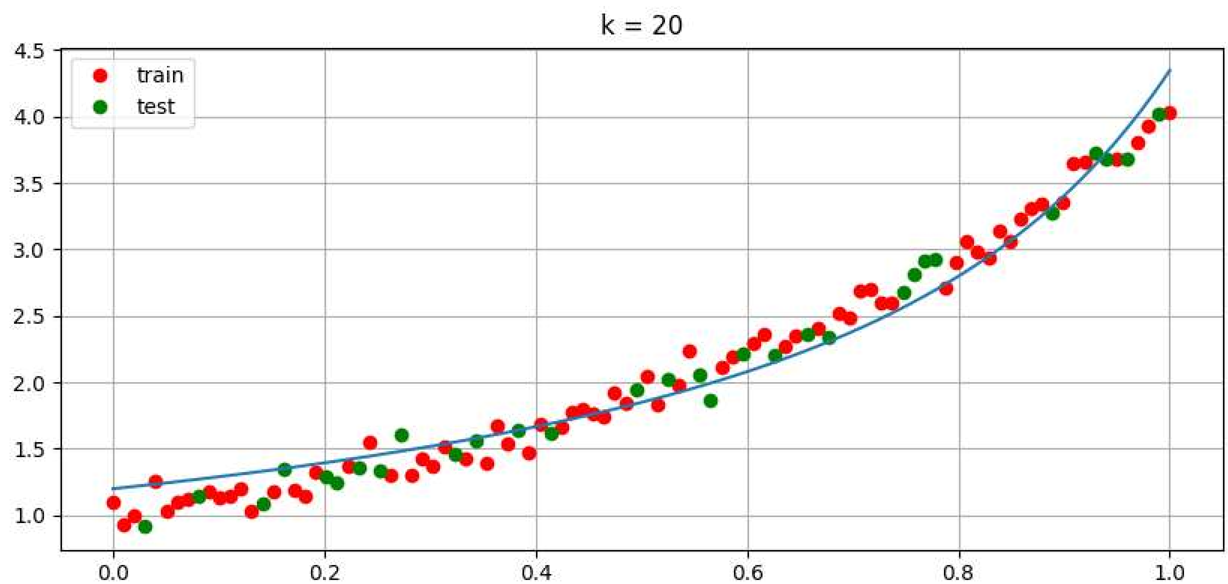
For different values of K, plot the training datapoints and the test datapoints with different colors, and visualize (as a continuous line) the learnt regression model $f_{theta_{MLE}}(x)$.

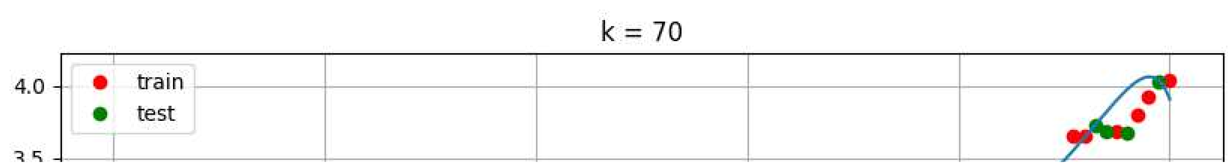
In [ ]:
```python
D_train, D_test = split_data(D[0], D[1], 70)
X_train, Y_train = D_train
X_test, Y_test = D_test

k_vector = [2, 4, 20, 70]
theta_mle_vector = []

for k in k_vector:
    theta_mle = MLE(D_train, k, "GD")
    theta_mle_vector.append(theta_mle)

fig, ax = plt.subplots(len(k_vector), figsize = (10, 20))

for i in range(len(k_vector)):

    theta = theta_mle_vector[i]
    k = k_vector[i]

    XX = np.linspace(a, b, 1000)
    YY = polynomial_regression(XX, k, theta)

    ax[i].set_title(f'k = {k}')
    ax[i].plot(X_train, Y_train, 'ro')
```

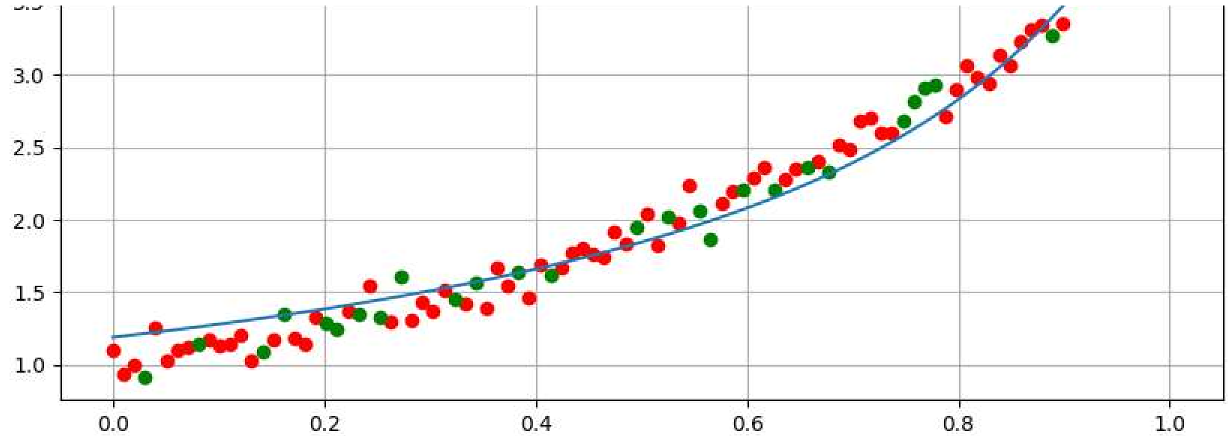
```
    ax[i].plot(X_test, Y_test, 'go')
    ax[i].legend(['train', 'test'])
    ax[i].plot(XX, YY)
    ax[i].grid()

plt.show()
```


k = 2
train
test


k = 4
train
test


k = 20
train
test


k = 70
train
test



For increasing values of K cmpute the training and test error. Plot the two errors with respect to K.

In [ ]:
```
D_train, D_test = split_data(D[0], D[1], 70)

X_test, Y_test = D_test

k_vector = [2, 4, 20, 70]

theta_vector = []
errors_train = []
errors_test = []

for k in k_vector:
    theta_mle = MLE(D_train, k, "GD")
    theta_vector.append(theta_mle)

    training_error = error(D_train, k, theta_mle)
    errors_train.append(training_error)

    test_error = error(D_test, k, theta_mle)
    errors_test.append(test_error)

plt.figure(figsize=(7,5))
plt.title("Training and test errors")
plt.plot(k_vector, errors_train, color='red')
plt.plot(k_vector, errors_test, color='green')
plt.legend(['train', 'test'])
plt.grid()
plt.plot()
```

Out[ ]: []

MAP

Write a function that returns the MAP solution. Note that the loss function can be optimized by GD, SGD or Normal Equations.

In [ ]:
```
def MAP(D, k_trial, lmbda, mod):
    X, Y = D

    Phi = vander(X, k_trial)

    if mod[0] == 'N':
        #Normal equation
        A = (Phi.T @ Phi) + (lmbda * np.identity(k_trial))
        b = Phi.T @ Y

        try:
            L = scipy.linalg.cholesky(A, lower = True)
            y = scipy.linalg.solve_triangular(L, b, lower = True)
            theta_MAP = scipy.linalg.solve_triangular(L.T, y)
        except:
            theta_MAP = np.linalg.solve((Phi.T @ Phi) + (lmbda * np.identity(k_trial))

    elif mod[0] == 'G':
        #Gradient Descent
        theta_MAP = GD(grad_f_MAP(X, Y, lmbda), np.zeros((k_trial, )))

    else:
```

```python
        #Stochastic Gradient Descent
        theta_MAP = SGD(grad_f_MAP, np.zeros((k_trial, )), D, 5, 10, lmbda)

    return theta_MAP
```

In [ ]:

```python
theta_map_normal = MAP(D, k_chosen, 1, 'Normal equation')
theta_map_gd = MAP(D, k_chosen, 1, 'GD')
theta_map_sgd = MAP(D, k_chosen, 1, 'SGD')

print("Theta, MAP, lambda = 1 - Normal equation: ", theta_map_normal)
print("Theta, MAP, lambda = 1 - GD: ", theta_map_gd)
print("Theta, MAP, lambda = 1 - SGD: ", theta_map_sgd)
```

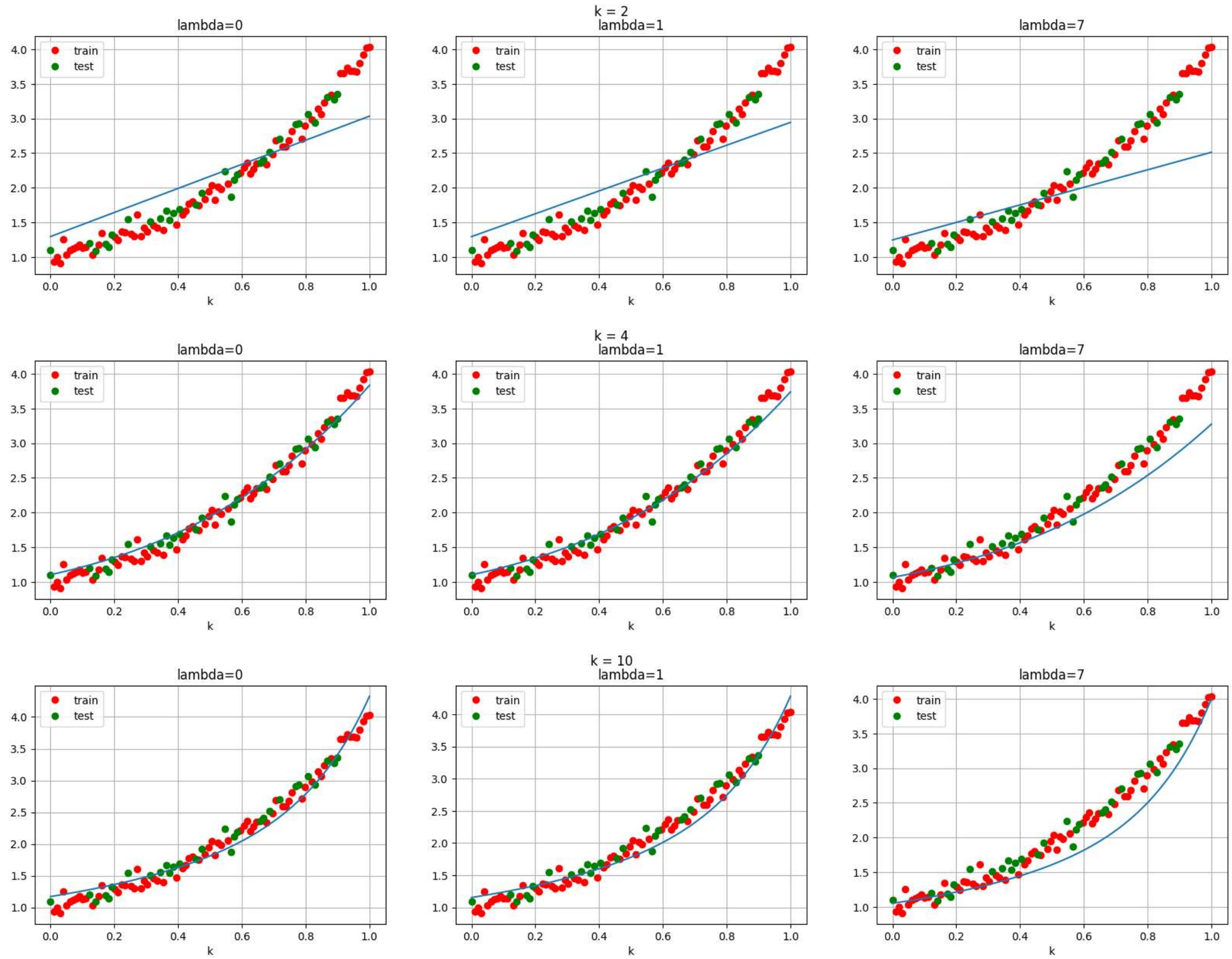
```
Theta, MAP, lambda = 1 - Normal equation:  [1.02196468 1.03863013 0.96799729 0.900860
68]
Theta, MAP, lambda = 1 - GD:  [1.08163532 1.03081442 0.89270064 0.7787599 ]
Theta, MAP, lambda = 1 - SGD:  [0.91644105 0.59557124 0.45302824 0.36879968]
```

For K lower, equal and greater than the correct degree of the test polynomial, plot the training datapoints and the test datapoints with different colors, and visualize (as a continuous line) the learnt regression model $f_{theta_{MAP}}(x)$ with different values of lambda.

In [ ]:

```python
D_train, D_test = split_data(D[0], D[1], 70)
X_train, Y_train = D_train
X_test, Y_test = D_test

k_vector = [2, 4, 10]
l_vector = [0, 1, 7]

theta_tot_k = []

for k in k_vector:

    theta_tot_l = []

    for l in l_vector:
        theta_map = MAP(D_train, k, l, "GD")
        theta_tot_l.append(theta_map)

    theta_tot_k.append(theta_tot_l)

for i in range(len(k_vector)):
    k = k_vector[i]

    plt.figure(figsize=(20, 4))
    plt.suptitle(f'k = {k}')

    for j in range(len(theta_tot_k[i])):
        theta = theta_tot_k[i][j]

        plt.subplot(1, len(theta_tot_k[i]), j+1)
        plt.title(f"lambda={l_vector[j]}")
        plt.xlabel('k')

        XX = np.linspace(a, b, 1000)
        YY = polynomial_regression(XX, k, theta)
```

```
        plt.plot(X_train, Y_train, 'ro')
        plt.plot(X_test, Y_test, 'go')
        plt.legend(['train', 'test'])
        plt.plot(XX, YY)
        plt.grid()

plt.show()
```



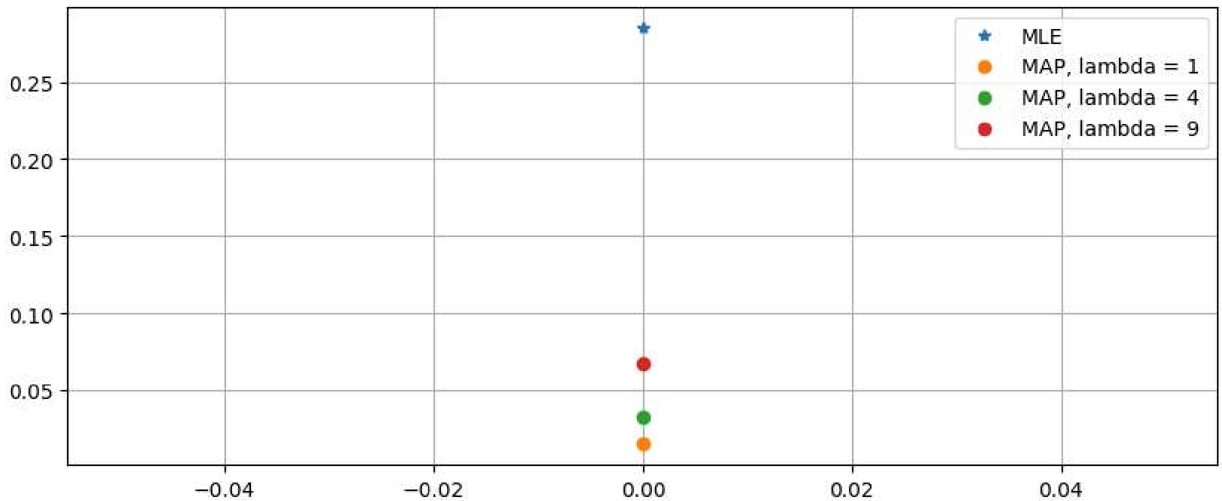
MLE and MAP

For K being way greater than the correct degree of the polynomial, compute the MLE and MAP solution. Compare the test error of the two, for different values of lambda (in the case of MAP).

In [ ]:

```
k_big = 70
l_vector = [1, 4, 9]
D_train, D_test = split_data(D[0], D[1], 70)

theta_mle = MLE(D_train, k_big, "NE")
thetas_map = [MAP(D_train, k_big, l, "NE") for l in l_vector]

mle_error = error(D_test, k_big, theta_mle)
map_errors = [error(D_test, k_big, theta_map) for theta_map in thetas_map]

legend_mle = ['MLE']
legend_map = ['MAP, lambda = ' + str(l) + '' for l in l_vector]
```

```python
legend = legend_mle + legend_map

plt.figure(figsize=(10,4))
plt.plot(mle_error, '*')
for i in range(len(l_vector)):
    plt.plot(map_errors[i], 'o')
plt.legend(legend)
plt.grid()
plt.show()
```



For K greater than the true degree of the polynomial, define the relative error and compute it for MLE and MAP for increasing values of K and different values of lambda.

In [ ]:
```python
def err_theta(theta, k):
    theta_true = np.ones((k,))
    diff = len(theta) - k
    if(diff > 0):
        theta_true = np.concatenate(theta_true, np.zeros(diff,))
    return np.linalg.norm(theta - theta_true) / np.linalg.norm(theta_true)
```

In [ ]:
```python
k_vector = [5, 7, 10]
l_vector = [1, 3, 7]

theta_mle_error_tot_k = []
theta_map_error_tot_k = []

for k in k_vector:

    theta_mle = MLE(D_train, k, "NE")
    error_mle = err_theta(theta_mle, k)
    theta_mle_error_tot_k.append(error_mle)

    theta_map_tot_l = []

    for l in l_vector:
        theta_map = MAP(D_train, k, l, "NE")
        error_map = err_theta(theta_map, k)
        theta_map_tot_l.append(error_map)

    theta_map_error_tot_k.append(theta_map_tot_l)
```

```python
legend_mle = ['MLE']
legend_map = ['MAP, lambda = ' + str(l) + '' for l in l_vector]
legend = legend_mle + legend_map

plt.figure(figsize=(20, 4))

for i in range(len(k_vector)):
    k = k_vector[i]

    plt.subplot(1, len(theta_map_error_tot_k[i]), i+1)
    plt.title(f'k = {k}')

    for j in range(len(theta_map_error_tot_k[i])):

        mle_error_ = theta_mle_error_tot_k[i]
        plt.plot(mle_error_, 'r*')

        map_error_ = theta_map_error_tot_k[i][j]
        plt.plot(map_error_, 'o')

    plt.legend(legend)
    plt.grid()

plt.show()
```