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1 contributor

473 lines (473 sloc) | 247 KB

In [1]:

```python
#import libraries needed
import numpy as np
import matplotlib.pyplot as plt
import scipy.linalg
```

Direct methods for the solution of Linear Systems

Given a matrix A ∈ nxn and the vector x_{true} = $(1, 1, \ldots, 1)^T$ ∈ R^n, write a script that:

- Computes the right-hand side of the linear system b = A x_{true}.
- Computes the condition number in 2-norm of the matrix A. It is ill-conditioned? What if we use the ∞-norm instead of the 2-norm?
- Solves the linear system Ax = b with the function np.linalg.solve().
- Computes the relative error between the solution computed before and the true solution x_{true}.

Remember that the relative error between x_true and x in R n can be computed as $E(x_{true}, x) = \frac{||x - x_{true}||_2}{||x_{true}||_2}$

- Plot a graph (using matplotlib.pyplot) with the relative errors as a function of n and (in a new

window) the condition number in 2-norm K2(A) and in ∞-norm, as a function of n.

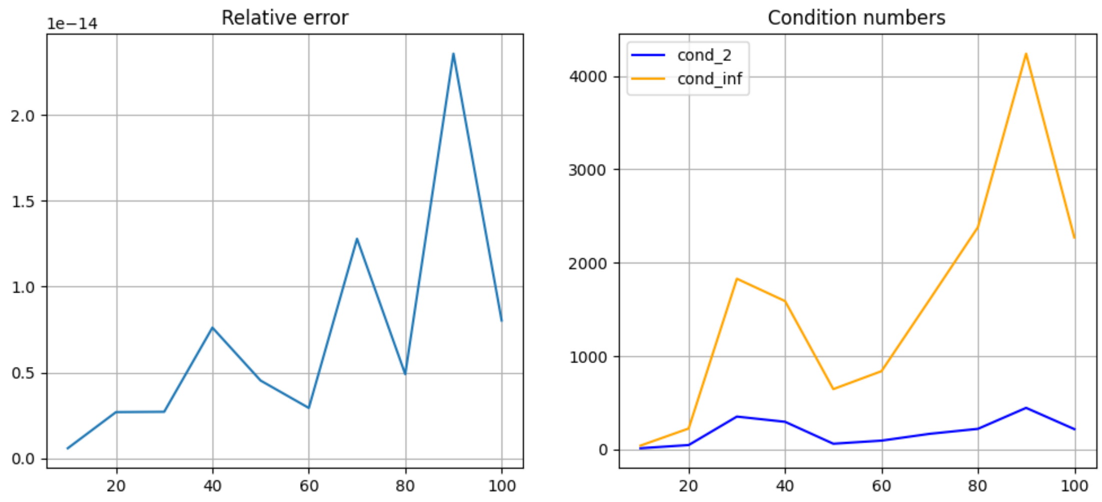
In [26]:

```python
def linear_system(A, x_true):
    b = A@x_true                #calculate the right hand side of the linear

    cond_2 = np.linalg.cond(A, 2)
    cond_inf = np.linalg.cond(A, np.Infinity)

    x = np.linalg.solve(A, b)   #solve the linear system Ax=b

    relative_error = np.linalg.norm(x-x_true, 2)/np.linalg.norm(x_true, 2)

    return relative_error, cond_2, cond_inf

def plot(relative_error, cond_2, cond_inf, n_vector):
    plt.figure(figsize=(12,5))

    plt.subplot(1,2,1)
    plt.title("Relative error")
    plt.plot(n_vector, relative_error)
    plt.grid()

    plt.subplot(1,2,2)
    plt.title("Condition numbers")
    plt.plot(n_vector, cond_2, '-', color='blue')
    plt.plot(n_vector ,cond_inf, '-', color='orange')
    plt.legend(['cond_2', 'cond_inf'])

    plt.grid()
    plt.show()
```

Test the program above with the following choices of A ∈ R nxn:

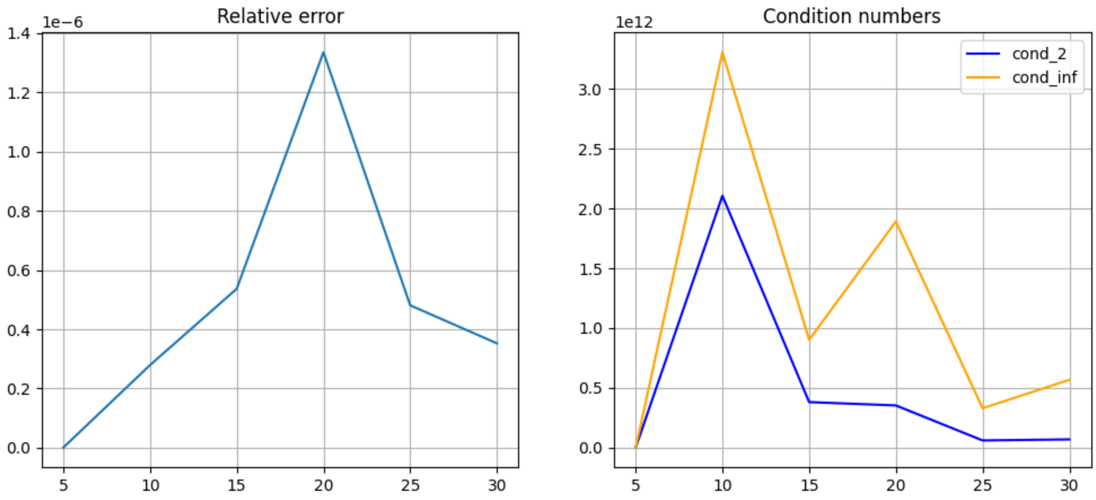
A random matrix (created with the function np.random.rand()) with size varying with n = {10, 20, 30, ... , 100}

In [27]:
```
n_vector = np.arange(10, 101, 10)

#Initialize lists for the results
error_vector = []
k2_vector = []
kinf_vector = []

for n in n_vector:
    A = np.random.randn(n,n)    #define the random transformation matrix
    x_true = np.ones((n,))
    err, k2, kinf = linear_system(A, x_true)
    error_vector.append(err)
    k2_vector.append(k2)
    kinf_vector.append(kinf)

plot(error_vector, k2_vector, kinf_vector, n_vector)
```



The Vandermonde matrix (np.vander) of dimension n = {5, 10, 15 20, 25, 30} with respect to the vector x = {1, 2, 3, ... , n}.

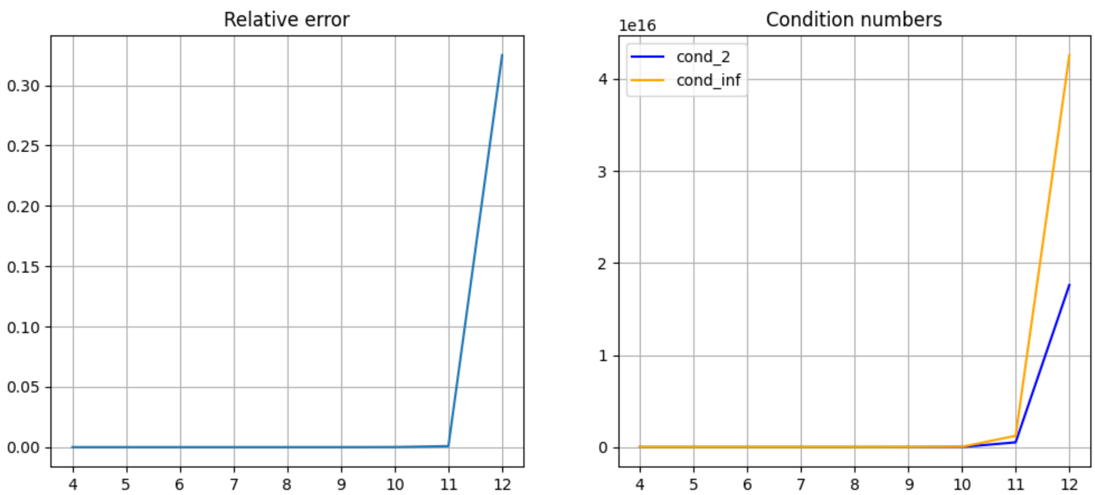
In [30]:
```
n_vector = np.array([5, 10, 15, 20, 25, 30])

#Initialize lists for thre results
error_vector = []
k2_vector = []
kinf_vector = []

for n in n_vector:
    x_vector = np.arange(1, n+1, 1)
    A = np.vander(x_vector)
    x_true = np.ones((n,))

    err, k2, kinf = linear_system(A, x_true)

    error_vector.append(err)
    k2_vector.append(k2)
    kinf_vector.append(kinf)

plot(error_vector, k2_vector, kinf_vector, n_vector)
```



The Hilbert matrix (scipy.linalg.hilbert) of dimension n = {4, 5, 6, ... , 12}.

In [31]:

```
n_vector = np.arange(4, 12+1, 1)

#Initialize lists for the results
error_vector = []
k2_vector = []
kinf_vector = []

for n in n_vector:

    A = scipy.linalg.hilbert(n)
    x_true = np.ones((n,))

    err, k2, kinf = linear_system(A, x_true)

    error_vector.append(err)
    k2_vector.append(k2)
    kinf_vector.append(kinf)

plot(error_vector, k2_vector, kinf_vector, n_vector)
```



Floating point arithmetic

1.

The Machine epsilon ϵ is the distance between 1 and the next floating point number.
Compute ϵ, which is defined as the smallest floating point number such that it holds:
$fl(1+\epsilon) > 1$

Tips: use a while structure.

In [32]:

```
machine_epsilon = np.asarray([1], np.float32)

while 1.0 + machine_epsilon > 1.0:
    machine_epsilon = machine_epsilon/2

machine_epsilon = machine_epsilon*2     #it was half of the machine precisio

print(f"Single precision: {machine_epsilon[0]}")

machine_epsilon = np.asarray([1],np.float64)

while 1.0 + machine_epsilon > 1.0:
    machine_epsilon = machine_epsilon/2

machine_epsilon = machine_epsilon*2     #it was half of the machine precisio

print(f"Double precision: {machine_epsilon[0]}")
```

```
Single precision: 1.1920928955078125e-07
Double precision: 2.220446049250313e-16
```

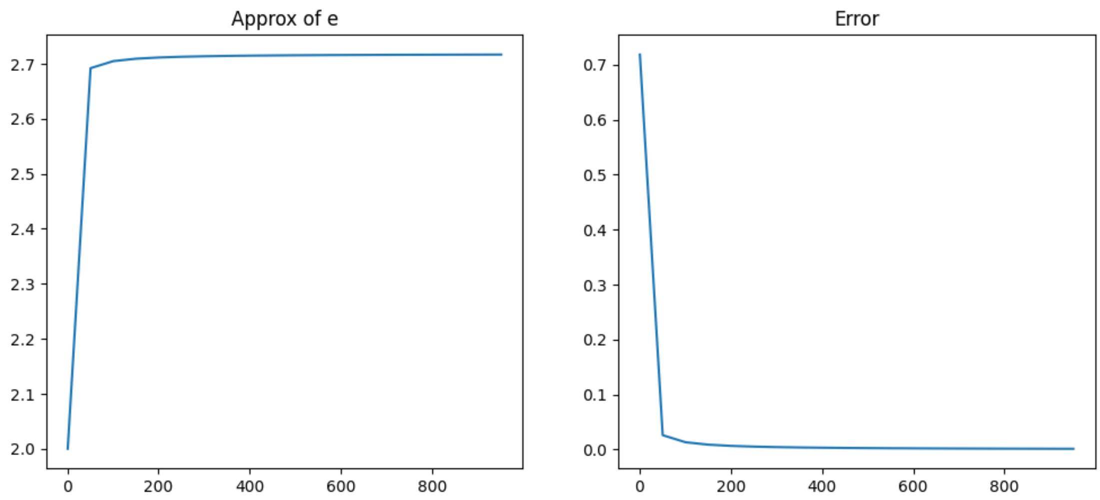
2.

Let's consider the sequence $a_n = (1 + 1/n)^n$. It is well known that: $\lim_{n\to\infty} a_n = e$ where e is the Euler costant.
Choose different values for n, compute a_n and compare it to the real value of the Euler costant.
What happens if you choose a large value of n? Guess the reason.

In [59]:

```
n_vector = np.arange(1, 1000, 50)     #initialize the array with all the val
errors = []
e_ = []

for n in n_vector:
    a_n = (1 + 1/n)**n
    e_.append(a_n)
    errors.append((np.e - a_n))

plt.figure(figsize=(12,5))

plt.subplot(1,2,1)
plt.title('Approx of e')
plt.plot(n_vector, e_)

plt.subplot(1,2,2)
plt.title('Error')
plt.plot(n_vector, errors)

plt.show()

#Large value of n
```

```
n_vector = np.arange(199900, 200000)      #initialize the array with all the
errors = []

for n in n_vector:
    a_n = (1 + 1/n)**n
    errors.append((np.e - a_n))


plt.figure(figsize=(7,5))
plt.title('Behaviour of the error for big n')
plt.plot(n_vector, errors)

plt.show()
```



3.

Let's consider the matrices:

$$A = \begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix},\ B = \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}$$

Compute the rank of A and B and their eigenvalues.

- Are A and B full-rank matrices?
- Can you infer some relationship between the values of the eigenvalues and the full-rank condition?

Please, corroborate your deduction with other examples.

Tips: Please, have a look at np.linalg.

In [61]:

```
A = np.array([[4, 2], [1, 3]])
B = np.array([[4, 2], [2, 1]])

rank_A = np.linalg.matrix_rank(A)
rank_B = np.linalg.matrix_rank(B)

eigenvalues_A = np.linalg.eigvals(A)
eigenvalues_B = np.linalg.eigvals(B)

print(f"Eigenvalues of A = {eigenvalues_A} - Rank of A = {rank_A}")
if rank_A == 2:
    print("A is a full-rank matrix")
else:
    print("A is not a full-rank matrix")

print(f"Eigenvalues of B = {eigenvalues_B} - Rank of B = {rank_B}")
if rank_B == 2:
    print("B is a full-rank matrix")
else:
    print("B is not a full-rank matrix")
```

```
Eigenvalues of A = [5. 2.] - Rank of A = 2
A is a full-rank matrix
Eigenvalues of B = [5. 0.] - Rank of B = 1
B is not a full-rank matrix
```

In [64]:

```
def full_rank(A):
    Eig = np.linalg.eigvals(A)
    r = np.linalg.matrix_rank(A)
    max_rank = A.shape[0]

    print(f"Eigenvalues = {Eig} - Rank = {r}")
    if r == max_rank:
        print("full-rank matrix")
    else:
        print("not a full-rank matrix")


A = np.array([[1,2,3],[0,0,0],[1,2,3]])
full_rank(A)

A = np.array([[1, 2], [1, 2]])
full_rank(A)

A = np.array([[3,2,1],[4,5,2],[7,6,1]])
full_rank(A)
```

```
Eigenvalues = [0. 4. 0.] - Rank = 1
not a full-rank matrix
Eigenvalues = [0. 3.] - Rank = 1
not a full-rank matrix
```

```
not a full-rank matrix
Eigenvalues = [ 9.29150262  1.         -1.29150262] - Rank = 3
full-rank matrix
```