HOW TO MAKE DAVIES' THEOREM VISIBLE

MARIANNA CSÖRNYEI

Abstract

We prove that for an arbitrary measurable set $A \subset \mathbb{R}^2$ and a σ-finite Borel measure μ on the plane, there is a Borel set of lines L such that for each point in A, the set of directions of those lines from L containing the point is a residual set, and, moreover, $\mu(A) = \mu\left(\left\{\bigcup \ell : \ell \in L\right\}\right)$. We show how this result may be used to characterise the sets of the plane from which an invisible set is visible. We also characterise the rectifiable sets C_1, C_2 for which there is a set which is visible from C_1 and invisible from C_2.

1. *Introduction*

In 1952, R. O. Davies proved that an arbitrary measurable set $A \subset \mathbb{R}^2$ can be covered by lines in such a way that the set of all points covered by these lines has the same Lebesgue measure as A (see [**2**]).

The natural identification between lines and points in the projective plane allows us to define a set of lines to be measurable, Borel, of measure zero, if its (dual) set of points in the projective plane is measurable, Borel, of measure zero, respectively.

Given a set of lines L and a direction d, we denote by L_d the set of lines with direction d which belong to L. We say that L contains zero many lines in direction d if the set of points $\bigcup L_d$ is of Lebesgue measure zero. By Fubini's theorem, this holds if and only if on taking a line ℓ of direction orthogonal to d, the intersection of ℓ and the elements of L_d forms a set of linear measure zero on ℓ.

Since a measurable set A has measure zero if and only if it intersects almost every line through a given point in a set of (linear) measure zero, it is easy to see that a set of lines L has measure zero if and only if it contains only zero many lines in almost every direction.

Using this remark, we can state the dual of the theorem of Davies mentioned above: for an arbitrary measurable set of lines L, there exists a set of points P such that every line of L intersects P, and the set of all lines through the points of P not belonging to L has measure zero, that is, in almost every direction there are only zero many lines intersecting P which do not belong to L. A higher-dimensional version of this result can be found in [**3**].

The main result of Section 2 is that Davies' theorem (and its dual) holds not only for Lebesgue measure, but for every σ-finite Borel measure of the plane. (That is, for every set $A \subset \mathbb{R}^2$ and for every σ-finite Borel measure μ of the plane, there exists a set of lines L such that the set $L^* \stackrel{\text{def}}{=} \{\bigcup \ell : \ell \in L\}$ covers A, and $\mu(A) = \mu(L^*)$.)

Received 10 June 1999; revised 18 December 1999.

2000 *Mathematics Subject Classification* 28A75.

Research supported by the Hungarian National Foundation for Scientific Research Grants F029768, T019476, and FKFP Grant 0192/1999.

Moreover, we also show that this set L can be chosen in such a way that, in addition, for each point in A, the set of directions of those lines from L containing the point is a residual set. (Recall that a set is called residual if its complement is of first category.) Similarly, in the dual case, for every set of lines L, there exists a residual set of points on each line of L with the required property.

In Section 3 we give some applications of Davies' theorem to invisible sets. A set is said to be invisible if almost every projection of it has measure zero; a set is visible if it is not invisible. A set is invisible from a point if almost every line through this point does not hit the set.

J. M. Marstrand proved that the set of points from which an invisible set of finite length is visible has Hausdorff dimension at most 1. He also gave an example which shows that it can be 1 (see [**4**]).

As a generalisation of this result, P. Mattila in [**5**] proved that for an arbitrary invisible set (without assuming finite length), the set of points from which the set is visible has capacity zero, and it is purely unrectifiable. Mattila asked whether the set of the points from which an invisible set is visible is invisible. This question is answered in the negative in [**1**].

In this paper we characterise the sets of the plane from which an invisible set is visible, and show how easily the previous results (purely unrectifiable, not necessarily invisible, and so on) follow from our characterisation.

One can see immediately that a set is invisible if and only if it is invisible from almost every point of the ideal line of the projective plane. Using this observation, we can define invisibility from rectifiable sets: a set is said to be invisible from a rectifiable set C if it is invisible from $\mathscr{H}^1$-almost every point of C, where $\mathscr{H}^1$ denotes the 1-dimensional Hausdorff measure. By the theorem of Mattila above, if a set is invisible from the ideal line, then it must be invisible from every rectifiable set C. K. J. Falconer posed the problem of finding those rectifiable sets C_1, C_2 for which there exists a set which is visible from C_1 and invisible from C_2. In Theorem 13 we characterise the rectifiable sets C_1, C_2 with this property.

2. *Davies' theorem for Borel measures*

For a given parallelogram P and an interval of directions I, let $A_{P,I}$ denote the set of lines through the points of P in directions belonging to I. For a set of lines L, let L^* denote the set of all points covered by the lines of L, and for a set of points A, let A^* be the set of lines through the points of A.

Our main result is the following theorem.

THEOREM 1. *Let μ be a σ-finite Borel measure on the plane. Then for an arbitrary measurable set $A \subset \mathbb{R}^2$, there exists a Borel set of lines L such that*:

(i) *through each point of A there are residually many lines of L* (*that is*, *the set of directions of those lines in L containing the point forms a residual set*);

(ii) $\mu(L^*) = \mu(A)$.

Similarly, *for an arbitrary σ-finite Borel measure μ on the space of lines*, *and for an arbitrary measurable set of lines L*, *there exists a Borel set of points A such that*:

(i) *each line of L intersects A in a residual set*;

(ii) $\mu(A^*) = \mu(L)$.

We remark that if L is Borel, then L^* is analytic, thus L^* is measurable with respect to any Borel measure. Similarly, if A is Borel, then A^* is measurable with respect to the measure μ.

It is enough to prove the first part of the theorem, since the second part is just its dual. Moreover, it is enough to prove the first part for an open set A. Indeed, any measurable set A can be covered by a G_δ set $\bigcap_{n=1}^{\infty} G_n$ of the same measure (where $G_1 \supset G_2 \supset \cdots$ are open), and if the conditions of the first part of Theorem 1 are satisfied for the sets $G_1, G_2, \ldots$ and for the sets of lines $L_1, L_2, \ldots$, then the conditions are also satisfied for $A = \bigcap_{n=1}^{\infty} G_n$ and $L = \bigcap_{n=1}^{\infty} L_n$.

We use the following lemma, which is a sharpened version of Davies' theorem: not only does the covering set L^* have the same measure as the measure of A, but $L^* \setminus A$ intersects every line through a given point in a set of (Lebesgue) measure zero.

LEMMA 2. *Let A be an open set of the plane, and let x be a point not belonging to A. Then there exists a Borel set of lines L such that*:

(i) *L contains residually many lines through each point of A*;

(ii) *$L^* \setminus A$ intersects each line through x in a set of* (*Lebesgue*) *measure zero.*

The dual of this lemma is the following.

LEMMA 3. *Let L be an open set of lines, and let X be a line not belonging to L. Then there exists a Borel set of points A for which*:

(i) *each line of L intersects A in a residual set*;

(ii) *through each point of X there are only zero many lines from $A^* \setminus L$.*

Without loss of generality, we can assume that X is the ideal line of the projective plane. Then the points of X are the directions, and we can formulate Lemma 3 more easily as follows.

LEMMA 4. *For an arbitrary open set of lines L, there exists a Borel set of points A for which*:

(i) *each line of L intersects A in a residual set*;

(ii) *in each direction there are only zero many lines which intersect A and do not belong to L.*

To prove this, we shall use the following lemma, a proof of which may be found in [**1**].

LEMMA 5. *For every parallelogram P, positive number ε and intervals of directions I, J with* $\mathrm{cl}(I) \subset \mathrm{int}(J)$, *there are subparallelograms $P_1, P_2, \ldots$ such that*:

(i) *each line which intersects P and whose direction belongs to I also intersects one of the subparallelograms* (*that is,* $\bigcup_n A_{P_n,I} = A_{P,I}$);

(ii) *the projection of $\bigcup_n P_n$ has Lebesgue measure less than ε in each direction not belonging to J.*

Proof of Lemma 4. We choose positive numbers $\varepsilon_{\mathbf{n}}$ for every finite sequence of natural numbers $\mathbf{n} = n_1 n_2 \cdots n_k$ such that $\sum_{\mathbf{n}=n_1 n_2 \cdots n_k} \varepsilon_{\mathbf{n}} < 1/2^k$. Let $P_1, P_2, \ldots$ be rational open parallelograms, and let $I_1, I_2, \ldots$ be rational open intervals of directions, for which $A_{P_j,I_j} \subset L$ for every j, and for every line $\ell \in L$, the system of parallelograms

$$\{P_j : \ell \in A_{P_j,I_j}\}$$

gives a Vitali covering of ℓ, that is, all the points of ℓ are covered by an arbitrarily small parallelogram from this set.

If $P = P_{\mathbf{n}}$, $I = I_{\mathbf{n}}$ and $\varepsilon = \varepsilon_{\mathbf{n}}$ have been defined for a finite sequence of natural numbers $\mathbf{n} = n_1 n_2 \cdots n_k$, then we define $P_{\mathbf{n}j}$, $I_{\mathbf{n}j}$ for every $j \in \mathbb{N}$ by the following procedure.

First we choose rational open subparallelograms $P^1, P^2, \dots$ of $P = P_{\mathbf{n}}$ and rational open subintervals $I^1, I^2, \dots$ of $I = I_{\mathbf{n}}$ such that for every $\ell \in A_{P,I}$, the system of subparallelograms

$$\{P^j : \ell \in A_{P^j,I^j}\}$$

is a Vitali covering of the segment $\ell \cap P$. Then we choose positive numbers ε^j for which $\sum_j \varepsilon^j < \varepsilon = \varepsilon_{\mathbf{n}}$, and we choose rational intervals of directions J^j for which $\mathrm{cl}(I^j) \subset \mathrm{int}(J^j)$ and $\sum_j |J^j \setminus I^j| < \varepsilon$. Then we apply Lemma 5 for all parallelograms P^j, intervals I^j, J^j and numbers ε^j. The result is a countable system of subparallelograms of P, say $P_{\mathbf{n}1}, P_{\mathbf{n}2}, \dots$. For every m for which the parallelogram $P_{\mathbf{n}m}$ is the result of Lemma 5 applied for $P^j, I^j, J^j, \varepsilon^j$, let $I_{\mathbf{n}m}$ be the interval I^j.

The set $\{P^j : \ell \in A_{P^j,I^j}\}$ is a Vitali covering of $\ell \cap P$ for every $\ell \in A_{P^j,I^j}$, and, by conclusion (i) of Lemma 5, ℓ must intersect one of the subparallelograms obtained by applying the lemma for $P^j, I^j, J^j, \varepsilon^j$. Thus the set

$$\{P_{\mathbf{n}m} : \ell \in A_{P_{\mathbf{n}m}, I_{\mathbf{n}m}}\}$$

covers a dense open subset of the segment $\ell \cap P$. From this it follows immediately that $\bigcup_{\mathbf{n}=n_1 n_2 \cdots n_k} P_{\mathbf{n}}$ intersects every line $\ell \in L$ in a dense open set, hence the G_δ set

$$A \stackrel{\text{def}}{=} \bigcap_k \bigcup_{\mathbf{n}} P_{\mathbf{n}=n_1 n_2 \cdots n_k}$$

intersects every line $\ell \in L$ in a residual set, thus it satisfies (i) of Lemma 4.

On the other hand, if $\ell \notin L$, then for every fixed sequence $\mathbf{n}$, either ℓ does not hit $P_{\mathbf{n}}$, or the direction of ℓ does not belong to $I_{\mathbf{n}}$. In the latter case, the direction of ℓ does not belong to $\bigcup_{m_1 m_2 \cdots m_l} J_{\mathbf{n}m_1 \cdots m_l}$ provided that l is large enough, and hence $\bigcup_{m_1 m_2 \cdots m_l} P_{\mathbf{n}m_1 \cdots m_l}$ intersects at most $\sum_{m_1 m_2 \cdots m_l} \varepsilon_{\mathbf{n}m_1 \cdots m_l}$ many lines parallel to ℓ. This implies (ii) and completes the proof of Lemma 4, and hence also the proofs of Lemmas 2 and 3.

Proof of Theorem 1. Let $K_1 \subset K_2 \subset \cdots$ be compact subsets of $\mathbb{R}^2 \setminus A$ for which $\mu\left((\mathbb{R}^2 \setminus A) \setminus \bigcup_n K_n\right) = 0$, and let μ_n be the restriction of μ to the set K_n. It is enough to prove that there is a set of lines L_n such that assumptions (i) and (ii) of the first part of Theorem 1 hold for L_n and μ_n instead of L and μ; then (i) and (ii) also hold for $\bigcap_{n=1}^{\infty} L_n$ and μ. That is, we can assume that μ has a compact support K disjoint from A.

We choose countably many open balls $B(x_n, r_n)$ for which

$$A \subset \bigcup_n \big(B(x_n, r_n) \setminus \{x_n\}\big)$$

and $B(x_n, 2r_n) \cap K = \varnothing$ for every n. It is enough to prove that there exists a set of lines L_n such that (i) and (ii) hold for $B(x_n, r_n) \setminus \{x_n\}$ and μ instead of A and μ. That is, we can assume that $A = B(x, r) \setminus \{x\}$ and $B(x, 2r)$ is disjoint from K.

For the sake of brevity, we assume that $A = B(0, 1) \setminus \{0\}$ and $B(0, 2) \cap K = \emptyset$. We apply Lemma 2 for this set A and $x = 0$, and let M be the set of lines thus obtained.

Let M^* denote the analytic set of points covered by the lines of M, and let $M^{**} = M^* \setminus B(0, 1)$. We consider the space

$$(\mathbb{R}^2, \mu) \times (\mathbb{R}, \lambda),$$

where λ is the Lebesgue measure on the line, and we consider the subset

$$\{(x, t) \in \mathbb{R}^2 \times \mathbb{R} : tx \in M^{**}\}.$$

Since M satisfies condition (ii) of Lemma 2, all the vertical sections of this subset have Lebesgue measure zero. This means that almost every horizontal section has measure zero. Therefore we can choose a number u such that $1/2 < u < 1$ and $\mu\big(\{x : ux \in M^{**}\}\big) = 0$, and put $t = 1/u$. So $1 < t < 2$ and $\mu(t\,M^{**}) = 0$. We claim that $t\,M$ satisfies the conditions of Theorem 1.

Since M contains residually many lines through the points of $B(0, 1) \setminus \{0\}$, $t\,M$ contains residually many lines through the points of

$$t\,B(0, 1) \setminus \{0\} \supset B(0, 1) \setminus \{0\} = A,$$

thus (i) holds. On the other hand, since $B(0, 2) \cap K = \emptyset$ and $t < 2$, we have

$$\mu(t\,M^*) = \mu(t\,M^* \cap K) = \mu\big(t\,M^* \setminus B(0, t)\big)$$

and

$$t\,M^* \setminus B(0, t) = t\big(M^* \setminus B(0, 1)\big) = t\,M^{**}.$$

Since $\mu(t\,M^{**}) = 0$, statement (ii) is proved.

3. *Invisible sets*

In this section, all the sets mentioned are analytic, and all the measures are σ-finite Borel measures.

DEFINITION 6. An analytic set is said to be *invisible* if almost every projection of it has measure zero; a set is *visible* if it is not invisible. A set is *invisible from a point* if almost every line through this point does not hit the set.

REMARK 7. Our definition for invisible sets is slightly different from the standard one. Usually, a set is said to be invisible from a point if and only if almost every line through this point does not hit the set, *except possibly for the point itself*. Thus a set is not necessarily visible from its points, while by our definition it must be visible. However, we shall study how large the set from which an invisible set is visible can be, and an invisible set is 'small' (it has capacity zero, is purely unrectifiable, and so on), thus it will make no difference in our results whether or not we count the points of the invisible set itself.

Note that a set A is invisible if and only if A^* has measure zero, where A^* denotes the set of all lines through the points of A. From this observation, the next theorem follows very easily.

THEOREM 8. *For every set $B \subset \mathbb{R}^2$, the following two statements are equivalent.*

(i) *There exists an invisible set A which is visible from all the points of B.*

(ii) *It is possible to put lines through the points of B such that there are lines in positively many directions through all the points of B, but in every direction we have only zero many lines.*

Proof. Condition (i) implies condition (ii) immediately, since if an invisible set A is visible from B, then the set of all lines connecting A and B has this property. On the other hand, if condition (ii) holds for a certain set of lines L, then by the dual of Davies' theorem, there exists a set of points A which intersects all the lines of L (that is, A is visible from B), but the set of all lines intersecting A has the same measure as the measure of L, that is, it is a null set (thus A is invisible).

This theorem implies the following.

THEOREM 9. *For an arbitrary set $B \subset \mathbb{R}^2$ and measure μ on B, the following two statements are equivalent.*

(i) *There is no invisible set which is visible from positively many points of B with respect to μ.*

(ii) *Almost every marginal of μ is absolutely continuous with respect to the Lebesgue measure.*

(Recall that the marginals of a planar measure μ are the linear measures μ_d defined by $\mu_d(A) = \mu(\{P \in \mathbb{R}^2 : \mathrm{Pr}_d(P) \in A\})$, where $A \subset \mathbb{R}$ and Pr_d denotes the projection to a line of direction d.)

COROLLARY 10. *If an invisible set is visible from the points of a set B, then B is purely unrectifiable and has capacity zero.*

Proof. If an invisible set is visible from the points of B, then there is no measure μ on B whose marginals are absolutely continuous. Now, it is easy to see that for every rectifiable set C, the marginals of the 1-dimensional Hausdorff measure on C are absolutely continuous in almost every direction (in fact, they are absolutely continuous in all but countably many directions), so B must be purely unrectifiable. Similarly, all the measures of finite energy have absolutely continuous marginals in almost every direction (see, for example, [**6**, Chapter 9]), thus A also has capacity zero.

COROLLARY 11. *There exists an invisible set A and a visible set B for which A is visible from all the points of B.*

Proof. Let D be a Borel set of directions for which both D and its complement have positive measure, and let L_1 be the set of all lines of directions belonging to D. By the dual of Davies' theorem, there exists a set B which intersects all the lines of L_1, thus B is not invisible. On the other hand, in directions not belonging to D there are only zero many lines intersecting B; let the set of these lines be L_2. Now, B and L_2 satisfy condition (ii) of Theorem 8, thus there exists an invisible set A which is visible from all the points of B.

For the characterisation of those rectifiable sets C_1, C_2 for which there exists a set which is visible from C_1 and invisible from C_2, we first require a definition.

DEFINITION 12. A point P is said to be *surrounded by* C if almost every line through P intersects C. We say that a rectifiable set A is surrounded by C if $\mathscr{H}^1$-almost every point of A is surrounded by C.

THEOREM 13. *Let C_1, C_2 be rectifiable sets. Then the following two statements are equivalent.*

(i) *There exists a Borel set A which is visible from C_1 and invisible from C_2.*

(ii) *C_1 is not surrounded by C_2.*

It is easy to see that all the points of the plane are surrounded by the infinite line, hence this theorem implies that there is no rectifiable set from which an invisible set is visible.

The proof of Theorem 13 is based on the following definition.

DEFINITION 14. Let C be a rectifiable set on the plane, and let $\mathscr{H}^1$ denote the 1-dimensional Hausdorff measure. We denote by D the set of directions, and let λ be the natural measure on D. Let L be a set of lines. We define the measure $\mu_C(L) \stackrel{\text{def}}{=} (\mathscr{H}^1 \times \lambda)(L_*)$, where

$$L_* \stackrel{\text{def}}{=} \{(x, d) \in C \times D : \text{ the line through } x \text{ in direction } d \text{ belongs to } L\}.$$

REMARK 15. One can see immediately that a set A is invisible from a rectifiable set C if and only if $\mu_C(A^*) = 0$, where A^* denotes the set of lines through the points of A. So we have to characterise the rectifiable sets C_1, C_2 for which there exists a Borel set A such that A^* has positive measure with respect to μ_{C_1} and is a null set with respect to μ_{C_2}. First, we characterise the rectifiable sets C_1, C_2 for which there exists a set of lines which has positive measure with respect to μ_{C_1} and zero measure with respect to μ_{C_2}.

THEOREM 16. *Let C_1 and C_2 be rectifiable sets. Then $\mu_{C_1} \ll \mu_{C_2}$ if and only if C_1 is surrounded by C_2.*

Proof. If C_1 is not surrounded by C_2, then through positively many points of C_1 there are lines in positively many directions which do not hit C_2. This is a set of lines which has positive measure with respect to μ_{C_1} and zero measure with respect to μ_{C_2}.

On the other hand, suppose that C_1 is surrounded by C_2, and for a set of lines L we have $\mu_{C_1}(L) > 0$. We show that $\mu_{C_2}(L) > 0$.

Since C_1 is surrounded by C_2, the set of all lines which do not intersect C_1 or C_2 has measure zero with respect to μ_{C_1}, thus we can assume that every element of L intersects both C_1 and C_2. Similarly, there are only countably many lines which intersect C_1 or C_2 in a set of positive length, thus we can assume that all the lines of L intersect C_1 and C_2 in a set of zero length only. Finally, there are only countably many points $c_2 \in C_2$ through which there are lines of L in zero many directions which cover a positive subset of C_1, thus we can assume that C_2 does not contain any point with this property. Now we consider the set

$$\{(c_1, c_2) \in C_1 \times C_2 : \text{ there is an element of } L \text{ through the points } c_1,\, c_2\}$$

with the measure $\mathscr{H}^1 \times \mathscr{H}^1$. Since $\mu_{C_1}(L) > 0$, there are positively many points $c_1 \in C_1$ through which there are lines of L in positively many directions, and since almost every point of C_1 is surrounded by C_2, these intersect C_2 in a set of positive length. That is, the set above has positively many vertical sections of positive measure. Thus there are positively many horizontal sections of positive measure, that is, there are positively many $c_2 \in C_2$ for which the set of lines through c_2 hits C_1 in a positive set. By our assumption above, this implies that we have lines through these points in positively many directions, which is a subset of L of positive measure with respect to μ_{C_2}. This completes our proof.

Proof of Theorem 13. By Remark 15 and Theorem 16, it is enough to show that whenever $\mu_{C_1}(L) > 0$ and $\mu_{C_2}(L) = 0$ for a set of lines L, we have a set A for which the set of lines A^* satisfies $\mu_{C_1}(A^*) > 0$ and $\mu_{C_2}(A^*) = 0$. But this is immediate by applying the first part of Theorem 1 for L and μ_{C_2}: we obtain a set A for which $L \subset A^*$ (thus $\mu_{C_1}(A^*) > 0$) and $\mu_{C_2}(A^*) = \mu_{C_2}(L) = 0$.

References

1. M. Csörnyei, 'On the visibility of invisible sets', *Ann. Acad. Sci. Fenn.*, Math. 25 (2000) 417–421.
2. R. O. Davies, 'On accessibility of plane sets and differentiation of functions of two real variables', *Proc. Cambridge Philos. Soc.* 48 (1952) 215–232.
3. K. J. Falconer, 'Sets with prescribed projections and Nikodym sets', *Proc. London Math. Soc.* (3) 53 (1986) 48–64.
4. J. M. Marstrand, 'Some fundamental geometrical properties of plane sets of fractal dimensions', *Proc. London Math. Soc.* (3) 4 (1954) 257–302.
5. P. Mattila, 'Integralgeometric properties of capacities', *Trans. Amer. Math. Soc.* 266 (1981) 539–554.
6. P. Mattila, *Geometry of sets and measures in Euclidean spaces* (Cambridge University Press, 1995) 130–131.

Department of Mathematics
University College London
Gower Street
London WC1E 6BT

e-mail: mari@math.ucl.ac.uk